ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE Mathématiques

Eric Bedford, Kyounghee Kim

Linear Fractional Recurrences: Periodicities and Integrability

Tome XX, nº S2 (2011), p. 33-56.

<http://afst.cedram.org/item?id=AFST_2011_6_20_S2_33_0>

Annales de la Faculté des Sciences de Toulouse Vol. XX, n° Spécial, 2011
pp. 33–56

Linear Fractional Recurrences: Periodicities and Integrability

Eric Bedford[(1)], Kyounghee Kim[(2)]

Abstract. — Linear fractional recurrences are given as $z_{n+k} = A(z)/B(z)$, where $A(z)$ and $B(z)$ are linear functions of $z_n, z_{n+1}, \ldots, z_{n+k-1}$. In this article we consider two questions about these recurrences: (1) Find $A(z)$ and $B(z)$ such that the recurrence is periodic; and (2) Find (invariant) integrals in case the induced birational map has quadratic degree growth. We approach these questions by considering the induced birational map and determining its dynamical degree. The first theorem shows that for each k there are k-step linear fractional recurrences which are periodic of period $4k$. The second theorem shows that the Lyness process, $A(z) = a + z_{n+1} + z_{n+2} + \cdots + z_{n+k-1}$ and $B(z) = z_{n+1}$ has quadratic degree growth. The Lyness process is integrable, and we discuss its known integrals.

Résumé. — Les récurrences fractionnaires linéaires sont données par $z_{n+k} = A(z)/B(z)$, où $A(z)$ et $B(z)$ sont des fonctions linéaires de $z_n, z_{n+1}, \ldots, z_{n+k-1}$. Dans cet article nous considérons deux questions concernant ces récurrences: (1) Trouver $A(z)$ et $B(z)$ telles que la récurrence soit périodique; (2) Trouver des intégrales invariantes dans le cas où le degré de l'application birationnelle induite a une croissance quadratique. L'approche de ces questions se fait en considérant l'application birationnelle induite et en déterminant ses degrés dynamiques. Le premier théorème montre que pour tout k il y a des récurrences fractionnaires linéaires à k pas qui sont périodiques de période $4k$. Le second théorème montre que le degré du procédé de Lyness $A(z) = a + z_{n+1} + z_{n+2} + \cdots + z_{n+k-1}$ et $B(z) = z_{n+1}$ est à croissance quadratique. Le procédé de Lyness est intégrable, et nous en discuterons les intégrales connues.

(1) Department of Mathematics, Indiana University, Bloomington, IN 47405 USA
bedford@indiana.edu

(2) Department of Mathematics, Florida State University, Tallahassee, FL 32306 USA
kim@math.fsu.edu

0. Introduction

Let $k \geqslant 3$, and let $\alpha = (\alpha_0, \ldots, \alpha_k)$, $\beta = (\beta_0, \ldots, \beta_k)$ be $(k+1)$-tuples of complex numbers. We consider a k-step linear fractional recurrence

$$x_{n+k+1} = \frac{\alpha_0 + \alpha_1 x_{n+1} + \cdots + \alpha_k x_{n+k}}{\beta_0 + \beta_1 x_{n+1} + \cdots + \beta_k x_{n+k}} \tag{0.1}$$

Given a k-tuple $(x_1, \ldots, x_k)$, the relation (0.1) generates a sequence $\{x_j, j \geqslant 1\}$ as long as the denominator does not vanish. The question has been raised (see [GL] and [CL]) to find the α and β for which (0.1) is periodic. By "periodic" we mean that the sequence $\{x_j, j \geqslant 0\}$ is periodic for every starting point $(x_1, \ldots, x_k)$. There are a number of works in the literature that have considered this question under the hypothesis that all numbers are positive. Here we consider it natural to examine this question over the field of complex numbers.

The case $k = 2$ was considered in [BK1] for general α and β, and it was shown that the only possible nontrivial periods are 6, 5, 8, 12, 18, and 30. (Here, "nontrivial" means that the map cannot be reduced to a simpler map, e.g. linear or 1-dimensional.) McMullen [M] observed that these periods are the orders of the Coxeter elements of certain Coxeter groups. The case of dimension 3 is determined in [BK2]: the only possible nontrivial periods are 8 and 12. The 3-step, period 8 maps had been found previously; there are two essentially different maps, one is in [L], and the other is in [CsLa]. Here we show that the period 12 corresponds to a phenomenon that holds for k-step recurrences for all k:

Theorem 0.1. — *For each k, there are k different recurrences of the form (0.1) with α, β as in (5.3), which have period $4k$.*

Our approach is similar to that of [BK1,2]: we consider (0.1) in terms of the associated birational map

$$f_{\alpha,\beta}(x_1, \ldots, x_k) = \left(x_2, \ldots, x_k, \frac{\alpha_0 + \alpha_1 x_1 + \cdots + \alpha_k x_k}{\beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k} \right) \tag{0.2}$$

of k-dimensional space. We may consider $f_{\alpha,\beta}$ as a birational map of complex projective space $\mathbf{P}^k$, as well as any space X which is birationally equivalent to $\mathbf{P}^k$. For a rational map f of X, there are well-defined pull-back maps f^* on the cohomology groups $H^{p,q}(X)$, as well as on the Picard group $Pic(X)$. We may define a notion of growth by

$$\delta(f) := \lim_{n \to \infty} ||(f^n)^*||^{\frac{1}{n}}.$$

Here we work on $Pic(X)$ (or $H^{1,1}$), where $\delta(f)$ is equivalent to degree growth. To determine $\delta(f)$, we replace $\mathbf{P}^k$ with a space X with the property that passage from f to f^* is compatible with iteration. Specifically, we "regularize" the map f in the sense that we replace $\mathbf{P}^k$ by an X such that $(f^n)^* = (f^*)^n$ holds on $Pic(X)$; so in this case we obtain $\delta(f)$ as the modulus of the largest eigenvalue of f^*. The way we find our space X is to analyze the "singular" behavior of f, by which we mean the behavior that prevents $f_{\alpha,\beta}$ from being a diffeomorphism. Namely, there are hypersurfaces E with the property that either $f(E)$ or $f^{-1}(E)$ has codimension > 1. Such a hypersurface is called *exceptional*. The existence/nonexistence of exceptional hypersurfaces depends on the choice of representative X for f, and the regularity of f_X is determined by the behavior of the orbits of exceptional hypersurfaces.

In §2 we show that for generic α and β we have $\delta(f_{\alpha,\beta}) = \Delta_k > 1$. In particular, we conclude that a generic $f_{\alpha,\beta}$ is not periodic. In order to prove Theorem 0.1, we find a space X for which f_X^* is periodic, and we use this to conclude that f is periodic. Although our map f_X is not an automorphism, de Fernex and Ein [dFE] have shown that since f is periodic there will exist a space Z so that the induced map f_Z is biholomorphic.

Next we consider the mappings

$$h(x_1,\ldots,x_k) = \left(x_2,\ldots,x_k,\frac{a+x_2+x_3+\cdots+x_k}{x_1}\right), \tag{0.3}$$

which have been discussed in several places, often under the name of "Lyness map" because of its origin in [L]. Except in two exceptional cases, these maps are not periodic, but they exhibit an integrability which has been studied by several authors: ([KLR], [KL], [Z], [CGM1–3], [GBM], [HKY], [GKI]). Applying our analysis to h we construct a rather different regularization and obtain:

Theorem 0.2. — *If $k > 3$, or if $a \neq 1$, the degree of h^n is quadratic in n.*

In dimension $k = 2$, there is a strong connection (see [DF] and [G]) between polynomial degree growth and integrability. Namely, if g is a birational surface map, then linear degree growth corresponds to preserving a rational fibration; and quadratic degree growth corresponds to preserving an elliptic fibration. In §6 we discuss the structure of rational functions that are invariant under f which were found in [GKI].

We wish to thank the referee for helpful comments on this paper.

1. Birational maps

Let us recall a few notions from algebraic geometry that we will use. The reader is referred to [H] for further details. A rational map of projective space $\mathbf{P}^k$ is given by a $k+1$-tuple of homogeneous polynomials $f = [f_0 : f_1 : \cdots : f_k]$ of a common degree $d = \deg(f_0) = \cdots = \deg(f_k)$. We refer to d as the degree of f. Without loss of generality we assume that the f_j's have no common factor, so the degree is well defined. The indeterminacy locus is $\mathcal{I}(f) = \{x : f_0(x) = \cdots = f_k(x) = 0\}$. Since the f_j's have no common factor, $\mathcal{I}(f)$ always has codimension at least 2. The map f is holomorphic at all points of $\mathbf{P}^k - \mathcal{I}(f)$ but cannot be extended to be continuous at any point of $\mathcal{I}$. If V is a variety for which no irreducible component is contained in $\mathcal{I}$ then the *strict transform* $f(V)$ is defined as the closure of $f(V - \mathcal{I})$. The strict transform is again a variety. We say that a map f is *dominant* if its image contains an open set. Given two rational maps f and g, there is a rational map $f \circ g$, and $f \circ g$ is equal to $f(g(x))$ for all $x \notin \mathcal{I}(g)$ such that $g(x) \notin \mathcal{I}(f)$. The map f is said to be *birational* if there is a rational map g such that $f \circ g$ and $g \circ f$ are both the identity.

If f is a rational map, we say that a subvariety E is *exceptional* if $E \not\subset \mathcal{I}$, and the dimension of $f(E)$ is strictly less than the dimension of E. We let $\mathcal{E}$ denote the set of exceptional hypersurfaces of f. We will say that f is a *pseudo-automorphism* if there is no exceptional hypersurface.

We will define manifolds by the procedure of blowing up. If $p \in X$ is a point, then the *blowup* of X at p is given by a new manifold Y with a holomorphic projection $\pi : Y \to X$ such that $\pi : Y - \pi^{-1}p \to X - p$ is biholomorphic, and $\pi^{-1}p$ is equivalent to $\mathbf{P}^{k-1}$. Similarly, if S is a smooth submanifold of X, we may define a blowup of X along the center S. Given a blowup $\pi : Y \to X$, the preimage $\pi^{-1}S$ of S under π will be called the exceptional blowup fiber. If $f : X \dashrightarrow X$ is a rational map, then there is an induced rational map $f_Y := \pi^{-1} \circ f_X \circ \pi$ on Y.

We refer to a variety of pure codimension 1 as a hypersurface. We say that two hypersurfaces S_1 and S_2 are *linearly equivalent* if there is a rational function r such that the divisor of $\{r = 0\}$ is $S_1 - S_2$. By $Pic(X)$ we denote the set of all divisors modulo linear equivalence. The spaces X that we will deal with all arise from $\mathbf{P}^k$ by blowups, so that in fact $Pic(X)$ is isomorphic to the Dolbeault cohomology group $H^{1,1}(X)$. If $\pi : Y \to X$ is a blowup along a smooth center S, and if $\mathcal{S}$ denotes the exceptional blowup fiber over S, then $Pic(Y)$ is generated by $Pic(X)$, together with the class of $\mathcal{S}$.

Suppose that $f : X \dashrightarrow X$ is a rational map. Given a hypersurface $S = \{p = 0\}$ we define the pullback f^*S as the closure of $\{p(f) = 0\} - \mathcal{I}(f)$.

