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On the approximation of functions on a Hodge manifold

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Abstract. — If (M, ω) is a Hodge manifold and $f \in C^\infty(M, \mathbb{R})$ we construct a canonical sequence of functions f_N such that $f_N \to f$ in the C^∞ topology. These functions have a simple geometric interpretation in terms of the moment map and they are real algebraic, in the sense that they are regular functions when M is regarded as a real algebraic variety. The definition of f_N is inspired by Berezin-Toeplitz quantization and by ideas of Donaldson. The proof follows quickly from known results of Fine, Liu and Ma.

Résumé. — Soit (M, ω) une variété de Hodge et soit $f \in C^\infty(M, \mathbb{R})$. Nous definissons une suite canonique de fonctions f_N telle que $f_N \to f$ dans la topologie C^∞. Cette construction admet une interprétation très simple du point de vue de l'application moment. En plus les fonctions f_N sont algébriques réelles, c'est-à-dire qu'elles sont des fonctions régulières sur M vue comme variété algébrique réelle. La définition des f_N est inspirée de la quantification de Berezin-Toeplitz et s'appuie sur des idées de Donaldson. La preuve découle très vite de certains résultats dus à Fine, Liu et Ma.

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1. Introduction

Let M be a complex projective algebraic manifold and let ω be a Hodge metric, i.e. a Kähler metric such that $[\omega/2\pi]$ lies in the image of $H^2(M,\mathbb{Z})$ inside $H^2(M,\mathbb{R})$. Using ideas of Donaldson [6, §4], Fine, Liu and Ma [7] have already introduced a canonical way to approximate smooth functions on M. More precisely, using the Fourier modes of the Szegö kernel they defined a sequence of operators

$$Q_N : C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R}) \qquad N = 1, 2, 3, \ldots$$

with the property that for any $f \in C^\infty(M,\mathbb{R})$ one has $Q_N f \to f$ in the topology of $C^\infty(M,\mathbb{R})$, see (3.7) below and Theorem 3.4. The importance of these operators comes from the fact that Q_N is the derivative of the nonlinear map that associates to a Kähler form ω its N-th Bergman metric ω_N, see [7, pp. 495-496].

The purpose of this note is to introduce another sequence of operators, denoted P_N, which are a variant of the Q_N. Theorem 3.5, which is the main result in the paper, asserts that these operators still have the property that $P_N f \to f$ in $C^\infty(M,\mathbb{R})$. But they have two other nice features. First, if L is a polarization with $2\pi c_1(L) = [\omega]$, the operator P_N admits a simple geometric interpretation in terms of the complete linear system of L^N and the moment map. Next recall that $\mathbb{P}^m(\mathbb{C})$ can be embedded in the space of Hermitian matrices by sending a line in $\mathbb{C}^{m+1}$ to the orthogonal projector onto it. This embedding is just the moment map of $\mathbb{P}^m(\mathbb{C})$ in disguise and it endows $\mathbb{P}^m(\mathbb{C})$ with a canonical structure of (affine) real algebraic variety, see e.g. [1, p. 73]. The same follows for every complex subvariety of $\mathbb{P}^m(\mathbb{C})$. Thus every complex projective variety has a canonical structure of affine algebraic variety over $\mathbb{R}$. The *real algebraic functions* on M are the functions which are regular with respect to this structure. The second nice feature of the operators P_N is that their range consists of real algebraic functions. Thus an arbitray function $f \in C^\infty(M,\mathbb{R})$ is approximated by functions $P_N f$ which are real algebraic.

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2. The geometric construction

Let W be a finite-dimensional complex vector space. Let $\mathbb{P}(W^*)$ denote the projective space of *lines* in W^*. Any $w \in W$ can be identified with a section $s_w \in H^0\left(\mathbb{P}(W^*), \mathcal{O}_{\mathbb{P}(W^*)}(1)\right)$ defined by $s_w([\lambda])(t\lambda) = t\lambda(w)$. Here $t\lambda$ is the generic element of $\mathcal{O}_{\mathbb{P}(W^*),[\lambda]}(-1)$. If $\langle\,,\rangle$ is a Hermitian product on W, the line bundle $\mathcal{O}_{\mathbb{P}(W^*)}(1)$ inherits a Hermitian metric h_{FS} such that

