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Stability for a certain class of numerical methods – abstract approach and application to the stationary Navier-Stokes equations

Elżbieta Motyl[(1)]

Abstract. — We consider some abstract nonlinear equations in a separable Hilbert space H and some class of approximate equations on closed linear subspaces of H. The main result concerns stability with respect to the approximation of the space H. We prove that, generically, the set of all solutions of the exact equation is the limit in the sense of the Hausdorff metric over H of the sets of approximate solutions, over some filterbase on the family of all closed linear subspaces of H. The abstract results are applied to the classical Galerkin method and to the Holly method for the stationary Navier-Stokes equations for incompressible fluid in 2 and 3-dimensional bounded domains.

Résumé. — On considère certaines équations non linéaires abstraites dans un espace de Hilbert séparable H et certaines classes d'équations approchées dans les sous-espaces vectoriels fermés de H. Le résultat principal concerne la stabilité relativement à l'approximation de l'espace H. On prouve que l'ensemble de toutes les solutions de l'équation exacte est la limite dans la métrique de Hausdorff des ensembles des solutions approchées, relativement à certaine base filtrée sur la famille des sous-espaces vectoriels fermés de H. Les résultats généraux sont appliqués à la méthode de Galerkin et à la méthode de Holly pour les équations de Navier-Stokes stationnaires dans domaines bornés de dimension 2 et 3.

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Article proposé par Jean-Michel Coron.

0. Introduction

We consider an abstract (nonlinear) equation of the form

$$\mu u + T(u) = g \tag{$*$}$$

in a real separable Hilbert space, where $(\mu, g) \in]0, \infty[\times H$ and a mapping $T : H \to H$ of class $\mathcal{C}^1$ are given while u is unknown.

On every closed linear subspace $M \subset H$ of H let us consider equation of the form

$$\mu w + T_M(w) = g_M, \tag{$*_M$}$$

where $g_M \in M$ and $T_M : M \to M$ of class $\mathcal{C}^1$ are given and w is looked for. Relations between mappings T and T_M are described in assumptions (A.1)-(A.6) in Section 2. If $H \neq M$, then equation $(*_M)$ will be interpreted as the approximate equation of $(*)$.

Let $\mathcal{S}(H)$ be the family of all closed linear subspaces of H. We consider the topology on $\mathcal{S}(H)$ induced by some filterbase $\mathcal{B}$ introduced by K. Holly in [7]. In this way we have the notion on convergence in $\mathcal{S}(H)$. We recall this construction in Preliminaries (see Section 1.3).

In the present paper, we investigate stability with respect to approximation of the space H. More precisely, let us denote

$$\mathfrak{R}(\mu, g) \text{ - the set of all solutions of the equation } (*)$$
$$\mathfrak{R}_M(\mu, g_M) \text{ - the set of all solutions of the equation } (*_M)$$

We prove that for the data (μ, g) from a certain set $\mathcal{O} \subset]0, \infty[\times H$

$$\lim_{M \succ \mathcal{B}} \mathfrak{R}_M(\mu, g_M) = \mathfrak{R}(\mu, g) \qquad \textit{in the Hausdorff metric over } H,$$

whenever $\lim_{M \succ \mathcal{B}} g_M = g$, where the limit is taken over the filterbase $\mathcal{B}$ on the family $\mathcal{S}(H)$ (see Theorem 2.10). Let us mention that the solutions of the considered equations may be non-unique. Set $\mathcal{O}$ is defined by

$$\begin{aligned} \mathcal{O} := \ & \{(\mu, g) \in]0, \infty[\times H : \ \ g \textit{ is a regular value} \\ & \textit{of the mapping } H \ni u \mapsto \mu u + T(u) \in H\}. \end{aligned}$$

Moreover, the set $\mathcal{O}$ is open and dense in $]0, \infty[\times H$ (see Theorem 2.9). This problem has been investigated in the paper [12], Section 3 and we recall it in Appendix C.

Then we say that the numerical method expressed in an abstract way as the family of the equations $\{(*_M),\ M \in \mathcal{S}(H)\}$ is **generically stable with respect to the approximation of the space H.**

The technique of analysis is based on the methods of functional analysis, especially on the theory of Fredholm mappings (see Appendix C). Moreover, the crucial point is the application of a certain version of the implicit function theorem for the space with filterbase (see Theorem 2.7).

The above abstract considerations arised on the base of investigation some numerical methods in the stationary Navier-Stokes equations. Main results of this paper are generalizations of the results of paper [12]. The present approach has been deduced from the concrete numerical methods and put into an abstract framework (see [7] and [12]).

In the second part of the present paper, we apply the abstract framework for

- the classical Galerkin method (see Section 4.2)
- and for the method introduced by Holly (see Section 6.2).

for stationary Navier-Stokes equations.

Consider the stationary Navier-Stokes equations for incompressible fluid filling a bounded domain $\Omega \subset \mathbb{R}^n$, where $n \in \{2, 3\}$, i.e.

$$\begin{cases} \sum_{i=1}^n v_i \frac{\partial v}{\partial x_i} = \nu \Delta v + f - \nabla p, \\ \mathrm{div} v = 0, \\ v_{|\partial\Omega} = 0. \end{cases}$$

Here $\nu > 0$ (viscosity) and $f : \Omega \to \mathbb{R}^n$ (external forces) are given while $v : \Omega \to \mathbb{R}^n$ (velocity) and $p : \Omega \to \mathbb{R}$ (pressure) are looked for. We are interested in weak solutions of the above problem (see Definition 3.1). Every internal approximation of the space V of all divergence-free vector fields enables us to look for a stationary velocity of the fluid with the aid of the Galerkin method without any limitations on viscosity and external forces (see e.g. [10], [19]). However, there arises the problem of numerical construction of the approximation of V. Difficulties with the approximation are an incentive to look for other methods (see [6], [19]).

K. Holly introduced a new numerical method of finding velocity v in the stationary Navier-Stokes problem. The construction of the solution is based on the internal approximation of the whole Sobolev space H_0^1, which is in practice well approximated, e.g. by splines in the finite element method (see [6], [19], [12]). We present this method in Section 5. Moreover, we provide an analysis of the pressure in this method.

Motivation of the present approach. Let us again consider the abstract equation (*) in the separable Hilbert space H. General ideas of construction of a solution of (*) by using a numerical metod are as follows

- Consider a sequence $(H_N)_{N\in\mathbb{N}}$ of finite-dimensional subspaces of H such that for every $h \in H$, the corresponding sequence of distances $dist(h, H_N)$, $N \in \mathbb{N}$ of h from H_N tends to zero as $N \to \infty$. The sequence $(H_N)_{N\in\mathbb{N}}$ is called *an internal approximation of* H.
- For each $N \in \mathbb{N}$, consider appropriate approximate equation (dependent on the chosen method) in the subspace H_N and prove existence of a solution $u_N \in H_N$.
- Prove that $(u_N)_{n\in\mathbb{N}}$ contains a convergent subsequence and that the limit is a solution od equation (*).

There arises one more problem which we call the **problem of stability with respect to approximation of the space H,** and which is the main topic of the present paper. This problem is important from the numerical point of view, because in practice, the internal approximation $(H_N)_{N\in\mathbb{N}}$ of H is numerically computed. For each $N \in \mathbb{N}$, the subspace H_N is determined by its Hamel base. Vectors of this base are usually numerically computed (for example, this base may be constructed with the aid of splines in the finite element method). Thus, even "very small" perturbation of this vectors changes the subspace H_N. This reflects in the perturbation of the set of solutions of the approximate equation corresponding to the perturbated subspace H_N. Roughly speaking, the question is whether "small" perturbations of the subspaces H_N call "small" perturbations of the corresponding sets of solutions. However, if we want to describe this effect precisely, we need some topology (and notion of convergence) on the family of linear subspaces of $\mathcal{S}(H)$. We consider topology induced by some filterbase on the family $\mathcal{S}(H)$ of all closed linear subspaces of H. It is described in Preliminaries. In the space of sets of solutions we consider the Hausdorff metric. The main result concernig stability with respect to approximation of the space states that if (μ, g) belong to some open and dense set $\mathcal{O}$, then the sets of approximate solutions corresponding to subpaces of the space H converge to the set $\mathfrak{R}(\mu, g)$ in the Hausdorff metric over H when the spaces converge over the filterbase. Then we say that the method is **generically stable with respect to approximation of the space H.**

The present paper is organised as follows. In Preliminaries, we recall the concept of the filterbase and the notion of convergence in the sense of the filterbase. Next, we deal with the construction and properties of the filterbase on the family of all closed linear subspaces of a separable

Hilbert space. Auxiliary results about filterbases are put in the Appendix A in Section 8. Section 2 contains the abstract framework concernig the problem of stability. In Sections 3, 4, 5 and 6 we consider the stationary Navier-Stokes equations. In Section 4 we illustrate the abstract approach of Section 2 on the example of the classical Galerkin method. Section 5 is devoted to the presentation of the Holly method and in Section 6 we apply the abstract framework to prove stability of this method. The last four sections are appendices. In Appendix B, we consider a certain version of the fixed point theorem in the finite-dimensional Hilbert space (Theorem 9.2) proved by J.L. Lions and its generalization to the case of the infinite-dimensional Hilbert space (Theorem 9.4). In Appendix C, we recall in details the problem of generic properties of the set of solutions of equation (*). We use these results in Section 2. At the end of this appendix, there are some generalizations, which we apply in Section 6 to the Holly method. In Appendix D, we recall the results about the divdiv*-operator and its inversion based on the von Neumann lemma. These results are of crucial importance in the Holly method.

Contents

1. Preliminaries

1.1. Notations

Let $(X, |\cdot|_X)$, $(Y, |\cdot|_Y)$ be real normed spaces. Then $K_X(x_0, r) := \{x \in X : |x - x_0| < r\}$ is the open ball with center at x_0 and radius r, and $\overline{K}_X(x_0, r)$ is the appropriate closed ball. Moreover, $K_X(r) := K_X(0, r)$. If no confusion seems likely, we omit the index X.

The symbol $\mathcal{L}(X, Y)$ stands for the linear space of all continuous linear operators from X to Y. $\mathcal{E}pi(X, Y)$ is the subspace of all *epimorphisms* ,i.e., the family of all $A \in \mathcal{L}(X, Y)$ such that $A(X) = Y$ and $\mathcal{M}ono(X, Y)$ – the subspace of all *monomorphisms* , i.e. the family of all injections in $\mathcal{L}(X, Y)$. Moreover,

$$\mathcal{I}so(X, Y) := \{A \in \mathcal{L}(X, Y) : \quad A \textit{ is bijective and } A^{-1} \in \mathcal{L}(Y, X)\}$$

is the family of all *isomorphisms.* In particular, the space $\mathcal{L}(X, X) =: \mathcal{E}ndX$ is called the space of *endomorphisms* of X and $\mathcal{I}so(X, X) =: \mathcal{A}utX$ is called the space of *authomorphisms* of X. Moreover, $\mathcal{E}piX := \mathcal{E}pi(X, X)$ and $\mathcal{M}onoX := \mathcal{M}ono(X, X)$. If $Y = \mathbb{R}$, then $X' := \mathcal{L}(X, \mathbb{R})$ is called the *dual* space of X and its elements are called *continuous linear functionals* of X. The identity mapping on a set X is denoted by id_X; if no confusion seems likely, we omit the index X.

The topology of a topological space Z is denoted by $\mathrm{top}Z$. The symbol $\mathrm{cotop}Z$ denotes the family of all closed subsets of Z, i.e. $\mathrm{cotop}Z := \{Z \setminus \mathcal{O} : \mathcal{O} \in \mathrm{top}Z\}$.

1.2. Filterbases – definitions and notations

Let S be a nonempty set. The symbol 2^S stands for the family of all subsets of S.

Definition 1.1. — *A subfamily $\mathcal{B} \subset 2^S$ is called a filterbase* on S iff it is nonempty, empty set does not belong to $\mathcal{B}$ and

$$\begin{aligned}&\text{for every } A, B \in \mathcal{B} \text{ there exists } C \in \mathcal{B} \\ &\text{such that } C \subset A \cap B.\end{aligned} \tag{1.1}$$

We will assume that $\bigcap \mathcal{B} := \bigcap_{B \in \mathcal{B}} B \neq \emptyset$.

Example. — Let $S = \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. Then the family

$$\{[N,\infty] \cap \overline{\mathbb{N}}; \quad N \in \mathbb{N}\}$$

is a filterbase. Notice that $\bigcap_{n\in\mathbb{N}}\big([N,\infty] \cap \overline{\mathbb{N}}\big) = \{\infty\}$.

Filterbase induces topology on S in the following way. For fixed $\omega_0 \in \bigcap \mathcal{B}$, the family

$$\mathcal{B}(\omega_0) := \{\{\omega\}; \quad \omega_0 \neq \omega \in S\} \cup \{\bigcap_{i=1}^{k} B_i; \qquad k \in \mathbb{N}, \quad B_i \in \mathcal{B}\}$$

has properties of the topological base, i.e. $S = \bigcup \mathcal{B}(\omega_0)$ and for every $A, B \in \mathcal{B}(\omega_0)$ and every $\omega \in A \cap B$ there exists $C \in \mathcal{B}(\omega_0)$ such that $\omega \in C \subset A \cap B$. Thus

$$\mathrm{top}S := \{\bigcup \mathcal{U}; \quad \mathcal{U} \subset \mathcal{B}(\omega_0)\}$$

is a topology on S.

Now, we recall notion of the convergence over the filterbase. Let $\psi : S \to Z$, where Z is a topological space.

Definition 1.2. — *An element $z_0 \in Z$ is a limit of the function ψ over the filterbase $\mathcal{B}$* iff for every $U \in \mathcal{F}(z_0)$ there exists $B \in \mathcal{B}$ such that $\psi(B) \subset U$. Then we write

$$\lim_{\omega \succ \mathcal{B}} \psi(\omega) = z_0 \qquad \textit{or} \qquad \psi(\omega) \to z_0 \quad \text{as} \quad \omega \succ \mathcal{B}.$$

(The symbol $\mathcal{F}(z_0)$ denotes the filter of all neighbourhoods of z_0.)

Definition 1.3. — *The filterbase $\mathcal{B}$ is of countable type* iff there exists a countable family $\mathcal{B}_0 = \{B_0^1, B_0^2, \ldots\}$ such that

$$\text{for every } B \in \mathcal{B} \text{ there exists } B_0 \in \mathcal{B}_0 \text{ such that } B_0 \subset B. \tag{1.2}$$

Then we write $\mathcal{B}_0 \succ \mathcal{B}$.

1.3. Filterbase on the family of all closed linear subspaces of a Hilbert space.

We recall construction of a filterbase introduced by K. Holly in [7] as well as some of its properties. Let $\big(H, (\cdot|\cdot)\big)$ be a real separable Hilbert space. The norm induced by the scalar product $(\cdot|\cdot)$ is denoted by $|\cdot|$. Consider the family $\mathcal{S}(H)$ of all closed linear subspaces of H. For a finite-dimensional subspace $W \in \mathcal{S}(H)$ and for $\delta > 0$ let us define

$$B_{W,\delta} := \{M \in \mathcal{S}(H) : \quad W \cap \sigma(1) \subset M + \overline{K}(\delta)\}, \tag{1.3}$$

where $\overline{K}(\delta) := \{x \in H : |x| \leqslant \delta$ and $\sigma(1) := \partial\overline{K}(1) := \{x \in H : |x| = 1\}$. Then the family

$$\mathcal{B} := \big\{B_{W,\delta}; \quad W \in \mathcal{S}(H) \cap \{\dim < \infty\}, \delta > 0\big\} \tag{1.4}$$

is a filterbase on $\mathcal{S}(H)$.

Let us note that condition (1.1) is satisfied, because for every $W_1, W_2 \in \mathcal{S}(H) \cap \{\dim < \infty\}$ and every $\delta_1, \delta_2 > 0$

$$B_{W,\delta} \subset B_{W_1,\delta_1} \cap B_{W_2,\delta_2},$$

where $W = W_1 + W_2$ and $\delta = \min\{\delta_1, \delta_2\}$. Indeed, let $M \in B_{W,\delta}$. Then $(W_1 + W_2) \cap \sigma(1) \subset M + \overline{K}(\delta)$. Thus, in particular

$$W_i \cap \sigma(1) \subset M + \overline{K}(\delta) \subset M + \overline{K}(\delta_i), \quad i = 1, 2$$

which means that $M \in B_{W_i,\delta_i}$, $i = 1, 2$.

For a subspace $M \in \mathcal{S}(H)$ let $P_M : H \to M$ denote the $(\cdot|\cdot)$ - orthogonal projection onto M.

Remark 1.4 (Remark 1.21 in [7]). — Let $M \in \mathcal{S}(H)$. Then

$$M \in B_{W,\delta} \;\Leftrightarrow\; |x - P_M x| \leqslant \delta|x| \quad \textit{for every } x \in W.$$

Proof. — Ad.“$\Rightarrow$”. Let $x \in W$. We may assume that $x \neq 0$. Then

$$\frac{x}{|x|} \in W \cap \sigma(1) \subset M + \overline{K}(\delta).$$

Thus $\operatorname{dist}\big(\frac{x}{|x|}, M\big) \leqslant \delta$. On the other hand,

$$\operatorname{dist}\Big(\frac{x}{|x|}, M\Big) = \Big|\frac{x}{|x|} - P_M\Big(\frac{x}{|x|}\Big)\Big|$$

In conclusion, $|x - P_M x| \leqslant \delta|x|$.

Ad.“$\Leftarrow$”. We have to check that $W \cap \sigma(1) \subset M + \overline{K}(\delta)$. Let $x \in W \cap \sigma(1)$. Then $\operatorname{dist}(x, M) = |x - P_M x| \leqslant \delta$. Hence

$$x \in \overline{K}(P_M x, \delta) = P_M x + \overline{K}(\delta) \subset M + \overline{K}(\delta).$$

Thus $M \in B_{W,\delta}$. $\square$

Corollary 1.5 *(Corollary 1.22 in [7]).* —

$$\lim_{M \succ \mathcal{B}} P_M(x) = x, \qquad x \in H.$$

Proof. — We will use Definition 1.2. Let us fix $x \in H$ and let

$$\psi(M) := P_M(x), \qquad M \in \mathcal{S}(H).$$

Let $\mathcal{U}$ be a neighbourhood of x in the space H. Then, there exists $\varepsilon > 0$ such that $\overline{K}(\varepsilon) \subset \mathcal{U}$. We will check that $\psi(B_{W,\delta}) \subset \mathcal{U}$ for $W := \mathbb{R} \cdot x$ and $\delta := \frac{\varepsilon}{|x|}$. Indeed, let $M \in B_{W,\delta}$. By Remark 1.4, $|z - P_M(z)| \leqslant \delta |z|$ for every $z \in W$. Since $W = \mathbb{R} \cdot x$, $z = rx$ for some $r \in \mathbb{R}$. Hence

$$|z - P_M(z)| = |rx - rP_M(x)| \leqslant \delta r|x| = \varepsilon r.$$

Thus $|x - P_M(x)| \leqslant \varepsilon$, which means that $\psi(M) = P_M(x) \in \overline{K}(x, \varepsilon) \subset \mathcal{U}$. □

Definition 1.6. — *A sequence $(W_k)_{k \in \mathbb{N}}$ of finite-dimensional linear subspaces of H is called an internal appoximation of H* iff

$$\lim_{k \to \infty} |x - P_{W_k}(x)| = 0, \qquad x \in H.$$

Corollary 1.7 *(Corollary 1.23 in [7]).* — *Let (W_k) be an internal approximation of H and let $(\delta_k)_{k \in \mathbb{N}}$ be a sequence of positive real numbers such that $\lim_{k \to \infty} \delta_k = 0$. Then*

(a) for every subspace $W \in \mathcal{S}(H) \cap \{\dim < \infty\}$ and every $\delta > 0$

$$B_{W_k,\delta_k} \subset B_{W,\delta} \qquad \textit{for almost all } k \in \mathbb{N};$$

(b) if $(M_k) \in \mathrm{X}_{k=1}^{\infty} B_{W_k,\delta_k}$, then

$$\lim_{k \to \infty} |x - P_{M_k}(x)| = 0, \qquad x \in H.$$

Proof. — Ad. *(a).* Let us fix a subspace $W \in \mathcal{S}(H) \cap \{\dim < \infty\}$ and a number $\delta > 0$. Since $\delta_k \to 0$, there exists $\tilde{k_1} \in \mathbb{N}$ such that $\delta_k < \frac{\delta}{2}$ for each $k \geqslant \tilde{k_1}$. Let the vectors $e_1, \dots, e_l$ form an orthonormal base in W. Since (W_k) is internal approximation of W, then for each $i \in \{1, \dots, l\}$ there exists $k_i \in \mathbb{N}$ such that

$$|e_i - P_{W_k}(e_i)| \leqslant \frac{\delta}{2l} \qquad \text{for each } k \geqslant k_i.$$

Let $k_0 := \max\{\tilde{k_1}, k_1, \ldots, k_l\}$. We assert that

$$B_{W_k,\delta_k} \subset B_{W,\delta} \qquad \text{for each } k \geqslant k_0.$$

Indeed, let $M \in B_{W_k,\delta_k}$. We have to prove that $W \cap \sigma(1) \subset M + \overline{K}(\delta)$. Let $x \in W \cap \sigma(1)$. By Remark 1.4, it is sufficient to show that $|x - P_M(x)| \leqslant \delta$ (because $|x| = 1$). Let us write the following inequality

$$|x - P_M(x)| \leqslant |x - P_{W_k}(x)| + |P_{W_k}(x) - P_M(x)|.$$

Since $x \in W \cap \sigma(1)$, $x = \sum_{i=1}^{l} \lambda_i e_i$ for some $\lambda_i \in \mathbb{R}$ and $\sum_{i=1}^{l} \lambda_i^2 = 1$. Hence

$$\begin{aligned} |x - P_{W_k}(x)| &= \Big|\sum_{i=1}^{l} \lambda_i\big(e_i - P_{W_k}(e_i)\big)\Big| \leqslant \sum_{i=1}^{l} |\lambda_i| \cdot |e_i - P_{W_k}(e_i)| \\ &\leqslant \left(\sum_{i=1}^{l} \lambda_i^2\right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{l} |e_i - P_{W_k}(e_i)|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^{l} |e_i - P_{W_k}(e_i)|^2\right)^{\frac{1}{2}} \leqslant \frac{\delta}{2}. \end{aligned}$$

Since $M \in B_{W_k,\delta_k}$, then by Remark 1.4,

$$|P_{W_k}(x) - P_M(x)| \leqslant \delta_k |P_{W_k}(x)| < \frac{\delta}{2}.$$

In conclusion, $|x - P_M(x)| \leqslant \delta$.

Ad. *(b)* Let $x \in H$ and let $\varepsilon > 0$. According to Corollary 1.5, $\lim_{M \succ \mathcal{B}} P_M(x) = x$. Thus, there exists $W \in \mathcal{S}(H) \cap \{\dim < \infty\}$ and $\delta > 0$ such that $\psi(B_{W,\delta}) \subset \overline{K}(x, \varepsilon)$, where $\psi(M) := P_M(x)$, $M \in \mathcal{S}(H)$. Due to assertion (a)

$$M_k \in B_{W_k,\delta_k} \subset B_{W,\delta} \qquad \text{for almost all } k \in \mathbb{N}.$$

Thus, in particular, $\psi(M_k) \in \overline{K}(x, \varepsilon)$ which means that $|x - P_{M_k}(x)| \leqslant \varepsilon$ for almost all $k \in \mathbb{N}$ and ends the proof. $\square$

Using this Corollary, we deduce that the filterbase $\mathcal{B}$ is of countable type, because condition (1.2) holds with $\mathcal{B}_0 = \{B_{W_k,\delta_k}, k \in \mathbb{N}\}$.

Since $W_k \in B_{W_k,\delta_k}$, Corollary 1.7 (a) yields the following

Corollary 1.8. — *If (W_k) is an internal approximation of the space H, then for every subspace $W \in \mathcal{S}(H) \cap \{\dim < \infty\}$ and every $\delta > 0$*

$$W_k \in B_{W,\delta} \qquad \textit{for almost all } k \in \mathbb{N}.$$

Remark 1.9. — Let (W_k), (δ_k) be like in Corrollary 1.7 and let $M_k \in B_{W_k,\delta_k}$, $k \in \mathbb{N}$. If $\psi : \mathcal{S}(H) \to Z$ is a mapping such that $\lim_{M \succ \mathcal{B}} \psi(M) = z_0$, where $z_0 \in Z$ and Z is a topological space, then

$$\lim_{k \to \infty} \psi(M_k) = z_0.$$

Proof. — Let us fix a neighbourhood $\mathcal{U} \in \mathcal{F}(z_0)$. Since $\psi(M) \to z_0$ as $M \succ \mathcal{B}$, there exists $B \in \mathcal{B}$ such that $\psi(B) \subset \mathcal{U}$. By the construction of the filterbase $\mathcal{B}$, we deduce that $B = B_{W,\delta}$ for some $W \in \mathcal{S}(H) \cap \{\dim < \infty\}$ and $\delta > 0$. Corollary 1.7 (a) yields that $M_k \in B_{W,\delta}$ for almost all $k \in \mathbb{N}$. Thus $\psi(M_k) \in \mathcal{U}$ for almost all $k \in \mathbb{N}$. $\square$

Further auxiliary results concerning filterbases are proven in Appendix A.

2. Abstract results

2.1. Statement of the problem

Let us consider the following equation in the space H

$$\mu u + T(u) = g, \tag{$*$}$$

where $\mu \in]0, \infty[$, $g \in H$ and $T : H \to H$ is a $\mathcal{C}^1$ - mapping.

For every subspace $M \in \mathcal{S}(H)$, we consider an analogous equation in M

$$\mu w + T_M(w) = g_M, \tag{$*_M$}$$

where $g_M \in M$ and $T_M : M \to M$ is a $\mathcal{C}^1$ - mapping. If $M \neq H$, then we interprete $(*_M)$ as the approximate equation of the equation (*).

For fixed data $(\mu, g) \in]0, \infty[\times H$ let us denote

$\mathfrak{R}(\mu, g)$ - the set of all solutions of the equation (*), i.e.

$$\mathfrak{R}(\mu, g) := \{u \in H : \quad \mu u + T(u) = g\}. \tag{2.1}$$

For fixed subspace $M \in \mathcal{S}(H) \setminus H$ and data $(\mu, g_M) \in]0,\infty[\times M$:

$$\mathfrak{R}_M(\mu, g_M) \text{ - the set of all solutions of the equation } (*_M)\text{, i.e.}$$
$$\mathfrak{R}_M(\mu, g_M) := \{w \in M : \quad \mu w + T_M(w) = g_M\}. \tag{2.2}$$

Assumptions:

(A.1) For every $M \in \mathcal{S}(H) \setminus H$ and every $(\mu, g_M) \in]0,\infty[\times M$ the set $\mathfrak{R}_M(\mu, g_M)$ is nonempty.

(A.2) For every $M \in \mathcal{S}(H)$ there exists a $\mathcal{C}^1$- mapping $\tilde{T}_M : H \to H$ such that $T_M \subset \tilde{T}_M$ (i.e. $\{(u, T_M(u)), u \in M\} \subset \{(u, \tilde{T}_M(u)), u \in H\}$), $\tilde{T}_H = T$ and $\mathfrak{R}_M(\mu, g_M) = \{u \in H : \mu u + \tilde{T}_M(u) = g_M\}$.

(A.3) There exists a continuous function $\kappa :]0,\infty[\times[0,\infty[\to [0,\infty[$ such that for every $M \in \mathcal{S}(H)$ and every $w \in \mathfrak{R}_M(\mu, g_M)$ the following inequality holds

$$|w| \leqslant \kappa(\mu, |g_M|).$$

(A.4) For every $u \in H$

$$\tilde{T}_M(u) \to T(u) \quad \text{in} \quad H \quad \text{as} \quad M \succ \mathcal{B}.$$

(A.5) If (W_k) is an internal approximation of H and (δ_k) is a sequence of positive real numbers such that $\lim_{k\to\infty} \delta_k = 0$, then for every (M_k) such that $M_k \in B_{W_k,\delta_k}$, $k \in \mathbb{N}$ and every (u_k) weakly convergent to u in H

$$\tilde{T}_{M_k}(u_k) \to T(u) \quad \textit{in } H \quad \textit{as } k \to \infty.$$

(A.6) For every $u_0 \in H$ the Fréchet differentials

$$\mathrm{d}_u\tilde{T}_M \to \mathrm{d}_{u_0}T \quad \text{in} \quad \mathcal{E}ndH \quad \text{as} \quad (M, u) \succ \mathcal{B}\underline{\times}\mathcal{F}(u_0),$$

where $\mathcal{F}(u_0)$ denotes the filter of all neighbourhoods of u_0 in the norm-topology of H.

Notice that in (A.1), we assume the existence of the solution of the equation $(*_M)$ for every $M \in \mathcal{S}(H)$ different from H, whereas condition (A.3) concernes a priori estimates of the solutions. Notice also that assumption (A.5) is satisfied in the case when

(A.5') for every $u_0 \in H$

$$\tilde{T}_M(u) \to T(u_0) \text{ in } H \qquad \text{as } (M,u) \succ \mathcal{B}\underline{\times}\mathcal{F}_{weak}(u_0),$$

where $\mathcal{F}_{weak}(u_0)$ denotes the filter of all neighbourhoods of u_0 in the weak topology of H.

Proof. — Assume that condition (A.5') holds. We will check that condition (A.5) is satisfied. Let (W_k) be an internal approximation of H and let $0 < \delta_k \to 0$. Suppose that $M_k \in B_{W_k,\delta_k}$, $k \in \mathbb{N}$ and $u_k \to u$ weakly in H. Putting $u_0 := u$ in (A.5'), we infer that given $\varepsilon > 0$, there exist $\mathcal{X} \in \mathcal{B}$ and $\mathcal{U} \in \mathcal{F}_{weak}(u)$ such that

$$|\tilde{T}_M(w) - T(u)| < \varepsilon, \qquad (M,w) \in \mathcal{X} \times \mathcal{U}.$$

Since $u_k \to u$ weakly in H, there exists $k_1 \in \mathbb{N}$ such that $u_k \in \mathcal{U}$ for $k \geqslant k_1$. From the construction of the filterbase $\mathcal{B}$,there follows that

$$\mathcal{X} = B_{W,\delta} \quad \text{for some } W \in \mathcal{S}(H) \cap \{\dim < \infty\} \text{ and } \delta > 0.$$

By Corollary 1.7, we infer that there exists $k_2 \in \mathbb{N}$ such that $B_{W_k,\delta_k} \subset B_{W,\delta}$ for each $k \geqslant k_2$. Hence, in particular,

$$|\tilde{T}_{M_k}(u_k) - T(u)| < \varepsilon \qquad \text{for each } k \geqslant k_0 := \max(k_1, k_2),$$

which ends the proof. $\square$

2.2. The convergence result

Now, we will prove some convergence result which states that from a sequence of approximate solutions we can choose a convergent subsequence and its limit is a solution of the equation (*).

THEOREM 2.1 (**Convergence**). — *Suppose that conditions (A.1) - (A.5) hold. Let (W_k) be an internal approximation of H, $0 < \delta_k \to 0$ and $M_k \in B_{W_k,\delta_k}$, $k \in \mathbb{N}$. If $g_k \in M_k$ and $g_k \to g$ in H and $u_k \in \mathfrak{R}_{M_k}(\mu, g_k)$, $k \in \mathbb{N}$ then, there exist an infinite subset $\mathcal{N} \subset \mathbb{N}$ and an element $u \in H$ such that*

$$\lim_{\mathcal{N} \ni k \to \infty} |u_k - u| = 0$$

and $u \in \mathfrak{R}(\mu, g)$.

Proof. — By condition (A.3), we deduce that

$$|u_k| \leqslant \kappa(\mu, |g_k|) \leqslant \max\{\kappa(\mu, |g|), \kappa(\mu, |g_l|), l = 1, 2, \ldots\} < \infty,$$

because the set $\{(\mu, g), (\mu, g_l), l = 1, 2, \ldots\}$ is compact and κ is continuous. Thus, the sequence (u_k) is bounded. By the Banach-Alaoglu theorem, there exist an infinite subset $\mathcal{N} \subset \mathbb{N}$ and an element $u \in H$ such that

$$u_k \to u \qquad \text{weakly in } H \text{ as } \mathcal{N} \ni k \to \infty.$$

We assert that the subsequence $(u_k)_{k \in \mathcal{N}}$ is strongly convergent to u and $u \in \mathfrak{R}(\mu, g)$. Indeed, since $u_k \in \mathfrak{R}_{M_k}(\mu, g_{M_k})$,

$$u_k = -\tilde{T}_{M_k}(u_k) + g_k. \tag{2.3}$$

From (A.5), there follows that

$$\tilde{T}_{M_k}(u_k) \to T(u) \quad \text{in } H \qquad \text{as } \mathcal{N} \ni k \to \infty. \tag{2.4}$$

Thus, taking into account equality (2.3), we infer that $(u_k)_{k \in \mathcal{N}}$ is convergent in the sense of norm to u and $u + T(u) = g$. $\square$

COROLLARY 2.2. — *The set $\mathfrak{R}(\mu, g)$ is nonempty for every $(\mu, g) \in]0, \infty[\times H$.*

Digression. — Let us note that directly from the proof of Theorem 2.1, there follows some weaker version of the convergence result if we replace condition (A.5) with the following one

$$\begin{aligned} &\textit{if a sequence } (M_k) \textit{ is an internal approximation of } H \\ &\textit{and } w_k \to w \textit{ weakly in } H, \textit{ then} \\ &\qquad \tilde{T}_{M_k}(w_k) \to T(w) \quad \textit{weakly in } H \quad \textit{as } k \to \infty. \end{aligned} \tag{2.5}$$

THEOREM 2.3. — *Assume that conditions (A.1) - (A.4) and (2.5) hold. Let (M_k) be an internal approximation of H, $g_k \in M_k$, $g_k \to g$ weakly in H and $u_k \in \mathfrak{R}_{M_k}(\mu, g_k)$. Then, there exist an infinite subset $\mathcal{N} \subset \mathbb{N}$ and an element $u \in H$ such that*

$$u_k \to u \qquad \textit{weakly in } H \textit{ as } \mathcal{N} \ni k \to \infty$$

and $u \in \mathfrak{R}(\mu, g)$.