This gives a well defined linear map f^* of $Pic(X)$. We say that f is *1-regular* if $(f^n)^* = (f^*)^n$ holds on $Pic(X)$. The (first) dynamical degree of f_X is defined by the growth of the iterates on $Pic(X)$:

$$\delta(f_X) := \lim_{n\to\infty} ||(f_X^n)^*|_{Pic(X)}||^{\frac{1}{n}}$$

This is independent of the choice of norm $||\cdot||$. And since $\pi : X \to \mathbf{P}^k$ is holomorphic, we have $(f_X)^n = (\pi^{-1}\circ f\circ\pi)^n = \pi^{-1}\circ f^n\circ\pi = (f^n)_X$, so $\delta(f) = \delta(f_X)$.

2. Linear Fractional Recurrences

We may interpret equation (0.2) as a rational map $f : \mathbf{P}^k \dashrightarrow \mathbf{P}^k$ by writing it in homogeneous coordinates as

$$f[x_0 : \cdots : x_k] = [x_0\beta\cdot x : x_2\beta\cdot x : \ldots : x_k\beta\cdot x : x_0\alpha\cdot x] \tag{2.1}$$

where $\alpha\cdot x = \alpha_0x_0 + \alpha_1x_1 + \cdots \alpha_kx_k$. Let us set $B = (-\alpha_1, 0, \ldots, 0, \beta_1)$, $\alpha' = (\alpha_0, \alpha_2, \ldots, \alpha_k, 0)$ and $\beta' = (\beta_0, \beta_2, \ldots, \beta_k, 0)$. The inverse of f is given by the map

$$f^{-1}[x_0 : \cdots : x_k] = [x_0B\cdot x : x_0\alpha'\cdot x - x_k\beta'\cdot x : x_1B\cdot x : \cdots : x_{k-1}B\cdot x] \tag{2.2}$$

We assume that

$$\begin{aligned}&(\beta_1, \beta_2, \ldots, \beta_k) \neq (0, 0, \ldots, 0),\\ &\alpha \text{ is not a multiple of } \beta, \text{ and}\\ &(\alpha_i, \beta_i) \neq (0, 0) \text{ for } i = 1 \text{ and some } 1 < i \leqslant k.\end{aligned} \tag{2.3}$$

If $(\alpha_1, \beta_1) = (0, 0)$ then f does not depend on x_1 and thus f can be realized as a $k-1$ step recurrence relation. If $(\alpha_i, \beta_i) = (0, 0)$ for all $i = 2, \ldots, k$ then f^k is an essentially 1-dimensional mapping.

Let us set $\gamma = \beta_1\alpha - \alpha_1\beta$ and $C = \beta_1\alpha' - \alpha_1\beta'$. For $0 \leqslant i \leqslant k$ we use notation $\Sigma_i = \{x_i = 0\}$ and $e_i = [0 : \cdots : 0 : 1 : 0 : \cdots : 0]$, the point whose i-th coordinate is nonzero and everything else is zero. We also use $\Sigma_\beta = \{\beta\cdot x = 0\}, \Sigma_\gamma = \{\gamma\cdot x = 0\}, \Sigma_B = \{B\cdot x = 0\}$, and $\Sigma_C = \{C\cdot x = 0\}$. To indicate the intersection we combine their subscripts, for example $\Sigma_{0\beta} = \{x_0 = \beta\cdot x = 0\}$ and $\Sigma_{01} = \{x_0 = x_1 = 0\}$. The Jacobian of f is a constant multiple of $x_0(\beta\cdot x)^{k-1}(\gamma\cdot x)$. The Jacobian vanishes on three hypersurfaces $\Sigma_0, \Sigma_\beta, \Sigma_\gamma$; these hypersurfaces are exceptional and are mapped to the lower dimensional linear subspaces:

$$f(\Sigma_0) = \Sigma_{0B} = \Sigma_{0k}, \quad f(\Sigma_\beta) = e_k, \quad f(\Sigma_\gamma) = \Sigma_{BC}.$$

The Jacobian of f^{-1} is a constant multiple of $x_0(B\cdot x)^{k-1}(C\cdot x)$ and we have

$$f^{-1}:\Sigma_0\mapsto\Sigma_{0\beta},\quad \Sigma_B\mapsto e_1,\quad \text{and}\quad \Sigma_C\mapsto\Sigma_{\beta\gamma}.$$

The indeterminacy locus of f is $\mathcal{I}^+=\{e_1,\Sigma_{0\beta},\Sigma_{\beta\gamma}\}$ and f^{-1} is $\mathcal{I}^+=\{e_k,\Sigma_{0B},\Sigma_{BC}\}$.

Let us consider the maps which satisfy (2.3) and the following:

$$\beta_1\neq 0\qquad\text{and}\qquad \beta_1\alpha_j-\alpha_1\beta_j\neq 0\ \text{ for all }\ j=2,\ldots,k \tag{2.4}$$

For every choice of parameters α,β satisfying (2.3–4), we have

$$f:\Sigma_0\mapsto\Sigma_{0k}\mapsto\Sigma_{0\,k-1\,k}\mapsto\cdots\mapsto\Sigma_{0\,3\ldots k}\mapsto e_1\rightsquigarrow\Sigma_B \tag{2.5}$$

We first modify the orbit of Σ_0. Let $\pi:Y\to\mathbf{P}^k$ be the complex manifold obtained by blowing up e_1 and then $\Sigma_{0\,3\ldots k}$ and continuing successively until we reach $\Sigma_{0\,k}$. That is, we let $\Sigma_{3\ldots k}$ denote its strict transform in the space obtained by blowing up e_1, and then we blow up along the center $\Sigma_{3\ldots k}$, etc. We let $E_1=\pi^{-1}e_1$ denote the exceptional fiber over e_1, and let $\mathcal{S}_{0,j}$ denote the exceptional fiber over $\Sigma_{0\,j\ldots k}$ for all $j\geqslant 3$. Let us set $\alpha^{(j)}=(0,\alpha_1,\ldots,\alpha_j,0,\ldots,0)$, and $\beta^{(j)}=(0,\beta_1,\ldots,\beta_j,0,\ldots,0)$. For $3\leqslant j\leqslant k$ we use local coordinates near $\mathcal{S}_{0\,j}$

$$\pi_{sj}:(s,x_2,\ldots,x_{j-1},\xi_j,\ldots,\xi_k)_{s_j}\mapsto[s:1:x_2:\cdots:x_{j-1}:s\xi_j:\cdots:s\xi_k]\in\mathbf{P}^k$$

and for the neighborhood of the exceptional divisor E_1 we use

$$\pi_{e_1}:(s,\xi_2,\ldots,\xi_k)_{e_1}\mapsto[s:1:s\xi_2:\cdots:\xi_k]\in\mathbf{P}^k.$$

Let us use subscripts to indicate which local coordinates we are using; for example $(\cdots)_{s_j}$ is a coordinate of a point in local coordinates near $\mathcal{S}_{0\,j}$. Working with the induced birational map $f_Y:=\pi^{-1}\circ f\circ\pi$ we have

$$f_Y:\Sigma_0\ni[0:x_1:x_2:\cdots,x_k]\mapsto\left(0,x_2,\ldots,x_k,\frac{\alpha^{(k)}\cdot x}{\beta^{(k)}\cdot x}\right)_{s_k}\in\mathcal{S}_{0\,k}$$

Similarly we have for all $j=2,\ldots,k-1$

$$\begin{aligned}f_Y:\mathcal{S}_{0,j+1}\ni&(0,x_2,\ldots,x_j,\xi_{j+1}\ldots,\xi_k)_{s_{j+1}}\\ &\mapsto\left(0,x_3/x_2,\ldots,x_j/x_2,\xi_{j+1},\ldots,\xi_k,\frac{\alpha^{(k)}\cdot x}{\beta^{(k)}\cdot x}\right)_{s_j}\in\mathcal{S}_{0\,j}\end{aligned}$$

For the points of E_1, we have

$$f_Y:E_1\ni(0,\xi_2,\ldots,\xi_k)_{e_1}\mapsto[\beta_1:\beta_1\xi_2:\cdots:\beta_1\xi_k:\alpha_1]\in\Sigma_B.$$

By condition (2.4), we see that $\alpha^{(j)}$ is not a constant multiple of $\beta^{(j)}$ for all $2 \leqslant j \leqslant k$. It follows that

Lemma 2.1. — *The map f_Y is a local diffeomorphism at generic points of Σ_0, E_1, and $S_{0,j+1}$ for $2 \leqslant j \leqslant k-1$.*

Since the induced map f_Y is a local diffeomorphism at points of $\Sigma_0 \cup E_1 \cup \bigcup_{j=3}^{k} \mathcal{S}_{0,j}$, Σ_0 and all the exceptional (blowup) divisors E_1 and $\mathcal{S}_{0,j}$ for $j = 3, \ldots, k$ are not exceptional for f_Y. Thus the exceptional set for f_Y consists of two divisors: $\mathcal{E}_Y^+ = \{\Sigma_\beta, \Sigma_\gamma\}$. The indeterminacy locus for f_Y is $\mathcal{I}_Y^+ = \{\Sigma_{0\beta}, \Sigma_{\beta\gamma}\}$. For the inverse map f_Y^{-1} we have $\mathcal{E}_Y^- = \{\Sigma_0, \Sigma_C\}$ and $\mathcal{I}_Y^- = \{\{e_k\}, \Sigma_{BC}\}$.

Let us consider the ordered basis of $Pic(Y) : H, E_1, \mathcal{S}_{0,3}, \ldots, \mathcal{S}_{0,k}$. Using the discussion above, we have

Lemma 2.2. — *With the given ordered basis the action of f_Y^* on $Pic(Y)$ is given by*

$$\begin{aligned} f_Y^* :\ & E_1 \mapsto \mathcal{S}_{0,3} \mapsto \cdots \mapsto \mathcal{S}_{0,k} \\ & \mathcal{S}_{0,k} \mapsto \{\Sigma_0\} = H - E_1 - \mathcal{S}_{0,3} - \cdots - \mathcal{S}_{0,k} \\ & H \mapsto 2H - E_1. \end{aligned}$$

Under the same ordered basis the action of f_Y^{-1} is:

$$\begin{aligned} f_Y^{-1^*} :\ & \mathcal{S}_{0,k} \mapsto \mathcal{S}_{0,k-1} \mapsto \cdots \mapsto \mathcal{S}_{0,3} \mapsto E_1 \\ & E_1 \mapsto \{\Sigma_B\} = H - E_1 - \mathcal{S}_{0,3} - \cdots - \mathcal{S}_{0,k} \\ & H \mapsto 2H - E_1 - \mathcal{S}_{0,3} - \cdots - \mathcal{S}_{0,k}. \end{aligned}$$

Now let us consider the following condition on exceptional hypersurfaces

$$\begin{array}{lll} f_Y^n \mathcal{E} \not\subset \Sigma_{\beta\gamma} \cup \Sigma_{0\beta} & \text{for all } n \geqslant 0 & \mathcal{E} = \Sigma_\beta, \Sigma_\gamma \\ f_Y^{-n} \mathcal{E} \not\subset \Sigma_{BC} \cup \{e_k\} & \text{for all } n \geqslant 0 & \mathcal{E} = \Sigma_0, \Sigma_C \end{array} \tag{2.6}$$

When (2.6) holds, f_Y is regular in the sense that $(f_Y^*)^n = (f_Y^n)^*$ holds on $Pic(Y)$, and similarly for f_Y^{-1}. Thus the dynamical degree $\delta(f)$ is the spectral radius of f_Y^* acting on $Pic(Y)$.