$$\left|s_w([\lambda])\right|_{h_{FS}} = \frac{|\lambda(w)|}{|\lambda|}.$$

(With slight abuse we denote by $|\lambda(w)|$ the absolute value of a complex number and by $|\lambda|$ the norm of a vector in W^*.) If $\Theta(h_{FS})$ denotes the curvature of h_{FS}, then $\omega_{FS} = i\Theta(h_{FS})$ is by definition the Fubini-Study metric on $\mathbb{P}(W^*)$ induced by the product $\langle\,,\rangle$. The group $\mathrm{SU}(W, \langle\,,\rangle)$ acts on $(\mathbb{P}(W^*), \omega_{FS})$ both holomorphically and isometrically and

$$\Phi : \mathbb{P}(W^*) \to \mathfrak{su}(W) \qquad \Phi([\lambda]) = i\left(\frac{\langle\cdot, w_\lambda\rangle w_\lambda}{|w_\lambda|^2} - \frac{\mathrm{id}_W}{\dim W}\right)$$

is the moment map. (Here w_λ is the vector such that $\lambda(\cdot) = \langle\cdot, w_\lambda\rangle$.) This means that it is equivariant and that for any $A \in \mathfrak{su}(W)$

$$d\,(\Phi, A) = -i_{\xi_A}\omega_{FS},$$

where $(X, Y) := -\mathrm{tr}\, XY$ is the Killing product on $\mathfrak{su}(W)$ and ξ_A is the fundamental vector field corresponding to A. Notice that

$$\left(\Phi\left([\lambda]\right), A\right) = -i\frac{\langle Aw_\lambda, w_\lambda\rangle}{|w_\lambda|^2} \qquad A \in \mathfrak{su}(W). \tag{2.1}$$

(For the proof see for example [9, p. 24].)

Let M^m be a projective manifold with an ample line bundle $L \to M$ and let h be a Hermitian metric on L. Denote by $\Theta(h)$ the curvature of the Chern connection of (L, h) and assume that $\omega := i\Theta(h) \in 2\pi\mathrm{c}_1(L)$ is a Kähler form. Let N be a natural number. Using h and the volume form $dV_M := \omega^m/m!$ one can endow the space $V_N := H^0(M, L^N)$ with the L^2-Hermitian product

$$\langle s_1, s_2\rangle_{L^2} := \int_M h^{\otimes N}(s_1(z), s_2(z))dV_M(z) \quad s_1, s_2 \in V_N.$$

Correspondingly, the projective space $\mathbb{P}(V_N^*)$ is endowed with the Fubini-Study metric induced from $\langle\,,\rangle_{L^2}$. Denote by

$$\varphi_N : M \to \mathbb{P}(V_N^*).$$

the map associated to the complete linear system of L^N, by ω_N the Fubini-Study metric on $\mathbb{P}(V_N^*)$ induced by the Hermitian product $\langle\, ,\rangle_{L^2}$ and by $\Phi_N : \mathbb{P}(V_N^*) \to \mathfrak{su}(V_N)$ the moment map of $(\mathbb{P}(V_N^*), \omega_N)$ as described above.

Let $L^2(M, L^N)$ be the Hilbert space of L^2 sections of L^N. For any N let $\Pi_N : L^2(M, L^N) \to V_N$ be the orthogonal projector onto $V_N = H^0(M, L^N)$. A function $f \in C^\infty(M, \mathbb{C})$ induces a sequence of *Toeplitz* operators

$$T_{f,N} : L^2(M, L^N) \to L^2(M, L^N)$$

$$T_{f,N}s := \Pi_N(f \cdot \Pi_N s) \qquad s \in L^2(M, L^N).$$

Since the range of $T_{f,N}$ is contained in V_N and $T_{f,N} \equiv 0$ on $V_N^\perp$, we will identify $T_{f,N}$ with its restriction to V_N considered as an endomorphism of V_N. If $s \in V_N$, then $T_{f,N}s := \Pi_N(fs)$. Since f is *real valued*, the operator $T_{f,N}$ is self-adjoint. Set

$$d_N = h^0(M, L^N) - 1$$

and let $T^0_{f,N}$ be the trace-free part of $T_{f,N}$:

$$T^0_{f,N} := T_{f,N} - \frac{\operatorname{tr} T_{f,N}}{d_N + 1} \mathrm{id}_{V_N}.$$

Then $iT^0_{f,N} \in \mathfrak{su}(V_N)$ and $(\,\cdot\,, iT^0_{f,N})$ is a linear function on $\mathfrak{su}(V_N)$. The pull-back of this function to M via the map $\Phi_N \circ \varphi_N : M \to \mathfrak{su}(V_N)$ is a real valued smooth function on M, that we denote by $P^0_N f$. More explicitly

$$(P^0_N f)(z) := \left(\Phi_N\left(\varphi_N(z)\right), iT^0_{f,N}\right). \tag{2.2}$$

So we have defined a sequence of operators $P^0_N : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$. These operators are linear since the map $f \mapsto T_{f,N}$ is linear.