Thus, here we have weak convergence only. Condition (A.5) guaranties convergence in the norm of some sequence of approximate solutions. Condition (A.5) will be also crucial in the further investigations about stability.

2.3. Properties of the operator T and T_M

We will use the technique of Fredholm mappings. Results investigated in [12], Section 3 will be of great importance. For the convenience of the reader we recall them in Appendix C.

Now, we will concentrate on some properties of the mappings T and T_M and of the sets $\mathfrak{R}(\mu,g)$ and $\mathfrak{R}_M(\mu,g_M)$. Since $\tilde{T}_H = T$ and $H \in B_{W,\delta}$ for all $W \in \mathcal{S}(H) \cap \{\dim < \infty\}$ and $\delta > 0$, condition (A.5) implies that

$$\textit{if } u_k \to u \textit{ weakly in } H, \textit{ then } T(u_k) \to T(u) \textit{ in } H \textit{ as } k \to \infty. \tag{2.6}$$

Thus, in particular,

$$\textit{the mapping } T \textit{ is completely continuous.} \tag{2.7}$$

By (A.3) (with $M := H$), we deduce that for every $u \in \mathfrak{R}(\mu,g)$, the following estimate holds

$$|u| \leqslant \kappa(\mu, |g|). \tag{2.8}$$

Thus, by (2.6) and (2.8), mapping T satisfies assumptions (10.4) - (10.5) in Appendix C.

We will use the following notations

$$\begin{aligned} \mathcal{E}_\mu &: H \ni u \mapsto \mu u + T(u) \in H, && \mu \in]0,\infty[\\ \mathcal{E}_{\mu,M} &: M \ni u \mapsto \mu u + T_M(u) \in M, && \mu \in]0,\infty[,\ M \in \mathcal{S}(H) \setminus H. \end{aligned}$$

Let us note that for every $(\mu, g) \in]0,\infty[\times H$:

$$\mathfrak{R}(\mu,g) = \mathcal{E}_\mu^{-1}(\{g\}) \tag{2.9}$$

and for every $M \in \mathcal{S}(H) \setminus H$ and every $(\mu, g_M) \in]0,\infty[\times M$:

$$\mathfrak{R}_M(\mu,g_M) = \mathcal{E}_{\mu,M}^{-1}(\{g_M\}). \tag{2.10}$$

By (10.7) and Proposition 10.8 mapping $\mathcal{E}_\mu$ has the following properties.

Remark 2.4. — The mapping $\mathcal{E}_\mu$

(1) is a Fredholm mapping of index 0,

(2) is proper, i.e. the preimage of a compact subset is compact.

By Remark 2.4 (1), we infer that for every $u \in H$:

$$d_u\mathcal{E}_\mu \in \mathcal{E}piH \;\;\Leftrightarrow\;\; d_u\mathcal{E}_\mu \in \mathcal{M}onoH \;\;\Leftrightarrow\;\; d_u\mathcal{E}_\mu \in \mathcal{A}utH. \tag{2.11}$$

In view of the relation (2.9) and Remark 2.4 (2), we have

COROLLARY 2.5. — *The set $\mathfrak{R}(\mu, g)$ is a compact subset of H for every pair $(\mu, g) \in]0, \infty[\times H$.*

By the continuity of the mapping $\mathcal{E}_{\mu,M}$, inequality (2.8) and relation (2.10), we infer that

COROLLARY 2.6. — *The set $\mathfrak{R}_M(\mu, g_M)$ is a closed bounded subset of M for every subspace $M \in \mathcal{S}(H)$ and every pair $(\mu, g_M) \in]0, \infty[\times M$.*

Proof. — Since $\{g_M\}$ is closed and $\mathcal{E}_{\mu,M}$ is continuous, thus $\mathfrak{R}_M(\mu, g_M)$ is closed as the preimage of a closed set by continuous mapping. By the inequality in assumption (A.3),

$$|w| \leqslant \kappa(\mu, |g_M|)$$

for every $w \in \mathfrak{R}_M(\mu, g_M)$. Thus $\mathfrak{R}_M(\mu, g_M)$ is bounded. $\square$

2.4. The implicit function theorem – version for the space with filterbase

In the sequel we will use the following version of the implicit function theorem proven in [7].

THEOREM 2.7. — *(Th. 1.20 in [7]). Let $\mathcal{B}$ be a filterbase on a set X and let $x_0 \in \bigcap \mathcal{B}$. Consider Banach spaces Y, Z and a point $y_0 \in Y$. Suppose that a mapping $F : X \times Y \to Z$ satisfies the following conditions*

(i) $F(x_0, y_0) = 0$ *;*

(ii) *for every $(x, y) \in X \times Y$ there exists the Fréchet differential*

$$d^{II}_{(x,y)}F := d_y F(x, \cdot) \in \mathcal{L}(Y, Z);$$

(iii) $d^{II}_{(x_0,y_0)}F \in \mathcal{I}so(Y, Z)$*;*

(iv) *for every $y \in Y$: $\lim_{x \succ \mathcal{B}} F(x, y) = F(x_0, y)$;*

(v) $d^{II}_{(x,y)}F \to d^{II}_{(x_0,y_0)}F$ *in $\mathcal{L}(Y, Z)$ as $(x, y) \succ \mathcal{B}\underline{\times}\mathcal{F}(y_0)$, where $\mathcal{B}\underline{\times}\mathcal{F}(y_0) := \{B \times \mathcal{U},\ B \in \mathcal{B},\ \mathcal{U} \in \mathcal{F}(y_0)\}$.*

Then, there exist $\mathcal{X} \in \mathcal{B}$ *and* $\mathcal{Y} \in \mathcal{F}(y_0)$ *such that the relation*

$\eta := \{F = 0\} \cap (\mathcal{X} \times \mathcal{Y})$ *is a function from* $\mathcal{X}$ *to* $\mathcal{Y}$ *and* $\lim_{x \succ \mathcal{B} \bar{\cap} \mathcal{X}} \eta(x) = y_0$,

where $\mathcal{B} \bar{\cap} \mathcal{X} := \{B \in \mathcal{B} : \;\; B \subset \mathcal{X}\}$.

($\mathcal{F}(y_0)$ *denotes the filter of all neighbourhoods of* y_0 *in* Y*.)*

2.5. The stability problem

Using the above version of the implicit function theorem with

$$F : \mathcal{S}(H) \times H \ni (M, u) \mapsto \mu u + \tilde{T}_M(u) - g_M \in H \tag{2.12}$$

and $(x_0, y_0) := (H, u_0)$, where $u_0 \in \mathfrak{R}(\mu, g)$, we obtain the following lemma.

Lemma 2.8. — *Assume that the conditions (A.1) - (A.6) hold. Let* $(\mu, g) \in]0, \infty[\times H$, $u_0 \in \mathfrak{R}(\mu, g)$ *and* $\mathrm{d}_{u_0}\mathcal{E}_\mu \in \mathcal{E}piH$*. Let* $g_M \to g$ *as* $M \succ \mathcal{B}$*. Then, there exist* $\mathcal{X} \in \mathcal{B}$ *and* $\mathcal{Y} \in \mathcal{F}(u_0)$ *such that*

- **(i)** $\#\big(\mathcal{Y} \cap \mathfrak{R}_M(\mu, g_M)\big) = 1$ *for every* $M \in \mathcal{X}$*;*
- **(ii)** $\lim_{M \succ \mathcal{B} \bar{\cap} \mathcal{X}} |u_M - u_0| = 0$, *where* $\{u_M\} := \mathcal{Y} \cap \mathfrak{R}_M(\mu, g_M)$.

Proof. — We will check that the mapping (2.12) satisfies the assumptions of Theorem 2.7. Indeed, since $u_0 \in \mathfrak{R}(\mu, g)$,

$$F(H, u_0) = \mu u_0 + T(u_0) - g = 0.$$

For every $(M, u) \in \mathcal{S}(H) \times H$:

$$\mathrm{d}^{II}_{(M,u)} F = \mu \mathrm{id} + \mathrm{d}_u \tilde{T}_M \in \mathcal{E}ndH.$$

Hence, by (A.2) and (2.11), we infer that

$$\mathrm{d}^{II}_{(H,u_0)} F = \mu \mathrm{id} + \mathrm{d}_{u_0} T = \mathrm{d}_{u_0} \mathcal{E}_\mu \in \mathcal{A}utH,$$

i.e. condition (iii) is fulfielled. Condition (iv) is satified due to assumption (A.4) and condition (v) follows from (A.6).

Thus, there exist $\mathcal{X} \in \mathcal{B}$ and $\mathcal{Y} \in \mathcal{F}(y_0)$ such that the relation $\eta := \{F = 0\} \cap (\mathcal{X} \times \mathcal{Y})$ is a function from $\mathcal{X}$ to $\mathcal{Y}$ and $\lim_{M \succ \mathcal{B} \bar{\cap} \mathcal{X}} |\eta(M) - \eta(H)| = 0$. In particular, for every $M \in \mathcal{X}$ there exists the unique $u_M \in \mathcal{Y}$ such that $\eta(M) = u_M$. Since $\eta \subset \{F = 0\}$, $F(M, u_M) = 0$. Thus $u_M \in \mathcal{Y} \cap \mathfrak{R}_M(\mu, g_M)$, by (A.2). To infer (ii), it is sufficient to note that $\eta(H) = u_0$. $\square$

In the forthcoming considerations we will need set $\mathcal{O}$ introduced while investigating generic properties of the set of solutions $\mathfrak{R}(\mu, g)$. Here we collect properties of this set $\mathcal{O}$ (see Appendix C).

Let us consider the set

$$\mathcal{O} := \{(\mu, g) \in]0, \infty[\times H : g \textit{ is a regular value of the mapping } H \ni u \mapsto \mu u + T(u) \in H\}. \tag{2.13}$$

In Appendix C, we have proven that the function

$$\mathcal{O} \ni (\mu, g) \mapsto \mathfrak{R}(\mu, g) \subset H$$

is continuous if we consider the Hausdorff metric on the family of all nonempty closed and bounded subsets of H. Moreover, $\#\mathfrak{R}(\mu, g) < \infty$ for $(\mu, g) \in \mathcal{O}$ (see Theorem 10.11 in Appendix C). Furthermore, we have the following

THEOREM 2.9. — *The set $\mathcal{O}$ defined by (2.13) is open and dense in $]0, \infty[\times H$.*

(See Theorem 10.12 in Appendix C.) Thus, we can say that the set $\mathfrak{R}(\mu, g)$, generically, depends continuously on the data (μ, g).

Now, we move to the stability problem. We prove that for tha data (μ, g) from the same set $\mathcal{O}$, the set $\mathfrak{R}(\mu, g)$ can be approximated by the sets $\mathfrak{R}_M(\mu, g_M)$ in the Hausdorff metric over H, i. e. that

$$\mathfrak{R}_M(\mu, g_M) \to \mathfrak{R}(\mu, g) \quad \text{in the Hausdorff metric over } H$$

as $M \succ \mathcal{B}$. Then we say that the method, understood as the class of equations $\{(*)_M,\ M \in \mathcal{S}(H)\}$ is, generically, stable with respect to approximation. The main result concerning stability with respect to approximation is expressed in the following

THEOREM 2.10 (**stability**). — *Assume that conditions (A.1)-(A.6) hold. Let $(\mu, g) \in \mathcal{O}$ and $g_M \to g$ as $M \succ \mathcal{B}$. Then for every $\varepsilon > 0$ there exist $W \in \mathcal{S}(H) \cap \{\dim < \infty\}$ and $\delta > 0$ such that*

(i) $d\big(\mathfrak{R}_M(\mu, g_M), \mathfrak{R}(\mu, g)\big) \leqslant \varepsilon$,

(ii) $\#\mathfrak{R}_M(\mu, g_M) = \#\mathfrak{R}(\mu, g) < \infty$,

whenever $M \in B_{W,\delta}$.

(The letter d stands for the Hausdorff metric over H.)

Proof. — Let $(\mu, g) \in \mathcal{O}$. Due to Remark 2.4 (1), $\mathcal{E}_\mu$ is a Fredholm mapping of index 0. Hence, the Smale theorem yields that the set $\mathfrak{R}(\mu, g)$ is discrete. On the other hand, it is compact, by Corollary 2.5. Thus, it is finite.

Let us fix $u \in \mathfrak{R}(\mu, g)$. Then $(\mu\mathrm{id} + \mathrm{d}_u T) \in \mathcal{E}piH$. By Lemma 2.8, there exist (dependent on u) subset $\mathcal{X}(u) \in \mathcal{B}$ and a neighbourhood $\mathcal{Y}(u)$ of u such that

(1) $\#\big(\mathcal{Y}(u) \cap \mathfrak{R}(\mu, g)\big) = 1$ for every $M \in \mathcal{X}(u)$;

(2) $\lim_{M \succ \mathcal{B} \bar{\cap} \mathcal{X}(u)} |u_M - u| = 0$, where $\{u_M\} := \mathcal{Y}(u) \cap \mathfrak{R}_M(\mu, g_M)$.

Since the set $\mathfrak{R}(\mu, g)$ is finite, there exists a number $r > 0$ such that the closed balls

$$\{\bar{K}_H(u, r),\ u \in \mathfrak{R}(\mu, g)\} \quad \text{are pairwise disjoint.}$$

Moreover, by (1.1), there exists $\mathcal{X}_1 \in \mathcal{B}$ such that $\mathcal{X}_1 \subset \bigcap\{\mathcal{X}(u),\ u \in \mathfrak{R}(\mu, g)\}$. Taking into account (2), we infer that

$$\lim_{M \succ \mathcal{B} \bar{\cap} \mathcal{X}_1} \max\{|u_M - u|,\ u \in \mathfrak{R}(\mu, g)\} = 0. \tag{2.14}$$

In particular, there exists $\mathcal{X}_2 \in \mathcal{B} \bar{\cap} \mathcal{X}_1$ such that $u_M \in \bar{K}(u, r)$ whenever $M \in \mathcal{X}_2$. This implies that the function

$$\mathfrak{R}(\mu, g) \ni u \mapsto u_M \in \mathfrak{R}_M(\mu, g_M)$$

is injective for $M \in \mathcal{X}_2$. In particular, $\#\mathfrak{R}(\mu, g) \leqslant \#\mathfrak{R}_M(\mu, g_M)$. We assert that

$$\begin{aligned} &\text{there exists } \tilde{\mathcal{X}} \in \mathcal{B} \bar{\cap} \mathcal{X}_2 \text{ such that} \\ &\#\mathfrak{R}(\mu, g) = \#\mathfrak{R}_M(\mu, g_M) \quad \text{for } M \in \tilde{\mathcal{X}}. \end{aligned} \tag{2.15}$$

Suppose, contrary to our claim, that

$$\begin{aligned} &\text{for every } \mathcal{Z} \in \mathcal{B} \bar{\cap} \mathcal{X}_2 \text{ there exists } M \in \mathcal{Z} \\ &\text{such that } \#\mathfrak{R}(\mu, g) < \#\mathfrak{R}_M(\mu, g_M). \end{aligned} \tag{2.16}$$

Let (W_k) be an internal approximation of the space H and let $0 < \delta_k \to 0$. By Corollary 1.7 (a), we infer that there exists $k_0 \in \mathbb{N}$ such that

$$B_{W_k, \delta_k} \subset \mathcal{Z} \qquad \text{for each } k \geqslant k_0.$$

Let $k \geqslant k_0$. By (2.16), there exists $M_k \in B_{W_k, \delta_k}$ such that

$$\#\mathfrak{R}(\mu, g) < \#\mathfrak{R}_{M_k}(\mu, g_{M_k}).$$

The set $\{u_{M_k},\ u \in \mathfrak{R}(\mu, g)\}$ has exactly $\#\mathfrak{R}(\mu, g)$ elements. Thus, it is not the whole set $\mathfrak{R}_{M_k}(\mu, g_{M_k})$. Let us select

$$a_k \in \mathfrak{R}_{M_k}(\mu, g_{M_k}) \setminus \{u_{M_k},\ u \in \mathfrak{R}(\mu, g)\}.$$

Then $a_k \notin \bigcup\{\mathcal{Y}(u),\ u \in \mathfrak{R}(\mu, g)$. Since $a_k \in \mathfrak{R}_{M_k}(\mu, g_{M_k})$ and $g_{M_k} \to g$, Theorem 2.1 yields that there exist an infinite subset $\mathcal{N} \subset \mathbb{N}$ and an element $a \in H$ such that $a_k \to a$ as $\mathcal{N} \ni k \to \infty$. Moreover, $a \in \mathfrak{R}(\mu, g)$. This leads to a contradiction

$$\bigcup\{\mathcal{Y}(u),\ u \in \mathfrak{R}(\mu, g)\} \supset \mathfrak{R}(\mu, g) \ni a \notin \bigcup\{\mathcal{Y}(u),\ u \in \mathfrak{R}(\mu, g)\}$$

Thus, (2.15) holds. At the same time equality in assertion (ii) holds for $M \in \tilde{\mathcal{X}}$.

To prove (i), let us fix $\varepsilon > 0$. From (2.14) and (2.15), there follows that there exists $\mathcal{X} \in \mathcal{B}\bar{\cap}\tilde{\mathcal{X}}$ such that

$$\max\{|u_M - u|,\ u \in \mathfrak{R}(\mu, g)\} \leqslant \varepsilon$$

whenever $M \in \mathcal{X}$. Thus

$$\mathfrak{R}(\mu, g) \subset \mathfrak{R}_M(\mu, g_M) + \bar{K}(\varepsilon), \qquad M \in \mathcal{X}. \tag{2.17}$$

We will show that

$$\mathfrak{R}_M(\mu, g_M) \subset \mathfrak{R}(\mu, g) + \bar{K}(\varepsilon), \qquad M \in \mathcal{X}. \tag{2.18}$$

Indeed, let $w \in \mathfrak{R}_M(\mu, g_M)$. Then $w = u_M$ for some $u \in \mathfrak{R}(\mu, g)$. Hence $|w - u| = |u_M - u| \leqslant \varepsilon$ and

$$w \in \{u\} + \bar{K}(\varepsilon) \subset \mathfrak{R}(\mu, g) + \bar{K}(\varepsilon).$$

Inclusions (2.17), (2.18) mean that the Hausdorff distance between $\mathfrak{R}(\mu, g)$ and $\mathfrak{R}_M(\mu, g_M)$ is not greater than ε. To complete the proof, let us remark that from the construction of the filterbase $\mathcal{B}$, there follows that $\mathcal{X} = B_{W,\delta}$ for some $W \in \mathcal{S}(H) \cap \{\dim < \infty\}$ and $\delta > 0$. □

3. The stationary Navier-Stokes equations.

We consider the stationary Navier-Stokes equations for viscous incompressible fluid filling the bounded domain $\Omega \subset \mathbb{R}^n$ with the Lipschitian boundary $\partial\Omega$, where $n \in \{2, 3, 4\}$

$$\partial^v v = \nu \Delta v + f - \nabla p \tag{3.1}$$

$$\mathrm{div} v = 0 \tag{3.2}$$

with the homogeneous boundary condition

$$v_{\partial\Omega} = 0. \tag{3.3}$$

For any differentiable vector fields

$$u = (u_1, \dots, u_n) : \Omega \to \mathbb{R}^n, \qquad w = (w_1, \dots, w_n) : \Omega \to \mathbb{R}^n$$

the symbol $\partial^u w$ stands for the vector field

$$\sum_{i=1}^{n} u_i \frac{\partial w}{\partial x_i}.$$

Let us also recall that

$$\mathrm{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i}.$$

Vector fields satisfying (3.2) are called *solenoidal* or *divergence-free.* The number $\nu \in]0, \infty[$ (kinematic viscosity) and $f : \Omega \to \mathbb{R}^n$ (external forces) are given, while $v : \Omega \to \mathbb{R}^n$ (velocity) and $p : \Omega \to \mathbb{R}$ (pressure) are looked for. We will consider weak solutions of the problem (3.1) - (3.3).

Sobolev spaces. Let $Y \in \{\mathbb{R}, \mathbb{R}^n\}$. The symbol $\mathcal{D}(\Omega, Y)$ stands for the space of all *test functions* $\phi : \Omega \to Y$, i.e., $\mathcal{C}^\infty$-mappings with compact support contained in Ω. We will consider the Sobolev space

$$H^1(\Omega, Y) := \{u \in L^2(\Omega, Y) : \text{ there exist } \tfrac{\partial u}{\partial x_i} \text{ in the weak sense}$$
$$\text{and } \tfrac{\partial u}{\partial x_i} \in L^2(\Omega, Y) \text{ for each } 1 \leqslant i \leqslant n\},$$

which is a Hilbert space with the scalar product

$$(u, w) \mapsto (u|w)_{L^2(\Omega,Y)} + ((u|w)),$$

where $((u|w)) := \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} | \frac{\partial w}{\partial x_i}\right)_{L^2(\Omega,Y)}$. The symbol $H^1_0(\Omega, Y)$ stands for the closure of $\mathcal{D}(\Omega, Y)$ in $H^1(\Omega, Y)$. From the well-known Poincaré inequality, it follows that the form $((\cdot|\cdot))$ is a scalar product in $H^1_0(\Omega, Y)$ inducing the topology inherited from $H^1(\Omega, Y)$. It is called the *Dirichlet scalar product .*

In the sequal, we will consider $H^1_0 := H^1_0(\Omega, \mathbb{R}^n)$ equipped with the Dirichlet scalar product $((\cdot|\cdot))$.

From the Sobolev embedding theorem (see Th. 5.4 in [1]), it follows that

$H^1_0(\Omega, Y) \subset L^{r(n)}(\Omega, Y)$ and the embedding
$H^1_0(\Omega, Y) \hookrightarrow L^{r(n)}(\Omega, Y)$ is continuous,

where

$$r(n) := \begin{cases} \frac{2n}{n-2} & \text{if } n \geqslant 3, \\ \textit{any number in }]1,\infty[& \text{if } n = 2. \end{cases}$$

In particular, for $n \in \{2,3,4\}$, the embedding

$$H_0^1(\Omega, Y) \hookrightarrow L^4(\Omega, Y) \tag{3.4}$$

is well defined and continuous. By the Rellich-Kondrashev theorem (see Th. 6.2 in [1]), the embedding $H_0^1(\Omega, Y) \hookrightarrow L^2(\Omega, Y)$ is completely continuous. If $n \in \{2,3\}$, then the embedding

$$H_0^1(\Omega, Y) \hookrightarrow L^6(\Omega, Y) \tag{3.5}$$

is well defined and continuous and the embedding

$$\iota : H_0^1 \hookrightarrow L^4 \tag{3.6}$$

is completely continuous (see Th. 6.2 in [1]).

Let $\mathcal{V} := \mathcal{D}(\Omega, \mathbb{R}^n) \cap \{\mathrm{div} = 0\}$ denote the space of all divergence-free test vector fields on Ω, and let V be its closure in the Hilbert space $\left(H_0^1, ((\cdot|\cdot))\right)$. Let us recall the weak formulation of the problem (3.1)-(3.3) due to J. Leray.

Definition 3.1. — *Suppose that $n \in \{2,3,4\}$ and $f \in (H_0^1)'$. A vector field $v \in V$ is a (weak) solution of the problem (3.1) - (3.3) iff for all $\phi \in V$:*

$$\int_\Omega (\partial^v v)\phi \, dm = -\nu((v|\phi)) + f(\phi). \tag{3.7}$$

It is well-known that there exists at least one solution of the problem (3.1)-(3.3). For example, J.L. Lions, using the Galerkin method, has proven the existence of a weak solution (see [10], Sect. I, Th. 7.1 and [19], Ch.II, Th. 1.2).

The Leray idea of the choice of divergence-free test vector fields $\phi \in V$ separates the problem of finding the velocity v and the pressure p. However, it is well known that the pressure can be recovered, in general, as a distribution, by applying the de Rham theorem, see Temam [19]. To be more specific, there exists a scalar-valued distribution $P \in \mathcal{D}'(\Omega)$ such that the pair (v, P) satisfies the Navier-Stokes equation

$$\partial^v v = \nu \Delta v + f - \nabla P$$

in the distribution sense. In fact, P is a regular distribution generated by a unique $p \in L^2(\Omega)$ with $\int_\Omega p(x)\, dx = 0$, i.e. $P = [p]$.

4. Stability of the Galerkin method with respect to the approximation of the space V

4.1. Basic facts and notations.

Let us consider the following three-linear form

$$b : V^3 \ni (u, w, \phi) \mapsto \int_\Omega (\partial^u w)\phi \, dm \in \mathbb{R}. \tag{4.1}$$

Since $\mathrm{div} u = 0$ for $u \in V$, we have

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}(u_i w) = \Big(\sum_{i=1}^n \frac{\partial u_i}{\partial x_i}\Big) w + \sum_{i=1}^n u_i \frac{\partial w}{\partial x_i} = (\mathrm{div} u) w + \partial^u w = \partial^u w,$$

Hence, by the integration by parts formula,

$$\begin{aligned} b(u, w, \phi) &= \int_\Omega (\partial^u w)\phi \, dx = \sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i}(u_i w)\phi \, dx = -\sum_{i=1}^n \int_\Omega u_i w \frac{\partial \phi}{\partial x_i} \, dx \\ &= -\int_\Omega \Big(\sum_{i=1}^n u_i \frac{\partial \phi}{\partial x_i}\Big) w \, dx = -\int_\Omega (\partial^u \phi) w \, dx = -b(u, \phi, w). \end{aligned}$$

Thus

$$b(u, w, \phi) = -b(u, \phi, w), \qquad u, w, \phi \in V. \tag{4.2}$$

In particular,

$$b(u, \phi, \phi) = 0 \qquad u, \phi \in V. \tag{4.3}$$

(See [19], Chapter II, Lemma 1.3). By the Sobolev embedding theorem and the Hőlder inequality, it is easy to obtain the following inequalities

$$\begin{aligned} |b(u, w, \phi)| = |b(u, \phi, w)| &\leqslant \|u\|_{L^4} \|w\|_{L^4} \|\phi\|_V && (4.4) \\ &\leqslant |\iota|^2 \|u\|_V \|w\|_V \|\phi\|_V, && (4.5) \end{aligned}$$

where $|\iota|$ stands for the norm of the embedding $\iota : H_0^1(\Omega, \mathbb{R}^n) \hookrightarrow L^4(\Omega, \mathbb{R}^n)$. Thus, the form b is continuous. (See [19], Chapter II, Lemma 1.2). Moreover, if $B(u, w) := b(u, w, \cdot) \in V'$, then by (4.4) and (4.5), we have the following inequalities

$$|B(u, w)|_{V'} \leqslant \|u\|_{L^4} \|w\|_{L^4} \leqslant |\iota|^2 \|u\|_V \|w\|_V, \qquad u, v \in V. \tag{4.6}$$

Thus, the mapping $B : V \times V \to V'$ is bilinear and continuous.

Since for fixed $v \in V$, $b(v, v, \cdot) \in V'$, thus by the Riesz representation theorem, there exists a unique element $Q(v) \in V$ such that

$$b(v, v, \phi) = ((Q(v)|\phi)) \qquad \text{for all } \phi \in V.$$

Using $\mathcal{R}_V$ - the Riesz isomorphism in the space V, we have the following relation

$$Q(v) = \mathcal{R}_V^{-1} B(v,v), \qquad v \in V. \tag{4.7}$$

Similarly, there exists a unique element $c \in V$ such that

$$f_{|V}(\phi) = ((c|\phi)), \qquad \phi \in V.$$

In this way, the variational equality (3.7) can be written in the form

$$\nu v + Q(v) = c. \tag{4.8}$$

Now, let us concentrate on some properties of the mapping Q. By (4.3), we have

$$((Q(v)|v)) = 0, \qquad v \in V. \tag{4.9}$$

It is easy to verify that in the case of $n \in \{2,3,4\}$ the mapping Q maps weakly convergent sequences into weakly convergent sequences, i.e.

$$\text{if } v_k \to v \text{ weakly in } V, \text{ then} \tag{4.10}$$

$$Q(v_k) \to Q(v) \text{ weakly in } V \text{ as } k \to \infty. \tag{4.11}$$

(see [19], Chapter II, Lemma 1.5). However, if $n \in \{2,3\}$, then we can prove a stronger result. In fact, we have the following

Lemma 4.1. — *Assume that $n \in \{2,3\}$. If two sequences (u_k) and (w_k) tend weakly in V to u and w, respectively as $k \to \infty$, then*

$$\lim_{k\to\infty} |B(u_k,w_k) - B(u,w)|_{V'} = 0. \tag{4.12}$$

Proof. — Using the first inequality in (4.6), we obtain

$$\begin{aligned} |B(u_k,w_k) - B(u,w)|_{V'} &\leqslant |B(u_k,w_k-w)|_{V'} + |B(u_k-u,w)|_{V'} \\ &\leqslant \|u_k\|_{L^4} \cdot \|w_k - w\|_{L^4} + \|u_k - u\|_{L^4} \cdot \|w\|_{L^4}. \end{aligned}$$

Since the embedding $\iota : H_0^1 \hookrightarrow L^4$ is completely continuous, thus ι maps sequences weakly convergent in H_0^1 into sequences convergent in the norm of L^4. The proof is thus complete. $\square$

Corollary 4.2. — *Assume that $n \in \{2,3\}$. If sequence (v_k) tends weakly in V to v as $k \to \infty$, then*

$$\lim_{k\to\infty} \|Q(v_k) - Q(v)\|_V = 0. \tag{4.13}$$

In particular, the mapping Q is completely continuous.

Proof. — By the relation (4.7) the assertion follows directly from Lemma 4.1. $\square$

4.2. Stability of the Galerkin method – application of the abstract framework.

Assume that $n \in \{2,3\}$. Recall that using the Riesz representations of appropriate functionals on Hilbert space V, the variational equality (3.7) in Definition 3.1 has been written as the following equation in the space V

$$\nu v + Q(v) = c. \tag{4.14}$$

In particular, the set of solutions of the Navier-Stokes problem (3.1) - (3.3) coincides with the set of solutions of equation (4.14).

Let M be a closed linear subspace of V. Using the $(\!(\cdot|\cdot)\!)$ - orthogonal projection $P_M : V \to M$, consider the Galerkin equation induced by (4.14) on the subspace M, i.e.

$$u + P_M Q(u) = P_M c \tag{4.15}$$

and let us denote

$$\begin{aligned} \mathfrak{S}(\nu, c) &:= \text{ the set of all solutions of equation (4.14)} && (4.16)\\ \mathfrak{S}_M(\nu, c) &:= \text{ the set of all solutions of equation (4.15),} && (4.17) \end{aligned}$$

where $\nu > 0$ and $c \in V$ are given.

We begin with some auxiliary result. Using the fixed point theorem in the version of Theorem 9.4 in Appendix B, we will prove the following

Proposition 4.3. — *Let M be a closed linear subspace of V. Then for every $\mu > 0$ and every $g_M \in M$ there exists $w \in M$ such that*

$$\mu w + P_M Q(w) = g_M. \tag{4.18}$$

Moreover,

$$\|w\|_V \leqslant \frac{\|g_M\|_V}{\mu}. \tag{4.19}$$

Proof. — Let us fix $\mu > 0$ and $g_M \in M$. We begin with proving inequality (4.19). Suppose that $w \in M$ satisfies equation (4.18). Multiply the equation (4.18) scalarly in V by w to obtain

$$\mu \|w\|_V^2 + \big(\!\big(P_M Q(w)|w\big)\!\big) = \big(\!\big(g_M|w\big)\!\big).$$

Since the projection P_M is selfadjoint and $P_M w = w$, we obtain

$$\mu \|w\|_V^2 + \big(\!\big(Q(w)|w\big)\!\big) = \big(\!\big(g_M|w\big)\!\big).$$

Since $\big(\!\big(Q(w)|w\big)\!\big) = 0$, then by the Schwarz inequality, we get

$$\mu\|w\|_V^2 = \big(\!\big(g_M|w\big)\!\big) \leqslant \|g_M\|_V \cdot \|w\|_V.$$

Thus $\|w\|_V \leqslant \frac{\|g_M\|_V}{\mu}$, i.e. inequality (4.19) holds.

To prove the first part of the statement, let us consider the ball $\bar{K}_M(R) := \{x \in M : \ \|x\|_V \leqslant R\}$. We assert that the mapping

$$F : M \supset \bar{K}_M(R) \ni u \mapsto u + \frac{1}{\mu}P_M Q(u) - \frac{1}{\mu}g_M \in M$$

satisfies the assumptions of Theorem 9.4 in Appendix B with $R := \frac{\|g_M\|_V}{\mu}$.