Theorem 2.3. — *For generic parameters, the dynamical degrees satisfy the properties:*

(i) $\delta(f) = \Delta_k^+$ is the largest root of $x^k - (x^k - 1)/(x - 1)$,

(ii) $\delta(f^{-1}) = \Delta_k^-$ is the largest root of $x^k - x^{k-1} + (-1)^{k-1}$.

Furthermore we have

$$\lim_{k\to\infty} \Delta_k^+ = 2, \quad \text{and} \quad \lim_{k\to\infty} \Delta_k^- = 1.$$

Proof. — It is clear that for a generic map f, the three exceptional hypersurfaces $\{\Sigma_0, \Sigma_\beta, \Sigma_\gamma\}$ are in general position, that is, the intersection of all three is a point. Next we claim that (2.6) holds for a generic map. Since (2.6) defines the complement of countably many varieties inside $\mathbf{C}^{2k+2}$, it suffices to show that there is one parameter value (α, β) for which (2.6) holds. For each $k \geqslant 3$ let us consider $\alpha = (0, 1, 0, \dots, 0)$ and $\beta = (0, 1, \dots, 2-k)$. In this case $\gamma = (0, 0, -1, \dots, k-2)$ and $B = (1, 0, \dots, 0, -1)$ and $C = (0, -1, \dots, -1, k-2, 0)$. It follows that

$$f_Y \Sigma_\gamma = \Sigma_{B\,C} \ni [1:1:\cdots:1] \quad \text{and} \quad f_Y[1:\ldots:1] = [1:\ldots:1]$$

We also have $f_Y^{k+1} : \Sigma_\beta = [1:0:\cdots:0:1]$ and

$$\begin{aligned} f_Y : [1:0:\cdots:0:1] &\mapsto [1:0:\cdots:0:1:0] \mapsto \cdots \\ &\mapsto [1:1:0:\cdots:0] \mapsto [1:0:\cdots:0:1] \end{aligned}$$

Thus Σ_β is pre-periodic and there exists a fixed point in $\Sigma_\gamma \setminus \Sigma_\beta$. It follows that this mapping satisfies the the first half of (2.6). The second part of condition (2.6) follows similarly.

The matrix representation of f_Y^* and f_Y^{-1*} is given by $k \times k$ matrices:

$$f_Y^* = \begin{pmatrix} 2 & 0 & \cdots & 0 & 1 \\ -1 & 0 & & & -1 \\ 0 & 1 & & & \vdots \\ & 0 & \ddots & & \vdots \\ 0 & & 0 & 1 & -1 \end{pmatrix}, \quad f_Y^{-1*} = \begin{pmatrix} 2 & 1 & & & 0 \\ -1 & -1 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ \vdots & \vdots & & & 1 \\ -1 & -1 & 0 & 0 & 0 \end{pmatrix} \tag{2.7}$$

The dynamical degrees Δ_k^+ and Δ_k^- are given by the spectral radius of the above matrix representations. Now (2.6) implies that f_Y is regular, so the spectral radius of f_Y^* gives the dynamical degree. The polynomials in statements (i) and (ii) are the characteristic polynomials of these matrices. $\square$

Corollary 2.4. — *For every f of the form (0.2), we have $\delta(f) \leqslant \Delta_k^+$ and $\delta(f^{-1}) \leqslant \Delta_k^-$.*

Proof. — Let f_Y^* be the matrix representation given in (2.7) and let m_n denote the $1,1$ entry of $(f_Y^*)^n$. For generic α, β we have that degree$(f_{\alpha,\beta}^n) = m_n$. By the lower semicontinuity of the degree, we have for all $n \geqslant 1$

$$(\text{degree}(f_{\alpha,\beta}^n))^{1/n} \leqslant m_n^{1/n} \quad \text{for all } \alpha, \beta.$$

It follows that $\delta(f_{\alpha,\beta}) \leqslant \lim_{n\to\infty} m_n^{1/n} = \Delta_k^+$ for all α, β. $\square$

3. Non-Periodicity

We call the set of exceptional hypersurfaces $\{\Sigma_0, \Sigma_\beta, \Sigma_\gamma\}$ the critical triangle. When these three hypersurfaces are distinct, we say the critical triangle is nondegenerate.

LEMMA 3.1. — *If $\beta_1 \neq 0$ and $\beta_1\alpha_j - \alpha_1\beta_j = 0$ for all $j \neq 0$ then there is unique exceptional hypersurface. If $\beta_1 = 0$, then there are only two distinct exceptional hypersurfaces. Otherwise the critical triangle is nondegenerate.*

Proof. — Using the condition (2.3) we see that $\Sigma_0 \neq \Sigma_\beta$ and $\Sigma_B \neq \Sigma_C$. It follows that f and f^{-1} has at least two distinct exceptional hypersurfaces. If $\beta_1 = 0$, we have $\gamma = -\alpha_1\beta$ and $B = (-\alpha_1, 0, \ldots, 0)$. It follows that $\Sigma_\beta = \Sigma_\gamma$ and $\Sigma_0 = \Sigma_B$. If $\beta_1 \neq 0$ and $\beta_1\alpha_j - \alpha_1\beta_j = 0$ for all $j \neq 0$, then $\gamma = C = \beta_1(\alpha_0, 0, \ldots, 0)$ and therefore $\Sigma_0 = \Sigma_\gamma = \Sigma_C$. $\square$

For $j = 2, \ldots, k$ let us consider the codimension $k-j+2$ linear subspaces

$$\mathcal{L}_j := \Sigma_{0\beta} \cap \bigcap_{\ell=j}^{k} \{\beta_1 x_{k-\ell+1} + \beta_2 x_{k-\ell+2} + \cdots \beta_j x_{k-\ell+k} = 0\}.$$

LEMMA 3.2. — *Let j_* be the largest integer such that $\beta_j \neq 0$. If $j_* > 1$ then $e_k \notin \mathcal{L}_{j_*}$ and Σ_0 is pre-fixed under f^{-1}:*

$$f^{-(k-j_*+1)}\Sigma_0 = \mathcal{L}_{j_*}, \quad f^{-1}\mathcal{L}_{j_*} = \mathcal{L}_{j_*}.$$

In case $j_* = 1$ we have

$$f^{-1} : \Sigma_0 \mapsto \Sigma_{0\,1} \mapsto \Sigma_{0\,1\,2} \mapsto \cdots \mapsto e_k \rightsquigarrow \Sigma_\beta.$$

Proof. — If $j_* > 1$, then $e_k \notin \mathcal{L}_{j_*}$ is a consequence of the condition $\beta_{j_*} \neq 0$. The second part is an immediate consequence of (2.5).To show that $\mathcal{L}_{j_*}$ is fixed under f^{-1}, first notice that $\mathcal{L}_{j_*}$ has codimension $2+k-j_*$. A generic point $p \in \mathcal{L}_{j_*}$ can be written in terms of $x_{k-j_*+2}, \ldots, x_k$ and $x_{k-j_*+1} = -(\beta_2 x_{k-j_*+2} + \cdots + \beta_{j_*} x_k)$. It follows that the image of this point $f^{-1}p$ is $[0 : y_1 : \cdots : y_k]$ where $y_i = x_{i-1}$ for $i \geqslant 2$ and there for codimension of $f^{-1}\mathcal{L}_{j_*}$ is $2 + k - j_*$. When $j_* = 1$, $\mathcal{L}_j = \Sigma_{0\,1\cdots\,k+1-j}$ for $j = 1, \ldots, k$. $\square$

If the mapping f is periodic with period p then f^{-1} is also periodic and for every hypersurface H in $\mathbf{P}^k$ $f^p H = H$ and therefore the codimension of $f^p H$ has to be equal to 1. Thus we have

COROLLARY 3.3. — *If $\beta_j \neq 0$ for some $j \geqslant 2$, then f is not periodic.*

4. Critical Case

We say f is *critical* if $\beta_j = 0$ for all $j > 1$ and the critical triangle is non-degenerate. Using (2.4) we may also set $\alpha_k = 1$, and by Lemma 2.1, we may assume that

$$\alpha = (\alpha_0, 0, \alpha_2, \ldots, \alpha_{k-1}, 1) \quad \text{and} \quad \beta = (\beta_0, 1, 0, \ldots, 0), \alpha_2 \cdots \alpha_{k-1} \neq 0. \tag{4.1}$$

Let us consider the involution $\tau[x_0 : x_1 : \cdots : x_k] = [x_0 : x_k : \cdots : x_1]$ gotten by interchanging the variables $x_j \leftrightarrow x_{k-j+1}$, $1 \leqslant j \leqslant k$. We see that f is reversible in the sense that $f^{-1} = \tau \circ f \circ \tau$. If (4.1) holds, we have

$$\gamma = \beta_1 \alpha - \alpha_1 \beta = \alpha, \quad B = (0, \ldots, 0, 1), \quad \text{and} \quad C = \beta_1 \alpha' - \alpha_1 \beta' = \alpha'.$$

When the mapping is critical, we use the conjugacy by τ and apply Lemma 2.2 to f^{-1} to obtain:

$$f_Y : \Sigma_\beta \mapsto e_k \rightsquigarrow \Sigma_{0\,1\ldots\,k-2} \rightsquigarrow \Sigma_{0\,1\ldots,k-3} \rightsquigarrow \cdots \rightsquigarrow \Sigma_{0\,1} = \Sigma_{0\beta} \rightsquigarrow \Sigma_0 \tag{4.2}$$

where e_k, $\Sigma_{0\,1\ldots\,k-2}$, ... , $\Sigma_{0\,1}$ are the strict transforms in Y of the corresponding linear subspaces in $\mathbf{P}^k$.