3. Relation with the Szegö kernel

The operators P^0_N that we have just defined (and the related operators P_N to be defined below) admit a simple description in terms of the Fourier modes of the Szegö kernel. This will enable us to deduce very quickly some of their analytic properties from known results on the Szegö kernel. We start by recalling some important facts regarding the Szegö kernel. (It should be noted that all the constructions in the paper could be rephrased in terms of the Bergman kernel of the line bundle L, as in [4] and [12], rather than using the Szegö kernel of the circle bundle. The two approaches are completely equivalent.)

Set $\mathrm{Vol}(M) = \int_M dV_M = (2\pi)^m \mathrm{c}_1(L)^m/m!$. Let h^* denote the Hermitian metric on L^* induced from h and let $|\cdot|_{h^*}$ denote the corresponding norm. Define $\rho : L^* \to \mathbb{R}$ by $\rho(\lambda) := |\lambda|^2_{h^*} - 1$ and set

$$D := \{\lambda \in L^* : \rho(\lambda) < 0\} = \{\lambda \in L^* : |\lambda|_{h^*} < 1\}$$

$$X := \{\lambda \in L^* : \rho(\lambda) = 0\} = \{\lambda \in L^* : |\lambda|_{h^*} = 1\}.$$

D is a strictly pseudoconvex domain in L^* since $\omega = i\Theta(h)$ is positive. $X = \partial D$ is a smooth hypersurface in L^* and is a principal S^1-bundle over M. Set $\alpha := i\bar{\partial}\rho|_X$ and

$$dV_X := \frac{\alpha \wedge (d\alpha)^m}{2\pi m!} = \frac{\alpha \wedge \pi^* dV_M}{2\pi}.$$

For $N \in \mathbb{Z}$, let $\rho_N : S^1 \to \mathrm{Gl}(1) = \mathbb{C}^*$ be the representation $\rho_N(e^{i\theta}) = e^{iN\theta}$. Recall that the associated bundle $X \times_{\rho_{-N}} \mathbb{C}$ is the quotient $(X \times \mathbb{C})/S^1$, where S^1 acts on $X \times \mathbb{C}$ according to the rule

$$e^{i\theta} \cdot (\lambda, z) = (\lambda e^{i\theta}, \rho_{-N}(e^{-i\theta})z) = (e^{i\theta}\lambda, e^{iN\theta}z).$$

The map

$$[\lambda, z] \mapsto z \cdot \left(\lambda^{-1}(1)\right)^{\otimes N} \qquad u \mapsto [\lambda, \lambda^{\otimes N}(u)] \tag{3.1}$$

is an isomorphism of $X \times_{\rho_{-N}} \mathbb{C}$ onto L^N. Therefore a section s of L^N corresponds to an equivariant function $\hat{s} : X \to \mathbb{C}$ defined by $\hat{s}(x) = x^{\otimes N}(s(\pi(x)))$. The equivariance means that

$$\hat{s}(e^{i\theta}\lambda) = \rho_{-N}(e^{-i\theta})\hat{s}(\lambda) = e^{iN\theta}\hat{s}(\lambda). \tag{3.2}$$

Denote by $\mathcal{H}^2(X)$ the space of L^2 functions on X that are annihilated by the Cauchy-Riemann operator $\bar{\partial}_b$ (see [19, p. 321]). $\mathcal{H}^2(X)$ coincides with the space of L^2–boundary values of holomorphic functions on D. It is a closed S^1-invariant subspace of $L^2(X)$, hence splits as a direct sum $\mathcal{H}^2(X) = \oplus_N \mathcal{H}^2_{N \in \mathbb{Z}}(X)$, where $\mathcal{H}^2_N(X)$ is the set of functions $\hat{s}$ in $\mathcal{H}^2(X)$ that satisfy (3.2). Via the correspondence (3.1) the holomorphic sections of L^N correspond to elements of $\mathcal{H}^2_N(X)$. If $s_1, s_2 \in C^\infty(M, L^N)$, then

$$\langle s_1, s_2 \rangle_{L^2} = \int_X \hat{s}_1 \overline{\hat{s}_2} dV_X = \langle \hat{s}_1, \hat{s}_2 \rangle_{L^2(X)}.$$