Indeed, let $\zeta \in \partial\bar{K}_M(R)$, i.e. $\|\zeta\|_V = R$. We calculate

$$\begin{aligned}((F(\zeta)|\zeta)) \ &= \ \big(\!\big(\zeta + \frac{1}{\mu}P_M Q(\zeta) - \frac{g_M}{\mu}|\zeta\big)\!\big) = \|\zeta\|_V^2 + \frac{1}{\mu}\big(\!\big(Q(\zeta)|\zeta\big)\!\big) - \frac{1}{\mu}\big(\!\big(g_M|\zeta\big)\!\big)\\ &= \ \|\zeta\|_V^2 - \frac{1}{\mu}\big(\!\big(g_M|\zeta\big)\!\big).\end{aligned}$$

Since $\|\zeta\|_V = \frac{\|g_M\|_V}{\mu}$ and $\frac{1}{\mu}\big(\!\big(g_M|\zeta\big) \leqslant \frac{\|g_M\|_V}{\mu}\|\zeta\|_V = \|\zeta\|_V^2$, we infer that

$((F(\zeta)|\zeta)) \geqslant 0.$

By Corollary 4.2, the mapping Q is completely continuous. Thus, also the mapping

$$\mathrm{id}_M - F = \frac{1}{\mu}g_M - \frac{1}{\mu}P_M \circ Q$$

is completely continuous. In particular, the set $(\mathrm{id}_M - F)(\bar{K}_M(R))$ is relatively compact in M. Consequently, Theorem 9.4 implies that the set $\{F = 0\}$ is nonempty, or equivalently, that the set of solutions of equation (4.18) is nonempty. $\square$

Consider the set

$$\begin{aligned}\mathbf{G} := \{(\nu, c) \in]0, \infty[\times V : \ &\textit{c is a regular value of the}\\ &\textit{mapping } V \ni \phi \mapsto \nu\phi + Q(\phi) \in V\}\end{aligned} \tag{4.20}$$

Consider the family $\mathcal{S}(V)$ of all closed linear subspaces of V and let $\mathcal{B}$ denote the filterbase on $\mathcal{S}(V)$ described in Preliminaries (see (1.3) and (1.4)). Then we have the corresponding family of equations (4.15) for every subspace $M \in \mathcal{S}(V)$. Using the abstract framework from Section 2, we prove the following result.

Theorem 4.4 **(stability of the Galerkin method)**. — *Assume that* $(\nu, c) \in \mathbf{G}$. *Then, for every* $\varepsilon > 0$ *there exist* $W \in \mathcal{S}(V) \cap \{\dim < \infty\}$ *and* $\delta > 0$ *such that*

(i) $d\big(\mathfrak{S}_M(\nu, c), \mathfrak{S}(\nu, c)\big) \leqslant \varepsilon$,

(ii) $\#\mathfrak{S}_M(\nu, c) = \#\mathfrak{S}(\nu, c) < \infty$,

whenever $M \in B_{W,\delta}$. *(Here* d *stands for the Hausdorff metric over* V*.)*

In particular, assertion **(i)** quarantees that

$$\lim_{M \succ \mathcal{B}} \mathfrak{S}_M(\nu, c) = \mathfrak{S}(\nu, c) \qquad \text{in the Hausdorff metric over } V,$$

i.e. that the sets of solutions of the Galerkin equations tend to the set of the Navier-Stokes equation in the Hausdorff metric over V as M approaches V in the sense of the filterbase $\mathcal{B}$.

Proof of Theorem 4.4. — We apply the abstract framework from Section 2 to the Hilbert space $\big(V, ((\cdot|\cdot))\big)$ and the mappings

$$T(u) := Q(u), \qquad u \in V$$

and

$$T_M(w) := P_M Q(w), \qquad w \in M,$$

where $M \in \mathcal{S}(V)$.

To apply Theorem 2.10, we will check that the mappings T and T_M satisfy conditions (A.1)-(A.6) of Section 2.1.

Ad. (A.1). Condition (A.1) is satisfied due to Proposition 4.3.

Ad. (A.2). It is sufficient to take

$$\tilde{T}_M(u) := P_M Q(w), \qquad u \in V,$$

i.e. $\tilde{T}_M$ is given by the same formula as T_M, but $\tilde{T}_M$ is considered as the mapping on the whole space V.

Let $g_M \in M$ and denote

$$\begin{aligned} \mathfrak{R}_M(\mu, g_M) &:= \{w \in M : \quad \mu w + T_M(w) = g_M\}, \\ \tilde{\mathfrak{R}}_M(\mu, g_M) &:= \{u \in V : \quad \mu u + \tilde{T}_M(u) = g_M\}. \end{aligned}$$

It is clear that $\mathfrak{R}_M(\mu, g_M) \subset \tilde{\mathfrak{R}}_M(\mu, g_M)$. On the other hand, since $g_M \in M$ and $\tilde{T}_M(V) = (P_M \circ Q)(V) \subset M$, we infer that also $\tilde{\mathfrak{R}}_M(\mu, g_M) \subset M \cap \mathfrak{R}_M(\mu, g_M) \subset \mathfrak{R}_M(\mu, g_M)$.

Ad. (A.3). By inequality (4.19) in Proposition 4.3, condition (A.3) holds with

$$\kappa :]0,\infty[\times[0,\infty[\ni(\mu,r)\mapsto\frac{r}{\mu}\in[0,\infty[.$$

Ad. (A.4). Let $u\in V$. By Corollary 1.5

$$\lim_{M\succ\mathcal{B}}\|P_MQ(u)-Q(u)\|_V=0,$$

thus condition (A.4) is satisfied.

Ad. (A.5). Let (W_k) be an internal approximation of V, (see Definition 1.6), let $0<\delta_k\to 0$ and $M_k\in B_{W_k,\delta_k}$, $k\in\mathbb{N}$. Suppose that $u_k\to u$ weakly in V. We have

$$\begin{aligned}\|\tilde{T}_{M_k}(u_k)-T(u)\|_V &= \|P_{M_k}Q(u_k)-Q(u)\|_V\\ &\leqslant \|P_{M_k}\big(Q(u_k)-Q(u)\big)\|_V+\|P_{M_k}Q(u)-Q(u)\|_V\\ &\leqslant \|Q(u_k)-Q(u)\|_V+\|P_{M_k}Q(u)-Q(u)\|_V.\end{aligned}$$

By Corollary 4.2, $\lim_{k\to\infty}\|Q(u_k)-Q(u)\|_V=0$ and by Corollary 1.7 (b), $\lim_{k\to\infty}\|P_{M_k}Q(u)-Q(u)\|_V=0$. Thus $\lim_{k\to\infty}\|\tilde{T}_{M_k}(u_k)-T(u)\|_V=0$ and condition (A.5) is satisfied.

Ad. (A.6). Let us fix $u_0\in V$ and let $u\in V$. Let us calculate the Fréchet differentials

$$\begin{aligned}\mathrm{d}_u\tilde{T}_M &= P_M\circ d_uQ, \qquad M\in\mathcal{S}(V)\\ \mathrm{d}_{u_0}T &= d_{u_0}Q.\end{aligned}$$

We have to prove that

$$\mathrm{d}_u\tilde{T}_M\to\mathrm{d}_{u_0}T\quad\text{in}\quad\mathcal{E}ndV\quad\text{as}\quad(M,u)\succ\mathcal{B}\times\mathcal{F}(u_0),$$

where $\mathcal{F}(u_0)$ denotes the filter of all neighbourhoods of u_0 in the norm topology of V. We have

$$\begin{aligned}\big|\mathrm{d}_u\tilde{T}_M-\mathrm{d}_{u_0}T\big|_{\mathcal{E}ndV} &= \big|P_Md_uQ-d_{u_0}Q\big|_{\mathcal{E}ndV}\\ &\leqslant \big|P_M\big(d_uQ-d_{u_0}Q\big)\big|_{\mathcal{E}ndV}+\big|P_Md_{u_0}Q-d_{u_0}Q\big|_{\mathcal{E}ndV}\\ &\leqslant \big|d_uQ-d_{u_0}Q\big|_{\mathcal{E}ndV}+\big|P_Md_{u_0}Q-d_{u_0}Q\big|_{\mathcal{E}ndV}.\end{aligned}$$

By Corollary 4.2, the mapping Q is completely continuous, thus its Fréchet differential $d_{u_0}Q$ is a completely continuous linear operator. Since the projections P_M tend to the identity operator id_V pointwise on V as $M\succ\mathcal{B}$

(see Corollary 1.5), thus by Lemma 8.4 in Appendix A $\lim_{M\succ\mathcal{B}}\big|P_M d_{u_0}Q - d_{u_0}Q\big|_{\mathcal{E}ndV} = 0$. It is sufficient to prove that

$$\lim_{u\to u_0}\big|d_uQ - d_{u_0}Q\big|_{\mathcal{E}ndV} = 0,$$

or equivalently,

$$\lim_{u\to u_0}\big|\mathcal{R}_V\circ d_uQ - \mathcal{R}_V\circ d_{u_0}Q\big|_{\mathcal{L}(V,V')} = 0. \tag{4.21}$$

By (4.7), $\mathcal{R}_V\circ d_uQ = d_u\big(\mathcal{R}_V\circ Q\big) = d_{(u,u)}B$. Since the mapping

$$B : V\times V \ni (u,w)\mapsto B(u,w)\in V'$$

is bilinear and continuous (see (4.6)), thus

$$d_u\big(\mathcal{R}_V\circ Q\big)(h) = B(u,h) + B(h,u), \qquad h\in V.$$

Then by (4.6)

$$\begin{aligned}
&\big|d_u\big(\mathcal{R}_V\circ Q\big)(h) - d_{u_0}\big(\mathcal{R}_V\circ Q\big)(h)\big|_{V'}\\
&\qquad = \big|B(u,h) + B(h,u) - B(u_0,h) - B(h,u_0)\big|_{V'}\\
&\qquad \leqslant \big|B(u-u_0,h)\big|_{V'} + \big|B(h,u-u_0)\big|_{V'} \leqslant 2|\iota|^2\|u-u_0\|_V\|h\|_V.
\end{aligned}$$

Thus

$$\big|d_u\big(\mathcal{R}_V\circ Q\big) - d_{u_0}\big(\mathcal{R}_V\circ Q\big)\big|_{\mathcal{L}(V,V')} \leqslant 2|\iota|^2\|u-u_0\|_V.$$

Hence

$$\lim_{u\to u_0}\big|d_u\big(\mathcal{R}_V\circ Q\big) - d_{u_0}\big(\mathcal{R}_V\circ Q\big)\big|_{\mathcal{L}(V,V')} = 0$$

and (4.21) holds. At the same time, this guaranties that condition (A.6) is fulfilled.

Let us fix $(\nu,c)\in\mathbf{G}$ and note that the sets $\mathfrak{S}(\nu,c)$ and $\mathfrak{S}_M(\nu,c)$ correspond to the following sets from the abstract setting

$$\mathfrak{S}(\nu,c) = \mathfrak{R}(\mu,g) \quad\text{ and }\quad \mathfrak{S}_M(\nu,c) = \mathfrak{R}_M(\mu,g_M) \tag{4.22}$$

for $\mu := \nu$, $g := c$ and $g_M := P_Mc$ (compare (2.1) and (2.2) in Section 2.1 with (4.16) and (4.17)). Now, the assertion follows from Theorem 2.10.

Directly by Theorem 2.9, we obtain

Corollary 4.5. — *The set* $\mathbf{G}$ *defined by (4.20) is open and dense in* $]0,\infty[\times V$.

In view of Theorem 4.4 and Corollary 4.5, we can say that the Galerkin method is generically stable with respect to approximation of the space V.

5. The Holly method

Holly introduced a new method of finding approximate velocity in the problem (3.1) - (3.3). The approximate solutions are constructed in the subspaces of the whole Sobolev space H_0^1. In the sequel, we will write the integral identity (3.7) in the equivalent form, as some operator equation of the type $(*)$ of Section 2.1, in the space H_0^1. To this aim, we consider first the acceleration functional and its properties.

5.1. The acceleration functional

PROPOSITION 5.1 (Lemma 2.7 in [12]). — *Assume that* $n \in \{2, 3, 4\}$. *Then*

(a) *For any* $u, w \in H_0^1$ *the functional*

$$\mathcal{A}_{u,w} : H_0^1 \ni \phi \mapsto \int_\Omega \big(\partial^u w + \frac{\operatorname{div} u}{2} w\big) \phi \, dx \in \mathbb{R} \tag{5.1}$$

is well-defined linear continuous and the following inequality holds

$$|\mathcal{A}_{u,w}|_{(H_0^1)'} \leqslant \frac{3}{2} |\iota|^2 \|u\|_{H_0^1} \|w\|_{H_0^1} \tag{5.2}$$

The symbol $|\iota|$ *stands for the norm of the embedding* $\iota : H_0^1 \hookrightarrow L^4$. *Moreover,*

$$\mathcal{A}_{u,w}(\phi) = -\mathcal{A}_{u,\phi}(w) \qquad \textit{for any } \phi \in H_0^1. \tag{5.3}$$

In particular

$$\mathcal{A}_{u,w}(w) = 0. \tag{5.4}$$

(b) *The mapping*

$$\mathcal{A} : H_0^1 \times H_0^1 \ni (u, w) \mapsto \mathcal{A}_{u,w} \in (H_0^1)' \tag{5.5}$$

is bilinear and continuous.

The functional defined by (5.1) is called *the acceleration functional.*

Proof. — By the Schwarz and the Hölder inequalities, we obtain

$$\begin{aligned}
|\mathcal{A}_{u,w}(\phi)| &= \Big|\int_\Omega \big(\partial^u w + \frac{\mathrm{div} u}{2} w\big)\phi dx\Big| \\
&\leqslant \int_\Omega |(\partial^u w)\phi| dx + \frac{1}{2}\int_\Omega |(\mathrm{div} u) w \phi| dx \\
&\leqslant \int_\Omega \sum_{i=1}^n |u_i \frac{\partial w}{\partial x_i}\phi| dx + \frac{1}{2}\int_\Omega |(\mathrm{div} u) w \phi| dx \\
&\leqslant \int_\Omega |u| (\sum_{i=1}^n |\frac{\partial w}{\partial x_i}|^2)^{\frac{1}{2}} |\phi| dx + \frac{1}{2} ||\mathrm{div} u||_{L^2} \|w\|_{L^4} \|\phi\|_{L^4} \\
&\leqslant \|u\|_{L^4} \|w\|_{H_0^1} \|\phi\|_{L^4} + \frac{1}{2}\|u\|_{H_0^1} \|w\|_{L^4} \|\phi\|_{L^4}.
\end{aligned}$$

Because of the continuity of the embedding $\iota : H_0^1 \to L^4$, we have

$$|\mathcal{A}_{u,w}(\phi)| \leqslant \frac{3}{2}|\iota| \|u\|_{H_0^1} \|w\|_{H_0^1} \|\phi\|_{L^4} \leqslant \frac{3}{2}|\iota|^2 \|u\|_{H_0^1} \|w\|_{H_0^1} \|\phi\|_{H_0^1}. \tag{5.6}$$

Thus $\mathcal{A}_{u,w} \in (H_0^1)'$ and

$$|\mathcal{A}_{u,w}|_{(H_0^1)'} \leqslant \frac{3}{2}|\iota|^2 \|u\|_{H_0^1} \|w\|_{H_0^1}.$$

Integrating by parts, we can write the functional (5.1) in the form

$$\begin{aligned}
\mathcal{A}_{u,w}(\phi) &= \int_\Omega \big(\partial^u w + \frac{\mathrm{div} u}{2} w\big)\phi dx = \int_\Omega (\partial^u w)\phi dx + \frac{1}{2}\int_\Omega \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} w \phi dx \\
&= \int_\Omega (\partial^u w)\phi dx - \frac{1}{2}\int_\Omega \sum_{i=1}^n u_i \frac{\partial w}{\partial x_i}\phi dx - \frac{1}{2}\int_\Omega \sum_{i=1}^n u_i \frac{\partial \phi}{\partial x_i} w dx \\
&= \frac{1}{2}\int_\Omega \big((\partial^u w)\phi - (\partial^u \phi) w\big) dx.
\end{aligned}$$

As a consequence of the above equality, we obtain (5.3) and (5.4).

Assertion (b) follows immediately from inequality (5.2). □

Using the acceleration functional we can rewrite the weak formulation of the Navier-Stokes problem (see (3.7)) as the operator equation in the space H_0^1.

Let $P_V : H_0^1 \to V$ denote the $((\cdot|\cdot))$-orthogonal projection and let $\mathcal{R}$ denote the Riesz isomorphism in the space $\big(H_0^1, ((\cdot|\cdot))\big)$, i.e.

$$\mathcal{R} : H_0^1 \ni u \mapsto ((u|\cdot)) \in (H_0^1)'$$

We have the following

PROPOSITION 5.2. — *Suppose $\nu \in]0,\infty[$, $f \in (H_0^1)'$, $v \in H_0^1$. Then the following conditions are equivalent*

(i) *$v \in V$ and v satisfies identity (3.7) for every $\phi \in V$, i.e.*

$$\int_\Omega (\partial^v v)\phi \, dm = -\nu((v|\phi)) + f(\phi).$$

(ii) *v satisfies the following equation*

$$\nu v + P_V \mathcal{R}^{-1} \mathcal{A}_{v,P_V v} = P_V \mathcal{R}^{-1} f.$$

Proof. — To prove the implication (i) $\Rightarrow$ (ii), let $\psi \in H_0^1$. Putting $\phi = P_V \psi$ in the identity (i), and using the fact that $v \in V$, thus $P_V v = v$, we have

$$\begin{aligned} 0 &= \int_\Omega (\partial^v v)\phi dx - f(\phi) + \nu((v|\phi)) = ((\mathcal{R}^{-1}(\mathcal{A}_{v,v} - f)|\phi)) + ((\nu v|\phi)) \\ &= ((P_V \mathcal{R}^{-1}(\mathcal{A}_{v,v} - f)|\psi)) + ((\nu v|\psi)) = ((P_V \mathcal{R}^{-1}(\mathcal{A}_{v,P_V v} - f) + \nu v|\psi)). \end{aligned}$$

Since ψ was chosen in an arbitrary way, we obtain (ii).

To prove that (i) follows from (ii), let us first remark that v is an element of V, because v belongs to the image of the projection P_V. Scalar multiplication of the equation in (ii) by $\phi \in V$ yields

$$\begin{aligned} \nu((v|\phi)) &= ((P_V \mathcal{R}^{-1}(f - \mathcal{A}_{v,P_V v})|\phi)) = ((\mathcal{R}^{-1}(f - \mathcal{A}_{v,v})|\phi)) \\ &= f(\phi) - \mathcal{A}_{v,v}(\phi) = f(\phi) - \int_\Omega (\partial^v v)\phi dx. \end{aligned}$$

Thus v satisfies the integral identity in (i). $\square$

Proposition 5.2 enables us to write the integral identity (3.7) in the definition of the solution of the Navier-Stokes equations in an equivalent form as the equation

$$\nu u + P_V \mathcal{R}^{-1} \mathcal{A}_{u,P_V u} = P_V \mathcal{R}^{-1} f \tag{5.7}$$

in the space H_0^1. This means in particular that the set of elements satisfying identity (3.7) is equal to the set of solutions to the equation (5.7). Note that there is one subtle point here, i.e. we consider (5.7) as the equation in H_0^1. However, its solutions belong, in fact, to the subspace V. This follows from the fact that, if u is a solution of (5.7), then

$$u = \frac{1}{\nu} P_V \mathcal{R}^{-1}(f - \mathcal{A}_{u,P_V u}).$$

Hence u lies in the image of the projection P_V, thus in V.

5.2. Further properties of the acceleration functional

In this subsection, we will prove that mapping $\mathcal{A}$ introduced in Proposition 5.5 is completely continuous. First, we prove the following lemma.

Lemma 5.3. — *Assume that $n \in \{2,3\}$. Then*

(a) *the following inequality holds*

$$|\mathcal{A}_{u,w} - \mathcal{A}_{u^0,w^0}|_{(H_0^1)'} \leqslant \frac{3}{2}|\iota| \|u\|_{H_0^1} \|w - w^0\|_{L^4} + |\iota| \|w^0\|_{H_0^1} \|u - u^0\|_{L^4} \tag{5.8}$$

for any $u, w, u^0, w^0 \in H_0^1$, where $\iota : H_0^1 \hookrightarrow L^4$.

(b) *If two sequences (u^k) and (w^k) tend weakly in H_0^1 to u^0 and w^0, respectively as $k \to \infty$, then*

$$\lim_{k\to\infty} \left|\mathcal{A}_{u^k,w^k} - \mathcal{A}_{u^0,w^0}\right|_{(H_0^1)'} = 0.$$

Proof. — Ad.**(a)**. It is easy to see that

$$\sum_{i=1}^n a_i \frac{\partial b}{\partial x_i} + \frac{\mathrm{div} a}{2} b = \sum_{i=1}^n \frac{\partial}{\partial x_i}(a_i b) - \frac{\mathrm{div} a}{2} b$$

for every vector fields $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \in H_0^1$. Let $\phi \in H_0^1$. Using the above equality and the integration by parts formula, we obtain

$$\begin{aligned}
&\mathcal{A}_{u,w}(\phi) - \mathcal{A}_{u^0,w^0}(\phi) \\
&= \int_\Omega \Big(\sum_{i=1}^n \frac{\partial}{\partial x_i}(u_i w) - \frac{\mathrm{div} u}{2} w \Big) \phi \, dx - \int_\Omega \Big(\sum_{i=1}^n \frac{\partial}{\partial x_i}(u_i^0 w^0) - \frac{\mathrm{div} u^0}{2} w^0 \Big) \phi \, dx \\
&= \int_\Omega \Big(\sum_{i=1}^n \frac{\partial}{\partial x_i}(u_i w - u_i^0 w^0) \Big) \phi \, dx - \frac{1}{2} \int_\Omega \Big((\mathrm{div} u) w - (\mathrm{div} u^0) w^0 \Big) \phi \, dx \\
&= - \int_\Omega \sum_{i=1}^n \Big(u_i (w - w^0) + (u - u^0)_i w^0 \Big) \frac{\partial \phi}{\partial x_i} dx \\
&\quad - \frac{1}{2} \int_\Omega \Big((\mathrm{div} u)(w - w^0) + \big(\mathrm{div}(u - u^0)\big) w^0 \Big) \phi \, dx \\
&= - \int_\Omega \sum_{i=1}^n u_i (w - w^0) \frac{\partial \phi}{\partial x_i} dx - \int_\Omega \sum_{i=1}^n (u - u^0)_i w^0 \frac{\partial \phi}{\partial x_i} dx \\
&\quad - \frac{1}{2} \int_\Omega (\mathrm{div} u)(w - w^0) \phi \, dx - \frac{1}{2} \int_\Omega \mathrm{div}(u - u^0) w^0 \phi \, dx.
\end{aligned}$$

Next, integrating by parts the last term in the above equality, we obtain

$$\begin{aligned}\int_\Omega \operatorname{div}(u-u^0)w^0\phi\,dx &= \int_\Omega \sum_{i=1}^n \frac{\partial(u-u^0)_i}{\partial x_i} w^0\phi\,dx\\ &= -\int_\Omega \sum_{i=1}^n (u-u^0)_i \frac{\partial}{\partial x_i}(w^0\phi)\,dx\\ &= -\int_\Omega \sum_{i=1}^n (u-u^0)_i \frac{\partial w^0}{\partial x_i}\phi\,dx\\ &\qquad -\int_\Omega \sum_{i=1}^n (u-u^0)_i w^0 \frac{\partial \phi}{\partial x_i}\,dx.\end{aligned}$$

Hence

$$\begin{aligned}&-\int_\Omega \sum_{i=1}^n (u-u^0)_i w^0 \frac{\partial \phi}{\partial x_i} dx - \frac{1}{2}\int_\Omega \operatorname{div}(u-u^0)w^0\phi\,dx\\ &= \frac{1}{2}\int_\Omega \sum_{i=1}^n (u-u^0)_i \frac{\partial w^0}{\partial x_i}\phi\,dx - \frac{1}{2}\int_\Omega \sum_{i=1}^n (u-u^0)_i w^0 \frac{\partial \phi}{\partial x_i}\,dx.\end{aligned}$$

Consequently

$$\begin{aligned}\mathcal{A}_{u,w}(\phi)-\mathcal{A}_{u^0,w^0}(\phi) = &-\int_\Omega \sum_{i=1}^n u_i(w-w^0)\frac{\partial \phi}{\partial x_i} dx - \frac{1}{2}\int_\Omega (\operatorname{div} u)(w-w^0)\phi\,dx\\ &+\frac{1}{2}\int_\Omega \sum_{i=1}^n (u-u^0)_i \frac{\partial w^0}{\partial x_i}\phi\,dx\\ &\qquad -\frac{1}{2}\int_\Omega \sum_{i=1}^n (u-u^0)_i w^0 \frac{\partial \phi}{\partial x_i}\,dx.\end{aligned}$$

The Hőlder inequality and continuity of the embedding $\iota : H_0^1 \hookrightarrow L^4$ yields the following inequalities

$$\begin{aligned}|\mathcal{A}_{u,w}(\phi)-\mathcal{A}_{u^0,w^0}(\phi)| &\leqslant \|u\|_{L^4}\|w-w^0\|_{L^4}\|\phi\|_{H_0^1} + \frac{1}{2}\|\operatorname{div} u\|_{L^2}\|w-w^0\|_{L^4}\|\phi\|_{L^4}\\ &\quad +\frac{1}{2}\|u-u^0\|_{L^4}\|w^0\|_{H_0^1}\|\phi\|_{L^4} + \frac{1}{2}\|u-u^0\|_{L^4}\|w^0\|_{L^4}\|\phi\|_{H_0^1}\\ &\leqslant \big(\frac{3}{2}|\iota|\|u\|_{H_0^1}\|w-w^0\|_{L^4} + |\iota|\|w^0\|_{H_0^1}\|u-u^0\|_{L^4}\big)\|\phi\|_{H_0^1}.\end{aligned}$$

Hence

$$|\mathcal{A}_{u,w}-\mathcal{A}_{u^0,w^0}|_{(H_0^1)'} \leqslant \tfrac{3}{2}|\iota|\|u\|_{H_0^1}\|w-w^0\|_{L^4} + |\iota|\|w^0\|_{H_0^1}\|u-u^0\|_{L^4}.$$

Ad.**(b)**. Since the embedding $\iota : H_0^1 \hookrightarrow L^4$ is completely continuous, we infer that

$$\textit{if} \quad v^k \to 0 \quad \textit{weakly} \quad \textit{in} \quad H_0^1 \quad \textit{then} \quad \iota(v^k) \to 0 \quad \textit{in} \quad L^4 \quad \textit{as} \quad k \to \infty.$$

Thus, by virtue of the estimate in assertion **(a)** we conclude that

$$\lim_{k\to\infty} |\mathcal{A}_{u^k,w^k} - \mathcal{A}_{u^0,w^0}|_{(H_0^1)'} = 0,$$

because $\|u^k - u^0\|_{L^4} \to 0$, $\|w^k - w^0\|_{L^4} \to 0$ as $k \to \infty$ and the sequence $(\|u^k\|_{H_0^1})_{k\in\mathbb{N}}$ is bounded. $\square$

By assertion (b) of Lemma 5.3, we have the following

Corollary 5.4. — *If $n \in \{2,3\}$, then the mapping*

$$\mathcal{A} : H_0^1 \times H_0^1 \ni (u,w) \mapsto \mathcal{A}_{u,w} \in (H_0^1)'$$

defined by (5.5) is completely continuous.

5.3. First step of the approximation of the Navier-Stokes equations

The Holly method is split into two steps. In the first step, equation (5.7), i.e.

$$\nu u + P_V \mathcal{R}^{-1} \mathcal{A}_{u,P_V u} = P_V \mathcal{R}^{-1} f$$

in the space H_0^1 is approximated by some equation in H_0^1. The second step involves discretization. However, first we have to consider some operators. To be more specific, we will approximate the projection P_V. The crucial point is the equality

$$P_V u = u - \mathrm{div}^* (\mathrm{div}\mathrm{div}^*)^{-1} \mathrm{div} u$$

(see (11.11) in Appendix D) and inversion of the divdiv^*-operator recalled with details in Appendix D (see Theorem 11.17). Here, $\mathrm{div}^* : L^2(\Omega) \to H_0^1$ is the adjoint of the divergence operator $\mathrm{div} : H_0^1 \to L^2(\Omega)$.

Operators $\mathbf{P_V^s}$. For $s \in \mathbb{N}$, let

$$P_V^s u := u - \mathrm{div}^* \Big(\sum_{j=0}^{s} (\mathrm{id} - \mathrm{div}\mathrm{div}^*)^j \Big) \mathrm{div} u, \qquad u \in H_0^1,$$

Directly from the above formula, it follows that $P_V^s \psi = \psi$ for every $\psi \in V$.

Remark 5.5 Operators P_V^s, $s \in \mathbb{N}$ have the following properties

(a) *P_V^s is a selfadjoint operator of the space H_0^1.*

(b) $|P_V^s - P_V|_{\mathcal{E}ndH_0^1} \leqslant \frac{1}{\theta}(1-\theta)^{s+1}$ *and* $P_V \leqslant P_V^s \leqslant \mathrm{id}_{H_0^1}$,

(c) $||\mathrm{div}P_V^s u||_{L^2(\Omega)} \leqslant \frac{1}{\theta}(1-\theta)^{s+1}||\mathrm{div}u||_{L^2(\Omega)} \leqslant \frac{1}{\theta}(1-\theta)^{s+1}||u||_{H_0^1}$ *for every* $u \in H_0^1$

for some constant $\theta \in]0,1[$ dependent on Ω only.

Proof. — Ad. **(a)** Assertion (a) follows directly from the definitions of P_V^s.

Ad. **(b)** By (11.11), we have

$$\begin{aligned} P_V^s - P_V &= P_{V^\perp} - P_{V^\perp}^s \\ &= \mathrm{div}^*\left((\mathrm{divdiv}^*)^{-1} - \sum_{i=1}^n (\mathrm{id} - \mathrm{divdiv}^*)^i\right)\mathrm{div} \end{aligned} \tag{5.9}$$

By Theorem 11.12 and Theorem 11.17, we obtain

$$\begin{aligned} &|P_V^s - P_V|_{\mathcal{E}nd\{\int=0\}} \\ &\leqslant \left|(\mathrm{divdiv}^*)^{-1} - \sum_{i=1}^n (\mathrm{id} - \mathrm{divdiv}^*)^i\right|_{\mathcal{E}nd\{\int=0\}} \leqslant \frac{1}{\theta}(1-\theta)^{(s+1)}. \end{aligned}$$

The second part of assertion (b) is a consequence of the inequality $0 \leqslant \mathrm{divdiv}^* \leqslant \mathrm{id}$ (see Corollary 11.13).

Ad. **(c)** Since $\mathrm{div}(P_V u) = 0$, thus (5.9), Theorem 11.12 and Theorem 11.17 (b) yield

$$\begin{aligned} \|\mathrm{div}P_V^s(u)\|_{L^2(\Omega)} &= \|\mathrm{div}(P_V^s - P_V)(u)\|_{L^2(\Omega)} \leqslant \|(P_V^s - P_V)(u)\|_{H_0^1} \\ &\leqslant \frac{1}{\theta}(1-\theta)^{(s+1)}\|\mathrm{div}u\|_{L^2(\Omega)} \leqslant \frac{1}{\theta}(1-\theta)^{(s+1)}\|u\|_{H_0^1}, \end{aligned}$$

which completes the proof. □

Let us recall that Proposition 5.2 enables us to write the integral identity (3.7) in the definition of the solution of the Navier-Stokes equations in an equivalent form as the equation

$$\nu u + P_V \mathcal{R}^{-1}\mathcal{A}_{u,P_V u} = P_V \mathcal{R}^{-1} f \tag{5.10}$$

in the space H_0^1, where $\nu > 0$ and $f \in (H_0^1)'$ are given. This means in particular that the set of elements satisfying identity (3.7) is equal to the set of solutions to the equation (5.10).

In the first step of the Holly method, the equation (5.10) is approximated by the equations of the form

$$\nu u + P_V^s \mathcal{R}^{-1} \mathcal{A}_{u, P_V^s u} = P_V^s \mathcal{R}^{-1} f_s \tag{5.11}$$

for $s \in \mathbb{N}$. Here $f_s \in (H_0^1)'$ and f_s tends to f in the dual space $(H_0^1)'$ as $s \to \infty$.

Note that the equation (5.11) is still an equation in the space H_0^1. It needs discretization. This will be done in the second step which involves approximation of the equation (5.11) for fixed s, by some equations on the closed linear subspaces M of H_0^1 (in particular, on finite-dimensional ones).

5.4. Second step of the approximation of the Navier-Stokes equations – discretization.

Let M be a closed linear subspace of H_0^1 and let $P_M : H_0^1 \to M$ be the $((\cdot|\cdot))$-orthogonal projection . We introduce some operators div_M^*, $\mathcal{R}_M^{-1}$ and $P_V^{s,M}$ which will approximate, respectively, the operators div^*, $\mathcal{R}^{-1}$ and P_V^s.

Operators $\operatorname{div}^*_{\mathbf{M}}$. Let div_M^* be the adjoint operator to the restriction

$$\operatorname{div}_{|M} : M \to L^2(\Omega).$$

of the divergence operator to the subspace M.

Consider the family $\mathcal{S}(H_0^1)$ of all closed linear subspaces of the Sobolev space H_0^1 and let $\mathcal{B}$ denote the filterbase on $\mathcal{S}(H_0^1)$ defined in Preliminaries (see (1.3) and (1.4)).