Let us consider a complex manifold $\pi_X : X \to Y$ obtained by a successive blowing up the sets e_k, $\Sigma_{0\,1\ldots\,k-2}$, ..., $\Sigma_{0\,1}$. We denote the exceptional divisors over e_k, $\Sigma_{0\,1\ldots\,k-2}$, ..., $\Sigma_{0\,1}$ be E_k, $\mathcal{P}_{0,k-2}$, ..., $\mathcal{P}_{0,1}$. We will show that the induced maps on the blowup fibers are dominant:

$$f_X : \Sigma_\beta \to E_k \mapsto \mathcal{P}_{0,k-2} \mapsto \cdots \mapsto \mathcal{P}_{0,1} \mapsto \Sigma_0 \mapsto \mathcal{S}_{0,k} \mapsto \cdots \mapsto \mathcal{S}_{0,3} \mapsto E_1 \mapsto \Sigma_B. \tag{4.3}$$

For this, we work with local coordinates. Near E_k we use

$$\pi_{e_k} : (s, \xi_1, \ldots, \xi_{k-1})_{e_k} \mapsto [s : s\xi_1 : \cdots : s\xi_{k-1} : 1]$$

and near $\mathcal{P}_{0,j}$, $1 \leqslant j \leqslant k-2$, we use:

$$\pi_{pj} : (s, \xi_1, \ldots, \xi_j, x_{j+1}, \ldots, x_{k-1})_{p_j} \mapsto [s : s\xi_1 : \cdots : s\xi_j : x_{j+1} : \cdots : x_{k-1} : 1]$$

so that $\{s = 0\} = \mathcal{P}_{0,j}$. Then the induced birational map f_X acts on Σ_β and the exceptional divisors as follows:

$$f_X : \Sigma_\beta \ni [x_0 : -\beta_0 x_0 : x_2 : \cdots : x_k] \mapsto (0, x_2/x_0, \ldots, x_k/x_0)_{e_k} \in E_k$$
$$E_k \ni (0, \xi_1, \ldots, \xi_k)_{e_k} \mapsto (0, \xi_2, \ldots, \xi_{k-1}, \beta_0 + \xi_1)_{p_{k-2}} \in \mathcal{P}_{0,k-2}$$

For all $2 \leqslant j \leqslant k-2$

$$f_X : \mathcal{P}_{0,j} \ni (0, \xi_1, \ldots, \xi_j, x_{j+1}, \ldots, x_{k-1})_{p_j}$$
$$\mapsto (0, \xi_2, \ldots, \xi_j, \frac{\beta \cdot \xi \, x_{j-1}}{\alpha \cdot x}, \ldots, \frac{\beta \cdot \xi \, x_{k-1}}{\alpha \cdot x})_{p_{j-1}} \in \mathcal{P}_{0,j-1}$$

where $\beta \cdot \xi = \beta_0 + \xi_1$ and $\alpha \cdot x = \alpha_{j+1}x_{j+1} + \cdots + \alpha_{k-1}x_{k-1} + \alpha_k$. And we also have

$$\begin{aligned} f_X : \mathcal{P}_{0,1} \ni & (0, \xi_1, x_2, \ldots, x_{k-1})_{p_1} \\ & \mapsto [0 : x_2\beta \cdot \xi : \cdots : x_{k-1}\beta \cdot \xi : \beta \cdot \xi : \alpha_2 x_2 + \cdots + \alpha_{k-1}x_{k-1} + \alpha_k] \in \Sigma_0 \end{aligned}$$

On the other hand the induced map f_X^{-1} acts as follows:

$$\begin{aligned} f_X^{-1} : \ & \Sigma_0 \ni [0 : x_1 : \cdots : x_k] \mapsto (0, (\alpha' \cdot x - \beta_0 x_k)/x_k, x_1/x_{k-1}, \ldots, x_{k-2}/x_{k-1})_{p_1} \\ & \in \mathcal{P}_{0,1} \\ & E_k \ni (0, \xi_1, \ldots, \xi_{k-1})_{e_k} \mapsto [1 : -\beta_0 : \xi_1 : \cdots : \xi_{k-1}] \in \Sigma_\beta \end{aligned}$$

Hence f_X is a local diffeomorphism at points of $\Sigma_\beta \cup E_k \cup \bigcup_{j=1}^{k-2} \mathcal{P}_{0,j}$ and f_X^{-1} is a local diffeomorphism at points of $\Sigma_0 \cup E_k \cup \bigcup_{j=1}^{k-2} \mathcal{P}_{0,j}$. It follows that

Lemma 4.1. — *In the critical case, the induced map f_X has only one exceptional hypersurface Σ_γ; and $\Sigma_{\beta\gamma}$ is the only component of the indeterminacy locus $\mathcal{I}(f_X)$ which blows up to a hypersurface.*

Lemma 4.2. — *Suppose that f is critical. Then with the ordered basis of $Pic(X)$: H, E_1, $\mathcal{S}_{0,3}$, …, $\mathcal{S}_{0,k}$, $\mathcal{P}_{0,1}$, …, $\mathcal{P}_{0,k-2}$ E_k, the action of f_X^* on $Pic(X)$ is given by*

$$\begin{aligned} f_X^* : \ & E_1 \mapsto \mathcal{S}_{0,3} \mapsto \cdots \mapsto \mathcal{S}_{0,k} \\ & \mathcal{S}_{0,k} \mapsto \{\Sigma_0\} = H - E_1 - \mathcal{S}_{0,3} - \cdots - \mathcal{S}_{0,k} - \mathcal{P}_{0,1} - \cdots - \mathcal{P}_{0,k-2} - E_k \\ & \mathcal{P}_{0,1} \mapsto \mathcal{P}_{0,2} \mapsto \cdots \mapsto \mathcal{P}_{0,k-2} \mapsto E_k \\ & E_k \mapsto \{\Sigma_\beta\} = H - \mathcal{P}_{0,1} - \cdots - \mathcal{P}_{0,k-2} - E_k \\ & H \mapsto 2H - E_1 - \mathcal{P}_{0,1} - \cdots - \mathcal{P}_{0,k-2} - E_k. \end{aligned} \tag{4.4}$$

The action on cohomology ${f_X^{-1}}^*$ is similar. In fact, the matrix representations for f_X^* and ${f_X^{-1}}^*$ are the same up to the order of basis. Furthermore the spectral radius is given by the largest root of $x^{2k-1} - (x^k - 1)/(x - 1)$.

Let Δ_k^c denote the largest root of the polynomial $x^{2k-1} - (x^k - 1)/(x - 1)$. We note that Δ_k^c decreases to 1 as $k \to \infty$. Using the fact that the degree of n-th iterate is lower semi-continuous, as in the proof of Corollary 2.4, we have the following:

Theorem 4.3. — *If f is critical, then $\delta(f), \delta(f^{-1}) \leqslant \Delta_k^c$.*

5. Periodic Mappings

Let us suppose f is critical for the rest of the paper. If f is periodic, then for every exceptional hypersurface E, there is an $n \geqslant 0$ such that the codimension of $f^{n+1}E = 1$, that is, $f^{n+1}E \subset \mathcal{I}(f)$ and $f^{n+1}E = f^{-1}E'$ where E' is an exceptional hypersurface of f^{-1}. Since f is critical and periodic, then by Lemma 4.1 there is an $n \geqslant 0$ such that $f_X^{n+1}\Sigma_\gamma = \Sigma_{\beta\gamma}$. Let us note that it is conceivable that there are $0 \leqslant n_1 < \ldots < n_\mu = n$ such that $f_X^{n_j+1}\Sigma_\gamma \subset \mathcal{I}(f_X)$ and $f_X^{n_j+1}\Sigma_\gamma \neq \Sigma_{\beta\gamma}$ for $j = 1, \ldots, \mu - 1$. Such possibilities are discussed in [BK2]; in this paper we only consider the simplest possibility, $\mu = 1$.

In this section, we consider the induced birational mapping f_X such that the orbit of Σ_γ ends up with $\Sigma_{\beta\gamma}$, that is for some $n_\star \geqslant 0$

$$f_X : \Sigma_\gamma \mapsto \Sigma_{BC} \mapsto f_X\Sigma_{BC} \mapsto \cdots \mapsto f^{n_\star}\Sigma_{BC} = \Sigma_{\beta\gamma}$$

and $f_X^j\Sigma_{BC} \not\subset \Sigma_0 \cup \Sigma_\beta \cup \Sigma_\gamma$ for $j = 0, \ldots, n_\star - 1$. Let $\pi_Z : Z \to X$ be the complex manifold obtained by blowing up the orbit of $\Sigma_{BC} = f_X\Sigma_\gamma$, and let $\mathcal{F}_j$ denote the exceptional divisor over $f_X^{j-1}\Sigma_{BC}$ for $j = 1, \ldots, n_\star + 1$.

Lemma 5.1. — *If there exists a positive integer $n_\star$ such that $f_X^{n_\star}\Sigma_{BC} = \Sigma_{\beta\gamma}$ and codimension($f_X^j\Sigma_{BC}$) = 2 for all $j = 1, \ldots, n_\star$, the dynamical degree is given by the largest root of the polynomial*

$$\chi_{k,n_\star}(x) = (x^{1+2k+n_\star} - x^{2k+n_\star} - x^{1+k+n_\star} + x^{1+n_\star} + x^{2k} - x^k - x + 1)/(x-1). \tag{5.1}$$

Proof. — Since f_X is well defined on $\Sigma_{BC}, \ldots, f^{n_\star-1}\Sigma$, it suffices to check the mapping on Σ_γ and $\Sigma_{\beta\gamma}$. By the induced map f_Z the generic point on Σ_γ map to a point on $\mathcal{F}_1$:

$$\begin{aligned} f_Z : \Sigma_\gamma \ni\ & [x_0 : x_1 : \cdots : x_{k-1} : -\alpha_0 x_0 - \alpha_2 x_2 - \cdots - \alpha_{k-1}x_{k-1}] \\ & \mapsto (x_2/x_0, \ldots, x_{k-1}/x_0, 0, x_0/(\beta_0 x_0 + x_1))_{\mathcal{F}_1} \in \mathcal{F}_1 \end{aligned}$$

where we use local coordinates near $\mathcal{F}_1$:

$$\begin{aligned} \pi_{\mathcal{F}_1} : & (x_1, \ldots, x_{k-2}, s, \xi_k)_{\mathcal{F}_1} \\ & \mapsto [1 : x_1 : \cdots : x_{k-2} : -\alpha_0 - \alpha_2 x_1 - \cdots - \alpha_{k-1}x_{k-2} + s : s\xi_k] \end{aligned}$$

Also under the inverse map f_Z^{-1} we have

$$\begin{aligned} f_Z^{-1} : \quad \Sigma_C \ni\ & [x_0 : \cdots : x_{k-2} : -\alpha_0 x_0 - \alpha_2 x_1 - \cdots - \alpha_{k-1}x_{k-2} : x_k] \\ & \mapsto (x_0, 0, x_2, \ldots, x_{k-2}, x_k/x_0)_{\mathcal{F}_{n_\star}} \in \mathcal{F}_{n_\star} \end{aligned}$$

where we use a local coordinates near $\mathcal{F}_{n_\star}$:

$$\pi_{\mathcal{F}_{n_\star}} : (x_0, s, x_2, \dots, x_{k-2}, \xi_k)_{\mathcal{F}_{n_\star}}$$
$$\mapsto [x_0 : -\beta_0 x_0 + s : x_2 : \cdots : x_{k-2} : -\alpha_0 - \alpha_2 x_2 - \cdots - \alpha_{k-1} x_{k-1} : s\xi_k].$$