If $\pi : X \to M$ denotes the projection of the circle bundle and $x \in X$, then the linear functional

$$V_N \longrightarrow \mathbb{C} \qquad s \mapsto x^{\otimes N}\left(s\left(\pi(x)\right)\right) = \hat{s}(x)$$

is represented by a section $e_{x,N} \in V_N$. Equivalently, $\hat{s}(x) = \langle s, e_{x,N}\rangle_{L^2} = \langle \hat{s}, \hat{e}_{x,N}\rangle_{L^2(X)}$ for any $s \in V_N$. The sections $e_{x,N}$ are called *coherent states* in the mathematical physics literature.

We use Π_N also to denote the projection $\Pi_N : L^2(X) \to \mathcal{H}^2_N(X)$. Let $\Pi_N(x, x')$ be the Schwartz kernel of Π_N (we use dV_X to identify functions with densities). In the following Proposition we gather various well-known and elementary properties of Π_N.

Proposition 3.1. — *(a) $\Pi_N \in C^\infty(X \times X)$. (b) $\Pi_N(x, x') = \overline{\Pi_N(x', x)}$. (c) For any $x \in X$, $\hat{e}_{x,N} = \Pi_N(\cdot, x)$. (d) There are functions $K_N \in C^\infty(M \times M)$ and $E_N \in C^\infty(M)$ such that if $x, x' \in X$, $z = \pi(x), z' = \pi(x')$, then*

$$K_N(z, z') = |\Pi_N(x, x')|^2 \qquad E_N(z) = \Pi_N(x, x).$$

(e) $K_N(z, z') = K_N(z', z)$. (f) If $\pi(x) = z$ and $\{s_j\}_{j=0}^{d_N}$ is an orthonormal basis of V_N, then

$$E_N(z) = \sum_{j=0}^{d_N} |s_j(z)|^2_h = |e_{x,N}|^2_{L^2}. \tag{3.3}$$

(g) For N large enough the function E_N is strictly positive.

(E_N coincides with the kernel of the Bergman projector $L^2(M, L^N) \to H^0(M, L^N)$ restricted to the diagonal.)

Proof. — Let $\{s_j\}_{j=0}^{d_N}$ be an orthonormal basis of V_N. Then for any $x, x' \in X$

$$\Pi_N(x, x') = \sum_{j=0}^{d_N} \hat{s}_j(x)\overline{\hat{s}_j(x')} \tag{3.4}$$

This proves (a) and (b). If $e_{x,N} = \sum_j a_j s_j$, then

$$\overline{a_j} = \langle s_j, e_{x,N}\rangle_{L^2} = \hat{s}_j(x).$$

Hence $e_{x,N} = \sum_j \overline{\hat{s}_j(x)} s_j$. Together with (3.4) this proves (c). Observe that by (3.2)

$$\Pi_N(e^{i\theta}x, e^{i\theta'}x') = e^{iN(\theta-\theta')}\Pi_N(x, x').$$

Hence the function $|\Pi_N|^2$ is $S^1 \times S^1$–invariant on $X \times X$ and descends to a smooth function $K_N \in C^\infty(M \times M)$. Similarly $\Pi_N(x, x)$ is S^1-invariant on X and descends to a smooth function $E_N \in C^\infty(M)$. This proves (d). (e) follows from (b). To prove (f) it is enough to observe that if $x \in X$, then $|x^{\otimes N}|_{h^*} = 1$, hence $|s_j(z)|_h = |x^{\otimes N}(s_j(z))| = |\hat{s}_j(x)|$. This yields immediately the first equality. For the same reason

$$|e_{x,N}|^2_{L^2} = \sum_{j=0}^{d_N} \left|\langle e_{x,N}, s_j\rangle_{L^2}\right|^2 = \sum_{j=0}^{d_N} \left|x^{\otimes N}(s_j(z))\right|^2 = \sum_{j=0}^{d_N} |s_j(z)|^2_h.$$