Remark 5.6. —

(a) *For every $M \in \mathcal{S}(H_0^1)$:*

$$\operatorname{div}_M^* = P_M \circ \operatorname{div}^* \quad \textit{and} \quad \operatorname{div}_M^* \in \mathcal{L}(L^2(\Omega), H_0^1).$$

(b) *For every $q \in L^2(\Omega)$:*

$$\lim_{M \succ \mathcal{B}} \|\operatorname{div}_M^* q - \operatorname{div}^* q\|_{H_0^1} = 0. \tag{5.12}$$

Proof. — Assertion (a) follows from the fact that for $q \in L^2(\Omega)$, $\mathrm{div}_M^* q$ is the $((\cdot|\cdot))$ - Riesz representation of the functional

$$M \ni \phi \mapsto \int_\Omega q(\mathrm{div}\phi)\, dm \in \mathbb{R}.$$

Thus

$$(\mathrm{div}\phi|q)_{L^2(\Omega)} = ((\phi|\mathrm{div}_M^* q))$$

On the other hand

$$(\mathrm{div}\phi|q)_{L^2(\Omega)} = ((\phi|\mathrm{div}^* q)) = ((P_M\phi|\mathrm{div}^* q)) = ((\phi|P_M \mathrm{div}^* q)),$$

which completes the proof of (a). Assertion (b) follows immediately from (a) and Corollary 1.5. ☐

Operators $\mathcal{R}_{\mathbf{M}}^{-1}$. For a functional $l \in (H_0^1)'$, let $\mathcal{R}_M^{-1}(l)$ denote the $((\cdot|\cdot))$ - Riesz representation of the restriction of l to the subspace M.

Remark 5.7. —

(a) $\mathcal{R}_M^{-1} = P_M \circ \mathcal{R}^{-1}$ *and* $\mathcal{R}_M^{-1} \in \mathcal{L}((H_0^1)', H_0^1)$.

(b) *For every* $l \in (H_0^1)'$

$$\lim_{M \succ \mathcal{B}} \|\mathcal{R}_M^{-1}(l) - \mathcal{R}^{-1}(l)\|_{H_0^1} = 0. \tag{5.13}$$

The proof is based on similar reasoning as in the proof of Remark 5.6.

Operators $\mathbf{P}_{\mathbf{V}}^{\mathbf{s},\mathbf{M}}$. For fixed $s \in \mathbb{N}$, let us denote

$$\begin{aligned} P_{V^\perp}^{s,M} &:= \mathrm{div}_M^* \circ \Big(\sum_{j=0}^{s} (\mathrm{id} - \mathrm{div}\mathrm{div}_M^*)^j\Big) \circ \mathrm{div} \circ P_M, \\ P_V^{s,M} &:= \mathrm{id} - P_{V^\perp}^{s,M}. \end{aligned}$$

In the following Remark, we collect properties of the operators $P_V^{s,M}$.

Remark 5.8. —

(a) $P_V^{s,M}$ *is a selfadjoint endomorphism of the space* H_0^1.

(b) *Subspace* M *is invariant with respect to* $P_V^{s,M}$, *i.e.*

$$P_V^{s,M}(M) \subset M. \tag{5.14}$$

(c) *The family of norms* $\{|P_V^{s,M}|_{\mathcal{E}ndH_0^1};\ M\in\mathcal{S}(H_0^1)\}$ *is bounded, i.e., there exists a constant* $C>0$ *such that*

$$|P_V^{s,M}|_{\mathcal{E}ndH_0^1}\leqslant C,\qquad M\in\mathcal{S}(H_0^1). \tag{5.15}$$

(d) *For every* $u\in H_0^1$

$$\lim_{M\succ\mathcal{B}}\|P_V^{s,M}(u)-P_V^s(u)\|_{H_0^1}=0. \tag{5.16}$$

Proof. — Assertions (a) and (b) follow directly from the definition of $P_V^{s,M}$ and from Remark 5.6 (a).

Since, by Remark 5.6 (a) $\operatorname{div}_M^*=P_M\circ\operatorname{div}^*$ and since $|P_M|_{\mathcal{E}ndH_0^1}=1$, we infere that (c) holds.

By Remark 5.6 (b), the operators div_M^* tend to div^* pointwise over the filterbase $\mathcal{B}$. Due to Corollary 1.5, the projections P_M tend over $\mathcal{B}$ pointwise to the identity mapping. Then, assertion (d) follows from Lemma 8.3 in Appendix A, because $P_V^{s,M}$ is a finite sum of compositions of operators pointwise convergent over $\mathcal{B}$. $\square$

Using the above operators, we consider the following equation

$$\nu w+P_V^{s,M}\mathcal{R}_M^{-1}\mathcal{A}_{w,P_V^{s,M}w}=P_V^{s,M}\mathcal{R}_M^{-1}f_s. \tag{5.17}$$

on the subspace M, which approximates equation (5.11) for fixed $s\in\mathbb{N}$.

6. Stability of the Holly method - application of the abstract framework

Assume that $\nu>0$, $f\in(H_0^1)'$ and $(f_s)_{s\in\mathbb{N}}$ tends to f in $(H_0^1)'$ as $s\to\infty$. Let us recall that by Proposition 5.2 the integral identity (3.7) in the definition of the solution of the Navier-Stokes equations is equivalent to the equation

$$\nu u+P_V\mathcal{R}^{-1}\mathcal{A}_{u,P_Vu}=P_V\mathcal{R}^{-1}f \tag{6.1}$$

in the space H_0^1. Let $s\in\mathbb{N}$. In the first step, the above equation is approximated by the equation

$$\nu u+P_V^s\mathcal{R}^{-1}\mathcal{A}_{u,P_V^su}=P_V^s\mathcal{R}^{-1}f_s. \tag{6.2}$$

in the space H_0^1. Next, let $M \in \mathcal{S}(H_0^1)$. In the second step, equation (6.2) (for fixed s) is approximated by the equation

$$\nu w + P_V^{s,M} \mathcal{R}_M^{-1} \mathcal{A}_{w, P_V^{s,M} w} = P_V^{s,M} \mathcal{R}_M^{-1} f_s. \tag{6.3}$$

in the subspace M. Let us denote

$$\mathbb{S}(\nu, f) \text{ - the set of all solutions of the equation (6.1),} \tag{6.4}$$

$$\mathbb{S}_s(\nu, f_s) \text{ - the set of all solutions of equation (6.2),} \tag{6.5}$$

$$\mathbb{S}_{s,M}(\nu, f_s) \text{ - the set of all solutions of equation (6.2).} \tag{6.6}$$

In this section we investigate behaviour of the sequence of sets $\mathbb{S}_s(\nu, f_s)$ as $s \to \infty$ which is called the problem of stability with respect to s. Next, we fix s sufficiently large and consider the convergence of the family of sets $\mathbb{S}_{s,M}(\nu, f_s)$ over the filter base $\mathcal{B}$ on $\mathcal{S}(H_0^1)$.

We will prove that the Holly method is stable if the data (viscosity, external forces) belong to the set

$$\begin{aligned} \mathcal{G} := \{(\nu, f) \in]0, \infty[\times (H_0^1)' : P_V \mathcal{R}^{-1} f \textit{ is a regular value} \\ \textit{of the mapping } H_0^1 \ni \phi \mapsto \nu\phi + P_V \mathcal{R}^{-1} \mathcal{A}_{\phi, P_V \phi} \in H_0^1\}. \end{aligned} \tag{6.7}$$

At last we prove that the set $\mathcal{G}$ is open and dense in $]0, \infty[\times (H_0^1)'$.

6.1. Stability with respect to "s"

We will prove that that for $(\nu, f) \in \mathcal{G}$ the sequence of sets $\mathbb{S}_s(\nu, f_s)$ tends to $\mathbb{S}(\nu, f)$ in the Hausdorff metric over H_0^1. Observe that here we have some approximations of the mappings and no approximation of the space; all the equations are in the space H_0^1.

THEOREM 6.1. — *Assume that $(\nu, f) \in \mathcal{G}$ and $(f_s)_{s \in \mathbb{N}}$ tends to f in $(H_0^1)'$ as $s \to \infty$. Then*

(i) *for almost all $s \in \mathbb{N}$:* $\quad \#\mathbb{S}_s(\nu, f_s) = \#\mathbb{S}(\nu, f) < \infty$.

(ii) $\lim_{s \to \infty} \mathbb{S}_s(\nu, f_s) = \mathbb{S}(\nu, f)$ *in the Hausdorff metric over H_0^1.*

Proof. — We apply the abstract results of Appendix C with

$$\begin{aligned} &H := H_0^1 \\ &T(v) := P_V \mathcal{R}^{-1} \mathcal{A}_{v, P_V v}, \qquad v \in H_0^1 \\ &T_s(u) := P_V^s \mathcal{R}^{-1} \mathcal{A}_{u, P_V^s u}, \qquad u \in H_0^1, \qquad s \in \mathbb{N}. \end{aligned}$$

We will use Theorem 10.15. First, let us check that the mapping T satisfies conditions (10.4)-(10.5) in Appendix C. To verify condition (10.4), let $u_k \to u$ weakly in H_0^1 as $k \to \infty$. Then by Lemma 5.3 (b) and continuity of the projection P_V

$$\mathcal{A}_{u_k, P_V u_k} \to \mathcal{A}_{u, P_V u} \quad \text{in} \quad (H_0^1)'.$$

Thus by continuity of P_V and of the Riesz isomorphism $\mathcal{R}$, $P_V \mathcal{R}^{-1} \mathcal{A}_{u_k, P_V u_k}$ tends to $P_V \mathcal{R}^{-1} \mathcal{A}_{u, P_V u}$ in H_0^1. This means that

$$\lim_{k \to \infty} \|T(u_k) - T(u)\|_{H_0^1} = 0,$$

i.e. condition (10.4) is fulfilled.

To verify condition (10.5), let us fix $\mu > 0$ and $g \in H_0^1$. Multiply the equation

$$\mu v + P_V \mathcal{R}^{-1} \mathcal{A}_{v, P_V v} = g$$

scalarly in H_0^1 by v to obtain

$$\mu \|v\|^2_{H_0^1} + (\!(P_V \mathcal{R}^{-1} \mathcal{A}_{v, P_V v} | v)\!) = (\!(g|v)\!).$$

Since by (5.4) in Proposition 5.1 (a)

$$(\!(P_V \mathcal{R}^{-1} \mathcal{A}_{v, P_V v} | v)\!) = \quad (\!(\mathcal{R}^{-1} \mathcal{A}_{v, P_V v} | P_V v)\!) = \mathcal{A}_{v, P_V v}(P_V v) = 0,$$

we infer that $\|v\|_{H_0^1} \leqslant \frac{\|g\|_{H_0^1}}{\mu}$. Thus, condition (10.5) holds for

$$\kappa(\mu, r) = \frac{r}{\mu}, \qquad \mu > 0, \quad r > 0. \tag{6.8}$$

Let us verify that the mappings T_s satisfy conditions (10.24)-(10.26) in Appendix C. To verify condition (10.24), let $u_s \to u$ weakly in H_0^1 as $s \to \infty$. We claim that $P_V^s u_s \to P_V u$ weakly in H_0^1. Indeed, it is sufficient to check that for every $w \in H_0^1$

$$(\!(P_V^s u_s | w)\!) \to (\!(P_V u | w)\!) \tag{6.9}$$

as $s \to \infty$. We have

$$\begin{aligned} |(\!(P_V^s u_s - P_V u | w)\!)| &\leqslant |(\!((P_V^s - P_V) u_s | w)\!)| + |(\!(P_V(u_s - u) | w)\!)| \\ &\leqslant |P_V^s - P_V|_{\mathcal{E}nd H_0^1} \cdot \|u_s\|_{H_0^1} \cdot \|w\|_{H_0^1} + |(\!(u_s - u | P_V w)\!)|. \end{aligned}$$

Then (6.9) follows, because by Remark 5.5 (b), $\lim_{s \to \infty} |P_V^s - P_V|_{\mathcal{E}nd H_0^1} = 0$. By Lemma 5.3 (b),

$$\mathcal{A}_{u_s, P_V^s u_s} \to \mathcal{A}_{u, P_V u} \quad \text{in} \quad (H_0^1)'.$$

Thus, using again the fact that P_V^s tends to P_V in the space of endomorphisms of H_0^1, we conclude that

$$P_V^s \mathcal{R}^{-1} \mathcal{A}_{u_s, P_V^s u_s} \to P_V \mathcal{R}^{-1} \mathcal{A}_{u, P_V u} \quad \text{in} \quad H_0^1,$$

i.e. $\lim_{s\to\infty} \|T_s(u_s) - T(u)\|_{H_0^1} = 0$, which completes the proof of (10.24).

Let us move to verifying (10.25). Let $w_s \to w$ in H_0^1 as $s \to \infty$. We calculate the Fréchet differentials

$$\begin{aligned} d_w T &= P_V \circ \mathcal{R}^{-1} \circ d_w\big(\mathcal{A} \circ (\mathrm{id} \bigtriangleup P_V)\big) \\ d_{w_s} T_s &= P_V^s \circ \mathcal{R}^{-1} \circ d_{w_s}\big(\mathcal{A} \circ (\mathrm{id} \bigtriangleup P_V^s)\big) \end{aligned}$$

Since $P_V^s \mathcal{R}^{-1} \to P_V \mathcal{R}^{-1}$ in $\mathcal{L}\big((H_0^1)', H_0^1\big)$, it is sufficient to prove that

$$d_{w_s}\big(\mathcal{A} \circ (\mathrm{id} \bigtriangleup P_V^s)\big) \to d_w\big(\mathcal{A} \circ (\mathrm{id} \bigtriangleup P_V)\big) \quad \text{in} \quad \mathcal{L}\big(H_0^1, (H_0^1)'\big) \tag{6.10}$$

as $s \to \infty$. Let $h \in H_0^1$. By Proposition 5.1 (b), the mapping $\mathcal{A}$ is bilinear and continuous; thus

$$\begin{aligned} &\big|d_{w_s}\big(\mathcal{A} \circ (\mathrm{id} \bigtriangleup P_V^s)\big)(h) - d_w\big(\mathcal{A} \circ (\mathrm{id} \bigtriangleup P_V)\big)(h)\big|_{(H_0^1)'} \\ &\quad = \big|\mathcal{A}(w_s, P_V^s h) + \mathcal{A}(h, P_V^s w_s) - \mathcal{A}(w, P_V h) - \mathcal{A}(h, P_V w)\big|_{(H_0^1)'} \\ &\quad \leqslant \big|\mathcal{A}_{w_s - w, P_V^s h}\big|_{(H_0^1)'} + \big|\mathcal{A}_{w, (P_V^s - P_V)h}\big|_{(H_0^1)'} + \big|\mathcal{A}_{h, P_V^s w_s - P_V w}\big|_{(H_0^1)'} \\ &\quad \leqslant \frac{3}{2}|\iota|^2 \big(\|w_s - w\|_{H_0^1} + \|w\|_{H_0^1} |P_V^s - P_V|_{\mathcal{E}nd H_0^1} + \|P_V^s w_s - P_V w\|_{H_0^1}\big) \|h\|_{H_0^1} \end{aligned}$$

for any $s \in \mathbb{N}$. To obtain the last inequality we have applied inequality (5.2) in Proposition 5.1 (a) . Hence

$$\begin{aligned} &\big|d_{w_s}\big(\mathcal{A} \circ (\mathrm{id} \bigtriangleup P_V^s)\big) - d_w\big(\mathcal{A} \circ (\mathrm{id} \bigtriangleup P_V)\big)\big|_{\mathcal{L}\left(H_0^1, (H_0^1)'\right)} \\ &\qquad \leqslant \frac{3}{2}|\iota|^2 \big(\|w_s - w\|_{H_0^1} + \|w\|_{H_0^1} |P_V^s - P_V|_{\mathcal{E}nd H_0^1} + \|P_V^s w_s - P_V w\|_{H_0^1}\big) \end{aligned}$$

Passing to the limit as $s \to \infty$, we obtain (6.10). Thus

$$\lim_{s\to\infty} \big|d_{w_s} T_s - d_w T\big|_{\mathcal{E}nd H_0^1} = 0,$$

i.e. condition (10.25) is satisfied.

To check condition (10.26), let us fix $\mu > 0$ and $g_s \in H_0^1$. Scalar multiplication $(\!(\cdot|u)\!)$ of the equation

$$\mu u + P_V^s \mathcal{R}^{-1} \mathcal{A}_{u, P_V^s u} = g_s$$

yields

$$\mu\|u\|^2_{H^1_0} + ((P^s_V \mathcal{R}^{-1}\mathcal{A}_{u,P^s_V u}|u)) = ((g_s|u)).$$

According to Remark 5.5 (a), P^s_V is selfadjoint; thus

$$\mu\|u\|^2_{H^1_0} + \mathcal{A}_{u,P^s_V u}(P^s_V u) = ((g_s|u)).$$

Since $\mathcal{A}_{u,P^s_V u}(P^s_V u) = 0$ (by (5.4) in Proposition 5.1 (a)), we conclude that

$$\|u\|_{H^1_0} \leqslant \frac{\|g_s\|_{H^1_0}}{\mu}.$$

Hence, condition (10.26) holds for the same function κ defined by (6.8).

Let us fix $(\nu, f) \in \mathcal{G}$ and observe that the sets $\mathbb{S}(\nu, f)$ and $\mathbb{S}_s(\nu, f_s)$ correspond to the following sets in the abstract setting

$$\mathbb{S}(\nu, f) = \mathfrak{R}(\mu, g) \qquad \text{and} \qquad \mathbb{S}_s(\nu, f_s) = \mathfrak{R}(s, \mu, g_s)$$

for $\mu := \nu$, $g := P_V \mathcal{R}^{-1} f$ and $g_s := P^s_V \mathcal{R}^{-1} f_s$ (Compare (6.4) and (6.5) with (10.3) and (10.23)). Now the assertion follows from Theorem 10.15 in Appendix C. $\square$

6.2. Stability of the Holly method with respect to the approximation of the space H^1_0

This problem of stability with respect to the approximation of the space H^1_0 concerns the second step. Here we will investigate behaviour of the family of sets $\mathbb{S}_{s,M}(\nu, f_s)$, when $s \in \mathbb{N}$ is fixed and M various over the family of all closed linear subspaces of H^1_0.

We begin with two auxiliary results. Using the fixed point theorem in the version of Theorem 9.4 in Appendix B, we will prove the following

Proposition 6.2. — *Let $s \in \mathbb{N}$ and M be a closed linear subspace of H^1_0. Then for every $\mu > 0$ and every $g_M \in M$ there exists $w \in M$ such that*

$$\mu w + P^{s,M}_V \mathcal{R}^{-1}_M \mathcal{A}(w, P^{s,M}_V w) = g_M. \tag{6.11}$$

Moreover,

$$\|w\|_{H^1_0} \leqslant \frac{\|g_M\|_{H^1_0}}{\mu}. \tag{6.12}$$

Proof. — Let us fix $\mu > 0$ and $g_M \in M$. We begin with proving inequality (6.12). Suppose that $w \in M$ satisfies equation (6.11). Multiply the equation

$$\mu w + P_V^{s,M} \mathcal{R}_M^{-1} \mathcal{A}(w, P_V^{s,M} w) = g_M$$

scalarly in H_0^1 by w to obtain

$$\mu \|w\|_{H_0^1}^2 + \big(\big(P_V^{s,M} \mathcal{R}_M^{-1} \mathcal{A}(w, P_V^{s,M} w) | w\big)\big) = \big(\big(g_M | w\big)\big)$$

By Remark 5.8 (a), $P_V^{s,M}$ is selfadjoint. Thus, by Remark 5.7 (a) and Remark 5.8 (b), we calculate

$$\begin{aligned} \big(\big(P_V^{s,M} \mathcal{R}_M^{-1} \mathcal{A}(w, P_V^{s,M} w) | w\big)\big) &= \big(\big(P_M \mathcal{R}^{-1} \mathcal{A}(w, P_V^{s,M} w) | P_V^{s,M} w\big)\big) \\ &= \big(\big(\mathcal{R}^{-1} \mathcal{A}(w, P_V^{s,M} w) | P_M P_V^{s,M} w\big)\big) \\ &= \mathcal{A}_{w, P_V^{s,M} w}(P_V^{s,M} w). \end{aligned}$$

Thus

$$\mu \|w\|_{H_0^1}^2 + \mathcal{A}_{w, P_V^{s,M} w}(P_V^{s,M} w) = \big(\big(g_M | w\big)\big).$$

Since $\mathcal{A}_{w, P_V^{s,M} w}(P_V^{s,M} w) = 0$ (by (5.4) in Proposition 5.1 (a)), we obtain

$$\mu \|w\|_{H_0^1}^2 = \big(\big(g_M | w\big)\big) \leqslant \|g_M\|_{H_0^1} \cdot \|w\|_{H_0^1}.$$

Thus $\|w\|_{H_0^1} \leqslant \frac{\|g_M\|_{H_0^1}}{\mu}$, i.e. inequality (6.12) holds.

To prove the first part of the statement, let us consider $\bar{K}_M(R) := \{x \in M : \|x\|_{H_0^1} \leqslant R\}$. We assert that the mapping

$$F : M \supset \bar{K}_M(R) \ni u \mapsto u - \frac{1}{\mu} g_M + \frac{1}{\mu} P_V^{s,M} \mathcal{R}_M^{-1} \mathcal{A}(u, P_V^{s,M} u) \in M$$

satisfies the assumptions of Theorem 9.4 in Appendix B with $R := \frac{\|g_M\|_{H_0^1}}{\mu}$.

Indeed, let $\zeta \in \partial \bar{K}_M(R)$. Property (5.4) of the operator $\mathcal{A}$ together with selfadjointness of $P_V^{s,M}$ (see Remark 5.8 (a)) imply that

$$((P_V^{s,M} \mathcal{R}_M^{-1} \mathcal{A}(\zeta, P_V^{s,M} \zeta) | \zeta)) = ((\mathcal{R}_M^{-1} \mathcal{A}_{\zeta, P_V^{s,M} \zeta} | P_V^{s,M} \zeta)) = \mathcal{A}_{\zeta, P_V^{s,M} \zeta}(P_V^{s,M} \zeta) = 0.$$

Moreover,

$$\frac{1}{\mu}((g_M | \zeta)) \leqslant \frac{1}{\mu^2} \|g_M\|_{H_0^1}^2 = \|\zeta\|_{H_0^1}^2.$$

Thus
$$((F(\zeta)|\zeta)) = \|\zeta\|^2_{H^1_0} - \frac{1}{\mu}((g_M|\zeta)) \geqslant 0.$$
By complete continuity of mapping $\mathcal{A}$ (see Corollary 5.4), the mapping
$$\mathrm{id}_M - F = \frac{1}{\mu} g_M - \frac{1}{\mu} P_V^{s,M} \mathcal{A} \circ (\mathrm{id} \triangle P_V^{s,M})$$
is completely continuous. In particular, the set $(\mathrm{id}_M - F)(\bar{K}_M(R))$ is relatively compact. Consequently, Theorem 9.4 implies that the set $\{F = 0\}$ is nonempty, or equivalently, that the set of solutions of equation (6.11) is nonempty. □

Remark 6.3. — If $(\nu, f) \in \mathcal{G}$, then $P_V^s \mathcal{R}^{-1} f_s$ is a regular value of the mapping
$$\nu \mathrm{id} + P_V^s \mathcal{R}^{-1} \circ \mathcal{A} \circ (\mathrm{id} \triangle P_V^s) : H^1_0 \to H^1_0,$$
for almost all $s \in \mathbb{N}$.

Proof. — Let us fix $(\nu, f) \in \mathcal{G}$. Let us denote
$$\mathcal{K}_\nu(u) := \nu u + P_V \mathcal{R}^{-1} \mathcal{A}(u, P_V u), \qquad u \in H^1_0$$
$$\mathcal{K}_{\nu,s}(u) := \nu u + P_V^s \mathcal{R}^{-1} \mathcal{A}(u, P_V^s u), \qquad u \in H^1_0, \quad s \in \mathbb{N}.$$
Observe that
$$\mathbb{S}(\nu, f) = \mathcal{K}_\nu^{-1}(\{P_V \mathcal{R}^{-1} f\}), \tag{6.13}$$
$$\mathbb{S}_s(\nu, f_s) = \mathcal{K}_{\nu,s}^{-1}(\{P_V^s \mathcal{R}^{-1} f_s\}). \tag{6.14}$$
(see (6.1), (6.2), (6.4) and (6.5)) According to the definition of the set $\mathcal{G}$, for every $v \in \mathbb{S}(\nu, f)$, the Fréchet differential $d_v \mathcal{K}_\nu \in \mathcal{E}pi H^1_0$. Since $\mathcal{K}_\nu$ is a Fredholm mapping (by Corollary 5.4), we infer that $d_v \mathcal{K}_\nu \in \mathcal{A}ut H^1_0$. Fix $\varepsilon > 0$. By Theorem 6.1, there exists $s_0 \in \mathbb{N}$ such that for $s \geqslant s_0$

(i) $\#\mathbb{S}_s(\nu, f_s) = \#\mathbb{S}(\nu, f) < \infty$;

(ii) the Hausdorff distance: $d\big(\mathbb{S}_s(\nu, f_s), \mathbb{S}(\nu, f)\big) < \varepsilon$, i.e.
$$\mathbb{S}_s(\nu, f_s) \subset \mathbb{S}(\nu, f) + K(0, \varepsilon) \quad \text{and} \quad \mathbb{S}(\nu, f) \subset \mathbb{S}_s(\nu, f_s) + K(0, \varepsilon).$$

Therefore for each $v \in \mathbb{S}(\nu, f)$ there exists a unique $v_s \in \mathbb{S}_s(\nu, f_s)$ such that $\|v - v_s\|_{H^1_0} < \varepsilon$. This means that v_s tends to v in H^1_0 as $s \to \infty$.

Let us fix $v \in \mathbb{S}(\nu, f)$. Since the space of authomorphisms of H^1_0 is open in the space of all endomorphisms of H^1_0,

there exists an open neighbourhood $\mathcal{W}(v)$ in $\mathcal{E}nd H^1_0$
of the differential $d_v \mathcal{K}_\nu$ such that $\mathcal{W}(v) \subset \mathcal{A}ut H^1_0$.

Thus, if we show that

$$d_{v_s}\mathcal{K}_{\nu,s} \to d_v\mathcal{K}_\nu \quad \text{in the space } \mathcal{E}nd H_0^1 \quad \text{as } s\to\infty, \tag{6.15}$$

then $d_{v_s}\mathcal{K}_{\nu,s} \subset \mathcal{A}ut H_0^1$ for sufficiently large $s \in \mathbb{N}$.

We calculate the Fréchet differentials

$$\begin{aligned} d_v\mathcal{K}_\nu &= \nu \mathrm{id} + P_V \circ \mathcal{R}^{-1} \circ d_v\big(\mathcal{A} \circ (\mathrm{id} \triangle P_V)\big) \\ d_{v_s}\mathcal{K}_{\nu,s} &= \nu \mathrm{id} + P_V^s \circ \mathcal{R}^{-1} \circ d_{v_s}\big(\mathcal{A} \circ (\mathrm{id} \triangle P_V^s)\big) \end{aligned}$$

Since $P_V^s\mathcal{R}^{-1} \to P_V\mathcal{R}^{-1}$ in $\mathcal{L}\big((H_0^1)', H_0^1\big)$ and by (6.10)

$$d_{v_s}\big(\mathcal{A} \circ (\mathrm{id} \triangle P_V^s)\big) \to d_v\big(\mathcal{A} \circ (\mathrm{id} \triangle P_V)\big) \quad \text{in} \quad \mathcal{L}\big(H_0^1, (H_0^1)'\big)$$

as $s \to \infty$, we infer that (6.15) holds. Since v was chosen in an arbitrary way, the proof is complete. □

Consider the family $\mathcal{S}(H_0^1)$ of all closed linear subspaces of H_0^1 and let $\mathcal{B}$ denote the filterbase on $\mathcal{S}(H_0^1)$ defined in Preliminaries (see (1.3) and (1.4)). Applying Theorem 2.10 of Section 2 we will prove the following theorem expressing stability with respect to approximation of the space H_0^1 for the Holly method.

Theorem 6.4 **(stability with respect to approximation of the space H_0^1).** — *Assume that $(\nu, f) \in \mathcal{G}$. Then, for almost all $s \in \mathbb{N}$ and every $\varepsilon > 0$ there exist $W_s \in \mathcal{S}(H_0^1) \cap \{\dim < \infty\}$ and $\delta_s > 0$ such that*

(i) $d\big(\mathbb{S}_{s,M}(\nu, f_s), \mathbb{S}_s(\nu, f_s)\big) \leqslant \varepsilon$,

(ii) $\#\mathbb{S}_{s,M}(\nu, f_s) = \#\mathbb{S}_s(\nu, f_s) < \infty$,

whenever $M \in B_{W_s,\delta_s}$. (Here d stands for the Hausdorff metric over H_0^1.)

In particular, assertion **(i)** quarantees that

$$\lim_{M \succ \mathcal{B}} \mathbb{S}_{s,M}(\nu, f_s) = \mathbb{S}_s(\nu, f_s) \qquad \text{in the Hausdorff metric over } H_0^1.$$

Proof of Theorem 6.4. — Let $(\nu, f) \in \mathcal{G}$. Let us fixed $s \in \mathbb{N}$ such that $P_V^s\mathcal{R}^{-1}f_s$ is a regular value of the mapping

$$\nu \mathrm{id} + P_V^s\mathcal{R}^{-1} \circ \mathcal{A} \circ (\mathrm{id} \triangle P_V^s) : H_0^1 \to H_0^1.$$

We apply the abstract framework of Section 2 in the Hilbert space $\big(H_0^1, ((\cdot|\cdot))\big)$ with

$$T(u) := P_V^s \mathcal{R}^{-1} \mathcal{A}(u, P_V^s u), \qquad u \in H_0^1$$

and

$$T_M(w) := P_V^{s,M} \mathcal{R}_M^{-1} \mathcal{A}(w, P_V^{s,M} w), \qquad w \in M,$$

where $M \in \mathcal{S}(H_0^1)$. To apply Theorem 2.10, we will check that the mappings T and T_M satisfy conditions (A.1)-(A.6) in Section 2.

Ad. (A.1). Condition (A.1) is satisfied due to Proposition 6.2.

Ad. (A.2). It is sufficient to take

$$\tilde{T}_M(u) := P_V^{s,M} \mathcal{R}_M^{-1} \mathcal{A}(u, P_V^{s,M} u), \qquad u \in H_0^1,$$

i.e. $\tilde{T}_M$ is defined on the whole space H_0^1 by the same formula as T_M. Let $g_M \in M$ and denote

$$\begin{aligned} \mathfrak{R}_M(\mu, g_M) &:= \{w \in M: \quad \mu w + T_M(w) = g_M\}, \\ \tilde{\mathfrak{R}}_M(\mu, g_M) &:= \{u \in H_0^1: \quad \mu u + \tilde{T}_M(u) = g_M\}. \end{aligned}$$

It is clear that $\mathfrak{R}_M(\mu, g_M) \subset \tilde{\mathfrak{R}}_M(\mu, g_M)$. On the other hand, since $\tilde{T}_M(H_0^1) = P_V^{s,M} \mathcal{R}_M^{-1} \mathcal{A}(H_0^1 \times P_V^{s,M} H_0^1) \subset M$, we infer that also $\tilde{\mathfrak{R}}_M(\mu, g_M) \subset M \cap \mathfrak{R}_M(\mu, g_M) \subset \mathfrak{R}_M(\mu, g_M)$.

Ad. (A.3). By inequality (6.12) in Proposition 6.2, condition (A.3) holds with

$$\kappa :]0, \infty[\times [0, \infty[\ni (\mu, r) \mapsto \frac{r}{\mu} \in [0, \infty[.$$

Ad. (A.4). Let $u \in H_0^1$. For $M \in \mathcal{S}(H_0^1)$, we have

$$\begin{aligned} \|\tilde{T}_M(u) - T(u)\|_{H_0^1} &= \|P_V^{s,M} \mathcal{R}_M^{-1} \mathcal{A}(u, P_V^{s,M} u) - P_V^s \mathcal{R}^{-1} \mathcal{A}(u, P_V^s u)\|_{H_0^1} \qquad (6.16) \\ &\leqslant \|P_V^{s,M} \big(\mathcal{R}_M^{-1} \mathcal{A}(u, P_V^{s,M} u) - \mathcal{A}(u, P_V^s u)\big)\|_{H_0^1} \\ &\quad + \|P_V^{s,M} \mathcal{R}_M^{-1} \mathcal{A}(u, P_V^s u) - P_V^s \mathcal{R}^{-1} \mathcal{A}(u, P_V^s u)\|_{H_0^1} \\ &\leqslant |P_V^{s,M} \mathcal{R}_M^{-1}|_{\mathcal{L}((H_0^1)', H_0^1)} |\mathcal{A}(u, P_V^{s,M} u) - \mathcal{A}(u, P_V^s u)|_{(H_0^1)'} \\ &\quad + \|P_V^{s,M} \mathcal{R}_M^{-1} \mathcal{A}(u, P_V^s u) - P_V^s \mathcal{R}^{-1} \mathcal{A}(u, P_V^s u)\|_{H_0^1}. \end{aligned}$$

In view of (5.13) and (5.16), by Lemma 8.3 in Appendix A

$$\lim_{\omega \succ \mathcal{B}} \|P_V^{s,M} \mathcal{R}_M^{-1}(l) - P_V^s \mathcal{R}^{-1}(l)\|_{H_0^1} = 0, \qquad l \in (H_0^1)'. \qquad (6.17)$$

Thus, the last term in inequality (6.16) tends to zero as $M \succ \mathcal{B}$. Since the family of norms $\{|P_V^{s,M}\mathcal{R}_M^{-1}|_{\mathcal{L}((H_0^1)',H_0^1)};\quad M \in \mathcal{S}(H_0^1)\}$ is bounded, it is sufficient to check that

$$|\mathcal{A}(u, P_V^{s,M}u) - \mathcal{A}(u, P_V^s u)|_{(H_0^1)'} \to 0 \qquad \text{as} \quad M \succ \mathcal{B}. \tag{6.18}$$

However, by (5.2), we have

$$|\mathcal{A}(u, P_V^{s,M}u) - \mathcal{A}(u, P_V^s u)|_{(H_0^1)'} \leqslant \frac{3}{2}|\iota|^2 \|u\|_{H_0^1} \|P_V^{s,M}u - P_V^s u\|_{H_0^1}.$$

Thus, (5.16), assertion (6.18) holds .