It follows that f_Z is a local diffeomorphism at points on $\Sigma_\gamma \cup \bigcup_{j=1}^{n_\star} \mathcal{F}_j$. Furthermore f_Z doesn't have any exceptional hypersurfaces and therefore f_Z is 1-regular. To compute the action on $Pic(Z)$ let us use the ordered basis H, E_1, $\mathcal{S}_{0,3}$, …, $\mathcal{S}_{0,k}$, $\mathcal{P}_{0,1}$, …, $\mathcal{P}_{0,k-2}$ E_k, $\mathcal{F}_{n_\star}$, …, $\mathcal{F}_1$. The action of f_Z^* on $Pic(Z)$ is given by

$$\begin{aligned} f_X^* : \ & E_1 \mapsto \mathcal{S}_{0,3} \mapsto \cdots \mapsto \mathcal{S}_{0,k} \\ & \mathcal{S}_{0,k} \mapsto \{\Sigma_0\} = H - E_1 - \mathcal{S}_{0,3} - \cdots - \mathcal{S}_{0,k} - \mathcal{P}_{0,1} - \cdots - \mathcal{P}_{0,k-2} - E_k \\ & \mathcal{P}_{0,1} \mapsto \mathcal{P}_{0,2} \mapsto \cdots \mapsto \mathcal{P}_{0,k-2} \mapsto E_k \\ & E_k \mapsto \{\Sigma_\beta\} = H - \mathcal{P}_{0,1} - \cdots - \mathcal{P}_{0,k-2} - E_k \\ & \mathcal{F}_{n_\star} \mapsto \cdots \mapsto \mathcal{F}_1 \mapsto \{\Sigma_\gamma\} = H - E_1 - \mathcal{F}_{n_\star} \\ & H \mapsto 2H - E_1 - \mathcal{P}_{0,1} - \cdots - \mathcal{P}_{0,k-2} - E_k. \end{aligned} \tag{5.2}$$

The spectral radius of the action given by (5.2) is the largest root of $\chi_{k,n_\star}(x)$ □

Lemma 5.2. — *If $n_\star > (k^2+k)/(k-1)$ then f has exponential degree growth.*

Proof. — The derivative of χ at $x = 1$ is negative if $n_\star > (k^2+k)/(k-1)$. It follows that if $n_\star > (k^2+k)/(k-1)$ then χ has a real root which is strictly bigger than 1. □

Lemma 5.3. — *For a critical map, $n_\star \geqslant k-1$.*

Proof. — Since we have $\Sigma_{BC} \subset \Sigma_k$, $\Sigma_{\beta\gamma} \not\subset \Sigma_j$, for $j \geqslant 2$, and $f : \Sigma_k \mapsto \Sigma_{k-1} \mapsto \cdots \mapsto \Sigma_1$, it requires at least $k-1$ iterations for Σ_{BC} to be mapped to $\Sigma_{\beta\gamma}$. □

Lemma 5.4. — *If $k > 3$ and $n_\star > k+2$ then the dynamical degree is strictly bigger than 1. If $k = 3$, the dynamical degree for $n_\star = 6 = (k^2+k)/(k-1)$ is equal to 1 and the dynamical degree for $n_\star \geqslant 7$ is strictly bigger than 1.*

Proof. — The second derivative of $(x-1)\chi_{k,n_\star}(x)$ at $x = 1$ is $2((1-k)n_\star + k(k+1))$. It follows that $\chi'_{k,n_\star}(1) < 0$ and therefore the dynamical degree is strictly bigger than 1 if and only if $n_\star > (k^2+k)/(k-1)$. Since $k+3 = (k^2+k+(k-3))/(k-1)$, $k+3 > (k^2+k)/(k-1)$ if $k > 3$ and $k+3 = (k^2+k)/(k-1)$ if $k = 3$. □

To discuss the case $n_\star = k+2$, we will look at

$$\chi = \chi_{k,k+2} = x^{3k+2} + (1-x^k)x^k(x^2+x+1) - 1.$$

Using Φ_n to denote the n-th cyclotomic polynomial, we factor $\chi_{k,k+2}$ for small k: $\chi_{0,2} = (x-1)\Phi_2$, $\chi_{1,3} = (x-1)\Phi_8(x)$, $\chi_{2,4} = (x-1)\Phi_{18}$, $\chi_{3,5} = (x-1)\Phi_4 \cdot \Phi_{30}$.

THEOREM 5.5. — *If $k \geqslant 3$ and $n_\star = k+2$, then f is not periodic. In fact, if $k > 3$, and $n_\star = k+2$, then f has exponential degree growth.*

Using computer, we can check that $f^5\Sigma_{BC} \not\subset \Sigma_\beta$. Thus Theorem 5.5 is valid in the case $k=3$ because we do not have $n_\star = k+2$. Before we begin the proof, we make an observation about $\chi(x)$:

LEMMA 5.6. — *Suppose that η is a primitive m-th root of unity, and is a root of $\chi_{k,k+2}$ for $k \geqslant m$. Then, in the following cases, η is a simple root:*

(i) $m = 2$, $k \equiv 0$ *mod* m
(ii) $m = 8$, $k \equiv 1$ *mod* m
(iii) $m = 18$, $k \equiv 2$ *mod* m
(iv) $m = 5$, *and* $k \equiv 3$ *mod* m
(v) $m = 30$, $k \equiv 3$ *mod* m.

Proof. — Let us consider, for instance, case (v), i.e., $m = 30$ and show that $\chi'(\eta) \neq 0$. To do this, we may substitute $x = \eta^k = \eta^3$ into the formula for $\chi'(x)$, and we have $\chi'(\eta) = c_0(\eta) + c_1(\eta)k$, where c_0 and c_1 are polynomials of degree 11 in η. Using the computer, we can determine the minimal polynomials of $-c_0(\eta)/c_1(\eta)$, where η is any of the primitive 30-th roots of unity, and we see that these quotients are not rational. Thus $\chi'(\eta) \neq 0$. The other cases are analogous. □

Proof of Theorem 5.5. — It will suffice to show that when $k \geqslant 4$, χ has a root of unity of modulus > 1. χ is a reciprocal polynomial, so if λ is a root of χ, then λ^{-1} is also a root. Thus if there is a root which is not on the unit circle, then there is a root with modulus > 1. In this case, f has exponential degree growth and is not periodic. Since χ has integer coefficients, if all the roots of χ lie on the unit circle, then they must all be roots of unity. Thus to prove the Lemma, it suffices to show that not all roots of χ are roots of unity. We will proceed by induction on k, and we will show that, in fact, if η is a root of χ which is a root of unity, then η falls into one of the cases (i)–(v) in Lemma 5.6. We start by using the computer to show that Theorem 5.5 is valid for $4 \leqslant k \leqslant 25$.

Now consider a root of unity η which is a root of χ; η is a primitive m-th root of unity for some m. If $m \leqslant k$, then we see from the formula for χ that

$\chi_{k-m,k-m+2}(\eta) = 0$. Thus $\chi_{\hat{k},\hat{k}+2}(\eta) = 0$, where $0 \leqslant \hat{k} < m \leqslant k$, and $\hat{k} \equiv k \mod m$. By induction, then, we conclude that m and k are one of the cases in Lemma 5.6, so it follows that η is a simple root. Since χ has $3k+2$ roots, and $k \geqslant 25$, the roots with $m \leqslant k$ cannot all fall into cases (i)–(v), so there must be a root with $m > k$. Since the primitive m-th roots of unity are all Galois conjugates of each other, we may assume that $\zeta = e^{\theta} = exp(2\pi i/m)$. We will show that if $k > 25$, then $m > \pi k^{3/2}$. Then we use an elementary lower estimate on the size of the Euler φ-function: $\varphi(m) > .37m^{9/10}$. This is the number of Galois conjugates of ζ, which must be zeros of χ. But for $k > 25$, $.37(\pi k^{3/2})^{9/10}$ is strictly bigger than $3k+2$, the degree of χ, which is a contradiction.

Now we estimate the size of m or, equivalently, θ. By standard trigonometric identities applied to the formula for χ, we find that

$$\chi(\zeta) = 2i\zeta^{(3k+2)/2}\left[\sin\left(\frac{3k+2}{2}\theta\right) - \frac{\sin(3\theta/2)}{\sin(\theta/2)}\sin\left(\frac{k}{2}\theta\right)\right] = 0.$$

If we set $\xi = k\theta/2$, this equation becomes

$$\sin((3+2/k)\xi) = \frac{\sin(3\xi/k)}{\sin(\xi/k)}\sin(\xi)$$

with $0 < \xi < \pi k/m \leqslant \pi$. If we take k to be large, then the term $\sin(3\xi/k)/\sin(\xi/k)$ is approximately equal to 3. We rewrite this as $\sin((3+\delta)\xi) = (3-\epsilon)\sin(\xi)$, with $\delta = 2/k$ and $0 < \epsilon < \theta^2 + \theta^4/12 < 2\theta^2$, where the last inequality follows since $m \geqslant k \geqslant 2$. Since we wish to estimate ξ from above, we estimate $\sin((3+\delta)\xi)$ from above and estimate for $(3-\epsilon)\sin(\xi)$ from below. Thus we represent the terms in our equation as

$$\begin{aligned} \sin((3+2/k)\xi) \quad &\leqslant (3+2/k)\xi - \tfrac{1}{6}\left(3+\tfrac{2}{k}\right)^3\xi^3 + \tfrac{1}{120}\left(3+\tfrac{2}{k}\right)^5\xi^5 \\ (3-\epsilon)\sin(\xi) \quad &\geqslant (3-2\theta^2)\sin(\xi/k) \geqslant \left(3 - 2\left(\tfrac{2\xi}{k}\right)^2\right)\left(\xi - \tfrac{1}{6}\xi^3\right). \end{aligned}$$

We set the polynomial expressions equal, divide by ξ and subtract 3:

$$2/k - \frac{1}{6}(3+2/k)^3\xi^2 + \frac{1}{120}(3+2/k)^5\xi^4 = -\left(\frac{1}{2}+\frac{8}{k^2}\right)\xi^2 + \frac{4}{3k^2}\xi^4.$$

This is a quadratic equation in ξ^2, and using simple estimates we find that the smaller root satisfies $0 < \xi^2 < k^{-1}$ when $k > 25$. This gives the estimate: $\theta = 2\xi/k < 2k^{-3/2}$ and thus $\pi k^{3/2} < m$, which completes the proof. $\square$

Theorem 5.7. — *Let f be a critical map with $k \geqslant 3$. Suppose there exists a positive integer $n_\star$ such that $f^{n_\star}\Sigma_{BC} = \Sigma_{\beta\gamma}$ and $f^{j_\star}\Sigma_{BC} \not\subset \Sigma_{\beta\gamma}$*

for $1 \leqslant j \leqslant n-1$. *If* f *is periodic then one of the following must occur*

(*i*) If $n_\star = k-1$ then the mapping is periodic with period $3k-1$
(*ii*) If $n_\star = k$ then the mapping is periodic with period $4k$
(*iii*) If $n_\star = k+1$ then the mapping is periodic with period $3k(k+1)$.