The positivity of E_N is equivalent to the fact that L^N has no base points, so (g) follows from the ampleness of L. □

THEOREM 3.2. — *If $f \in C^\infty(M, \mathbb{R})$ and $z \in M$, then*

$$P^0_N f(z) = \int_M \frac{K_N(z,z')}{E_N(z)} f(z') dV_M(z') - \frac{\operatorname{tr} T_{f,N}}{d_N + 1}, \tag{3.5}$$

$$\operatorname{tr} T_{f,N} = \int_M E_N(z) f(z) dV_M(z). \tag{3.6}$$

Proof. — Let $x \in X$ be such that $z = \pi(x)$. Set $\lambda(\cdot) = \langle \cdot, e_{x,N}\rangle_{L^2}$. Then $\lambda \in V^*_N$ and $\varphi_N(z) = [\lambda]$. By (2.2) and (2.1)

$$P^0_N f(z) = \left(\Phi_N(\varphi_N(z)), iT^0_{f,N}\right) = -i \frac{\langle iT^0_{f,N} e_{x,N}, e_{x,N}\rangle_{L^2}}{|e_{x,N}|^2_{L^2}} =$$

$$= \frac{\langle T_{f,N} e_{x,N}, e_{x,N}\rangle_{L^2}}{|e_{x,N}|^2_{L^2}} - \frac{\operatorname{tr} T_{f,N}}{d_N + 1}$$

$$\langle T_{f,N} e_{x,N}, e_{x,N}\rangle_{L^2} = \langle f e_{x,N}, e_{x,N}\rangle_{L^2} = \int_M f(z') \left|e_{x,N}(z')\right|^2_h dV_M(z').$$

If $z' = \pi(x')$, then $|e_{x,N}(z')|^2_h = |\hat{e}_{x,N}(x')|^2 = |\Pi_N(x', x)|^2 = K_N(z', z) = K_N(z, z')$. Hence

$$\langle T_{f,N} e_{x,N}, e_{x,N}\rangle_{L^2} = \int_M K_N(z,z') f(z') dV_M(z').$$

By (3.3) $|e_{x,N}|^2_{L^2} = E_N(z)$, so

$$\frac{\langle T_{f,N} e_{x,N}, e_{x,N}\rangle_{L^2}}{|e_{x,N}|^2_{L^2}} = \frac{1}{E_N(z)} \int_M K_N(z,z') f(z') dV_M(z') =$$

$$= \int_M \frac{K_N(z,z')}{E_N(z)} f(z') dV_M(z').$$

This proves (3.5). To prove (3.6) observe that if $\{s_j\}_{j=0}^{d_N}$ is an orthonormal basis of V_N, then

$$\operatorname{tr} T_{f,N} = \sum_{j=0}^{d_N} \langle T_{f,N} s_j, s_j \rangle_{L^2} = \sum_{j=0}^{d_N} \langle f s_j, s_j \rangle_{L^2} =$$

$$= \sum_{j=0}^{d_N} \int_M f(z') |s_j(z')|_h^2 dV_M(z') = \int_M f(z') E_N(z') dV_M(z').$$

In the last line we have used (3.3). $\square$

Definition 3.3. — *For $N = 1, 2, 3, \dots$ let $P_N : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ be the linear operator defined by*

$$P_N f(z) := \int_M \frac{K_N(z, z')}{E_N(z)} f(z') dV_M(z').$$

It follows from (3.5) that for any $f \in C^\infty(M, \mathbb{R})$ the functions $P_N f$ and $P_N^0 f$ differ only by a constant (depending on N). We will prove now that as $N \to \infty$ this constant approaches the average of f, while $P_N f \to f$. It follows that $P_N^0 f \to f$ for any function f with

$$\int_M f(z) dV_M(z) = 0$$

This result follows immediately from known results that we now recall. In [6, §4] Donaldson introduced operators $Q_N : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$, defined by

$$Q_N f(z) = \frac{\operatorname{Vol}(M)}{d_N + 1} \int_M K_N(z, z') f(z') dV_M(z'). \tag{3.7}$$

(His definition is actually more general since integration is performed with respect to a measure that does not necessarily coincide with dV_M.) In [7, Appendix, Thm. 26] Kefeng Liu and Xiaonan Ma proved the following result.