Ad. (A.5). Let (W_k) be an internal approximation of H_0^1, $0 < \delta_k \to 0$ and $M_k \in B_{W_k,\delta_k}$, $k \in \mathbb{N}$. Suppose that $u_k \to u$ weakly in H_0^1. We have

$$\begin{aligned}
&\|\tilde{T}_{M_k}(u_k) - T(u)\|_{H_0^1} \\
&\qquad = \|P_V^{s,M_k}\mathcal{R}_{M_k}^{-1}\mathcal{A}(u_k, P_V^{s,M_k}) - P_V^s\mathcal{R}^{-1}\mathcal{A}(u, P_V^s u)\|_{H_0^1} \\
&\qquad \leqslant \|P_V^{s,M_k}\mathcal{R}_{M_k}^{-1}\big(\mathcal{A}(u_k, P_V^{s,M_k}u_k) - \mathcal{A}(u, P_V^s u)\big)\|_{H_0^1} \\
&\qquad +\|P_V^{s,M_k}\mathcal{R}_{M_k}^{-1}\mathcal{A}(u, P_V^s u) - P_V^s\mathcal{R}^{-1}\mathcal{A}(u, P_V^s u)\|_{H_0^1}.
\end{aligned} \tag{6.19}$$

By Remark 1.9 and (5.16), we deduce that

$$\lim_{k\to\infty} \|P_V^{s,M_k}\mathcal{R}_{M_k}^{-1}(l) - P_V^s\mathcal{R}^{-1}(l)\|_{H_0^1} = 0, \qquad \text{for every} \quad l \in (H_0^1)'.$$

Thus, the last term in inequality (6.19) tends to zero as $k \to \infty$. Since

$$\begin{aligned}
&\|P_V^{s,M_k}\mathcal{R}_{M_k}^{-1}\big(\mathcal{A}(u_k, P_V^{s,M_k}u_k) - \mathcal{A}(u, P_V^s u)\big)\|_{H_0^1} \\
&\leqslant |P_V^{s,M_k}\mathcal{R}_{M_k}^{-1}|_{\mathcal{L}((H_0^1)',H_0^1)} |\mathcal{A}(u_k, P_V^{s,M_k}u_k) - \mathcal{A}(u, P_V^s u)|_{(H_0^1)'}
\end{aligned}$$

and the sequence $\big(|P_V^{s,M_k}\mathcal{R}_{M_k}^{-1}|_{\mathcal{L}((H_0^1)',H_0^1)}\big)_{k\in\mathbb{N}}$ is bounded, it is sufficient to prove that

$$|\mathcal{A}(u_k, P_V^{s,M_k}u_k) - \mathcal{A}(u, P_V^s u)|_{(H_0^1)'} \to 0 \qquad \text{as} \quad k \to \infty.$$

Taking into consideration inequality (5.8) from Lemma 5.3, we obtain

$$\begin{aligned}
&|\mathcal{A}(u_k, P_V^{s,M_k}u_k) - \mathcal{A}(u, P_V^s u)|_{(H_0^1)'} \\
&\leqslant \frac{3}{2}|\iota| \|u_k\|_{H_0^1} \|P_V^{s,M_k}u_k - P_V^s u\|_{L^4} + |\iota| \|P_V^s u\|_{H_0^1} \|u_k - u\|_{L^4}.
\end{aligned}$$

Complete continuity of the embedding $\iota : H^1_0 \hookrightarrow L^4$ implies that

$\lim_{k\to\infty} \|u_k - u\|_{L^4} = 0$. From Remark 1.9 and (5.16), we deduce that

$$P_V^{s,M_k} \to P_V^s \quad \text{pointwise on } H^1_0 \quad \text{as} \quad k \to \infty.$$

Moreover, P_V^{s,M_k} and P_V^s are selfadjoint. Consequently,

$$P_V^{s,M_k} u_k \to P_V^s u \quad \text{weakly in } H^1_0 \quad \text{as} \quad k \to \infty.$$

Again, thanks to the complete continuity of embedding ι

$$\lim_{k\to\infty} \|P_V^{s,M_k} u_k - P_V^s u\|_{L^4} = 0.$$

Now the assertion follows.

Ad. (A.6). Let us fix $u_0 \in H^1_0$ and let $u \in H^1_0$. Let us calculate the Fréchet differentials

$$\begin{aligned} \mathrm{d}_u \tilde{T}_M &= P_V^{s,M} \mathcal{R}_M^{-1} \mathrm{d}_u \big(\mathcal{A} \circ (\mathrm{id} \triangle P_V^{s,M})\big), \qquad M \in \mathcal{S}(H^1_0) \\ \mathrm{d}_{u_0} T &= P_V^s \mathcal{R}^{-1} \mathrm{d}_{u_0} \big(\mathcal{A} \circ (\mathrm{id} \triangle P_V^s)\big). \end{aligned}$$

By (5.4), $\mathcal{A}$ is completely continuous, thus

$$\mathrm{d}_{u_0}\big(\mathcal{A} \circ (\mathrm{id} \triangle P_V^s)\big) \text{ is completely continuous.}$$

Hence, by (6.17) and Corollary 8.5 in Appendix A, it is sufficient to prove that

$$\mathrm{d}_u\big(\mathcal{A} \circ (\mathrm{id} \triangle P_V^{s,M})\big) \to \mathrm{d}_{u_0}\big(\mathcal{A} \circ (\mathrm{id} \triangle P_V^s)\big) \quad \text{in} \quad \mathcal{L}(H^1_0 (H^1_0)')$$

as $(M,u) \succ \mathcal{B} \underline{\times} \mathcal{F}(u_0)$. Since mapping $\mathcal{A} : H^1_0 \times H^1_0 \to (H^1_0)'$ is bilinear and it satisfies inequality (5.8) from Lemma 5.3, we obtain

$$\begin{aligned} &|\mathrm{d}_u\big(\mathcal{A} \circ (\mathrm{id} \triangle P_V^{s,M})\big)(h) - \mathrm{d}_{u_0}\big(\mathcal{A} \circ (\mathrm{id} \triangle P_V^s)\big)(h)|_{(H^1_0)'} \\ &= |\mathcal{A}(u, P_V^{s,M} h) + \mathcal{A}(h, P_V^{s,M} u) - \mathcal{A}(u_0, P_V^s h) - \mathcal{A}(h, P_V^s u_0)|_{(H^1_0)'} \\ &\leqslant |\mathcal{A}(u, P_V^{s,M} h) - \mathcal{A}(u_0, P_V^s h)|_{(H^1_0)'} + |\mathcal{A}(h, P_V^{s,M} u) - \mathcal{A}(h, P_V^s u_0)|_{(H^1_0)'} \\ &\leqslant \frac{3}{2} |\iota| \|u\|_{H^1_0} \|P_V^{s,M} h - P_V^s h\|_{L^4} + |\iota| \|P_V^s h\|_{H^1_0} \|u - u_0\|_{L^4} \\ &\quad + \frac{3}{2} |\iota| \|h\|_{H^1_0} \|P_V^{s,M} u - P_V^s u_0\|_{L^4} \\ &\leqslant \frac{3}{2} |\iota| \|u\|_{H^1_0} |P_V^{s,M} - P_V^s|_{\mathcal{L}(H^1_0, L^4)} \|h\|_{H^1_0} + |\iota| |P_V^s|_{\mathcal{E}nd H^1_0} \|h\|_{H^1_0} \|u - u_0\|_{L^4} \\ &\quad + \frac{3}{2} |\iota| \|h\|_{H^1_0} \|P_V^{s,M} u - P_V^s u_0\|_{L^4}, \qquad h \in H^1_0. \end{aligned}$$

Hence

$$\begin{aligned}&\big|\mathrm{d}_u\big(\mathcal{A}\circ(\mathrm{id}\triangle P_V^{s,M})\big)-\mathrm{d}_{u_0}\big(\mathcal{A}\circ(\mathrm{id}\triangle P_V^{s})\big)\big|_{\mathcal{L}(H_0^1,(H_0^1)')}\\&\leqslant\frac{3}{2}|\iota|\|u\|_{H_0^1}|P_V^{s,M}-P_V^s|_{\mathcal{L}(H_0^1,L^4)}+|\iota||P_V^s|_{\mathcal{E}nd H_0^1}\|u-u_0\|_{L^4}\\&\quad+\frac{3}{2}|\iota|\|P_V^{s,M}u-P_V^s u_0\|_{L^4}.\end{aligned}$$

Applying Corollary 8.8 from Appendix A with

$$\psi:\mathcal{S}(H_0^1)\ni M\mapsto P_V^{s,M}\in\mathcal{E}nd H_0^1,\quad K:=\iota:H_0^1\hookrightarrow L^4\quad\text{and}\quad L:L^4\hookrightarrow L^2,$$

we deduce that

$$\lim_{M\succ\mathcal{B}}\big|P_V^{s,M}-P_V^s\big|_{\mathcal{L}(H_0^1,L^4)}=\lim_{M\succ\mathcal{B}}\big|\iota\circ(P_V^{s,M}-P_V^s)\big|_{\mathcal{L}(H_0^1,L^4)}=0.$$

Moreover, taking into consideration Lemma 8.9 from Appendix A, we infer that

$$\|P_V^{s,M}(u)-P_V^s(u_0)\|_{L^4}\to 0\qquad\text{as}\quad(M,u)\succ\mathcal{B}\underline{\times}\mathcal{F}(u_0).$$

Note that the sets $\mathbb{S}_s(\nu,f_s)$ and $\mathbb{S}_{s,M}(\nu,f_s)$ correspond to the following sets from the abstract setting

$$\mathbb{S}_s(\nu,f_s)=\mathfrak{R}(\mu,g)\qquad\text{and}\qquad\mathbb{S}_{s,M}(\nu,f_s)=\mathfrak{R}_M(\mu,g_M)$$

for $\mu:=\nu$, $g:=P_V^s\mathcal{R}^{-1}f_s$ and $g_M:=P_V^{s,M}\mathcal{R}^{-1}f_s$ (compare (2.1) and (2.2) in Section 2.1 with (6.5) and (6.6)). Now the assertion follows from Theorem 2.10.

From Theorems 6.1 and 6.4 we have the following result.

Theorem 6.5 **(stability)**. — *Assume that $(\nu,f)\in\mathcal{G}$ and $(f_s)_{s\in\mathbb{N}}$ tends to f in $(H_0^1)'$ as $s\to\infty$. Let $\varepsilon>0$. Then there exist $s_*\in\mathbb{N}$, a subspace $W\in\mathcal{S}(H_0^1)\cap\{\dim<\infty\}$ and $\delta>0$ such that*

(i) $d\big(\mathbb{S}_{s_*,M}(\nu,f_{s_*}),\mathbb{S}(\nu,f)\big)\leqslant\varepsilon$,

(ii) $\#\mathbb{S}_{s_*,M}(\nu,f_{s_*})=\#\mathbb{S}_{s_*}(\nu,f_{s_*})=\#\mathbb{S}(\nu,f)<\infty$,

whenever $M\in B_{W,\delta}$. (Here d stands for the Hausdorff metric over H_0^1.)

Proof. — Let us fix $(\nu,f)\in\mathcal{G}$ and $\varepsilon>0$. By Theorem 6.1 there exists $s_0\in\mathbb{N}$ such that for each $s\geqslant s_0$

$$d(\mathbb{S}_s(\nu,f_s),\mathbb{S}(\nu,f))\leqslant\frac{\varepsilon}{2}$$

and

$$\#\mathbb{S}_s(\nu, f_s) = \#\mathbb{S}(\nu, f) < \infty.$$

By Theorem 6.4, we can choose $s_* \geqslant s_0$, a finite-dimensional subspace $W \in \mathcal{S}(H_0^1)$ and $\delta > 0$ such that

$$d(\mathbb{S}_{s_*,M}(\nu, f_{s_*}), \mathbb{S}_{s_*}(\nu, f_{s_*})) \leqslant \frac{\varepsilon}{2}$$

and

$$\#\mathbb{S}_{s_*,M}(\nu, f_{s_*}) = \#\mathbb{S}_{s_*}(\nu, f_{s_*}) < \infty,$$

whenever $M \in B_{W,\delta}$. Then

$$\begin{gathered} d(\mathbb{S}_{s_*,M}(\nu, f_{s_*}), \mathbb{S}(\nu, f)) \\ \leqslant d(\mathbb{S}_{s_*,M}(\nu, f_{s_*}), \mathbb{S}_{s_*}(\nu, f_{s_*})) + d(\mathbb{S}_{s_*}(\nu, f_{s_*}), \mathbb{S}(\nu, f)) \leqslant \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon \end{gathered}$$

and

$$\#\mathbb{S}_{s_*,M}(\nu, f_{s_*}) = \#\mathbb{S}_{s_*}(\nu, f_{s_*}) = \#\mathbb{S}(\nu, f) < \infty,$$

whenever $M \in B_{W,\delta}$. □

Remark that in Theorems 6.1, 6.4 and 6.5, we have assumed that the data (ν, f) belong to the set $\mathcal{G}$ defined by (6.7). Now, we will concentrate on the properties of the set $\mathcal{G}$.

6.3. Properties of the set $\mathcal{G}$.

LEMMA 6.6. — *The set*

$$\begin{aligned} \mathcal{G}_1 := \{(\nu, c) \in]0,\infty[\times V :\ & c \text{ is a regular value of the} \\ & \text{mapping } V \ni \phi \mapsto \nu\phi + P_V \mathcal{R}^{-1}\mathcal{A}_{\phi,\phi} \in V\} \end{aligned}$$

is open and dense in $]0,\infty[\times V$.

Proof. — We claim that conditions (10.4)-(10.5) in Appendix C are satisfied if we take

$$\begin{aligned} &H := V \\ &T(\phi) := P_V \mathcal{R}^{-1}\mathcal{A}_{\phi,\phi}, \qquad \phi \in V. \end{aligned}$$

To check condition (10.4) let $\phi_k \to \phi$ weakly in $V \subset H_0^1$. Then $\mathcal{A}_{\phi_k,\phi_k} \to \mathcal{A}_{\phi,\phi}$ in $(H_0^1)'$, by Lemma 5.3 (b). Thus by continuity of P_V and of the Riesz isomorphism $\mathcal{R}$

$$P_V \mathcal{R}^{-1}\mathcal{A}_{\phi_k,\phi_k} \to P_V \mathcal{R}^{-1}\mathcal{A}_{\phi,\phi} \quad \text{in} \quad V.$$

To verify condition (10.5) let us fix $\mu > 0$ and $g \in V$. Next, multiply the equation

$$\mu\phi + P_V\mathcal{R}^{-1}\mathcal{A}_{\phi,\phi} = g$$

scalarly by $\phi \in V$ to obtain

$$\mu\|\phi\|^2_{H^1_0} + ((P_V\mathcal{R}^{-1}\mathcal{A}_{\phi,\phi}|\phi)) = ((g|\phi))$$
$$\mu\|\phi\|^2_{H^1_0} + \mathcal{A}_{\phi,\phi}(\phi) = ((g|\phi)).$$

Since $\mathcal{A}_{\phi,\phi}(\phi) = 0$, we infer that $\|\phi\|_{H^1_0} \leqslant \frac{\|g\|_{H^1_0}}{\mu}$. This means that condition (10.5) holds with

$$\kappa(\mu, r) := \frac{r}{\mu}, \qquad \mu > 0, \quad r > 0.$$

In conclusion, by Theorem 10.12, the set $\mathcal{G}_1$ is open and dense in $]0,\infty[\times V$. □

Let us denote

$$\mathcal{K}_\nu(u) := \nu u + P_V\mathcal{R}^{-1}\mathcal{A}(u, P_V u), \qquad u \in H^1_0. \tag{6.20}$$

Remark 6.7. — If $c \in V$, then the following conditions are equivalent

(i) *c is a regular value of the mapping $\mathcal{K}_\nu : H^1_0 \to H^1_0$,*

(ii) *c is a regular value of the mapping $(\mathcal{K}_\nu)_{|V} : V \to V$.*

Proof. — Since $\mathcal{K}_\nu(V) \subset V$, the mapping $(\mathcal{K}_\nu)_{|V}$ is well defined. Moreover, for $u \in V$

$$d_u(\mathcal{K}_\nu)_{|V} = (d_u\mathcal{K}_\nu)_{|V}. \tag{6.21}$$

By complete continuity of the mapping $\mathcal{A}$, both $\mathcal{K}_\nu$ and $(\mathcal{K}_\nu)_{|V}$ are Fredholm mappings of index 0. Hence, the Fréchet differentials of these mappings are epimorphisms if and only if they are monomorphisms. Since $c \in V$,

$$\{u \in H^1_0 : \mathcal{K}_\nu(u) = c\} = \{v \in V : (\mathcal{K}_\nu)_{|V}(v) = c\}. \tag{6.22}$$

Let $u \in \mathcal{K}_\nu^{-1}(\{c\})$.

To prove that (ii) follows from (i), it is sufficient to check that

$$d_u(\mathcal{K}_\nu)_{|V} \in MonoV.$$

Let $h \in V$ be such that $\big(d_u(\mathcal{K}_\nu)_{|V}\big)(h) = 0$. By (6.21), $(d_u\mathcal{K}_\nu)(h) = 0$. Since c is a regular value of $\mathcal{K}_\nu$, $d_u\mathcal{K}_\nu \in MonoH^1_0$, and hence $h = 0$.

Now, suppose that c is a regular value of the restriction $(\mathcal{K}_\nu)_{|V}$. Let $h \in H_0^1$ be such that $(d_u\mathcal{K}_\nu)(h) = 0$. Hence

$$\nu h = -(P_V \circ \mathcal{R}^{-1} \circ d_u\bigl(\mathcal{A} \circ (\mathrm{id} \bigtriangleup P_V))\bigr)(h).$$

In particular, $h \in V$, and $\bigl(d_u(\mathcal{K}_\nu)_{|V}\bigr)(h) = 0$. Since $d_u(\mathcal{K}_\nu)_{|V}$ is a monomorphism, $h = 0$. $\square$

Now, we prove some simple topological result which we will need in the proof of the main theorem of this subsection.

Remark 6.8. — Let X, Y be topological spaces. Assume that $P : X \to Y$ is an open mapping and Q is a dense subset of Y. Then $P^{-1}(Q)$ is a dense subset of X.

Proof. — It is sufficient to show that for any nonempty set $A \in topX$, the intersection $A \cap P^{-1}(Q)$ is nonempty. Since P is open, $P(A)$ is a nonempty open subset of Y. Then, the density of the set Q in the space Y yields that $P(A) \cap Q$ is nonempty. Hence, there exists an element $a \in A$ such that $P(a) \in Q$, or equivalently, $a \in A \cap P^{-1}(Q)$. $\square$

The main result concerning the properties of the set $\mathcal{G}$ is the following theorem.

THEOREM 6.9. — *The set $\mathcal{G}$ is open and dense in $]0,\infty[\times(H_0^1)'$.*

Proof. — Let us observe that

$$\mathcal{G} = (\mathrm{id} \times P_V\mathcal{R}^{-1})^{-1}(\mathcal{G}_1). \tag{6.23}$$

Indeed,

$$\begin{aligned}&(\nu, f) \in \mathcal{G} \Leftrightarrow (\nu, P_V\mathcal{R}^{-1}f) \in \mathcal{G}_1 \Leftrightarrow (\mathrm{id} \times P_V\mathcal{R}^{-1})(\nu, f) \in \mathcal{G}_1\\ &\Leftrightarrow (\nu, f) \in (\mathrm{id} \times P_V\mathcal{R}^{-1})^{-1}(\mathcal{G}_1).\end{aligned}$$

The mapping

$$(\mathrm{id} \times P_V\mathcal{R}^{-1}) :]0,\infty[\times(H_0^1)' \to]0,\infty[\times V \tag{6.24}$$

is open and continuous (its openness follows from the openness of the projection P_V). By Lemma 6.6, the set $\mathcal{G}_1$ is dense in $]0,\infty[\times V$. The openness of the mapping (6.24) and equality (6.23) yield, by Remark 6.8, that $\mathcal{G}$ is dense in $]0,\infty[\times(H_0^1)'$. Since $\mathcal{G}_1$ is open in $]0,\infty[\times V$ and the mapping (6.24) is continuous, we infer by (6.23) that $\mathcal{G}$ is also open in $]0,\infty[\times(H_0^1)'$. $\square$

6.4. Pressure in the Holly method

Calculation of the pressure p was considered in [13]. Theorem 3.1 in [13] states that if $v \in V$, then we have the following representation for $p \in L^2(\Omega)$ with $\int_\Omega p(x)\,dx = 0$

$$p = (\mathrm{divdiv}^*)^{-1}\mathrm{div}\mathcal{R}^{-1}\big(\mathcal{A}_{v,v} - f\big). \tag{6.25}$$

Let the pair $(v,p) \in V \times L^2(\Omega)$ be a solution od the N-S equations. With the double approximation of the velocity we can associate a double approximation of the pressure.

The first step of approximation of the pressure. Assume that $f_s \to f$ and $l_s \to \mathcal{A}_{v,v}$ in $(H_0^1)'$ as $s \to \infty$. We may put, e.g. $l_s := \mathcal{A}_{v_s P_V^s v_s,}$, where v_s is a solution of equation (5.11). Let

$$p_s := \sum_{j=0}^{s}(\mathrm{id} - \mathrm{divdiv}^*)^j \mathrm{div}\mathcal{R}^{-1}\big(l_s - f_s\big). \tag{6.26}$$

Then

$$\lim_{s\to\infty} \|p_s - p\|_{L^2(\Omega)} = 0,$$

see Theorem 3.2 in [13]. In conclusion, if we consider the sets of pairs (v,p) and (v_s,p_s), i.e.

$$\begin{aligned}\mathcal{P}(\nu,f) &:= \big\{(v,p):\ \ v \in \mathbb{S}(\nu,f)\big\},\\ \mathcal{P}_s(\nu,f_s) &:= \big\{(v_s,p_s);\ \ v_s \in \mathbb{S}_s(\nu,f_s)\quad \text{and } p_s \text{ is defined by (6.26)}\big\}, \quad s \in \mathbb{N},\end{aligned}$$

(see (6.4) and (6.5)), then by Theorem 6.1 we obtain the following corollary

Corollary 6.10. — *Under the assumptions of Theorem 6.1, we have*

$$\lim_{s\to\infty} \mathcal{P}_s(\nu,f_s) = \mathcal{P}(\nu,f) \tag{6.27}$$

in the Hausdorff metric over $H_0^1 \times L^2(\Omega)$.

The second step of approximation of the pressure. Let us now fix $s \in \mathbb{N}$ and let M be a closed linear subspace of H_0^1. Let us put

$$p_{s,M} := \sum_{j=0}^{s}(\mathrm{id} - \mathrm{divdiv}_M^*)^j \mathrm{div}\mathcal{R}_M^{-1}\big(l_s - f_s\big). \tag{6.28}$$

In particular, if $l_s := \mathcal{A}_{v_s P_V^s v_s,}$, then we may approximate $\mathcal{R}_M^{-1}(l_s)$ by $\mathcal{A}_{v_{s,M}, P_V^{s,M} v_{s,M},}$, where $v_{s,M}$ is a solution of equation (5.17), see [13].

By Remarks 5.6, 5.7, 5.8 and Theorem 6.4 we infer that

$$\lim_{M \succ \mathcal{B}} \|p_{s,M} - p_s\|_{L^2(\Omega)} = 0. \tag{6.29}$$

Let us consider the corresponding sets of pairs $(v_{s,M}, p_{s,M})$, i.e.

$\mathcal{P}_{s,M}(\nu, f_s) := \{(v_{s,M}, p_{s,M}); v_{s,M} \in \mathbb{S}_{s,M}(\nu, f_s)$ and $p_{s,M}$ is defined by (6.28)$\}$,

where $s \in \mathbb{N}$ and $M \in \mathcal{S}(H_0^1)$, see (6.6). By Theorem 6.4 we obtain the following conclusion

Corollary 6.11. — *Under the assumptions of Theorem 6.4, we have for sufficiently large $s \in \mathbb{N}$*

$$\lim_{M \succ \mathcal{B}} \mathcal{P}_{s,M}(\nu, f_s) = \mathcal{P}_s(\nu, f_s). \tag{6.30}$$

in the Hausdorff metric over $H_0^1 \times L^2(\Omega)$.

7. Summary

In this paper we have considered an abstract nonlinear equation in a real separable Hilbert space H and certain class of approximate equations on closed linear subspaces of H. In Section 2 we have provided certain approach to the problem of stability with respect to the approximation of the space H. We have proven that, generically, the set of all solutions of the exact equation is the limit in the Hausdorff metric of the sets of approximate solutions, over some filterbase on the family of all closed linear subspaces of H. The abstract results have been applied to the stationary Navier-Stokes equations in two and three dimensional bounded domains. Namely, we have proven that the classical Galerkin method is stable with respect to the approximation of the space V of divergence-free vector fields. Moreover, we have considered the Holly method of finding the velocity v in the Navier-Stokes problem. Using the general approach of Section 2 we have shown that this method is stable with respect to the approximation of the Sobolev space H_0^1. Moreover, referring to the results of paper [13] we have analysed behaviour of the pairs (v, p) and the corresponding sets of pairs of approximate velocities and approximate pressures.

The present paper contains theoretical analysis of the problem of stability. The computational aspect of the Holly method, especially reduction of calculation of some operators (i.e. div*, P_V^s and $P_V^{s,M}$) to the Dirichlet problem for the Poisson equation, and continuation of the results of papers [8] and [12] will be considered in the forthcoming paper.

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8. Appendix A: Auxiliary results about filterbases

Most of the presented results are generalizations of the results from functional analysis. We assume that we have a filterbase $\mathcal{B}$ of a countable type on a set S. Using the Baire theorem, we will prove the following version of the Banach-Steinhaus theorem.

THEOREM 8.1. — *Let X, Y be Banach spaces. Assume that a mapping $\psi : \mathcal{S} \to \mathcal{L}(X, Y)$ satisfies the following condition*

$$\begin{aligned} &\textit{for every } x \in X \textit{ there exists } B_x \in \mathcal{B} \textit{ such that} \\ &\textit{the set } \{\psi(\omega)(x),\ \omega \in B_x\} \textit{ is bounded}\,. \end{aligned} \tag{8.1}$$

Then there exists a set $B \in \mathcal{B}$ such that the set of norms $\{|\psi(\omega)|_{\mathcal{L}(X,Y)}$, $\omega \in B\}$ is bounded.

Proof. — Let

$$Z_{n,k} := \{x \in X :\ |\psi(\omega)(x)|_Y \leqslant n \ \text{ for } \ \omega \in B_0^k\}, \qquad k, n \in \mathbb{N}.$$

Then

$$X = \bigcup_{n,k=1}^{\infty} Z_{n,k}. \tag{8.2}$$

Indeed, it is sufficient to check that every element $x \in X$ belongs to the set on the right-hand side of (8.2) By (8.1), there exists $B_x \in \mathcal{B}$ and $m \in \mathbb{N}$ such that the set $\{\psi(\omega)(x),\ \omega \in B_x\}$ is contained in the ball $\overline{K}_Y(0, m)$. By (1.2) there exists $l \in \mathbb{N}$ such that $B_0^l \subset B_x$. Thus, for every $\omega \in B_0^l$: $|\psi(\omega)(x)|_Y \leqslant m$. This means that $x \in Z_{m,l}$.

Since X is complete, it is a set of the second category of Baire. Thus the Baire theorem yields that there exist $n_0, k_0 \in \mathbb{N}$ such that the set Z_{n_0,k_0} is not nowhere dense, i.e. int$\overline{Z}_{n_0,k_0}$ - interior of the closure of Z_{n_0,k_0}, is nonempty. By the continuity of the operators $\psi(\omega)$ and of the norm, the set Z_{n_0,k_0} is closed in X. Thus, there exist $x_0 \in X$ and $r_0 > 0$ such that $\overline{K}_X(x_0, r_0) \subset Z_{n_0,k_0}$, or equivalently,

$$\psi(\omega)\big(\overline{K}_X(x_0, r_0)\big) \subset \overline{K}_Y(0, n_0), \qquad \omega \in B_0^{k_0}. \tag{8.3}$$

We will prove that

$$\sup\{|\psi(\omega)|_{\mathcal{L}(X,Y)}, \quad \omega \in B_0^{k_0}\} < \infty. \tag{8.4}$$

Let $x \in \overline{K}_X(0,1)$ and $\omega \in B_0^{k_0}$. Since $\overline{K}_X(0,1) = \frac{1}{r_0}\big(-x_0 + \overline{K}_X(x_0, r_0)\big)$, $x = -\frac{1}{r_0}x_0 + \frac{1}{r_0}z$ for some $z \in \overline{K}_X(x_0, r_0)$. Hence, by (8.3),

$$\begin{aligned} |\psi(\omega)(x)|_Y \qquad &= |-\tfrac{1}{r_0}\psi(\omega)(x_0) + \tfrac{1}{r_0}\psi(\omega)(z)|_Y \\ &\leqslant |-\tfrac{1}{r_0}\psi(\omega)(x_0)|_Y + |\tfrac{1}{r_0}\psi(\omega)(z)|_Y \leqslant \tfrac{2n_0}{r_0}. \end{aligned}$$

Since x and ω are arbitrary, (8.4) holds. Thus, the assertion holds with $B := B_0^{k_0}$. $\square$

Corollary 8.2. — *Let X, Y be Banach spaces. If a mapping $\psi \to \mathcal{L}(X,Y)$ satisfies the following condition*

$$\lim_{\omega \succ \mathcal{B}} \psi(\omega)(x) = A(x), \qquad x \in X \tag{8.5}$$

for some $A \in \mathcal{L}(X,Y)$, then there exists a set $B \in \mathcal{B}$ such that the set of norms $\{|\psi(\omega)|_{\mathcal{L}(X,Y)}, \ \omega \in B\}$ is bounded.