Proof. — Using (5.1) it is not hard to show that $\chi_{k,k-1} = (x^{3k-1}-1)$, $\chi_{k,k} = (x^k-1)(x^{2k}+1)$ and $\chi_{k,k+1} = (x^{k+1}-1)(x^{2k}-x^k+1)$. In each case f is linear fractional. Suppose $f[x_0 : x_1 : \cdots : x_k] = [\sum_i a_{0,i}x_i : \sum_i a_{1,i}x_i : \cdots : \sum_i a_{k,i}x_i]$. Using the fact that E_1 and E_k is fixed under f_X we see that $a_{j,1} = 0$ for all $j \neq 1$ and $a_{j,k} = 0$ for all $j \neq k$. Since we are in the projective space we may assume that $a_{1,1} = a_{k,k} = 1$. For each fixed co-dimension j subspace we obtain $j+1$ equations on $a_{j,i}$. We continue this procedure for other fixed linear subspaces to conclude that the mapping is actually periodic. □

The case of dimension $k=2$ is not covered by Theorem 5.7; the numbers corresponding to the cases *(i–iii)* are 5, 8, and 18. These are all found to occur in [BK1], where it was shown that there are also the possibilities of period 6, 12, and 30. If $k=2$ then $\chi'_{2,n_\star}(1) = 6(6-n_\star)$ and therefore we have more possibilities for periodic mappings, that is $n_\star$ could be $k+2$ and $k+3$ which correspond to the cases of period 12 and period 30. The mapping with period 6 occurs when $\Sigma_{BC} = \Sigma_{\beta\gamma}$. This cannot happen in dimension 3 or higher, since Σ_{BC} and $\Sigma_{\beta\gamma}$ are linear spaces of positive dimension, and there exists a point $[1 : 1-\beta_0 : x_2 : \cdots : x_{k-2} : -\alpha_0 - \alpha_3 x_2 - \cdots - \alpha_{k-1}x_{k-2} : 0] \in \Sigma_{BC} \setminus \Sigma_{\beta\gamma}$. In the case of dimension $k=3$, [BK2] shows that the only possible periods are 8 and 12 (which correspond to cases *(i)* and *(ii)* in Theorem 5.7); the possibility $n_\star = k+1 = 4$ does not occur in dimension 3. In the case of dimension $k \geqslant 4$, we do not know whether cases *(i)* and *(iii)* of Theorem 5.7 actually occur.

Theorem 5.8. — *If* $a = (-1)^{1/k}$ *and*

$$\beta = (a^{k-1}, 1, 0, \ldots, 0) \quad \text{and} \quad \alpha = (a^{k-2}/(1-a), 0, a^{k-2}, \ldots, a^2, a, 1) \tag{5.3}$$

then $f_{\alpha,\beta}$ *is periodic with period* $4k$.

Proof. — It is suffices to show that with these choices of parameter values we have $f^k \Sigma_{BC} = \Sigma_{\beta\gamma}$. Let us set $\mathcal{A} := -(\alpha_0 x_0 + \alpha_2 x_1 + \cdots + \alpha_{k-1}x_{k-2})$, where the α_j's are the entries in α. The generic point $p \in \Sigma_{BC}$ can be written as $[x_0 : x_1 : \cdots : x_{k-2} : \mathcal{A} : 0]$. The last coordinate of $f(p)$ is given by

$$x_0(\alpha_0 x_0 + \alpha_2 x_2 + \cdots + \alpha_{k-2}x_{k-2} - \alpha_{k-1}\mathcal{A}).$$

Since $\alpha_{k-1}\alpha_j = \alpha_{j-1}$ for $j = 3, \ldots, k$ and $\alpha_0 - \alpha_{k-1}\alpha_0 = \alpha_{k-1}^{k-2}$, the last coordinate of $f(p)$ becomes

$$x_0(\alpha_{k-1}^{k-2}x_0 - \alpha_{k-1}^{k-1}x_1) = -\alpha_{k-1}^{k-1}(-\alpha_{k-1}^{-1}x_0 + x_1)x_0 = -\alpha_{k-1}^{k-1}x_0\beta \cdot x.$$

It follows that $f : \Sigma_{BC} \ni [x_0 : x_1 : \cdots : x_{k-2} : \mathcal{A} : 0] \mapsto [x_0 : x_2 : \cdots : x_{k-2} : \mathcal{A} : 0 : -\alpha_{k-1}^{k-1}x_0]$ and using that $\alpha_{k-1}^k = -1$ we have

$$f : \Sigma_{BC} \mapsto \{x_{k-1} = 0\} \cap \{x_0 - \alpha_{k-1}x_k = 0\}.$$

From (2.1) it is not hard to see that

$$\begin{aligned}
&f : \{x_{k-1} = 0\} \mapsto \{x_{k-2} = 0\} \mapsto \cdots \mapsto \{x_1 = 0\}\\
&f : \{x_0 - \alpha_{k-1}x_k = 0\} \mapsto \{x_0 - \alpha_{k-1}x_{k-1} = 0\} \mapsto \cdots \mapsto \{x_0 - \alpha_{k-1}x_2 = 0\}.
\end{aligned}$$

That is $f^{k-1}\Sigma_{BC} = \{x_1 = 0\} \cap \{x_0 - \alpha_{k-1}x_2 = 0\}$. Now let us map forward a point $p = [\alpha_{k-1}x_2 : 0 : x_2 : \cdots : x_k]$. Since $\beta \cdot p = -x_2$ we have

$$\begin{aligned}
f :\ &[\alpha_{k-1}x_2 : 0 : x_2 : \cdots : x_k] \mapsto\\
&[-\alpha_{k-1}x_2 : -x_2 : \cdots : -x_k : \alpha_{k-1}(\alpha_0\alpha_{k-1} + \alpha_2)x_2 + \alpha_{k-1}\alpha_3x_3\\
&\qquad\qquad\qquad\qquad\qquad\qquad + \ldots + \alpha_{k-1}x_k].
\end{aligned}$$

It follows that $\beta \cdot fp = -\alpha_{k-1}^k x_2 - x_2 = 0$ and $\gamma \cdot fp = \alpha_{k-1}(\alpha_0(\alpha_{k-1} - 1) + \alpha_2)x_2 + (-\alpha_2 + \alpha_{k-1}\alpha_3)x_3 + \cdots + (-\alpha_{k-1} + \alpha_{k-1})x_k = 0$ and therefore $f^k\Sigma_{BC} = \Sigma_{\beta\gamma}$. $\square$

6. Non-periodic maps; integrability

Let us consider the critical map given by $\alpha = (a, 0, 1, \ldots, 1)$ and $\beta = (0, 1, 0, \ldots, 0)$:

$$\begin{aligned}
f[x_0 : \cdots : x_k] &= [x_0x_1 : x_2x_1 : \cdots : x_kx_1 : x_0(ax_0 + x_2 + \cdots + x_k)]\\
f^{-1}[x_0 : \cdots : x_k] &= [x_0x_k : x_0(ax_0 + x_1 + \cdots + x_{k-1}) : x_1x_k : \cdots : x_{k-1}x_k].
\end{aligned} \tag{6.1}$$

It follows that we have

$$f : \quad \begin{aligned}
&\Sigma_\beta \mapsto e_k \rightsquigarrow \Sigma_{0\,1\,\ldots\,k-2} \rightsquigarrow \Sigma_{0\,1\,\ldots\,k-3} \rightsquigarrow \cdots \rightsquigarrow \Sigma_{0\,1} = \Sigma_{0\beta} \rightsquigarrow \Sigma_0\\
&\Sigma_0 \mapsto \Sigma_{0k} \mapsto \Sigma_{0\,k-1\,k} \mapsto \cdots \mapsto \Sigma_{0\,3\,\ldots\,k} \mapsto e_1 \rightsquigarrow \Sigma_B
\end{aligned} \tag{6.2}$$

In addition, since $\beta_0 = 0$ we have

$$f : \Sigma_B = \Sigma_k \mapsto \Sigma_{k-1} \mapsto \cdots \mapsto \Sigma_1 = \Sigma_\beta \tag{6.3}$$

so we expect to find (after blowing up) a closed orbit of hypersurfaces containing Σ_β, Σ_0, and Σ_B.

Our first task will be to make f 1-regular. For $j = 1, \dots, k-1$ let us set $q_j = [0 : \cdots : 0 : 1 : -1 : 0 : \cdots : 0]$, the point whose j-th coordinate is 1, whose $j+1$-th coordinate is -1, and every other coordinate is zero. Let us consider a complex manifold $\pi_1 : Z_1 \to \mathbf{P}^k$ obtained by blowing up the $k+1$ points e_1, e_k, and q_j, $j = 1, \dots, k-1$. We denote by $\mathcal{Q}_j$ the exceptional divisor over the point q_j. We also denote by E_1 and E_k the exceptional divisors over the points e_1 and e_k.

Lemma 6.1. — *The induced map f_{Z_1} is a local diffeomorphism at generic points of $\mathcal{Q}_j, j = 2, \dots, k-1$. Furthermore we have dominant maps*

$$f_{Z_1} : \mathcal{Q}_{k-1} \mapsto \mathcal{Q}_{k-2} \mapsto \cdots \mapsto \mathcal{Q}_2 \mapsto \mathcal{Q}_1$$

Proof. — Let us consider the local coordinates near $\mathcal{Q}_{k-1}$ and $\mathcal{Q}_{k-2}$

$$\begin{aligned} \pi_{k-1} : Z_1 \ni (s, \xi_1, \cdots, \xi_{k-1})_{k-1} & \\ & \mapsto [s : s\xi_1 : \cdots : s\xi_{k-2} : 1 + s\xi_{k-1} : -1] \in \mathbf{P}^k \\ \pi_{k-2} : Z_1 \ni (s, \xi_1, \cdots, \xi_{k-2}, \xi_k)_{k-2} & \\ & \mapsto [s : s\xi_1 : \cdots : s\xi_{k-3} : 1 + s\xi_{k-2} : -1 : s\xi_k] \in \mathbf{P}^k. \end{aligned}$$

Note that in those coordinates, $\{(s, \xi_1, \cdots, \xi_{k-1})_{k-1} : s = 0\} = \mathcal{Q}_{k-1}$, and we see that

$$f_{Z_1} : \mathcal{Q}_{k-1} \ni (0, \xi_1, \cdots, \xi_{k-1})_{k-1} \mapsto (0, \xi_2, \dots, \xi_{k-1}, \frac{a}{\xi_1} + \xi_2 + \cdots \xi_{k-1})_{k-2} \in \mathcal{Q}_{k-2}.$$

It follows that f_{Z_1} is locally diffeomorphic at generic points of $\mathcal{Q}_{k-1}$. For $j = 2, \dots, k-2$, the proof is identical. □

By constructing Z_1, we create three new exceptional hypersufaces including E_1 and E_k for each f_{Z_1} and $f_{Z_1}^{-1}$. Let us consider the local coordinates $\pi_1 : (s, \xi_1, \xi_3 \dots, \xi_k)_1 \mapsto [s : 1 + s\xi_1 : -1 : s\xi_3 : \cdots : s\xi_k]$ near $\mathcal{Q}_1$. We also use the local coordinates $\pi_{e_1} : (s, \xi_2, \dots, \xi_k)_{e_1} \mapsto [s : 1 : s\xi_2 : \cdots : s\xi_k]$ near E_1. With these coordinates, we see that

$$f_{Z_1} : \mathcal{Q}_1 \ni (0, \xi_1, \xi_3 \dots, \xi_k)_1 \mapsto (0, \xi_3, \dots, \xi_k, -1)_{e_1} \in E_1 \cap \{x_0 + x_k = 0\} \subset E_1.$$