Theorem 3.4 (Liu-Ma). — *For every k there is a constant C_k such that*

$$||Q_N f - f||_{C^k(M)} \leqslant \frac{C_k}{N} ||f||_{C^{k+1}(M)}. \tag{3.8}$$

In [7, p. 519] the estimate is given in terms of $||f||_{C^k(M)}$ on the right hand side, due to a small inaccuracy in the computations. The correct formulation, as above, is given in [13, p. 1 note 1]. I thank Professor Ma for pointing this out to me. It is important to notice that the operator Q_N represents the derivative of the map $\omega \mapsto \omega_N$, see [7, pp. 495-496].

THEOREM 3.5. — *(a) There is a positive constant C such that*

$$\left|\frac{\operatorname{tr} T_{f,N}}{d_N+1} - \frac{1}{\operatorname{Vol}(M)}\int_M f(z)dV_M(z)\right| \leqslant \frac{C}{N}||f||_\infty.$$

(b) For any k there is a positive constant C_k such that

$$||P_N f - f||_{C^k(M)} \leqslant \frac{C_k}{N}||f||_{C^{k+1}(M)}.$$

Proof. — The first statement follows from a fundamental result of Zelditch [19]: the sequence $\{E_N\}_N$ admits an asymptotic expansion in $C^\infty(M)$ of the form

$$E_N = \left(\frac{N}{2\pi}\right)^m + O(N^{m-1}). \tag{3.9}$$

(Apply [19, Cor. 2] with $G = \operatorname{Ric}(h)$.) By Riemann-Roch

$$\begin{aligned} d_N + 1 = h^0(M, L^N) = \frac{\mathrm{c}_1(L)^m}{m!}\,N^m + O(N^{m-1}) = \\ = \frac{\operatorname{Vol}(M)}{(2\pi)^m}\,N^m + O(N^{m-1}). \end{aligned} \tag{3.10}$$

Using (3.9) we get

$$\frac{E_N}{d_N+1} = \frac{1}{\operatorname{Vol}(M)} + O\left(\frac{1}{N}\right)$$

in $C^\infty(M)$. Integrating against f yields (a). To prove (b) it is enough to observe that

$$P_N f = \frac{d_N+1}{\operatorname{Vol}(M)\cdot E_N} Q_N f.$$

Therefore (3.8) and (3.10) yield the result. $\square$

4. Final remarks

1. Tian approximation theorem [17, 3, 19, 4] asserts that $\varphi_N^*\omega_N/N \to \omega$ in the C^∞ topology. The metrics $\varphi_N^*\omega_N$ are projectively induced hence of algebraic/polynomial character. Theorem 3.5 can be considered as an analogue for functions of Tian theorem. Indeed denote by $\mathscr{A}_N^0$ the set of functions of the form $(\Phi_N \circ \varphi_N)^*\lambda$ where λ varies in the dual of $\mathfrak{su}(V_N)$. $\mathscr{A}_N^0$ is a finite dimensional subspace of $C^\infty(M,\mathbb{R})$ whose elements are real algebraic since Φ_N is a real algebraic mapping. By construction the range of P_N^0 is contained in $\mathscr{A}_N^0$ and the range of P_N is contained in

$$\mathscr{A}_N := \mathbb{R} + \mathscr{A}_N^0 \subset C^\infty(M,\mathbb{R})$$

where $\mathbb{R}$ denotes the constant functions. Hence for any $f \in C^\infty(M, \mathbb{R})$, the function $P_N f$ is real algebraic function and projectively induced in a sense.

2. The elements of $\mathscr{A}^0_N$ are related to a particular kind of holomorphic functions $L^* \times \overline{L^*}$, the so-called Hermitian algebraic functions, see [18, Def. 2.1 p. 297].

3. The construction of the operators P^0_N and P_N is of course inspired by Berezin-Toeplitz quantization and especially by a paper of Bordemann, Meinrenken, Schlichenmaier [2] (see also [14]). In that paper the authors did not use explicitly the Toeplitz operators to construct new functions on M. Nevertheless what they showed in the proof of Theorem 4.1 is equivalent, in our notation, to the fact that $P_N f(z_0) \to f(z_0)$ at points $z_0 \in M$ where $|f|$ attains its maximum.