Proof. — Directly by Definition 1.2, condition (8.5) implies (8.1). Thus, the assertion is an immediate consequence of Theorem 8.1. $\square$

Lemma 8.3. — *Assume that X, Y, Z are Banach spaces. Let the following mappings be given*

$$\begin{aligned} &\psi_1 : S \to \mathcal{L}(X,Y) \quad \textit{such that} \quad \lim_{\omega \succ \mathcal{B}} \psi_1(\omega)(x) = A_1(x), \qquad x \in X, \\ &\psi_2 : S \to \mathcal{L}(Y,Z) \quad \textit{such that} \quad \lim_{\omega \succ \mathcal{B}} \psi_2(\omega)(y) = A_2(y), \qquad y \in Y \end{aligned}$$

for some $A_1 \in \mathcal{L}(X,Y)$, $A_2 \in \mathcal{L}(Y,Z)$. Then

$$\lim_{\omega \succ \mathcal{B}} \big(\psi_2(\omega) \circ \psi_1(\omega)\big)(x) = (A_2 \circ A_1)(x), \qquad x \in X.$$

Proof. — Let $x \in X$ and $\omega \in S$. Then

$$\begin{aligned}&\big(\psi_2(\omega) \circ \psi_1(\omega)\big)(x) - (A_2 \circ A_1)(x)\\ &= \big(\psi_2(\omega) \circ \psi_1(\omega)\big)(x) - \big(\psi_2(\omega) \circ A_1\big)(x) + \big(\psi_2(\omega) \circ A_1\big)(x) - (A_2 \circ A_1)(x)\\ &= \psi_2(\omega)\big(\psi_1(\omega)(x) - A_1(x)\big) + \big(\psi_2(\omega) - A_2\big)\big(A_1(x)\big).\end{aligned}$$

Thus, it is sufficient to check that

$$\psi_2(\omega)\big(\psi_1(\omega)(x) - A_1(x)\big) \to 0 \qquad \text{as} \quad \omega \succ \mathcal{B}. \tag{8.6}$$

By Corollary 8.2, there exist a set $B_0 \in \mathcal{B}$ and a constant $C > 0$ such that

$$\sup\{|\psi_2(\omega)|_{\mathcal{L}(Y,Z)}, \quad \omega \in B_0\} \leqslant C. \tag{8.7}$$

Let $\varepsilon > 0$. From the Definition 1.2, there follows that there exists $B_1 \in \mathcal{B}$ such that

$$|\psi_1(\omega)(x) - A_1(x)|_Y < \frac{\varepsilon}{C}, \qquad \omega \in B_1. \tag{8.8}$$

By (1.1), there exists $B \in \mathcal{B}$ such that $B \subset B_0 \cap B_1$. Then, by (8.7) and (8.8), we have

$$|\psi_2(\omega)\big(\psi_1(\omega)(x) - A_1(x)\big)|_Z \leqslant |\psi_2(\omega)|_{\mathcal{L}(Y,Z)} |\psi_1(\omega)(x) - A_1(x)|_Y < \varepsilon, \qquad \omega \in B,$$

which ends the proof. $\square$

LEMMA 8.4. — *Let X, Y, Z be Banach spaces. Assume that $A : X \to Y$ is a linear completely continuous operator. Let $\psi : S \to \mathcal{L}(Y,Z)$ be a mapping such that*

$$\lim_{\omega \succ \mathcal{B}} \psi(\omega)(y) = L(y), \qquad y \in Y \tag{8.9}$$

for some $L \in \mathcal{L}(Y,Z)$. Then

$$\lim_{\omega \succ \mathcal{B}} \big(\psi(\omega) \circ A\big) = L \circ A \quad \text{ in } \quad \mathcal{L}(X,Z).$$

Proof. — By Corollary 8.2, there exist $B_1 \in \mathcal{B}$ and $C > 0$ such that

$$\sup\{|\psi(\omega)|_{\mathcal{L}(Y,Z)}, \quad \omega \in B_1\} \leqslant C \quad \text{and} \quad |L|_{\mathcal{L}(Y,Z)} \leqslant C.$$

Let us fix $\varepsilon > 0$. Let $y_1, \dots, y_s \in Y$ be a $\frac{\varepsilon}{3C}$ - net of the set $A(\overline{K}_X(0,1))$. Let $x \in \overline{K}_X(0,1)$. Choose an index $j \in \{1, \dots, s\}$ such that $|A(x) - y_j|_Y \leqslant \frac{\varepsilon}{3C}$. Due to (8.9), there exists $B_2 \in \mathcal{B}$ such that

$$|\psi(\omega)(y_j) - L(y_j)|_Z < \frac{\varepsilon}{3}, \qquad \omega \in B_2.$$

By (1.1), there exists $B \subset B_1 \cap B_2$. Then

$$\begin{aligned}
&|\big(\psi(\omega)\circ A\big)(x) - (L\circ A)(x)|_Z \\
&\leqslant |\psi(\omega)(A(x)) - \psi(\omega)(y_j)|_Z + |\psi(\omega)(y_j) - L(y_j)|_Z + |L(y_j) - L(A(x))|_Z \\
&\leqslant |\psi(\omega)|_{\mathcal{L}(Y,Z)}|A(x)-y_j|_Y + |\psi(\omega)(y_j) - L(y_j)|_Z + |L|_{\mathcal{L}(Y,Z)}|y_j - A(x)|_Y < \varepsilon
\end{aligned}$$

for $\omega \in B$. Taking the supremum over all $x \in \overline{K}_X(0,1)$, we obtain

$$|\psi(\omega)\circ A - L\circ A|_{\mathcal{L}(X,Z)} \leqslant \varepsilon, \qquad \omega \in B,$$

which ends the proof. $\square$

Corollary 8.5 *Assume that X, Y, Z are Banach spaces. Let the following mappings be given*

$$\begin{aligned}
&\psi_1 : S \to \mathcal{L}(X,Y) \quad \text{such that} \quad \lim\nolimits_{\omega\succ\mathcal{B}} \psi_1(\omega) = A \quad \text{in } \mathcal{L}(X,Y),\\
&\psi_2 : S \to \mathcal{L}(Y,Z) \quad \text{such that} \quad \lim\nolimits_{\omega\succ\mathcal{B}} \psi_2(\omega)(y) = L(y), \qquad y \in Y
\end{aligned}$$

where $A : X \to Y$ is a linear completely continuous operator and $L \in \mathcal{L}(Y,Z)$. Then

$$\lim_{\omega\succ\mathcal{B}} \big(\psi_2(\omega)\circ\psi_1(\omega)\big) = L\circ A \quad \text{in } \mathcal{L}(X,Z).$$

Proof. — By virtue of the inequality

$$\begin{aligned}
&|\psi_2(\omega)\circ\psi_1(\omega) - L\circ A|_{\mathcal{L}(X,Z)} \\
&\leqslant |\psi_2(\omega)|_{\mathcal{L}(Y,Z)}|\psi_1(\omega) - A|_{\mathcal{L}(X,Y)} + |\psi_2(\omega)\circ A - L\circ A|_{\mathcal{L}(X,Z)}, \quad \omega \in S,
\end{aligned}$$

the assertion is a consequence of Corollary 8.2 and Lemma 8.4. $\square$

Lemma 8.6. — *Assume that H, H_1 are Hilbert spaces and H_1 is separable. Let $K : H \to H_1$ be a linear completely continuous operator and let $\psi : S \to \mathcal{E}nd H$ be a mapping such that*

(i) $\psi(\omega)$ *is selfadjoint ,* $\quad \omega \in S$,

(ii) $\lim_{\omega\succ\mathcal{B}} \psi(\omega)(x) = A(x), \quad x \in H$

for some $A \in \mathcal{E}nd H$. Then

$$\lim_{\omega\succ\mathcal{B}} \big(K\circ\psi(\omega)\big) = K\circ A \quad \text{in } \mathcal{L}(H,H_1).$$

Proof. — **Step 1^0.** Assume that $H_1 := \mathbb{R}$. Then K is a linear functional on H; denote $K =: \xi \in H'$.

By the Riesz theorem, $\xi = (\cdot|a)_H$ for some $a \in H$. Let $x \in \bar{K}_H(0,1)$. By selfadjointness of the operators $\psi(\omega)$, we obtain

$$\begin{aligned}
&|(\xi \circ \psi(\omega))(x) - (\xi \circ A)(x)| = |\xi(\psi(\omega)x) - \xi(Ax)| \\
&= |(\psi(\omega)x|a)_H - (Ax|a)_H| = |(x|\psi(\omega)a)_H - (x|Aa)_H| \\
&= |(x|\psi(\omega)a - Aa)_H| \leqslant |\psi(\omega)a - Aa|_H \quad \text{ for } \omega \in S.
\end{aligned}$$

Therefore

$$|\xi \circ \psi(\omega) - \xi \circ A|_{H'} \leqslant |\psi(\omega)a - Aa|_H \to 0 \qquad \text{as} \quad \omega \succ \mathcal{B}.$$

Step 2^0. Now, we assume that $K(H)$ is a finite-dimensional subspace of H_1. Let $b_1, \ldots, b_n$ form a base of $K(H)$, where $n = \dim K(H)$ and let $b_1^*, \ldots, b_n^*$ be the dual basis. Then

$$K \circ \psi(\omega) = \sum_{i=1}^{n} \Lambda_i \circ (\xi_i \circ \psi(\omega)), \qquad K \circ A = \sum_{i=1}^{n} \Lambda_i \circ (\xi_i \circ A), \tag{8.10}$$

where

$$\begin{aligned}
&\xi_i : H \ni x \mapsto (b_i^* \circ K)(x) \in \mathbb{R}, \\
&\Lambda_i : \mathbb{R} \ni r \mapsto r \cdot b_i \in H_1, \qquad i = 1, \ldots, n.
\end{aligned}$$

Indeed, let $x \in H$. Then

$$\begin{aligned}
\left(\sum_{i=1}^{n} \Lambda_i \circ (\xi_i \circ \psi(\omega))\right) x &= \sum_{i=1}^{n} \Lambda_i \big(\xi_i(\psi(\omega)x)\big) = \sum_{i=1}^{n} \Lambda_i \big(b_i^*((K \circ \psi(\omega))x)\big) \\
&= \sum_{i=1}^{n} b_i^*\big((K \circ \psi(\omega))x\big) \cdot b_i = (K \circ \psi(\omega))x.
\end{aligned}$$

Since $\psi(\omega) \to A$ pointwise as $\omega \succ \mathcal{B}$ and K is continuous, step 1^0 yields that

$$\xi_i \circ \psi(\omega) \to \xi_i \circ A \quad \text{in} \quad H' \qquad \text{as} \quad \omega \succ \mathcal{B}$$

and hence, by the continuity of the operators Λ_i

$$\Lambda_i \circ (\xi_i \circ \psi(\omega)) \to \Lambda_i \circ (\xi_i \circ A) \quad \text{in } \mathcal{L}(H, H_1) \qquad \text{as} \quad \omega \succ \mathcal{B} \tag{8.11}$$

for any $i \in \{1, \ldots, n\}$. Then, the assertion follows from (8.10) and (8.11) .

Step 3^0. The general case. By virtue of Corollary 8.2, there exists $B_1 \in \mathcal{B}$ such that $C := \sup\{|\psi(\omega)|_{\mathcal{E}ndH}; \quad \omega \in B_1\} < \infty$. Then also $|A|_{\mathcal{E}ndH} \leqslant C$. Let $\varepsilon > 0$. Since H_1 is a separable Hilbert space, the subspace of all finite-dimensional operators is dense in the space of all completely continuous operators. Therefore, there exists a finite-dimensional operator $L : H \to H_1$ such that $|K - L|_{\mathcal{L}(H,H_1)} \leqslant \frac{\varepsilon}{4C}$. Then

$$\begin{aligned}
&|K \circ \psi(\omega) - K \circ A|_{\mathcal{L}(H,H_1)} \leqslant |K \circ \psi(\omega) - L \circ \psi(\omega)|_{\mathcal{L}(H,H_1)} \\
&\quad + |L \circ \psi(\omega) - L \circ A|_{\mathcal{L}(H,H_1)} + |L \circ A - K \circ A|_{\mathcal{L}(H,H_1)} \\
&\leqslant |K - L|_{\mathcal{L}(H,H_1)} |\psi(\omega)|_{\mathcal{E}ndH} + |L \circ \psi(\omega) - L \circ A|_{\mathcal{L}(H,H_1)} \\
&\quad + |L - K|_{\mathcal{L}(H,H_1)} |A|_{\mathcal{E}ndH} \\
&\leqslant 2C|K - L|_{\mathcal{L}(H,H_1)} + |L \circ \psi(\omega) - L \circ A|_{\mathcal{L}(H,H_1)} \\
&\leqslant \frac{\varepsilon}{2} + |L \circ \psi(\omega) - L \circ A|_{\mathcal{L}(H,H_1)} \qquad \text{for} \quad \omega \in B_1.
\end{aligned}$$

Hence, by step 2^0, there exists $B_2 \in \mathcal{B}$ such that

$$|L \circ \psi(\omega) - L \circ A|_{\mathcal{L}(H,H_1)} \leqslant \frac{\varepsilon}{2} \qquad \text{for} \quad \omega \in B_2.$$

Since $\mathcal{B}$ is a filterbase, there exists $B \in \mathcal{B}$ such that $B \subset B_1 \cap B_2$. In conclusion,

$$|K \circ \psi(\omega) - K \circ A|_{\mathcal{L}(H,H_1)} < \varepsilon \qquad \text{for} \quad \omega \in B,$$

which end the proof. $\square$

We will use the following auxiliary Lemma.

Lemma 8.7 *(Lions). — Let E, E_1, E_2 be Banach spaces. Assume that $A : E_1 \to E$ is a linear completely continuous operator and $L : E \to E_2$ is a linear continuous monomorphism. Then for every $\alpha > 0$ there exists $\beta > 0$ such that for every $x \in E_1$*

$$|A(x)|_E \leqslant \alpha |x|_{E_1} + \beta |(L \circ A)x|_{E_2}.$$

In the case when A and L are embeddings, the above lemma is proved in [11] (Theorem 3.3 in Section III).

Assume that

E is a Banach space such that there exist a separable Hilbert space H_1 and a continuous linearmonomorphism $L : E \to H_1$. (8.12)

Remark that the above assumption is satisfied if the Banach space E is continuously imbedded in a separable Hilbert space.

Corollary 8.8 . — *Assume (8.12). Let $K : H \to E$ be a linear completely continuous operator and let $\psi : S \to \mathcal{E}ndH$ be a mapping such that*

(i) $\psi(\omega)$ *is selfadjoint* , $\qquad \omega \in S$,

(ii) $\lim_{\omega \succ \mathcal{B}} \psi(\omega)(x) = A(x), \qquad x \in H$

for some $A \in \mathcal{E}ndH$. Then

$$\lim_{\omega \succ \mathcal{B}} \big(K \circ \psi(\omega)\big) = K \circ A \quad \text{in } \mathcal{L}(H, E).$$

Proof. — By Corollary 8.2, there exists $B_1 \in \mathcal{B}$ such that $C := \sup\{|\psi(\omega)|_{\mathcal{E}ndH};\quad \omega \in B_1\} < \infty$. Then also $|A|_{\mathcal{E}ndH} \leqslant C$. Let $\varepsilon > 0$. Let $x \in \bar{K}_H(0,1)$. By Lemma 8.7, there exists $\beta > 0$ such that

$$\begin{aligned}
&|(K \circ \psi(\omega))x - (K \circ A)x|_E \\
&\leqslant \frac{\varepsilon}{4C} |\psi(\omega)x - Ax|_H + \beta |(L \circ K \circ \psi(\omega))x - (L \circ K \circ A)x|_{H_1} \\
&\leqslant \frac{\varepsilon}{2} + \beta |L \circ K \circ \psi(\omega) - L \circ K \circ A|_{\mathcal{L}(H,H_1)} \quad \text{for } \omega \in B_1.
\end{aligned}$$

Taking the supremum over $x \in \bar{K}_H(0,1)$, we obtain

$$\begin{aligned}
&|K \circ \psi(\omega) - K \circ A|_{\mathcal{L}(H,E)} \\
&\leqslant \frac{\varepsilon}{2} + \beta |L \circ K \circ \psi(\omega) - L \circ K \circ A|_{\mathcal{L}(H,H_1)} \quad \text{for } \omega \in B_1.
\end{aligned}$$

Since K is completely continuous, $L \circ K : H \to H_1$ is a completely continuous operator between Hilbert spaces. Thus, by Lemma 8.6

$$L \circ K \circ \psi(\omega) \to L \circ K \circ A \quad \text{in } \mathcal{L}(H, H_1) \quad \text{as} \quad \omega \succ \mathcal{B}.$$

Thus, there exists $B_2 \in \mathcal{B}$ such that

$$|L \circ K \circ \psi(\omega) - L \circ K \circ A|_{\mathcal{L}(H,H_1)} < \frac{\varepsilon}{2\beta} \qquad \text{for } \omega \in B_2$$

Since $\mathcal{B}$ is a filterbase, there exists $B \in \mathcal{B}$ such that $B \subset B_1 \cap B_2$. In conclusion

$$|K \circ \psi(\omega) - K \circ A|_{\mathcal{L}(H,E)} < \varepsilon \qquad \text{for } \omega \in B$$

which completes the proof. $\square$

Lemma 8.9. — *Let X, Y be Banach spaces and let $x_0 \in X$. Assume that $\psi : S \to \mathcal{L}(X,Y)$ is a mapping such that*

$$\psi(\omega) \to A \quad \text{in } \mathcal{L}(X,Y) \quad \text{as } \omega \succ \mathcal{B}. \tag{8.13}$$

Then

$$\psi(\omega)x \to Ax_0 \quad \text{as} \quad (\omega, x) \succ \mathcal{B}\underline{\times}\mathcal{F}(x_0).$$

(Let us recall that $\mathcal{B}\underline{\times}\mathcal{F}(x_0) := \{B \times U; \quad B \in \mathcal{B}, U \in \mathcal{F}(x_0)\}$.)

Proof. — By Corollary 8.2, there exist $B_1 \in \mathcal{B}$ and $C > 0$ such that

$$\sup\{|\psi(\omega)|_{\mathcal{L}(X,Y)}, \quad \omega \in B_1\} \leqslant C.$$

Let $\varepsilon > 0$. Let $U := K_X(x_0, \frac{\varepsilon}{2C})$. If $x_0 = 0$, then

$$|\psi(\omega)x - Ax_0|_Y = |\psi(\omega)x|_Y \leqslant |\psi(\omega)|_{\mathcal{L}(X,Y)}|x|_X < \varepsilon, \quad \omega \in B_1, \quad x \in U.$$

Assume that $x_0 \neq 0$. By the assumption (8.13), there exist $B_2 \in \mathcal{B}$ such that

$$|\psi(\omega) - A|_{\mathcal{L}(X,Y)} < \frac{\varepsilon}{2|x_0|_X}, \qquad \omega \in B_2.$$

By (1.1), there exists $B \in \mathcal{B}$ such that $B \in B_1 \cap B_2$. Then

$$\begin{aligned}
&|\psi(\omega)x - Ax_0|_Y \leqslant |\psi(\omega)x - \psi(\omega)x_0|_Y + |\psi(\omega)x_0 - Ax_0|_Y \\
&\leqslant |\psi(\omega)|_{\mathcal{L}(X,Y)}|x - x_0|_X + |\psi(\omega) - A|_{\mathcal{L}(X,Y)}|x_0|_X \\
&< C\frac{\varepsilon}{2C} + \frac{\varepsilon}{2|x_0|_X}|x_0|_X = \varepsilon
\end{aligned}$$

for $\omega \in B$ and $x \in U$. $\square$

9. Appendix B: A certain version of the Schauder fixed point theorem

Let $\bar{K} := \bar{K}(0, R)$ be a closed ball in a Hilbert space $(H, (\cdot|\cdot))$, with center at 0 and radius $R > 0$. The norm induced by the scalar product $(\cdot|\cdot)$ is denoted by $|\cdot|$.

Theorem 9.1 *(Brouwer). — Assume that* $\dim H < \infty$. *Then every continuous mapping $f : \bar{K} \to \bar{K}$ has a fixed point, i.e. there exists $x_0 \in \bar{K}$ such that $f(x_0) = x_0$.*

Using the Brouwer fixed point theorem, J.L. Lions has proven the following theorem.

THEOREM 9.2 *(Lemma 4.3 in [10]). — Assume that* $\dim H < \infty$ *and* $F : \bar{K} \to H$ *is a continuous mapping such that*

$$\big(F(\zeta)|\zeta\big) \geqslant 0, \qquad \zeta \in \partial\bar{K} := \{x \in H : |x| = R\}. \tag{9.1}$$

Then, there exists $z \in \bar{K}$ *such that* $F(z) = 0$.

Proof. — We recall the proof of Lions. The mapping

$$r : H \ni x \mapsto \begin{cases} x, & |x| \leqslant R \\ \frac{Rx}{|x|}, & |x| > R \end{cases}$$

is a continuous mapping of the space H in the ball $\bar{K}$. Thus, by the Brouwer fixed point theorem, the mapping

$$f := r \circ (\mathrm{id} - F) : \bar{K} \to \bar{K},$$

has a fixed point, i.e. there exists $x_0 \in \bar{K}$ such that

$$r\big(x_0 - F(x_0)\big) = x_0. \tag{9.2}$$

We assert that $(\mathrm{id} - F)(x_0) \in \bar{K}$. Indeed, suppose contrary to our claim that $|(\mathrm{id} - F)(x_0)| > R$. Then $|x_0| = \left|\frac{R(x_0 - F(x_0))}{|x_0 - F(x_0)|}\right| = R$. In particular $x_0 \in \partial\bar{K}$. Hence, by equality (9.2), we obtain

$$\frac{R}{|x_0 - F(x_0)|}\big(x_0 - F(x_0)|x_0\big) = |x_0|^2.$$

Thus

$$\big(F(x_0)|x_0\big) = \big(\frac{R}{|x_0 - F(x_0)|} - 1\big)|x_0|^2 < 0$$

which contradicts assumption (9.1). Thus $(\mathrm{id} - F)(x_0) \subset \bar{K}$ and equality (9.2) guarantees that $F(x_0) = 0$. □

If $\dim H = \infty$, we use the Schauder fixed point theorem.

THEOREM 9.3 *(Schauder). — Let* $f : \bar{K} \to \bar{K}$ *be a continuous mapping such that* $f(\bar{K})$ *is relatively compact. Then* f *has a fixed point.*

Using the Schauder fixed point theorem, we can prove analogous version of the Lions theorem for infinite dimensional Hilbert space.

Theorem 9.4. — *Let $(H, (\cdot|\cdot))$ be a real Hilbert space. Assume that $F : \bar{K} \to H$ is a continuous mapping such that the set $(\mathrm{id} - F)(\bar{K}) \subset H$ is relatively compact and*

$$\big(F(\zeta)|\zeta\big) \geqslant 0, \qquad \zeta \in \partial\bar{K}.$$

Then, there exists $z \in \bar{K}$ such that $F(z) = 0$.

Proof. — We assert that the mapping

$$f := r \circ (\mathrm{id} - F) : \bar{K} \to \bar{K},$$

where

$$r : H \ni x \mapsto \begin{cases} x, & |x| \leqslant R \\ \frac{Rx}{|x|}, & |x| > R \end{cases}$$

satisfies the assumptions of the Schauder fixed point theorem. It is clear that f is continuous. Thus, it is sufficient to establish that the set $f(\bar{K})$ is relatively compact. Remark that

$$f(\bar{K}) = r\big((\mathrm{id} - F)(\bar{K})\big) \subset r(Z),$$

where $Z := \overline{(\mathrm{id} - F)(\bar{K})}$. Since Z is compact and r is continuous, $r(Z)$ is compact. Thus $F(\bar{K})$ is relatively compact as a subset of the compact set $r(Z)$. The Schauder fixed point theorem implies the existence of $x_0 \in \bar{K}$ such that

$$f(x_0) = x_0.$$

Repeating the second part of the proof of Theorem 9.2, we deduce that $F(x_0) = 0$. □

10. Appendix C: Generic properties of some nonlinear problems - abstract approach

We recall some topological approach to the problem of generic properties of the set of solutions of an abstract (nonlinear) equation of the form

$$\mu u + T(u) = g$$

in a separable Hilbert space H. Here $\mu \in]0, \infty[$ and $g \in H$ are given while u is unknown. Under suitable assumptions, we point out some set $\mathcal{O}$ open and dense in the space $]0, \infty[\times H$ such that the set $\mathfrak{N}(\mu, g)$ of all solutions of the above equation is finite if $(\mu, g) \in \mathcal{O}$. Moreover, we prove that the function

$$\mathcal{O} \ni (\mu, g) \mapsto \mathfrak{N}(\mu, g) \subset H$$

is continuous, when we consider the Hausdorff metric on the family of all nonempty bounded and closed subsets of H.

Most of the presented results concerning generic properties have been proven in [12].

10.1. Fredholm mappings

Let X, Y be two real separable Banach spaces. An operator $L \in \mathcal{L}(X, Y)$ is called *a Fredholm operator* if

(a) $\dim \ker L < \infty$,

(b) $\mathrm{im} L := L(X) \in \mathrm{cotop} X$,

(c) $\mathrm{codim} L := \dim Y/\mathrm{im} L < \infty$.

If L is Fredholm, then its *index* is defined as follows: $\mathrm{ind} L := \dim \ker L$ - $\mathrm{codim} L$.

A $\mathcal{C}^1$- mapping $\mathcal{E} : X \to Y$ is called *a Fredholm mapping* if its Fréchet differential $d_x\mathcal{E} \in \mathcal{L}(X, Y)$ is a Fredholm operator for all $x \in X$. In such case, the index of $d_x\mathcal{E}$ is independent of x (Th.1.1 in [18]), and, by definition

$$\mathrm{ind}\, \mathcal{E} = \mathrm{ind}\, d_x\mathcal{E}.$$

A mapping $T : X \to Y$ is *completely continuous* if it is continuous and maps bounded subsets of X into relatively compact subsets of Y.

Remark 10.1. — Let $A : X \to Y$ be a linear operator. Then the following conditions are equivalent

(i) A is completely continuous ;

(ii) if $x_k \to x$ weakly in X, then $A(x_k) \to A(x)$ in Y.

Let us collect some results which follow from the theory of completely continuous linear operators ([15], Section IV).

Remark 10.2. — If $A \in \mathcal{E}nd X$ is completely continuous, then for any $\lambda \neq 0$

$$\lambda \mathrm{id} + A : X \to X$$

is a Fredholm operator of index 0.

Thus

$$\lambda \mathrm{id} + A \in \mathcal{E}piX \Leftrightarrow \lambda \mathrm{id} + A \in \mathcal{M}onoX \Leftrightarrow \lambda \mathrm{id} + A \in \mathcal{A}utX. \tag{10.1}$$

LEMMA 10.3 *(Lemma 2.7.1 in [14]). — If a $\mathcal{C}^1$-mapping $T : X \to Y$ is completely continuous, then the Fréchet differential $d_x T \in \mathcal{L}(X, Y)$ is completely continuous for all $x \in X$.*

By virtue of Remark 10.2 and Lemma 10.3, we have

COROLLARY 10.4. — *If a $\mathcal{C}^1$-mapping $T : X \to X$ is completely continuous, then*

$$\lambda \mathrm{id} + T : X \to X$$

is a Fredholm mapping of index 0 for $\lambda \neq 0$.

Let $\mathcal{E} : X \to Y$ be a $\mathcal{C}^1$-mapping. An element $x \in X$ is called *a regular point* of $\mathcal{E}$ if $d_x\mathcal{E} \in \mathcal{E}pi(X, Y)$ and x is *singular* if it is not regular. The images of all singular points under $\mathcal{E}$ are called *the singular values* or *critical values* and their complements - the *regular values.*

S. Smale has proved the following infinite-dimensional version of the Sard theorem for Fredholm mappings. We recall this theorem in version given by C.Foiaş and R.Temam in [4].

THEOREM 10.5 *(Smale). — Let $\mathcal{E} : X \to Y$ be a Fredholm $\mathcal{C}^q$-mapping, where $q > \max(\mathrm{ind}\,\mathcal{E}, 0)$. Then the regular values of $\mathcal{E}$ form a dense G_δ subset of Y.*

If $g \in Y$ is a regular value of $\mathcal{E}$, then $\mathcal{E}^{-1}(\{g\})$ is empty or it is a manifold of dimension $\mathrm{ind}\,\mathcal{E}$.

In particular, if $g \in Y$ is a regular value of $\mathcal{E}$ and $\mathrm{ind}\,\mathcal{E} = 0$, then $\mathcal{E}^{-1}(\{g\})$ is discrete.

Let us recall that a G_δ set is a countable intersection of open sets.

10.2. The implicit function theorem.

THEOREM 10.6. — *Let X, Y, Z be Banach spaces, $(x_0, y_0) \in X \times Y$ and $F : X \times Y \to Z$ be a $\mathcal{C}^k$-mapping, where $k \geqslant 1$, satisfying the following conditions*

(i) $F(x_0, y_0) = 0$,

(ii) $d^{II}_{(x_0,y_0)}F := d_{y_0}F(x_0,\cdot) \in \mathcal{I}so(Y,Z)$.

Then there exists neighbourhoods $\mathcal{X} \in \mathrm{top}X$ *of* x_0 *and* $\mathcal{Y} \in \mathrm{top}Y$ *of* y_0 *such that the relation* $\eta := \{F=0\} \cap (\mathcal{X} \times \mathcal{Y})$ *is a* $\mathcal{C}^k$*-mapping from* $\mathcal{X}$ *to* $\mathcal{Y}$ *and*

$$d_x\eta = -\big(d^{II}_{(x,\eta(x))}F\big)^{-1} \circ \big(d^{I}_{(x,\eta(x))}F\big) \quad \mathit{for} \quad x \in \mathcal{X}.$$

Here $\{F=0\} := \{(x,y) \in X \times Y : F(x,y) = 0\}$.

10.3. Generic properties of the set of solutions

We assume that $(H,(\cdot|\cdot))$ is a separable Hilbert space. Let $|\cdot|$ be the norm induced by the scalar product $(\cdot|\cdot)$. Let $T : H \to H$ be a $\mathcal{C}^1$-mapping. Consider the following equation

$$\mu u + T(u) = g \tag{10.2}$$

for a given $(\mu,g) \in]0,\infty[\times H$. Let

$$\mathfrak{R}(\mu,g) := \{u \in H : \mu u + T(u) = g\}, \tag{10.3}$$

i.e. $\mathfrak{R}(\mu,g)$ stands for the set of all solutions of equation (10.2) for a given $(\mu,g) \in]0,\infty[\times H$.

Assume that T satisfies the following conditions

$$\textit{If } u_k \to u \textit{ weakly in } H, \textit{ then } T(u_k) \to T(u) \textit{ in } H \textit{ as } k \to \infty \tag{10.4}$$

There exists a continuous function $\kappa :]0,\infty[\times[0,\infty[\to [0,\infty[$ (10.5)
such that for any $u \in \mathfrak{R}(\mu,g)$ *the following inequality holds* :
$|u| \leqslant \kappa(\mu,|g|)$.

Note that, if T maps weakly convergent sequences into the sequences convergent in the sense of norm, then T is completely continuous. Let us write this as the following

Remark 10.7. — Mapping T is completely continuous.

By Remark 10.1, for linear mappings, the condition (10.4) is equivalent to the complete continuity of T. In general, there exist (nonlinear) completely continuous mappings that do not satisfy the condition (10.4) (see Example 3.2 in [12]).

We will use the following notation

$$\mathcal{E}_\mu : H \ni u \mapsto \mu u + T(u) \in H.$$

Let us remark that for any $(\mu, g) \in]0, \infty[\times H$:

$$\mathfrak{R}(\mu, g) = \mathcal{E}_\mu^{-1}(\{g\}). \tag{10.6}$$

Since the mapping T is completely continuous, thus, by Corollary 10.4

$$\mathcal{E}_\mu \textit{ is the Fredholm mapping of index } 0. \tag{10.7}$$

Hence, by (10.1), for any $u \in H$

$$d_u\mathcal{E}_\mu \in \mathcal{E}piH \;\Leftrightarrow\; d_u\mathcal{E}_\mu \in \mathcal{M}onoH \;\Leftrightarrow\; d_u\mathcal{E}_\mu \in \mathcal{A}utH. \tag{10.8}$$

Proposition 10.8. — *The mapping $\mathcal{E}_\mu$ is proper, i.e. the preimage of a compact subset is compact.*

Proof. — Let K be a compact subset of H. It is sufficient to show that any sequence (u_k) of elements of the set $\mathcal{E}_\mu^{-1}(K)$ contains a subsequence convergent to some element of this set. Let $g_k = \mathcal{E}_\mu(u_k)$. By (10.6), $u_k \in \mathfrak{R}(\mu, g_k)$. Thus, by (10.5)

$$|u_k| \leqslant \kappa(\mu, |g_k|), \qquad k \in \mathbb{N}.$$

Since κ is continuous and K is compact, we infer that the sequence (u_k) is bounded. Thus, by the Banach-Alaoglu theorem

$$\begin{aligned}&\text{there exist an infinite subset } \mathcal{N}_1 \subset \mathbb{N} \text{ and an element } u \in H\\ &\text{such that } u_k \to u \text{ weakly in } H \text{ as } \mathcal{N}_1 \ni k \to \infty.\end{aligned} \tag{10.9}$$

Again, by the compactness of K,

$$\begin{aligned}&\text{there exist an infinite subset } \mathcal{N}_2 \subset \mathcal{N}_1 \text{ and an element } g \in K\\ &\text{such that } g_k \to g \text{ in } H \text{ as } \mathcal{N}_2 \ni k \to \infty.\end{aligned} \tag{10.10}$$

We will show that $(u_k)_{k \in \mathcal{N}_2}$ contains a subsequence convergent to u in the sense of norm and that $u \in \mathcal{E}_\mu^{-1}(\{g\})$. Indeed, since $(u_k)_{k \in \mathcal{N}_2}$ is bounded and T is completely continuous, the set $\{T(u_k)\}_{k \in \mathcal{N}_2}$ is relatively compact. Thus

$$\begin{aligned}&\text{there exist an infinite subset } \mathcal{N} \subset \mathcal{N}_2 \text{ such that}\\ &\big(T(u_k)\big)_{k \in \mathcal{N}} \text{ is convergent in } H.\end{aligned} \tag{10.11}$$

Since $u_k \in \mathfrak{R}(\mu, g_k)$, we have

$$\mu u_k = -T(u_k) + g_k. \tag{10.12}$$

Thus, by (10.11), (10.10), we infer that the subsequence $(u_k)_{k\in\mathcal{N}}$ is convergent in H (in the sense of norm). At the same time u is a weak limit of $(u_k)_{k\in\mathcal{N}}$. Hence $|u_k - u| \to 0$ as $\mathcal{N} \ni k \to \infty$. Since T is continuous, $T(u_k) \to T(u)$ as $\mathcal{N} \ni k \to \infty$. Thus passing to the limit in (10.12) as $\mathcal{N} \ni k \to \infty$ gives

$$\mu u = -T(u) + g,$$

which means that $u \in \mathfrak{R}(\mu, g) = \mathcal{E}_\mu^{-1}(\{g\})$. □

Combining Proposition 10.8 with the equality (10.6), we obtain the following

COROLLARY 10.9. — *The set $\mathfrak{R}(\mu, g)$ is a compact subset of H for any pair $(\mu, g) \in]0, \infty[\times H$.*

LEMMA 10.10 (Lemma 3.4 in [12]). — Assume that $(\mu_0, g_0) \in]0, \infty[\times H$,

$u_0 \in \mathfrak{R}(\mu_0, g_0)$ and the Fréchet differential $\mathrm{d}_{u_0}\mathcal{E}_{\mu_0}$ is an epimorphism of H. Then, there exist a neighbourhood $\mathcal{X} \in \mathrm{top}(]0, \infty[\times H)$ of (μ_0, g_0) and a neighbourhood $\mathcal{Y} \in \mathrm{top}H$ of u_0 such that

(i) $\#\big(\mathcal{Y} \cap \mathfrak{R}(\mu, g)\big)\big) = 1$ for $(\mu, g) \in \mathcal{X}$,

(ii) the function $\mathcal{X} \ni (\mu, g) \mapsto u_{\mu g} \in \mathcal{Y}$ is of class $\mathcal{C}^1$, where $\{u_{\mu g}\} := \mathcal{Y} \cap \mathfrak{R}(\mu, g)$.