Similarly with the local coordinates $\pi_{e_k} : (s, \xi_1, \dots, \xi_{k-1})_{e_k} \mapsto [s : s\xi_1 : \cdots : s\xi_{k-1} : 1]$ near E_k and the local coordinates near $\mathcal{Q}_{k-1}$ defined above, we have

$$\begin{aligned} f_{Z_1}^{-1} : \mathcal{Q}_{k-1} \ni (0, \xi_1, \xi_2 \dots, \xi_{k-1})_1 &\mapsto (0, -1, \xi_1, \dots, \xi_{k-2})_{e_k} \\ &= E_k \cap \{x_0 + x_1 = 0\} \in E_k. \end{aligned}$$

Thus we have

$$f_{Z_1} : E_k \cap \{x_0 + x_1 = 0\} \rightsquigarrow \mathcal{Q}_{k-1} \quad \text{and} \quad f_{Z_1}^{-1} : E_1 \cap \{x_0 + x_k = 0\} \rightsquigarrow \mathcal{Q}_1 \tag{6.4}$$

LEMMA 6.2. — *$f_{Z_1}^k \Sigma_{BC} = E_k \cap \{x_0 + x_1 = 0\}$ and $f_{Z_1}^{-k}\Sigma_{\beta\gamma} = E_1 \cap \{x_0 + x_k = 0\}$.*

Proof. — Let us consider the forward map first. A generic point p in Σ_{BC} can be written as $[x_0 : \cdots : x_{k-2} : -ax_0 - x_1 - \cdots - x_{k-2} : 0]$. Using (6.1) we have

$$\begin{aligned} f_X : &[x_0 : \cdots : x_{k-2} : -ax_0 - x_1 - \cdots - x_{k-2} : 0] \\ &\mapsto [x_0x_1 : x_2x_1 : \cdots : (-ax_0 - x_1 - \cdots - x_{k-2})x_1 : 0 : -x_0x_1]. \end{aligned}$$

It follows that $f_X\Sigma_{BC} \subset \{x_{k-1} = 0, x_0 + x_k = 0\}$. Since Σ_{BC} is not indeterminate for f we see that $f_X\Sigma_{BC} = \{x_{k-1} = 0, x_0 + x_k = 0\}$. Note that $f^{k-2}\{x_{k-1} = 0\} = \{x_1 = 0\}$, $f^{k-2}\{x_0 + x_k = 0\} = \{x_0 + x_2 = 0\}$ and therefore $f_X^{k-1}\Sigma_{BC} = \{x_1 = 0, x_0 + x_2 = 0\}$ Using the coordinates near E_k we see that

$$f_X : \Sigma_1 \ni [x_0 : 0 : x_2 : \cdots : x_k] \mapsto (0, \frac{x_2}{x_0}, \ldots, \frac{x_k}{x_0})_{e_k} \in E_k$$

Thus we have

$$f_X : f_X^{k-1}\Sigma_{BC} \mapsto (0, -1, \xi_2, \ldots, \xi_{k-1})_{e_k} \in E_k.$$

The argument for f^{-1} is essentially identical. □

Now let us construct a complex manifold $\pi_2 : Z_2 \to Z_1$ obtained by blowing up the sets $f_X^j\Sigma_{BC}, j = 0, \ldots, k$, $\mathcal{Q}_j, j = 1, \ldots, k-1$ and $f_X^{-j}\Sigma_{\beta\gamma}, j = k, \ldots, 0$. We denote $\mathcal{F}_j$ the exceptional divisor over the set $f_X^j\Sigma_{BC}$, and we also denote $\mathcal{H}_j$ the exceptional divisor over the set $f_X^{-j}\Sigma_{\beta\gamma}$.

LEMMA 6.3. — *The induced map f_{Z_2} is a local diffeomorphism at a generic points of $\bigcup_j \mathcal{F}_j \cup \bigcup_j \mathcal{H}_j$. Thus f_{Z_2} has four exceptional hypersurfaces Σ_0, Σ_β, E_1, and E_k.*

Proof. — It suffices to check at the points in $\Sigma_\gamma \cup \mathcal{F}_k \cup \mathcal{Q}_1 \cup \mathcal{H}_0$. Let us define local coordinates :

$$\begin{aligned} &(x_1, \ldots, x_{k-2}, s, \xi) \mapsto [1 : x_1 : \cdots : x_{k-2} : s - a - x_1 - \cdots - x_{k-2} : s\xi] \text{ near } \mathcal{F}_1 \\ &(s, \eta, \xi_2, \ldots, \xi_{k-1}) \mapsto (s, -1 + s\eta, \xi_2, \ldots, \xi_{k-1})_{e_k} \text{ near } \mathcal{F}_k \\ &(s, \xi_2, \ldots, \xi_{k-2}, \eta) \mapsto (s, \xi_2, \ldots, \xi_{k-2}, -1 + s\eta,)_{e_1} \qquad \text{near } \mathcal{H}_k \\ &(s, x_2, \ldots, x_{k-1}, \xi) \mapsto [1 : s : x_2 : \cdots : x_{k-1} : -a - x_2 - \cdots - x_{k-1} + s\xi] \text{ near } \mathcal{H}_1. \end{aligned}$$

Then we have

$$\begin{aligned} &\Sigma_\gamma \ni [x_0 : x_1 : \cdots : x_{k-1} : -ax_0 - x_2 - \cdots - x_{k-1}] \\ &\qquad\qquad\qquad\qquad \mapsto (x_2, \ldots, x_{k-1}, 0, x_0x_1^{-1}) \in \mathcal{F}_1 \\ f_{Z_2} : \; &\mathcal{F}_k \ni (0, \eta, \xi_2, \ldots, \xi_{k-1}) \mapsto (0, \xi_2, \ldots, \xi_{k-1}, -\eta + a + \xi_2 + \cdots + \xi_{k-1}) \in \mathcal{Q}_{k-1} \\ &\mathcal{Q}_1 \ni (0, \xi_1, \xi_3, \ldots, \xi_k) \mapsto (0, \xi_3, \ldots, \xi_k, a + \xi_1 + \xi_3 + \cdots + \xi_k) \in \mathcal{H}_k \\ &\mathcal{H}_1 \ni (0, x_2, \ldots, x_{k-1}, \xi) \mapsto [1 : x_2 : \cdots : x_{k-1} : -a - x_2 - \cdots - x_{k-1} : \xi] \in \Sigma_C. \end{aligned}$$

It follows that the induced map f_{Z_2} is local diffeomorphism on the orbit of Σ_γ. $\square$

For the other two exceptional hypersurfaces Σ_0 and Σ_β, we construct a blowup space $\pi_{\mathcal{Z}} : \mathcal{Z} \to Z_2$ obtained by blowing up the strict transforms of the sets $\Sigma_{0\,k}, \dots, \Sigma_{0\,3\,\dots\,k-1\,k}, \Sigma_{0\,\dots\,k-2}, \dots, \Sigma_{01}$ in (6.4). We use the same notation for the space X in §3. That is, the exceptional divisors over $\Sigma_{0\,1\,\dots\,k-2}, \dots, \Sigma_{0\,1}$ are $\mathcal{P}_{0,k-2}, \dots, \mathcal{P}_{0,1}$ and $\mathcal{S}_{0,j}$ are the exceptional divisors over $\Sigma_{0\,j\,\dots\,k}$ for all $j \geqslant 3$. Since are only consider a generic point on these exceptional divisors, the same computations as in §3 and §4 work and thus we conclude that $f_{\mathcal{Z}}$ is local diffeomorphic at a generic points on these new exceptional divisors as well as E_1, E_k, Σ_0 and Σ_β. It follows that:

Lemma 6.4. — *The induced map $f_{\mathcal{Z}}$ has no exceptional hypersurface and therefore $f_{\mathcal{Z}}$ is 1-regular.*

Now to compute the dynamical degree we use the following basis of $Pic(\mathcal{Z})$:

$$H, E_1, \mathcal{S}_{0,3}, \dots, \mathcal{S}_{0,k}, \mathcal{P}_{0,1}, \dots, \mathcal{P}_{0,k-2}, E_k, \mathcal{H}_0, \dots, \mathcal{H}_k, \mathcal{Q}_{k-1}, \dots, \mathcal{Q}_1, \mathcal{F}_k, \dots, \mathcal{F}_0.$$

Using (6.2), Lemma 6.1 and Lemma 6.3 we have:

Lemma 6.5. — *The action on cohomology $f^*_{\mathcal{Z}}$ is given by*

$$\begin{aligned} f^*_{\mathcal{Z}} :\quad & E_1 \mapsto \mathcal{S}_{0,3} \mapsto \cdots \mapsto \mathcal{S}_{0,k} \mapsto \{\Sigma_0\} \\ & \mathcal{P}_{0,1} \mapsto \mathcal{P}_{0,2} \mapsto \cdots \mapsto \mathcal{P}_{0,k-2} \mapsto E_k \mapsto \{\Sigma_\beta\} \\ & \mathcal{H}_{3k+1} \mapsto \mathcal{H}_{3k} \mapsto \cdots \mapsto \mathcal{H}_1 \mapsto \{\Sigma_\gamma\} \\ & H \mapsto 2H - E_1 - \mathcal{P}_{0,1} - \cdots - \mathcal{P}_{0,k-2} - E_k - \mathcal{H}_0 - \mathcal{H}_k - \sum_{j=2}^{k-1} \mathcal{Q}_j - \mathcal{F}_k. \end{aligned} \tag{6.5}$$

where

$$\begin{aligned} \{\Sigma_0\} &= H - E_1 - \mathcal{S}_{0,3} - \cdots - \mathcal{S}_{0,k} - \mathcal{P}_{0,1} \\ &\qquad - \cdots - \mathcal{P}_{0,k-2} - E_k - \mathcal{H}_k - \sum_{j=1}^{k-1} \mathcal{Q}_j - \mathcal{F}_k \\ \{\Sigma_\beta\} &= H - \mathcal{P}_{0,1} - \cdots - \mathcal{P}_{0,k-2} - E_k - \mathcal{H}_0 - \sum_{j=2}^{k-1} \mathcal{Q}_j - \mathcal{F}_k - \mathcal{F}_{k-1} \\ \{\Sigma_\gamma\} &= H - E_1 - \mathcal{H}_0 - \mathcal{H}_k - \sum_{j=2}^{k-1} \mathcal{Q}_j \end{aligned}$$

THEOREM 6.6. — *For every $k > 3$, the map f defined in* (6.1) *has quadratic degree growth.*

Proof. — The characteristic polynomial of the action on cohomology given in (6.5) is given by

$$\hat{\chi}_k(x) = \pm(x^k - 1)(x^{k+1} - 1)(x^{3k-1} - 1).$$

It follows that 1 is a zero of $\hat{\chi}_k(x)$ with multiplicity 3. Furthermore there is a unique (up to scalar multiple) eigenvector v corresponding to an eigenvalue 1:

$$v = -(k+1)H + (k-1)E_1 + \sum_{j=1}^{k-2} j\,\mathcal{S}_{0,k+1-j} + \sum_{j=1}^{k-2} j\,\mathcal{P}_{0,j} + (k-1)E_k + \\ + \sum_{j=0}^{k-1} \mathcal{H}_j + k\mathcal{H}_k + (k-1)\sum_{j-1}^{k-1} \mathcal{Q}_j + k\mathcal{F}_k + \sum_{j=0}^{k-1} \mathcal{F}_k$$

It follows that the Jordan decomposition has 3×3 block with 1 on the diagonal. Thus the powers of this matrix grow quadratically. □

We say that a rational function φ is an *integral* of f if $\varphi = \varphi \circ f$ at generic points. Some integrals of f are known (see [KLR], [KL] [CGM1], [GKI]). Here we will describe a method for finding integrals, which seems more systematic than the ones used in these references. For this, we start by finding homogeneous polynomials p which are invariant in the sense that

$$p \circ f = J \cdot p \tag{6.6}$$

where $J = x_0(\beta\cdot x)^{k-1}(\gamma\cdot x)$ is the Jacobian of f. This is the same as finding a meromorphic k-form η, written as $dx_1 \wedge \cdots \wedge dx_k/p(1, x_1, \ldots, x_k)$ on the affine coordinate chart $\{x_0 = 1\}$, and which is invariant in the sense that $f^*\eta = \eta$. If $p_1, \ldots, p_r$ satisfy (6.6), then $\sum \lambda_j p_j$ will also satisfy (6.6). And the quotient of any two of these polynomials $\sum \lambda_j p_j$ will give an integral.