4. Let $\hat{P}_N : C^\infty(X) \to C^\infty(X)$ be the operator with kernel

$$\hat{P}_N(x, x') = \frac{|\Pi_N(x, x')|^2}{\Pi_N(x, x)}.$$

Then $\hat{P}_N(\pi^* f) = \pi^* (P_N f)$ for any $f \in C^\infty(M)$ and the relation between $\hat{P}_N$ and Π_N is identical with that between the Poisson-Szegö and the Szegö kernels of a domain D, see e.g. [8, p. 79] and [11]. Since the Poisson-Szegö operator reproduces holomorphic functions in $\mathcal{H}^2(\partial D)$, one might wonder if something similar holds for $\hat{P}_N$. This is not the case. Indeed the trick used to prove the reproducing property for the Poisson-Szegö operator [10, p. 65] fails in our situation, since for $f \in \mathcal{H}^2_K(X)$ the function

$$u(x') = f(x') \frac{\overline{\Pi_N(x, x')}}{\Pi_N(x, x)}$$

belongs to $\mathcal{H}^2_{N+K}(X)$. So $\hat{P}_N f = f$ if and only if $f \in \mathcal{H}^2_0(X)$. This means that $\hat{P}_N$ reproduces only the constant functions.

5. One might try to prove (b) in Theorem 3.5 by another path, using the following result. Consider $T_{f,N}$ as an operator on $C^\infty(X)$ and denote by $T_{f,N}(x, x')$ its Schwartz kernel. The function $T_{f,N}(x, x)$ descends to a function t_N on M and

$$t_N = \left(\frac{N}{2\pi}\right)^m \cdot f + O(N^{m-1}) \tag{4.1}$$

in the topology of $C^\infty(M)$. This follows from the main result in [5] and can also be established using the scaling asymptotics of the Szegö kernel (established in [15, 16]), see [12, Lemma 7.2.4]. (See also [13] for a study of higher terms in the asymptotic expansion.) If $M_f : L^2(X) \to L^2(X)$ denotes the operator of multiplication by f, then $T_{f,N} = \Pi_N M_f \Pi_N$ so

$$T_{f,N}(x, x'') = \int_X \Pi_N(x, x') f\left(\pi(x')\right) \Pi_N(x', x'') dV_X(x')$$

$$\begin{aligned} t_N(z) = T_{f,N}(x, x) = \int_X |\Pi_N(x, x')|^2 f\left(\pi(x')\right) dV_X(x') = \\ = \int_M K_N(z, z') f(z') dV_M(z'). \end{aligned}$$

Therefore

$$P_N f(z) = \frac{t_N(z)}{E_N(z)}.$$

It follows from (3.9) and (4.1) that for every $f \in C^\infty(M)$ and any k there is C'_k such that $||P_N f - f||_{C^k(M)} \leqslant C'_k/N$. The constant C'_k depends on $f \in C^\infty(M)$ and we would like to show that it can be chosen in the form $C'_k = C_k ||f||_{C^{k+1}(M)}$. Unfortunately it is not clear how to accomplish this last step, so this method of proof is incomplete.

6. If $M = \mathbb{P}^1, L = \mathcal{O}_{\mathbb{P}^1}(2)$ and $\omega = 2\omega_{FS}$, i.e. if $M = S^2$ with the round metric, it would be interesting to relate this approximation procedure to more classical constructions. The Szegö kernel of $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1), \omega_{FS})$ is

$$\Pi_N(x, x') = \frac{N+1}{2\pi} \langle x, x' \rangle^N$$

where $x, x' \in X = S^3 \subset \mathbb{C}^2$ and $\langle\ ,\ \rangle$ denotes the Hermitian product on $\mathbb{C}^2$ (see e.g. [14, p. 65]). So for $(M, L, \omega) = \left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2), 2\omega_{FS}\right)$

$$\Pi_N(x, x') = \frac{2N+1}{4\pi} \langle x, x' \rangle^N$$

where $x, x' \in X = S^3/\{\pm 1\} = \mathrm{SO}(3)$. Hence $P_N = Q_N$ and the Schwartz kernel is

$$P_N(z, z') = \frac{2N+1}{4\pi} \left|\langle x, x' \rangle\right|^{2N} \qquad \pi(x) = z, \pi(x') = z'.$$