(For the set X, the symbol $\#X$ denotes its cardinal number).

Proof. — It is sufficient to apply the implicit function theorem (Th. 10.6) to the mapping

$$F : (]0, \infty[\times H) \times H \ni \big((\mu, g), u\big) \mapsto \mu u + T(u) - g \in H$$

with respect to the pair $\big((\mu_0, g_0), u_0\big)$. □

Let us consider a set

$$\begin{aligned} \mathcal{O} := \{(\mu, g) \in]0, \infty[\times H :\ & g \text{ is a regular value of} \\ & \text{the mapping: } H \ni u \mapsto \mu u + T(u) \to H\}. \end{aligned} \tag{10.13}$$

The result concerning the continuous dependence of the set of the solutions of the equation (10.2) is expressed in the following

Theorem 10.11. — *Assume (10.4)-(10.5). Let* $(\mu_0, g_0) \in \mathcal{O}$. *Then*

(i) *there exists a neighbourhood* $\mathcal{X} \in \operatorname{top}(]0,\infty[\times H)$ *of* (μ_0, g_0) *such that*

$$\#\mathfrak{R}(\mu, g) = \#\mathfrak{R}(\mu_0, g_0) < \infty \quad \text{for any } (\mu, g) \in \mathcal{X}.$$

(ii) $\lim \mathfrak{R}(\mu, g) = \mathfrak{R}(\mu_0, g_0)$ *in the Hausdorff metric over* H *as* $(\mu, g) \to (\mu_0, g_0)$ *in* $]0,\infty[\times H$.

Proof. — Due to (10.7), $\mathcal{E}_{\mu_0} : H \to H$ is a Fredholm mapping of index 0. By virtue of the Smale theorem (see Th. 10.5), the set $\mathfrak{R}(\mu, g) = \mathcal{E}_{\mu_0}^{-1}(\{g_0\})$ is discrete. On the other hand, by Corollary 10.9, this set is compact. Thus, it is finite.

Ad. **(i)**. Let $u \in \mathfrak{R}(\mu_0, g_0)$. By Lemma 10.10, there exist (dependent on u) neighbourhoods $\mathcal{X}(u) \in \operatorname{top}(]0,\infty[\times H)$ of (μ_0, g_0) and $\mathcal{Y}(u)$ of u such that

(1) $\#\big(\mathcal{Y}(u) \cap \mathfrak{R}(\mu, g)\big) = 1$ for $(\mu, g) \in \mathcal{X}(u)$,

(2) the function $\mathcal{X}(u) \ni (\mu, g) \mapsto u_{\mu g} \in \mathcal{Y}(u)$ is of class $\mathcal{C}^1$, where $\{u_{\mu g}\} := \mathcal{Y}(u) \cap \mathfrak{R}(\mu, g)$.

Let us choose $\varepsilon > 0$. Since the set $\mathfrak{R}(\mu_0, g_0)$ is finite, we may assume that ε is so small that the balls $\{\overline{B}(u, \varepsilon), u \in \mathfrak{R}(\mu_0, g_0)\}$ are pairwise disjoint and $\overline{B}(u, \varepsilon) \subset \mathcal{Y}(u)$ ($u \in \mathfrak{R}(\mu_0, g_0)$). By continuity of the function in (2), we infer that there exists a neighbourhood $\mathcal{X}_1(u) \subset \mathcal{X}(u)$ of (μ_0, g_0) in $]0,\infty[\times H$ such that $u_{\mu g} \in \overline{B}(u, \varepsilon)$, for $(\mu, g) \in \mathcal{X}_1(u)$. Thus putting $\mathcal{X}_1 := \bigcap\{\mathcal{X}_1(u), u \in \mathfrak{R}(\mu_0, g_0)\}$, we infer that if $(\mu, g) \in \mathcal{X}_1$, then the function

$$\mathfrak{R}(\mu_0, g_0) \ni u \mapsto u_{\mu g} \in \mathcal{Y}(u) \cap \mathfrak{R}(\mu, g)$$

is an injection and, as a consequence,

$$\#\mathfrak{R}(\mu_0, g_0) \leqslant \#\mathfrak{R}(\mu, g).$$

Analogously to the proof of Theorem 2.10 we show that there exists a neighbourhood $\mathcal{X} \in \operatorname{top}(]0,\infty[\times H)$ of (μ_0, g_0) such that

$$\#\mathfrak{R}(\mu_0, g_0) = \#\mathfrak{R}(\mu, g) \quad \text{for } (\mu, g) \in \mathcal{X}. \tag{10.14}$$

Ad. **(ii)**. Let us remark that from the previous considerations, it follows that for any $(\mu, g) \in \mathcal{X}$

$$\mathfrak{R}(\mu, g) = \{u_{\mu g}, u \in \mathfrak{R}(\mu_0, g_0)\} \subset \mathfrak{R}(\mu_0, g_0) + K(0, \varepsilon) \tag{10.15}$$

and
$$\mathfrak{R}(\mu_0, g_0) \subset \mathfrak{R}(\mu, g) + K(0, \varepsilon). \tag{10.16}$$
The inclusions (10.15),(10.16) mean that the Hausdorff distance between $\mathfrak{R}(\mu, g)$ and $\mathfrak{R}(\mu_0, g_0)$ is smaller than ε for $(\mu, g) \in \mathcal{X}$, which end the proof of Theorem. $\square$

Note that Theorem 10.11 states that the function
$$\mathcal{O} \ni (\mu, g) \mapsto \mathfrak{R}(\mu, g) \subset H$$
is continuous if we consider the Hausdorff metric over H. In particular, the function
$$\mathcal{O} \ni (\mu, g) \mapsto \#\mathfrak{R}(\mu, g) \in \mathbb{Z}$$
is constant on every connected component of $\mathcal{O}$.

As far as the properties of the set $\mathcal{O}$ are concerned we have the following theorem.

Theorem 10.12 (see Theorem 3.6 in [12]). — Assume (10.4)-(10.5). Then the set $\mathcal{O}$ defined by (10.13) is open and dense in $]0, \infty[\times H$.

Proof. — We begin with the openess of $\mathcal{O}$. Let us fix $(\mu_0, g_0) \in \mathcal{O}$. Directly from the definition of $\mathcal{O}$, it follows that $\mathrm{d}_u\mathcal{E}_{\mu_0} \in \mathcal{E}piH$ for any $u \in \mathfrak{R}(\mu_0, g_0)$. Then, by (10.8), $\mathrm{d}_u\mathcal{E}_{\mu_0} \in \mathcal{A}utH$.

Let us fix $u \in \mathfrak{R}(\mu_0, g_0)$. It is well known that the subspace $\mathcal{A}utH$ is open in the space $\mathcal{E}ndH$. Therefore
$$\begin{array}{l}\text{there exists a neighbourhood } \mathcal{W}(u) \in \mathrm{top}(\mathcal{E}ndH)\\ \text{of } \mathrm{d}_u\mathcal{E}_{\mu_0} \text{ such that } \mathcal{W}(u) \subset \mathcal{A}utH.\end{array} \tag{10.17}$$
Since the mapping T is of class $\mathcal{C}^1$, the mapping
$$(]0, \infty[\times H) \times H \ni ((\mu, g), u) \mapsto \mathrm{d}_u\mathcal{E}_\mu \in \mathcal{E}ndH$$
is, in particular, continuous at $((\mu_0, g_0), u)$. Hence
$$\begin{array}{l}\text{there exist a neighbourhood } \mathcal{X}_1(u) \in \mathrm{top}(]0, \infty[\times H)\\ \text{of } (\mu_0, g_0) \text{ and a neighbourhood } \mathcal{U}(u) \in \mathrm{top}H \text{ of } u\\ \text{such that } \mathrm{d}_w\mathcal{E}_\mu \in \mathcal{W}(u) \text{ for } ((\mu, g), w) \in \mathcal{X}_1(u) \times \mathcal{U}(u).\end{array} \tag{10.18}$$
By Theorem 10.11
$$\begin{array}{l}\text{there exists a neighbourhood } \mathcal{X}_2 \in \mathrm{top}(]0, \infty[\times H) \text{ of } (\mu_0, g_0)\\ \text{such that } \#\mathfrak{R}(\mu, g) = \#\mathfrak{R}(\mu_0, g_0) < \infty \text{ for } (\mu, g) \in \mathcal{X}_2\end{array}$$

and, by the proof of this theorem, we know that

$$\mathfrak{R}(\mu, g) = \{u_{\mu g}, u \in \mathfrak{R}(\mu_0, g_0)\}. \tag{10.19}$$

By Lemma 10.10 (*ii*), the functions

$$\mathcal{X}_2 \ni (\mu, g) \mapsto u_{\mu g} \in \mathcal{Y}(u), \qquad u \in \mathfrak{R}(\mu_0, g_0)$$

are continuous. Thus

$$\begin{aligned}&\text{there exists a neighbourhood } \mathcal{X}_3(u) \in \text{top}(]0,\infty[\times H) \text{ of } (\mu_0, g_0)\\ &\text{such that } u_{\mu g} \in \mathcal{U}(u) \text{ for } (\mu, g) \in \mathcal{X}_3(u).\end{aligned} \tag{10.20}$$

Putting $\mathcal{X} := \mathcal{X}_2 \cap \bigcap\{\mathcal{X}_1(u) \cap \mathcal{X}_3(u), u \in \mathfrak{R}(\mu_0, g_0)\}$, we infer by (10.6), (10.17), (10.18), (10.19) (10.20) that for $(\mu, g) \in \mathcal{X}$

$$\begin{aligned}&\mathcal{E}_\mu^{-1}(\{g\}) = \mathfrak{R}(\mu, g) = \{u_{\mu g}, u \in \mathfrak{R}(\mu_0, g_0)\}\\ &\text{and } \mathrm{d}_{u_{\mu g}}\mathcal{E}_\mu \in \mathcal{W}(u) \subset \mathcal{A}ut H\end{aligned}$$

This means that $\mathcal{X}$ is the neighbourhood of $(\mu_0, g_0) \in]0,\infty[\times H$ contained in $\mathcal{O}$.

We move to the proof of densiness. First, we establish that for a fixed $\mu \in]0,\infty[$

$$\begin{aligned}&\text{the set } \mathcal{O}_\mu := \{g \in H : \text{ g is a regular value}\\ &\text{of the mapping } \mathcal{E}_\mu\} \text{ is dense in } H.\end{aligned} \tag{10.21}$$

Indeed, by (10.7), $\mathcal{E}_\mu$ is a Fredholm mapping. Thus, by the Smale theorem, statement (10.21) holds. To prove densiness of the set $\mathcal{O}$ in $]0,\infty[\times H$, it is sufficient to show that

$$A \cap \mathcal{O} \neq \emptyset \qquad \text{for any } \emptyset \neq A \in \text{top}(]0,\infty[\times H).$$

We may assume that $A =]a,b[\times \mathcal{C}$ for some $a, b \in \mathbb{R}$ such that $0 < a < b$ and for some nonempty set $\mathcal{C} \in \text{top}H$. Let us choose an arbitrary $\mu \in]a,b[$. By (10.21), the intersection $\mathcal{C} \cap \mathcal{O}_\mu$ is nonempty. Let us choose $g \in \mathcal{C} \cap \mathcal{O}_\mu$. In particular, $(\mu, g) \in]a,b[\times \mathcal{C}$. Since $g \in \mathcal{O}_\mu$, g is a regular value of the mapping $\mathcal{E}_\mu$. By the definition of the set $\mathcal{O}$, this means that $(\mu, g) \in \mathcal{O}$. In conclusion, $(\mu, g) \in \mathcal{O} \cap (]a,b[\times \mathcal{C})$, which ends the proof. □

10.4. Some generalizations.

Let $T_s : H \to H$, $s \in \mathbb{N}$, be a sequence of $\mathcal{C}^1$-mappings. Consider the sequence of equations

$$\mu u + T_s(u) = g_s, \qquad s \in \mathbb{N} \tag{10.22}$$

in the Hilbert space H, where $g_s \in H$ and $g_s \to g$ in H as $s \to \infty$. Let

$$\mathfrak{R}(s, \mu, g_s) := \{u \in H : \quad \mu u + T_s(u) = g_s\}. \tag{10.23}$$

In addition to assumptions (10.4)-(10.5) on the mapping T , we assume the following conditions on T_s.

If $u_s \to u$ weakly in H, then $T_s(u_s) \to T(u)$ in H as $s \to \infty$ (10.24)

If $w_s \to u$ in H, then the Fréchet differential $d_{w_s} T_s \to d_w T$ in $\mathcal{E}nd H$ as $s \to \infty$ (10.25)

For each $s \in \mathbb{N}$ and for every $u \in \mathfrak{R}(s, \mu, g_s)$: $|u| \leqslant \kappa(\mu, |g_s|)$ for some continuous function $\kappa :]0, \infty[\times [0, \infty[\to [0, \infty[$ (10.26)

In the sequel, we will use the following version of the implicit function theorem.

Theorem 10.13 *(Th. 1.15 in [7]). — Consider a topological space X and Banach spaces Y, Z. Let $(x_0, y_0) \in X \times Y$. Assume that a mapping $F : X \times Y \to Z$ satisfies the following conditions*

(i) $F(x_0, y_0) = 0$,

(ii) *for all $(x, y) \in X \times Y$ there exists $d^{II}_{(x,y)} F := d_y F(x, \cdot) \in \mathcal{L}(Y, Z)$,*

(iii) $d^{II}_{(x_0,y_0)} F \in \mathcal{I}so(Y, Z)$,

(iv) *for all $y \in Y$, the mapping $F(\cdot, y) : X \to Z$ is continuous,*

(v) *the mapping $d^{II} F : X \times Y \ni (x, y) \mapsto d^{II}_{(x,y)} F \in \mathcal{L}(Y, Z)$ is continuous at (x_0, y_0).*

Then there exist neighbourhoods $\mathcal{X} \in \mathrm{top} X$ of x_0 and $\mathcal{Y} \in \mathrm{top} Y$ of y_0 such that the relation $\eta := \{F = 0\} \cap (\mathcal{X} \times \mathcal{Y})$ is a continuous function from $\mathcal{X}$ to $\mathcal{Y}$.

Let us denote

$$\begin{aligned} \mathcal{E}_\mu(u) &:= \mu u + T(u), \qquad u \in H \\ \mathcal{E}_{\mu,s}(u) &:= \mu u + T_s(u), \qquad u \in H. \end{aligned}$$

Observe that

$$\mathfrak{R}(\mu,g)=\mathcal{E}_{\mu}^{-1}(\{g\}) \quad \text{and} \quad \mathfrak{R}(s,\mu,g_s)=\mathcal{E}_{\mu,s}^{-1}(\{g_s\}).$$

LEMMA 10.14 *If $u\in\mathfrak{R}(\mu,g)$ and the Fréchet differential $d_u\mathcal{E}_\mu$ is epimorphism from H onto H, then there exists a neighbourhood $\mathcal{Y}\in \mathrm{top}H$ of u such that*

(i) $\#\big(\mathcal{Y}\cap\mathfrak{R}(s,\mu,g_s)\big)=1$ *for almost all $s\in\mathbb{N}$,*

(ii) $\lim_{s\to\infty}|u_s-u|=0$, *where $\{u_s\}=\mathcal{Y}\cap\mathfrak{R}(s,\mu,g_s)$.*

Proof. — The assertion follows from the Implicit Function Theorem 10.13 applied to $X=\overline{\mathbb{N}}:=\mathbb{N}\cup\{\infty\}$, $Y=Z=H$,

$$F:\overline{\mathbb{N}}\times H\ni(s,w)\mapsto\begin{cases}\mathcal{E}_{\mu,s}(w)-g_s & \text{for } s<\infty\\ \mathcal{E}_{\mu}(w)-g & \text{for } s=\infty\end{cases} \tag{10.27}$$

with respect to $(x_0,y_0):=(\infty,u)$. □

The main result about convergence of the sequence of sets $(\mathfrak{R}(s,\mu,g_s))_s$ as $s\to\infty$ is the following

THEOREM 10.15 *Assume (10.4)-(10.5) and (10.24)-(10.26). Let $g_s\to g$ in H as $s\to\infty$. If $(\mu,g)\in\mathcal{O}$, then*

(i) *for almost all $s\in\mathbb{N}$:* $\#\mathfrak{R}(s,\mu,g_s)=\#\mathfrak{R}(\mu,g)<\infty$;

(ii) $\lim_{s\to\infty}\mathfrak{R}(s,\mu,g_s)=\mathfrak{R}(\mu,g)$ *in the Hausdorff metric;*

Proof. — Let $(\mu,g)\in\mathcal{O}$ and let $u\in\mathfrak{R}(\mu,g)$. By the definition of the set $\mathcal{O}$, see(10.13), $d_u\mathcal{E}_\mu\in\mathcal{E}piH$. From Lemma 10.14, it follows that there exist $s(u)\in\overline{\mathbb{N}}$ and a neighbourhood $\mathcal{Y}(u)\in\mathrm{top}H$ – of u such that

(i) $\#\big(\mathcal{Y}(u)\cap\mathfrak{R}(s,\mu,g_s)\big)=1$ for $s>s(u)$,

(ii) $\lim_{s\to\infty}|u_s-u|=0$, where $\{u_s\}=\mathcal{Y}(u)\cap\mathfrak{R}(s,\mu,g_s)$.

By Theorem 10.11, the set $\mathfrak{R}(\mu,g)$ is finite. Thus, if $s>s_*:=\max\{s(u),\ u\in\mathfrak{R}(\mu,g)\}$, then $\#\big(\mathcal{Y}(u)\cap\mathfrak{R}(s,\mu,g_s)\big)=1$ for every $u\in\mathfrak{R}(\mu,g)$. Moreover, there exists $\delta>0$ such that the balls $\{\overline{K}(u,\delta),u\in\mathfrak{R}(\mu,g)\}$ (in the space H) are pairwise disjoint and $\overline{K}(u,\delta)\subset\mathcal{Y}(u)$ for every $u\in\mathfrak{R}(\mu,g)$.

Since, by (ii) and finiteness of the set $\mathfrak{R}(\mu, g)$,

$$\lim_{s\to\infty} \max\{|u_s - u|, \quad u \in \mathfrak{R}(\mu, g)\} = 0, \tag{10.28}$$

there exists $s(\delta) \in \overline{\mathbb{N}}$ such that $s(\delta) > s_*$ and for any $s > s(\delta)$

$$\max\{|u_s - u|, \quad u \in \mathfrak{R}(\mu, g)\} \leqslant \delta.$$

Hence, if $s > s(\delta)$, then the function

$$\mathfrak{R}(\mu, g) \ni u \mapsto u_s \in \mathfrak{R}(s, \mu, g_s)$$

is an injection. In particular

$$\#\mathfrak{R}(\mu, g) \leqslant \mathfrak{R}(s, \mu, g_s).$$

Arguing similarly to the proof of Theorem 2.10 we show that the set

$$\mathcal{S} := \{s \in \overline{\mathbb{N}} : s > s(\delta) \quad \text{and} \quad \#\mathfrak{R}(\mu, g) < \#\mathfrak{R}(s, \mu, g_s)\}$$

is finite. Then for any $s > s_{**} := \max\{s(\delta), \sup \mathcal{S}\}$:

$$\#\mathfrak{R}(s, \mu, g_s) = \#\mathfrak{R}(\mu, g). \tag{10.29}$$

Let us now prove statement (ii). Let $\varepsilon > 0$. There exists $\overline{s} \in \overline{\mathbb{N}}$ such that $\overline{s} > s_{**}$ and

$$\max\{|u_s - u|, \quad u \in \mathfrak{R}(\mu, g)\} \leqslant \varepsilon \quad \text{for } s > \overline{s}. \tag{10.30}$$

Let $s > \overline{s}$. Thus, by (10.29) and (10.30)

$$\mathfrak{R}(s, \mu, g_s) \subset \mathfrak{R}(\mu, g) + \overline{K}(\varepsilon) \tag{10.31}$$

and

$$\mathfrak{R}(\mu, g) \subset \mathfrak{R}(s, \mu, g_s) + \overline{K}(\varepsilon). \tag{10.32}$$

The above two inclusions mean that the Hausdorff distance between $\mathfrak{R}(s, \mu, g_s)$ and $\mathfrak{R}(\mu, g)$ is not greater than ε (for $s > \overline{s}$), which completes the proof. □

11. Appendix D: Inversion of the divdiv* operator and some auxiliary results

In this Appendix, we will be concerned with the divdiv* – operator considered in paper [8]. Properties of this operator will be of fundamental importance in the further consideration concerning construction of the approximate solution. For the convenience of the reader, we recall them. However, first we recall some elementary facts of the theory of distributions and of the Friedrichs theory.

11.1. Elements of the theory of distributions

Let $\Omega \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$. Let $Y \in \{\mathbb{R}, \mathbb{R}^n\}$. The symbol $\mathcal{D}(\Omega, Y)$ stands for the space of all *test functions* $\phi : \Omega \to Y$, i.e. $\mathcal{C}^\infty$ - mappings with compact support suppϕ contained in Ω. Let us recall that a linear mapping

$$\Lambda : \mathcal{D}(\Omega, \mathbb{R}) \to Y$$

is called a Y- *valued distribution* on Ω if it is continuous in the usual topology on $\mathcal{D}(\Omega, \mathbb{R})$. Then we write $\Lambda \in \mathcal{D}'(\Omega, Y)$. (Recall that the topology on $\mathcal{D}(\Omega, \mathbb{R})$ has the following property: A sequence $(\phi_k) \subset \mathcal{D}(\Omega, \mathbb{R})$ is convergent to ϕ in $\mathcal{D}(\Omega, \mathbb{R})$ iff

(i) there exists a compact subset $K \subset \Omega$ such that supp$\phi_k \subset K$, $k \in \mathbb{N}$.

(ii) (ϕ_k) converges to ϕ uniformly on K as $k \to \infty$.)

A locally integrable function $u \in L^1_{loc}(\Omega, Y)$ induces the *regular distribution* which we denote by $[u]$, i.e.

$$[u](\varphi) := \int_\Omega u(x)\,\varphi(x)\,dx, \qquad \varphi \in \mathcal{D}(\Omega, \mathbb{R}).$$

For a multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, the α -*derivative of* Λ is defined by

$$(D^\alpha \Lambda)(\varphi) := (-1)^{|\alpha|} \Lambda(D^\alpha \varphi), \qquad \varphi \in \mathcal{D}(\Omega, \mathbb{R}),$$

where $|\alpha| := \sum_{i=1}^n \alpha_i$.

Let $\Lambda \in \mathcal{D}'(\Omega, Y)$ and let $\phi \in \mathcal{D}(\Omega, Y)$. If $Y = \mathbb{R}^n$ then $\Lambda = (\Lambda_1, \dots, \Lambda_n)$ for some $\mathbb{R}$- valued distributions $\Lambda_i \in \mathcal{D}'(\Omega, \mathbb{R})$ and $\phi = (\phi_1, \dots, \phi_n)$ for some $\phi_i \in \mathcal{D}(\Omega, \mathbb{R})$ $(i = 1, \dots, n)$. We will use the following notation

$$(\Lambda|\phi)_{L^2} := \sum_{i=1}^n \Lambda_i(\phi_i). \tag{11.1}$$

The following proposition contains basic properties of the operation defined by (11.1).

Proposition 11.1. — *We have the following properties.*

(a) *The operation*

$$\mathcal{D}'(\Omega, Y) \times \mathcal{D}(\Omega, Y) \ni (\Lambda, \phi) \mapsto (\Lambda|\phi)_{L^2} \in \mathbb{R}$$

is bilinear.

(b) *If* $u \in L^1_{loc}(\Omega, Y)$ *then*

$$([u]|\phi)_{L^2} = \int_\Omega u(x) \cdot \phi(x)\, dx,$$

here, dot "$\cdot$" denotes the scalar product in $\mathbb{R}^n$.

(c) *For each* $\alpha \in \mathbb{N}^n$

$$(D^\alpha \Lambda|\phi)_{L^2} = (-1)^{|\alpha|}(\Lambda|D^\alpha \phi)_{L^2}$$

(d) *If* $V \in \mathcal{D}'(\Omega, \mathbb{R}^n)$ *then*

$$(\Lambda|\nabla\varphi)_{L^2} = -(\mathrm{div}\Lambda)(\varphi), \qquad \varphi \in \mathcal{D}(\Omega, \mathbb{R}),$$

where $\mathrm{div}\Lambda := \sum_{i=1}^n \frac{\partial}{\partial x_i}\Lambda_i$.

(e) *If* $V \in \mathcal{D}'(\Omega, \mathbb{R})$ *and* $\phi \in \mathcal{D}(\Omega, \mathbb{R}^n)$ *then*

$$(\nabla V|\phi)_{L^2(\Omega)} = -V(\mathrm{div}\phi).$$

Proofs of these facts follow immediately from definitions.

11.2. Elements of the Friedrichs theory

Let $\left(X, (\cdot|\cdot)\right)_X$ $\left(Z, (\cdot|\cdot)\right)_Z$ be a real Hilbert spaces and let $A : X \supset D(A) \to Z$ be a linear operator. Assume that A is densely defined, i.e. the closure $\overline{D(A)} = X$. Let us recall the notion of the adjoint operator. (Note that we do not assume that A is bounded). Let

$$D(A^*) := \{z \in Z : \quad \text{the functional } D(A) \ni x \mapsto (Ax|z) \text{ is continuous}\}.$$

(Note that $D(A^*) = Z$ if A is bounded.) Let $z \in D(A^*)$. By the Riesz theorem, there exists the unique $A^*z \in X$ such that

$$(Ax|z) = (x|A^*z), \qquad x \in D(A),.$$

Definition 11.2. — *The operator*

$$A^* : D(A^*) \ni z \mapsto A^*z \in X$$

is called the adjoint operator of A.

Assume that $X = Z$. We omit the index X at the symbol of the scalar product of X, i.e. $(\cdot|\cdot)_X = (\cdot|\cdot)$.

Definition 11.3. — *We say that operator A is*

- *symmetric* if $A \subset A^*$, i.e. $D(A) \subset D(A^*)$ and $(Ax|y) = (x|Ay)$ for every $x, y \in D(A)$;
- *selfadjoint* if $A = A^*$;
- *strictly positive* if it is symmetric and there exists a positive real number μ such that
$$(Ax|x) \geqslant \mu \|x\|^2. \tag{11.2}$$
The biggest of such numbers μ is called the *infimum* of A and is denoted by $\inf A$.

It is easy to see that

$$\inf A = \inf\{(Ax|x), \qquad x \in D(A), \quad \|x\| = 1\}. \tag{11.3}$$

We recall the notion of the Friedrichs space. These results are based on Section VI in [9].

Theorem 11.4. — *Assume that $A : D(A) \to X$ is strictly positive. Then*

(a) *there exists the unique Hilbert space $\big(X_A, (\cdot|\cdot)_A\big)$ such that*

  (i) $D(A) \subset X_A \subset X$;

  (ii) *$D(A)$ is dense in X_A and the inclusion $j : X_A \hookrightarrow X$ is continuous;*

  (iii) *for every $x \in D(A)$: $|x|_A^2 = (Ax|x)$, where $|\cdot|_A^2 = (\cdot|\cdot)_A$.*

(b) *The norm of the inclusion $j : X_A \hookrightarrow X$ is equal to $(\inf A)^{-\frac{1}{2}}$ and*
$$(Ax|y) = (x|y)_A, \qquad x \in D(A), \quad y \in X_A.$$

The pair $\big(X_A, (\cdot|\cdot)_A\big)$ is called the *Friedrichs space* of the operator A.

Now we recall the Friedrichs extension of the strictly positive operator A to the selfadjoint operator $\tilde{A}$. Let us consider the inclusion

$$j : X_A \hookrightarrow X$$

and its adjoint

$$j^* : X \to X_A.$$

Since A is strictly positive, the operator j^* is injective. Let $D(\tilde{A}) := j^*(X) \subset X_A$ and let

$$\tilde{A} := (j^*)^{-1} : D(\tilde{A}) \to X.$$

Operator $\tilde{A}$ is called the *Friedrichs extension* of A.

THEOREM 11.5 *(Friedrichs). — Operator* $\tilde{A} : \big(D(\tilde{A}), |\cdot|\big) \to X$ *is self-adjoint strictly positive and* $\inf \tilde{A} = \inf A$. *Moreover*

$$(\tilde{A}x|y) = (x|y)_A, \qquad x \in D(\tilde{A}), \quad y \in X_A.$$

11.3. The Friedrichs extension of the Laplace operator

Let $X = L^2(\Omega, Y)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $Y \in \{\mathbb{R}, \mathbb{R}^n\}$ and let

$$A_0 : L^2(\Omega, Y) \supset \mathcal{D}(\Omega, Y) \ni \phi \mapsto -\Delta\phi \in L^2(\Omega, Y).$$

Operator A_0 is strictly positive, because

$$(A_0\phi|\phi)_{L^2} = (\nabla\phi|\nabla\phi)_{L^2} = \|\phi\|^2_{H^1_0}, \qquad \phi \in \mathcal{D}(\Omega, Y).$$

From Theorem 11.4, there follows that the Sobolev space $\big(H^1_0(\Omega, Y), ((\cdot|\cdot))\big)$ is the Friedrichs space of A_0. Theorem 11.5 yields that

$$D(A) = \{u \in H^1_0(\Omega, Y) : \quad \text{there exists } \Delta u \text{ in the weak sense and } \Delta u \in L^2(\Omega, Y)\}$$

is the domain of the Friedrichs extension A of A_0. Moreover

$$Au = -\Delta u, \qquad u \in D(A). \tag{11.4}$$

For arbitrary $f \in L^2(\Omega, Y)$, the function $u := A^{-1}(f)$ is called the *generalized solution* of the boundary value problem

$$\begin{cases} -\Delta u = f \\ u_{|\partial\Omega} = 0. \end{cases}$$

11.4. The divdiv* operator

Assume that $\Omega \subset \mathbb{R}^n$ is an open and bounded subset of $\mathbb{R}^n$. Let us consider the divergence operator

$$\mathrm{div} : H^1_0 \to L^2(\Omega),$$

i.e. $\mathrm{div} u := \sum_{i=1}^n \frac{\partial u_i}{\partial x_i}$ for $u = (u_1, \ldots, u_n) \in H^1_0$. It is linear and continuous (see Theorem 11.12). Let

$$\mathrm{div}^* : L^2(\Omega) \to H^1_0$$

be its adjoint. Thus

$$(\mathrm{div} u|p)_{L^2(\Omega)} = ((u|\mathrm{div}^* p)), \qquad u \in H^1_0, \quad p \in L^2(\Omega).$$

Let $\mathcal{R}$ denote the *canonical Riesz isomorphism* in the space $\big(H_0^1, ((\cdot|\cdot))\big)$, i.e.

$$\mathcal{R} : H_0^1 \ni u \mapsto ((u|\cdot)) \in (H_0^1)'.$$

Let A denote the Friedrichs extension of the Laplace operator in the space $L^2(\Omega, \mathbb{R}^n)$.

For $p \in L^2(\Omega)$ the linear functional

$$\mathcal{D} \ni \phi \mapsto (\nabla[p]|\phi)_{L^2}$$

is H_0^1 - continuous. Its extension to H_0^1 is denoted by $\overline{(\nabla[p]|\cdot)_{L^2}}$. The operator

$$\mathfrak{N} : L^2(\Omega) \ni p \mapsto \overline{(\nabla[p]|\cdot)_{L^2}} \in (H_0^1)'$$

is called the Nečas operator.

We investigate some properties of the div^* operator.

Proposition 11.6. — *We have*

(a) $\mathcal{R}^{-1} \circ \mathfrak{N} = -\mathrm{div}^*$

(b) *If* $p \in H^1(\Omega)$*, then* $-\mathrm{div}^* p = A^{-1}(\nabla p)$*.*

Moreover, if $p \in L^2(\Omega)$ *and* $\mathrm{div}^* p \in D(A)$*, then* $p \in H^1(\Omega)$*.*

Proof. — Ad.**(a)**. Let $p \in L^2(\Omega)$ and let $u := (\mathcal{R}^{-1} \circ \mathfrak{N})(p)$. For every $\phi \in \mathcal{D}$, we have

$$\begin{aligned} ((u|\phi)) &= (((\mathcal{R}^{-1} \circ \mathfrak{N})(p)|\phi)) = \mathfrak{N}(p)(\phi) = \overline{(\nabla[p]|\cdot)_{L^2}}(\phi) = (\nabla[p]|\phi)_{L^2} \\ &= -[p](\mathrm{div}\phi) = -(p|\mathrm{div}\phi)_{L^2} = ((\mathrm{div}^* p|\phi)) \end{aligned}$$

Since $\mathcal{D}$ is dense in H_0^1, we have $u = -\mathrm{div}^* p$.

Ad.**(b)**. Assume that $p \in H^1(\Omega)$. It is sufficient to show that $(\mathcal{R}^{-1} \circ \mathfrak{N})(p) = A^{-1}(\nabla p)$. Let $u := A^{-1}(\nabla p)$. Then, for every $\phi \in \mathcal{D}$ we have

$$\mathcal{R}(u)(\phi) = ((u|\phi)) = (Au|\phi)_{L^2} = (\nabla p|\phi)_{L^2} = \mathfrak{N}(p)(\phi)$$

which ends the proof of assertion (b).