Since f has degree 2, J has degree $k+1$, so we look for polynomials p of degree $k+1$, so that the degrees of $p \circ f$ and Jp will both be $2(k+1)$. The invariant rational functions will then be given as quotients of invariant functions $h = p_1/p_2$. Recall that f maps Σ_β to e_k, and after (4.3) we showed that f_X is a local diffeormorphism at generic points Σ_β. Thus by (6.6) we see that p will vanish to order at least $k-1$ at e_k, since J vanishes to order $k-1$ at Σ_β. Similarly, since $f(\Sigma_0) = \Sigma_{0,k}$, we see that p must vanish at $\Sigma_{0,k}$. Now starting with a point $z \in \Sigma_{0,k}$, we have $f(z) \in \Sigma_{0,k-1,k}$, so by (6.6), p vanishes to order at least 2 on $\Sigma_{0,k-1,k}$. Continuing this way, we see that p

vanishes to order at least $k-j$ on $\Sigma_{j+1,j+2,\ldots,k}$ for $1 \leqslant j \leqslant k-1$. Finally, since J vanishes on Σ_γ, and $f(\Sigma_\gamma) = \Sigma_{BC}$, we see that p vanishes on Σ_{BC}. Iterating this, we see that p must vanish on $f^j(\Sigma_{BC})$ for $j = 0, \ldots, k$.

Next we give an organizational procedure for describing the integrals that were found in the references above. We will refine the equation (6.6) by using the notation $A = \alpha \cdot x$, $B = \beta \cdot x = x_1$ and $C = x_0$, so $J = AB^{k-1}C$. Looking for linear functions which vanish on certain of the sets above, we define three families:

$$\ell_j(x) := x_j,\ 0 \leqslant j \leqslant k,\ m_j(x) := x_0 + x_j,\ 1 \leqslant j \leqslant k,\ n_j := x_0 + x_j + x_{j+1},\ 1 \leqslant j \leqslant k-1,$$

and $n_0 = n_{k-1} \circ f$ and $m_0 = A + x_1 = ax_0 + x_1 + \cdots + x_k$. Thus we have a refined form of (6.6)

$$\begin{aligned} &\ell_j \circ f = B\ell_{j+1}, \quad m_j \circ f = Bm_{j+1}, \quad 1 \leqslant j \leqslant k-1, \\ &n_j \circ f = Bn_{j+1}, 1 \leqslant j \leqslant k-2 \ \ \ell_k \circ f = A\ell_0,\ \ell_0 \circ f = C\ell_1,\ m_k \circ f = Cm_0, \\ &m_1, m_0 \circ f = A \ \ n_0 \circ f = ABCn_1. \end{aligned} \tag{6.7}$$

By (6.7) it is evident that $p_0 := \ell_0\ell_1 \cdots \ell_k$ and $p_1 := m_0 m_1 \cdots m_k$ satisfy (6.6). If $k \geqslant 3$, then $p_2 := n_0 \cdots n_{k-1}$ also satisfies (6.6).

Now let us use the notation $\mathbf{j}$ for the product $\ell_j m_j$ and $(\mathbf{j+1})$ for $\ell_{j+1}m_{j+1}$. If $k \geqslant 5$ is odd, we define

$$\Phi_{\text{even}} := \mathbf{0\ 2\ 4} \cdots (\mathbf{k-1}) \qquad \Phi_{\text{odd}} := \mathbf{1\ 3\ 5} \cdots \mathbf{k}.$$

If q is a polynomial for which $q \circ f$ is divisible by J, we let T denote the operator $T(q) = q \circ f \cdot J^{-1}$, so that (6.6) holds exactly when p is a fixed point of T. By (6.7) we have $\mathbf{j} \circ f = (\mathbf{j+1})\ B^2$ for $1 \leqslant j \leqslant k-1$; and $\mathbf{k} \circ f = \mathbf{0}\ AC$, and $\mathbf{0} \circ f = \mathbf{1}\ AC$. Thus $T\Phi_{\text{even}} = \Phi_{\text{odd}}$, and $T\Phi_{\text{odd}} = \Phi_{\text{even}}$. We conclude that $p_3 := \Phi_{\text{even}} + \Phi_{\text{odd}}$ satisfies (6.6).

If $k > 5$ is even, we consider two functions:

$$\Psi_a := \mathbf{0}\ n_1\ \mathbf{3\ 5\ 7}\ \cdots (\mathbf{k-1}), \qquad \Psi_b := \mathbf{1\ 3\ 5\ 7}\ \cdots (\mathbf{k-1})\ \ell_k.$$

By (6.7) we see that $\Psi_b \circ f = A\ \ell_0\ \mathbf{2\ 4\ 6\ 8}\ \cdots\ \mathbf{k}\ B^k = J\ B\ \mathbf{2\ 4\ 6\ 8}\ \cdots\ \mathbf{k}$. Thus $T\Psi_b = \ell_1\ \mathbf{2\ 4\ 6\ 8}\ \cdots\ \mathbf{k}$. Applying T to Ψ_a, we have

$$T\Psi_a = \mathbf{1}\ n_2 \quad \mathbf{4\ 6\ 8}\ \cdots\ \mathbf{k}, \quad T^2\Psi_a = \mathbf{0\ 2}\ n_3\ \mathbf{5\ 7\ 9}\ \cdots\ (\mathbf{k-1})\mathbf{k}, \quad \ldots \quad T^{k-2}\Psi_a = \mathbf{0\ 2\ 4}\ \cdots (\mathbf{k-2})\ n_{k-1} \tag{6.8}$$

Now we claim that

$$p_3 = \left(\Psi_a + T\Psi_a + \cdots + T^{k-2}\Psi_a\right) + (\Psi_b + T\Psi_b)$$

satisfies (6.6). For this, it suffices to have

$$\Psi_a + \Psi_b = T^{k-1}\Psi_a + T^2\Psi_b. \tag{6.9}$$

Applying T to $T^{k-2}\Psi_a$ in (6.8), and using (6.7), we find $T^{k-1}\Psi_a = n_0\ \mathbf{1}\ \mathbf{3}\ \mathbf{5}\ \cdots\ (\mathbf{k}-\mathbf{1})$. Now apply T to the expression for $T\Psi_b$ found above, we find $T^2\Psi_b = \mathbf{0}\ \ell_2\ \mathbf{3}\ \mathbf{5}\ \cdots\ (\mathbf{k}-\mathbf{1})$. Thus (6.9) is a consequence of the simple identity

$$\mathbf{0}\ n_1 + \mathbf{1}\ \ell_k = n_0\ \mathbf{1} + \mathbf{0}\ \ell_2,$$

and we conclude that p_3 satisfies (6.6).

Bibliography

[BK1] Bedford (E.) and Kim K.. — Periodicities in linear fractional recurrences: degree growth of birational surface maps. Michigan Math. J. 54, no. 3, p. 647-670 (2006).

[BK2] Bedford (E.) and Kim K.. — Periodicities in 3-step linear fractional recurrences, preprint.

[CL] Camouzis (E.) and Ladas (G.). — *Dynamics of third-order rational difference equations with open problems and conjectures*. Advances in Discrete Mathematics and Applications, 5. Chapman & Hall/CRC, Boca Raton, FL, 2008.

[CGM1] Cima (A.), Gasull (A.), Mañosa (V.). — Dynamics of some rational discrete dynamical systems via invariants. Internat. J. Bifur. Chaos Appl. Sci. Engrg. 16, no. 3, p. 631-645 (2006).

[CGM2] Cima (A.), Gasull (A.), Mañosa (V.). — Dynamics of the third order Lyness' difference equation. J. Difference Equ. Appl. 13, no. 10, p. 855-884 (2007).

[CGM3] Cima (A.), Gasull (A.), Mañosa (V.). — Some properties of the k-dimensional Lyness' map. J. Phys. A 41, no. 28, 285205, 18 pp (2008).

[CsLa] Csörnyei (M.) and Laczkovich (M.). — Some periodic and non-periodic recursions, Monatsh. Math. 132, p. 215-236 (2001).

[dFE] de Fernex (T.) and Ein (L.). — Resolution of indeterminacy of pairs. *Algebraic geometry*, p. 165-177, de Gruyter, Berlin (2002).

[DF] Diller (J.) and Favre (C.). — Dynamics of bimeromorphic maps of surfaces, Amer. J. of Math., 123, p. 1135-1169 (2001).

[GKI] Gao (M.), Kato (Y.) and Ito (M.). — Some invariants for kth-order Lyness equation. Appl. Math. Lett. 17, no. 10, p. 1183-1189 (2004).

[GBM] Gardini (L.), Bischi (G.), and Mira (C.). — Invariant curves and focal points in a Lyness iterative process. Dynamical systems and functional equations (Murcia, 2000). Internat. J. Bifur. Chaos Appl. Sci. Engrg. 13, no. 7, p. 1841-1852 (2003).

[G] Gizatullin (M.). — Rational G-surfaces. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 44, no. 1, p. 110-144, 239 (1980).

[GL] Grove (E. A.) and Ladas (G.). — *Periodicities in nonlinear difference equations*, Adv. Discrete Math. Appl., 4, Chapman & Hall and CRC Press, Boca Raton, FL (2005).

[H] Hassett (B.). — *Introduction to Algebraic Geometry*, Cambridge U. Press (2007).

[HKY] Hirota (R.), Kimura (K.) and Yahagi (H.). — How to find the conserved quantities of nonlinear discrete equations, J. Phys. A: Math. Gen. 34, p. 10377-10386 (2001).

[KL] Kocic (V.I.) and Ladas (G.). — *Global Behaviour of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers 1993.

[KLR] Kocic (V.I.), Ladas (G.), and Rodrigues (I.W.). — On rational recursive sequences, J. Math. Anal. Appl 173, p. 127-157 (1993).

[L] Lyness (R. C.). — Notes 1581, 1847, and 2952, Math. Gazette 26 (1942), 62; 29 (1945), 231; 45 (1961), 201.

[M] McMullen (C. T.). — Dynamics on blowups of the projective plane. Publ. Math. Inst. Hautes Études Sci. No. 105, p. 49-89 (2007).

[Z] Zeeman (C.). — Geometric unfolding of a difference equation. Lecture.