To prove the last implication let $u := -\mathrm{div}^* p = \mathcal{R}^{-1}(\mathfrak{N}(p))$. Then we have

$$\overline{(\nabla[p]|\cdot)_{L^2}} = \mathfrak{N}(p) = \mathcal{R}(u) = \overline{(-\Delta[u]|\cdot)_{L^2}}.$$

Since $-\Delta u = Au \in L^2$, $-\Delta[u] = -[\Delta u]$. Thus

$$\nabla[p] = -\Delta[u] = -[\Delta u],$$

which means that there exists ∇p in the weak sense and $\nabla p \in L^2$. Thus $p \in H^1(\Omega)$. $\square$

Proposition 11.6 (b) states that computation of the values of $\mathrm{div}^* p$ for $p \in H^1(\Omega)$ can be reduced to the homogeneous Dirichlet boundary value problem for the Poisson equation

$$\begin{cases} \Delta u = \nabla p \\ u_{|\partial\Omega} = 0. \end{cases}$$

Let $\mathcal{S}$ be the family of all connected components of Ω. (It is at most countable).

Remark 11.7. — We have

(a) $\ker \mathrm{div}^* = \{p \in L^2(\Omega) : \quad p_S = \mathrm{const}$ *for every* $S \in \mathcal{S}\}$,

(b) $(\ker \mathrm{div}^*)^{\perp} = \{q \in L^2(\Omega) : \quad \int_S q dx = 0$ *for every* $S \in \mathcal{S}\}$

(the orthogonal complement in $L^2(\Omega)$*).*

Proof. — Ad.**(a)**. Since $\mathcal{R}^{-1} \circ \mathfrak{N} = -\mathrm{div}^*$, by the lemma of du Bois-Reymond we have

$$\begin{aligned} p \in \ker \mathrm{div}^* \quad &\Leftrightarrow \quad \mathrm{div}^* p = 0 \ \Leftrightarrow \ \mathfrak{N} p = \mathrm{const} \ \Leftrightarrow \ \nabla[p] = 0 \\ &\Leftrightarrow \quad p_S = \mathrm{const} \text{ for every } S \in \mathcal{S}. \end{aligned}$$

Ad.**(b)**. "$\supset$". Let $q \in L^2(\Omega)$ be an element such that $\int_S q dx = 0$ for every $S \in \mathcal{S}\}$. Let $p \in \ker \mathrm{div}^*$. Then $p_S = \mathrm{const}$ for every $S \in \mathcal{S}$. Thus

$$(q|p)_{L^2} = \sum_{S \in \mathcal{S}} \int_S p_S q \, dx = \sum_{S \in \mathcal{S}} p_S \int_S q \, dx = 0.$$

"$\subset$". Let $q \in (\ker \mathrm{div}^*)^{\perp}$. Let us fix a component $S_0 \in \mathcal{S}$. By assertion (a), $\chi_{S_0} \in \ker \mathrm{div}^*$. Hence

$$0 = \int_{\Omega} \chi_{S_0} q \, dx = \int_{S_0} q \, dx.$$

$\square$

Some notations. Let $\mathcal{V} := \mathcal{D}(\Omega, \mathbb{R}^n) \cap \{\mathrm{div} = 0\}$ denote the space of all divergence-free test vector fields on Ω, V denote its closure in the Hilbert space $\big(H_0^1, (\!(\cdot|\cdot)\!)\big)$ and $V^\perp$ - its $(\!(\cdot|\cdot)\!)$ - orthogonal complement in H_0^1. The symbols $P_V : H_0^1 \to V$, $P_{V^\perp} : H_0^1 \to V^\perp$ stand for the $(\!(\cdot|\cdot)\!)$ - orthogonal projections onto V and $V^\perp$, respectively. Moreover, $\{\textstyle\int = 0\} := \{q \in L^2(\Omega) : \int_\Omega q(x)\, dx = 0\}$ is a closed hyperplane in $L^2(\Omega)$. In particular, the pair $(\{\textstyle\int = 0\}, (\cdot|\cdot)_{L^2(\Omega)})$ is a Hilbert space.

Remark 11.8. — Assume that the image imdiv *of the divergence operator is closed, i.e.* $\mathrm{imdiv} \in \mathrm{cotop} L^2(\Omega)$.

(a) *If* Ω *is connected then* $\mathrm{imdiv} = \{\textstyle\int = 0\}$.

(b) $\mathrm{imdiv}^* = V^\perp \;\; \Leftrightarrow \;\; V = \ker \mathrm{div}$.

Proof. — Ad.**(a)**. "$\subset$". Let $\phi \in \mathcal{D}$. Then

$$\int_\Omega \mathrm{div}\phi\, dx = [1](\mathrm{div}\phi) = (-\nabla[1]|\phi)_{L^2} = 0.$$

Let $u \in H_0^1$. There exists a sequence $(\phi_k) \in \mathcal{D}^{\mathbb{N}}$ such that

$$\|\phi_k - u\|_{H_0^1} \to 0 \qquad \text{as } k \to \infty.$$

Then

$$\|\mathrm{div}\phi_k - \mathrm{div}u\|_{L^2(\Omega)} \to 0 \qquad \text{as } k \to \infty.$$

Hence

$$0 = \int_\Omega \mathrm{div}\phi_k\, dx \to \int_\Omega \mathrm{div}u\, dx \qquad \text{as } k \to \infty.$$

"$\supset$". Let X be the orthogonal complement of the image imdiv in the space $\{\textstyle\int = 0\}$. Let $p \in X$, $\varphi \in \mathcal{D}(\Omega)$ and $i \in \{1, \dots, n\}$. Then

$$0 \;\; = (p|\mathrm{div}(0, \dots, \varphi, \dots, 0))_{L^2} = \big(p|\tfrac{\partial \varphi}{\partial x_i}\big)_{L^2} = [p]\big(\tfrac{\partial \varphi}{\partial x_i}\big) = \big(-\tfrac{\partial}{\partial x_i}[p]\big)(\varphi)$$

Thus $\nabla[p] = 0$. Hence $p = \mathrm{const}$ by the lemma of du Bois-Reymond. Since $0 = \int_\Omega p\, dx = p\,|\Omega|$, where $|\Omega|$ stands for the n - dimensional measure in $\mathbb{R}^n$, thus $p = 0$ and $X = \{0\}$. Hence $\mathrm{imdiv} = \{\textstyle\int = 0\}$.

Ad.**(b)**. "$\Leftarrow$". Since $\mathrm{imdiv}^* \in \mathrm{cotop} H_0^1$,

$$\mathrm{imdiv}^* = \overline{\mathrm{imdiv}^*} = (\ker \mathrm{div}^{**})^\perp = (\ker \mathrm{div})^\perp.$$

"$\Rightarrow$". It is clear that $V \subset \ker \mathrm{div}$. In order to obtain the inverse inclusion, let $u \in \ker \mathrm{div} \cap V^\perp$. Since $u \in V^\perp$, there exists $p \in L^2(\Omega)$ such that $u = \mathrm{div}^* p$. Then

$$((u|u)) = ((u|\mathrm{div}^* p)) = (\mathrm{div}u|p)_{L^2(\Omega)} = 0.$$

Hence $u = 0$. $\square$

Now we will concentrate on some properties of the div and div* operators.

LEMMA 11.9. — *If $p \in L^2(\Omega)$, then $\Delta[p] = \Delta[\mathrm{div}\mathrm{div}^* p]$.*

In other words, *function $p - \mathrm{div}\mathrm{div}^* p$ induces a harmonic distribution.*

Proof. — Let $u := \mathrm{div}^* p$. By Proposition 11.6(a),

$$\mathcal{R}(u) = -\mathfrak{N}(p) = -\overline{(\nabla[p]|\cdot)_{L^2}}.$$

On the other hand

$$\mathcal{R}(u) = ((u|\cdot)) = (-\Delta[u]|\cdot)_{L^2}.$$

Hence $\Delta[u] = \nabla[p]$. Since $u \in H_0^1$,

$$\Delta[p] = \mathrm{div}\Delta[u] = \Delta[\mathrm{div} u] = \Delta[\mathrm{div}\mathrm{div}^* p].$$

□

LEMMA 11.10. — *Let q be a weakly differentiable element of the space $L^1_{loc}(\Omega)$. Moreover, assume that $\nabla q \in D(A)$ $(= \{u \in H_0^1 : \ \Delta u \in L^2\})$. Then*

$$p := -\Delta q \in H^1(\Omega) \qquad \textit{and} \qquad A^{-1}(\nabla p) = \nabla q. \tag{11.5}$$

In particular,

$$\mathrm{div}^* \mathrm{div} \nabla q = \nabla q. \tag{11.6}$$

Proof. — Since $\nabla q \in D(A) \subset H_0^1$, then $p = -\mathrm{div}(\nabla q) \in L^2(\Omega)$.

On the other hand, denote here by $\mathcal{A}$ the Friedrichs extension of the Laplace operator in the space $L^2(\Omega, \mathbb{R})$. Since $\nabla q \in D(A)$,

$$e_i^*(\nabla q) \in D(\mathcal{A}) \quad \text{and} \quad \mathcal{A}\big(e_i^*(\nabla q)\big) = (e_i^* \circ A)(\nabla q),$$

i.e. $\frac{\partial q}{\partial x_i} \in D(\mathcal{A})$ and $\mathcal{A}\big(\frac{\partial q}{\partial x_i}\big) = \big(A(\nabla q)\big)_i$ for each $i \in \{1, \ldots, \}$. Here $e_1^*, \ldots, e_n^*$ is the dual base of the canonical base $e_1, \ldots, e_n$ in $\mathbb{R}^n$. By standard calculation we obtain

$$\frac{\partial[p]}{\partial x_i} = \tfrac{\partial}{\partial x_i}[-\Delta q] = -\tfrac{\partial}{\partial x_i}\Delta[q] = -\Delta \tfrac{\partial}{\partial x_i}[q] = -\Delta\big[\tfrac{\partial q}{\partial x_i}\big] = \big[\mathcal{A}\big(\tfrac{\partial q}{\partial x_i}\big)\big].$$

This means that there exists $\frac{\partial p}{\partial x_i}$ in the weak sense and $\frac{\partial p}{\partial x_i} \in L^2(\Omega)$. Hence $p \in H^1(\Omega)$ and $A(\nabla q) = \nabla p$. Moreover, Prop. 11.6 (b) application yields

$$\mathrm{div}^* \mathrm{div}(\nabla q) = \mathrm{div}^*(\Delta q) = -\mathrm{div}^* p = A^{-1}(\nabla p) = \nabla q.$$

□

Theorem 11.11. — *The following statements hold.*

(a) *The operator* $\operatorname{divdiv}^* = \mathrm{id}$ *on the subspace* ΔH_0^2 *and*

$$\operatorname{divdiv}^*\{\Delta = 0\} \subset \big(\{\Delta = 0\} \cap \{\textstyle\int = 0\}\big).$$

(b) *Gradient* $\nabla : H_0^2(\Omega) \to \nabla H_0^2(\Omega) \subset H_0^1$ *is an isometry in the norm* $\|\Delta(\cdot)\|_{L^2(\Omega)}$ *in its domain and the norm* $\|\cdot\|_{H_0^1}$ *in the subspace* $\nabla H_0^2(\Omega)$.

(c) *Divergence* $\operatorname{div} : \nabla H_0^2(\Omega) \to \Delta H_0^2(\Omega)$ *is an isometry; its inverse is equal to* $\operatorname{div}^*{}_{|\Delta H_0^2(\Omega)}$.

Proof. — Ad.**(a)**. Let $q \in \mathcal{D}(\Omega)$. By Lemma 11.10, $\operatorname{div}^*(\Delta q) = \nabla q$. Applying the operator div to both sides of this equality, we obtain

$$\operatorname{divdiv}^*\Delta q = \Delta q.$$

Since the endomorphisms divdiv^* and id of the space $L^2(\Omega)$ are equal on the subspace $\Delta\mathcal{D}(\Omega)$ dense in $\Delta H_0^2(\Omega)$, the first part of statement (a) holds.

In order to prove the inclusion, choose $h \in \{\Delta = 0\}$. The proof of (a) will be completed as we show that $\operatorname{divdiv}^* h$ is harmonic function. Let $\varphi \in \mathcal{D}(\Omega)$. Then by using the fact that the operator divdiv^* is self-adjoint and by the first part of statement (a), we have

$$\begin{aligned}\big(\Delta[\operatorname{divdiv}^* h]\big)(\varphi) &= [\operatorname{divdiv}^* h](\Delta\varphi) = (\operatorname{divdiv}^* h|\Delta\varphi)_{L^2} \\ &= (h|\operatorname{divdiv}^*\Delta\varphi)_{L^2} = (h|\Delta\varphi)_{L^2} = (\Delta[h])(\varphi) = 0.\end{aligned}$$

Ad.**(b)**. Let $\varphi \in \mathcal{D}(\Omega)$. The integration by parts formula gives

$$\begin{aligned}\|\nabla\varphi\|^2_{H_0^1} &= \big(\nabla(\nabla\varphi)|\nabla(\nabla\varphi)\big)_{L^2} = -\big(\nabla\varphi|\Delta(\nabla\varphi)\big)_{L^2} = -\big(\nabla\varphi|\nabla(\Delta\varphi)\big)_{L^2} \\ &= (\Delta\varphi|\Delta\varphi)_{L^2(\Omega)} = \|\Delta\varphi\|^2_{L^2(\Omega)}.\end{aligned}$$

Thus gradient $\nabla : H_0^2(\Omega) \to \nabla H_0^2(\Omega)$ is an isometry as a continuous extension of an isometry on the subspace $\mathcal{D}(\Omega)$ dense in $\big(H_0^2(\Omega), \|\Delta(\cdot)\|_{L^2(\Omega)}\big)$.

Ad.**(c)**. Let us denote by g and L the following homeomorphisms

$$\begin{aligned} g &: H_0^2(\Omega) \ni \varphi \mapsto \nabla\varphi \in \nabla H_0^2(\Omega) \quad (\subset H_0^1) \\ L &: H_0^2(\Omega) \ni \varphi \mapsto \Delta\varphi \in \Delta H_0^2(\Omega) \quad (\subset L^2(\Omega)).\end{aligned}$$

We have

$$\operatorname{div} = L \circ g^{-1} : \nabla H_0^2(\Omega) \to \Delta H_0^2(\Omega).$$

According to Lemma 11.10, $\operatorname{div}^*(\Delta q) = \nabla q$ for every $q \in \mathcal{D}(\Omega)$. Thus the operators div^* and $\big(\operatorname{div}_{\nabla H_0^2(\Omega)}\big)^{-1}$ are equal on the subspace $\Delta\mathcal{D}(\Omega)$ dense in $\Delta H_0^2(\Omega)$. $\square$

Theorem 11.12. — *The divergence operator* div : $H_0^1 \to L^2(\Omega)$ *is of norm* 1, *i.e.* $|\mathrm{div}|_{\mathcal{L}(H_0^1, L^2(\Omega))} = 1$.

Proof. — Let $u = (u_1, \ldots, u_n) \in H_0^1$. We have

$$\begin{aligned}
\|\mathrm{div} u\|^2_{L^2(\Omega)} &= \left\| \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} \right\|^2_{L^2(\Omega)} = \left(\sum_{i=1}^n \frac{\partial u_i}{\partial x_i} | \sum_{j=1}^n \frac{\partial u_j}{\partial x_j} \right)_{L^2(\Omega)} \\
&= \sum_{i,j=1}^n \left(\frac{\partial u_i}{\partial x_i} | \frac{\partial u_j}{\partial x_j} \right)_{L^2(\Omega)} = \sum_{i,j=1}^n \left(\frac{\partial u_i}{\partial x_j} | \frac{\partial u_j}{\partial x_i} \right)_{L^2(\Omega)} \\
&\leqslant \sum_{i,j=1}^n \left\| \frac{\partial u_i}{\partial x_j} \right\|_{L^2(\Omega)} \cdot \left\| \frac{\partial u_j}{\partial x_i} \right\|_{L^2(\Omega)} \\
&\leqslant \left(\sum_{i,j=1}^n \left\| \frac{\partial u_i}{\partial x_j} \right\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} \cdot \left(\sum_{i,j=1}^n \left\| \frac{\partial u_j}{\partial x_i} \right\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} = \|u\|^2_{H_0^1}.
\end{aligned}$$

Hence $|\mathrm{div}|_{\mathcal{L}(H_0^1, L^2(\Omega))} \leqslant 1$. Now we will prove that $|\mathrm{div}|_{\mathcal{L}(H_0^1, L^2(\Omega))} \geqslant 1$. Indeed, by Theorem 11.11

$$\mathrm{div} : \left(\nabla H_0^2(\Omega), \| \cdot \|_{H_0^1} \right) \to \left(\Delta H_0^2(\Omega), \| \cdot \|_{L^2(\Omega)} \right)$$

is isometrical. Thus $\|\mathrm{div}(\nabla\psi)\|_{L^2(\Omega)} = \|\nabla\psi\|_{H_0^1}$ for every $\psi \in H_0^2(\Omega)$. Let us choose $\psi_0 \in H_0^2(\Omega)$ such that $\|\Delta\psi_0\|_{L^2(\Omega)} = 1$. Then

$$\|\nabla\psi_0\|_{H_0^1} = \|\mathrm{div}(\nabla\psi_0)\|_{L^2(\Omega)} = \|\Delta\psi_0\|_{L^2(\Omega)} = 1$$

and $|\mathrm{div}|_{\mathcal{L}(H_0^1, L^2(\Omega))} \geqslant 1$. □

Corollary 11.13. —

$$|\mathrm{divdiv}^*|_{\mathcal{L}(L^2(\Omega))} = 1.$$

Moreover $0 \leqslant \mathrm{divdiv}^* \leqslant \mathrm{id}_{L^2(\Omega)}$.

Proof. — Since $|\mathrm{div}^*|_{\mathcal{L}(L^2(\Omega), H_0^1)} = |\mathrm{div}|_{\mathcal{L}(H_0^1, L^2(\Omega))} = 1$,

$$|\mathrm{divdiv}^*|_{\mathcal{L}(L^2(\Omega))} \leqslant |\mathrm{div}|_{\mathcal{L}(H_0^1, L^2(\Omega))} \cdot |\mathrm{div}^*|_{\mathcal{L}(L^2(\Omega), H_0^1)} = 1.$$

On the other hand $\mathrm{divdiv}^* = \mathrm{id}$ on $\Delta H_0^2(\Omega)$ by Th. 11.11 (a). Thus

$|\mathrm{divdiv}^*|_{\mathcal{L}(L^2(\Omega))} = 1$ which ends the proof of the first part of the statement.

Let $p \in L^2(\Omega)$. Then

$$(\mathrm{divdiv}^* p|p)_{L^2(\Omega)} = ((\mathrm{div}^* p|\mathrm{div}^* p)) = \|\mathrm{div}^* p\|^2_{H^1_0} \geqslant 0.$$

On the other hand

$$\begin{aligned}(\mathrm{divdiv}^* p|p)_{L^2(\Omega)} &= \|\mathrm{div}^* p\|^2_{H^1_0} \leqslant |\mathrm{div}^*|^2_{\mathcal{L}(L^2(\Omega),H^1_0)} \cdot \|p\|^2_{L^2(\Omega)} \\ &= \|p\|^2_{L^2(\Omega)} = (p|p)_{L^2(\Omega)}.\end{aligned}$$

Thus

$$0 \leqslant (\mathrm{divdiv}^* p|p)_{L^2(\Omega)} \leqslant \big(\mathrm{id}_{L^2(\Omega)}(p)|p\big)_{L^2}.$$

□

Now, we will be concerned with the invariance of the space $H^1(\Omega)$ with respect to the divdiv* - operator.

PROPOSITION 11.14. — *Assume that* $D(A) \subset H^2(\Omega, \mathbb{R}^n)$. *Then*

(a) $\mathrm{divdiv}^*\big(H^1(\Omega)\big) \subset H^1(\Omega)$ *and* $\mathrm{divdiv}^*{}_{|H^1(\Omega)} \in \mathcal{E}nd H^1(\Omega)$.

(b) *If, in addition,* Ω *is connected and* $H^{\perp} = \nabla H^1(\Omega)$ *then*

$$\mathrm{divdiv}^*\big(H^1_0(\Omega) \cap \{\textstyle\int = 0\}\big) \in \mathrm{cotop} H^1(\Omega).$$

Proof. — Ad.**(a)**. It is clear that the operator div : $H^2 \to H^1(\Omega)$ is continuous, $D(A) = H^1_0 \cap H^2$ is a closed subset of H^2 and the injection $D(A) \hookrightarrow H^2$ is continuous. Finally, by Proposition 11.6 (b), $\mathrm{div}^*\big(H^1(\Omega)\big) \subset D(A)$ and $\|\mathrm{div}^* p\|_{D(A)} := \|A\mathrm{div}^* p\|_{L^2} = \|\nabla p\|_{L^2}$ for every $p \in H^1(\Omega)$. Since $\mathrm{divdiv}^*{}_{|H^1(\Omega)}$ is a composition of the following three operators

$$H^1(\Omega)\mathrm{div}^* \longrightarrow D(A) \hookrightarrow H^2 \mathrm{div} \longrightarrow H^1(\Omega),$$

the first part of the statement holds.

Ad.**(b)**. By Lemma 11.9, the function $h := \mathrm{divdiv}^* \varphi - \varphi$ is harmonic for every $\varphi \in \mathcal{D}(\Omega)$. Thus, integrating by parts, we obtain

$$\begin{aligned}(\nabla \mathrm{divdiv}^* \varphi|\nabla\varphi)_{L^2} &= (\nabla h|\nabla\varphi)_{L^2} + (\nabla\varphi|\nabla\varphi)_{L^2} \\ &= -(\Delta h|\varphi)_{L^2} + \|\nabla\varphi\|^2_{L^2} = \|\nabla\varphi\|^2_{L^2}.\end{aligned}$$

Hence

$$\|\nabla\varphi\|^2_{L^2} = |(\nabla \mathrm{divdiv}^* \varphi|\nabla\varphi)_{L^2}| \leqslant \|\nabla \mathrm{divdiv}^* \varphi\|_{L^2} \cdot \|\nabla\varphi\|_{L^2}$$

and

$$\|\nabla \mathrm{divdiv}^* \varphi\|_{L^2} \geqslant \|\nabla \varphi\|_{L^2}.$$

Since the Poincaré inner product $(\cdot|\cdot)_{\mathcal{P}}$ $((p|q)_{\mathcal{P}} := \int_\Omega p\,dx \cdot \int_\Omega q\,dx + (\nabla p|\nabla q)_{L^2})$ is admissible in $H^1(\Omega)$ and $\mathcal{D}(\Omega) \cap \{\int = 0\}$ is dense in $H_0^1(\Omega) \cap \{\int = 0\}$, the last inequality yields that the operator

$$\mathrm{divdiv}^* : H_0^1(\Omega) \cap \{\textstyle\int = 0\} \to H^1(\Omega)$$

is of positive infimum in the Poincaré norm. Thus its image is a closed subspace of $H^1(\Omega)$. $\square$

Assume that

$$\mathrm{imdiv} \in \mathrm{cotop} L^2(\Omega) \qquad \text{and} \qquad V = \ker \mathrm{div}. \tag{11.7}$$

Then by Remark 11.7 (b) $(\ker \mathrm{div}^*)^\perp = \{\int = 0\}$. Moreover, $\mathrm{imdiv} = \mathrm{div}(V^\perp) = \{\int = 0\}$ and $\mathrm{imdiv}^* = \mathrm{div}^*(\{\int = 0\}) = V^\perp$ by Remark 11.8. Since a linear operator is a monomorphism on the orthogonal complement of its kernel, we infer that $\mathrm{div}_{|V^\perp}$ and ${\mathrm{div}^*}_{|\{\int=0\}}$ are monomorphisms. Since their images form closed subspaces, $\mathrm{div}_{|V^\perp}$ and ${\mathrm{div}^*}_{|\{\int=0\}}$ are isomorphisms on their images, by the open mapping theorem. Thus

$$\begin{aligned} \mathrm{div}_{|V^\perp} &: V^\perp \tilde{\to} \{\textstyle\int = 0\} \\ {\mathrm{div}^*}_{|\{\int=0\}} &: \{\textstyle\int = 0\} \tilde{\to} V^\perp. \end{aligned}$$

Hence

$$\mathrm{divdiv}^* : \{\textstyle\int = 0\} \tilde{\to} \{\textstyle\int = 0\}. \tag{11.8}$$

The projections $P_{V^\perp}$ and P_V can be represented by using the authomorphism $(\mathrm{divdiv}^*)^{-1}$. Indeed, let $u \in H_0^1$. Since $P_{V^\perp} u \in V^\perp$, there exists $q \in \{\int = 0\}$ such that $P_{V^\perp} u = \mathrm{div}^* q$. Applying the divergence operator, we obtain

$$\mathrm{div} P_{V^\perp} u = \mathrm{divdiv}^* q. \tag{11.9}$$

Since $V \ni P_V u = u - P_{V^\perp} u$, $\mathrm{div} u = \mathrm{div} P_{V^\perp} u$. Hence, by (11.8),

$$(\mathrm{divdiv}^*)^{-1} \mathrm{div} u = q.$$

Thus

$$P_{V^\perp} u = \mathrm{div}^* (\mathrm{divdiv}^*)^{-1} \mathrm{div} u. \tag{11.10}$$

Because of the decomposition $u = P_V u + P_{V^\perp} u$, we obtain

$$P_V u = u - \mathrm{div}^* (\mathrm{divdiv}^*)^{-1} \mathrm{div} u. \tag{11.11}$$

11.5. Inversion of the divdiv* operator

Now we will be concerned with inverting the authomorphism divdiv* in the space $\{\int = 0\}$. We will use the von Neumann lemma so let us recall it as well as some other auxiliary results.

Lemma 11.15 *(von Neumann). — Assume that $(X, \|\cdot\|)$ is a Banach space, $A : X \to X$ is a linear bounded operator of norm $|A| < 1$ in the space of all endomorphisms of X. Then the operator $(\mathrm{id} - A)$ is invertible and its inversion is given by the von Neumann series*

$$(\mathrm{id} - A)^{-1} = \sum_{j=0}^{\infty} A^j$$

Moreover, for each $s \in \mathbb{N}$

$$\left| (\mathrm{id} - A)^{-1} - \sum_{j=0}^{s} A^j \right|_{\mathcal{E}ndX} \leqslant \frac{|A|^{s+1}}{1 - |A|}.$$

(as usual id *stands for the identity on X).*

We will use the following auxiliary lemma.

Lemma 11.16. — *Let $\big(X, (\cdot|\cdot)\big)$ be a Hilbert space. Let $A : X \to X$ be a bounded nonnegative selfadjoint linear operator such that $|A| = 1$ and $\inf A \leqslant 1$. Then $\mathrm{id} - A$ is selfadjoint and its norm is given by the formula*

$$|\mathrm{id} - A| = 1 - \inf A.$$

Proof. — Since $A = A^*$, thus $(\mathrm{id} - A)^* = \mathrm{id} - A$. Then

$$\begin{aligned}
|\mathrm{id} - A| &= \sup\{|((\mathrm{id} - A)x|x)|, \quad \|x\| = 1\} \\
&= \sup\{|\|x\|^2 - (Ax|x)|, \quad \|x\| = 1\} \\
&= \sup\{1 - (Ax|x), \quad \|x\| = 1\} \\
&= 1 - \inf\{(Ax|x), \quad \|x\| = 1\} = 1 - \inf A.
\end{aligned}$$

□

Theorem 11.17. — *Assume that Ω is connected and conditions (11.7) are satisfied. Then*

(a) *For every $s \in \mathbb{N}$ we have*

$$\left|(\mathrm{divdiv}^*)^{-1} - \sum_{j=0}^{s} (\mathrm{id} - \mathrm{divdiv}^*)^j \right|_{\mathcal{E}nd\{\int = 0\}} \leqslant \frac{1}{\theta}(1-\theta)^{s+1}$$

where the constant $\theta = \theta(\Omega) \in]0,1[$ depends on Ω only.

(b) *For every $s \in \mathbb{N}$ and for every $u \in H_0^1$ we have*

$$\left\| P_{V^\perp} u - \mathrm{div}^* \sum_{j=0}^{s} (\mathrm{id} - \mathrm{divdiv}^*)^j \mathrm{div} u \right\|_{H_0^1} \leqslant \frac{\|\mathrm{div} u\|_{L^2(\Omega)}}{\theta} (1-\theta)^{s+1}.$$

Proof. — Ad.**(a)**. By (11.8)

$$\mathrm{divdiv}^* : \{\textstyle\int = 0\} \xrightarrow{\sim} \{\textstyle\int = 0\},$$

i.e. $\mathrm{divdiv}^* \in \mathcal{A}ut(\{\int = 0\})$. Moreover divdiv^* is selfadjoint, $0 \leqslant \mathrm{divdiv}^* \leqslant \mathrm{id}_{L^2(\Omega)}$ and $|\mathrm{divdiv}^*|_{\mathcal{E}nd\{\int=0\}} = 1$. Thus

$$|\mathrm{id} - \mathrm{divdiv}^*|_{\mathcal{E}nd\{\int=0\}} = 1 - \inf(\mathrm{divdiv}^*),$$

by Lemma 11.16 . We will prove that $\theta := \inf(\mathrm{divdiv}^*) \in]0,1[$.

Indeed, since $\mathrm{divdiv}^* \in \mathcal{A}ut(\{\int = 0\})$,

$$\inf(\mathrm{divdiv}^*) = \frac{1}{|(\mathrm{divdiv}^*)^{-1}|_{\mathcal{E}nd\{\int=0\}}} > 0.$$

We will show that $\theta < 1$. Indeed, let

$$p(x) := x_1 - \frac{1}{|\Omega|} \int_\Omega y_1 \, dy.$$

Its clear that $p \in H^1(\Omega) \cap \{\int = 0\}$ and $\nabla p = e_1$, where e_i, $i = 1, \dots n$ form the canonical base of $\mathbb{R}^n$. Let $u := \mathrm{div}^* p$. We will show that $u_2 = \ldots = u_n = 0$. Indeed, for every $\varphi \in \mathcal{D}(\Omega)$ and for each $i \in \{2, \dots, n\}$ we have

$$\begin{aligned} ((\varphi | u_i)) &= ((\varphi \cdot e_i | u)) = ((\varphi \cdot e_i | \mathrm{div}^* p)) = (\mathrm{div}(\varphi \cdot e_i) | p)_{L^2(\Omega)} \\ &= \Big(\frac{\partial \varphi}{\partial x_i} | p\Big)_{L^2(\Omega)} = -\Big(\varphi | \frac{\partial p}{\partial x_i}\Big)_{L^2(\Omega)} = 0. \end{aligned}$$

Hence $u_i = 0$. Thus $\|\mathrm{div} u\|^2_{L^2(\Omega)} = \|\frac{\partial u_1}{\partial x_1}\|^2_{L^2(\Omega)}$. On the other hand

$$\|u\|^2_{H_0^1} = \|\nabla u_1\|^2_{L^2} = \sum_{i=1}^{n} \Big\|\frac{\partial u_1}{\partial x_i}\Big\|^2_{L^2} \geqslant \Big\|\frac{\partial u_1}{\partial x_1}\Big\|^2_{L^2} + \Big\|\frac{\partial u_1}{\partial x_2}\Big\|^2_{L^2}.$$

Suppose that $\|\frac{\partial u_1}{\partial x_2}\|_{L^2} = 0$. Then must be $\|u_1\|_{L^2(\Omega)} = 0$. However, then $u = 0$ and $p = 0$ which is not true, because $\|\nabla p\|_{L^2} > 0$. Thus $\|\frac{\partial u_1}{\partial x_2}\|_{L^2} > 0$ and

$$\|u\|^2_{H^1_0} > \|\mathrm{div} u\|^2_{L^2(\Omega)}.$$

The von Neumann Lemma application with $A := \mathrm{id} - \mathrm{divdiv}^*$ yields

$$(\mathrm{divdiv}^*)^{-1} = \sum_{j=0}^{\infty} (\mathrm{id} - \mathrm{divdiv}^*)^j$$

which completes proof of assertion (a).

Ad.**(b)**. Let $u \in H^1_0$. Using representation (11.10), we infer that

$$\begin{aligned}
&\left\| P_{V^\perp} u - \mathrm{div}^* \sum_{j=0}^{s} (\mathrm{id} - \mathrm{divdiv}^*)^j \mathrm{div} u \right\|_{H^1_0} \\
&= \left\| \mathrm{div}^* (\mathrm{divdiv}^*)^{-1} \mathrm{div} u - \mathrm{div}^* \sum_{j=0}^{s} (\mathrm{id} - \mathrm{divdiv}^*)^j \mathrm{div} u \right\|_{H^1_0} \\
&\leqslant \left\| (\mathrm{divdiv}^*)^{-1} \mathrm{div} u - \sum_{j=0}^{s} (\mathrm{id} - \mathrm{divdiv}^*)^j \mathrm{div} u \right\|_{L^2(\Omega)} \\
&\leqslant \left| (\mathrm{divdiv}^*)^{-1} - \sum_{j=0}^{s} (\mathrm{id} - \mathrm{divdiv}^*)^j \right|_{\mathcal{E}nd\{\int=0\}} \cdot \|\mathrm{div} u\|_{L^2(\Omega)} \\
&\leqslant \frac{\|\mathrm{div} u\|_{L^2(\Omega)}}{\theta} (1-\theta)^{s+1}.
\end{aligned}$$

□

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