If we identify $\mathbb{P}^1$ with the sphere $S^2 \subset \mathbb{R}^3$ in the standard way, this becomes

$$P_N(y, y') = \frac{2N+1}{4\pi} \left(\frac{1 + y \cdot y'}{2}\right)^N \qquad y, y' \in S^2$$

where $y \cdot y'$ denotes the scalar product in $\mathbb{R}^3$. Apart from this expression, which looks quite nice, one might try to express the operators P_N in terms of spherical harmonics on S^2. In fact this computation has already been carried out by Donaldson in [6, p. 612ff] (recall that $P_N = Q_N$ in this case). Since $L^2(S^2) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m$, where $\mathcal{H}_m$ is the space of degree k spherical harmonics, and the operators P_N are SO(3)-equivariant, $P_N(\mathcal{H}_m) \subset \mathcal{H}_m$ and P_N acts on $\mathcal{H}_m$ as multiplication by a scalar $\chi_{m,2N}$. The range of P_N is $\sum_{m=0}^{2N} \mathcal{H}_m$. One might expect (and the writer did hope) that P_N is just orthogonal projection onto its range. This is equivalent to $\chi_{m,2N} = 1$ for $m \leqslant 2N$, but Donaldson's computation shows this to be false. The only thing which is evident from Donaldson's formula is that $\chi_{m,2N} \to 1$ for $N \to \infty$, which is the equivalent to Theorem 5 in the case at hand.

Bibliography

[1] Bochnak (J.), Coste (M.), and Roy (M.-F.). — Real algebraic geometry, volume 36 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin (1998). Translated from the 1987 French original, Revised by the authors.

[2] Bordemann (M.), Meinrenken (E.), and Schlichenmaier (M.). — Toeplitz quantization of Kähler manifolds and gl(N), $N \to \infty$ limits. Comm. Math. Phys., 165(2), p. 281-296 (1994).

[3] Bouche (T.). — Convergence de la métrique de Fubini-Study d'un fibré linéaire positif. Ann. Inst. Fourier (Grenoble), 40(1), p. 117-130 (1990).

[4] Catlin (D.). — The Bergman kernel and a theorem of Tian. In Analysis and geometry in several complex variables (Katata, 1997), Trends Math., pages 1-23. Birkhäuser Boston, Boston, MA (1999).

[5] Charles. — Berezin-Toeplitz operators, a semi-classical approach. Comm. Math. Phys., 239(1-2), p. 1-28 (2003).

[6] Donaldson (S. K.). — Some numerical results in complex differential geometry. Pure Appl. Math. Q., 5(2, Special Issue: In honor of Friedrich Hirzebruch. Part 1), p. 571-618 (2009).

[7] Fine (J.). — Calabi flow and projective embeddings. J. Differential Geom., 84(3), p. 489-523 (2010). Appendix written by Kefeng Liu & Xiaonan Ma.

[8] Hua (L. K.). — Harmonic analysis of functions of several complex variables in the classical domains. Translated from the Russian by Leo Ebner and Adam Korányi. American Mathematical Society, Providence, R.I. (1963).

[9] Kirwan (F. C.). — Cohomology of quotients in symplectic and algebraic geometry, volume 31 of Mathematical Notes. Princeton University Press, Princeton, NJ (1984).

[10] Krantz (S. G.). — Function theory of several complex variables. The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, second edition (1992).

[11] Krantz (S. G.). — On a construction of L. Hua for positive reproducing kernels. Michigan Math. J., 59(1), p. 211-230 (2010).

[12] Ma (X.) and Marinescu (G.). — Holomorphic Morse inequalities and Bergman kernels, volume 254 of Progress in Mathematics. Birkhäuser Verlag, Basel (2007).

[13] Ma (X.) and Marinescu (G.). — Berezin-Toeplitz quantization on Kähler manifolds. arXiv:math.DG/1009.4405 (2010). Preprint.

[14] Schlichenmaier (M.). — Zwei Anwendungen algebraisch-geometrischer Methoden in der theoretischen Physik. 1996. Universität Mannheim. http://math.uni.lu/schlichenmaier/preprints/methoden.ps.gz.

[15] Shiffman (B.) and Zelditch (S.). — Asymptotics of almost holomorphic sections of ample line bundles on symplectic manifolds. J. Reine Angew. Math., 544, p. 181-222 (2002).

[16] Shiffman (B.) and Zelditch (S.). — Number variance of random zeros on complex manifolds. Geom. Funct. Anal., 18(4), p. 1422-1475 (2008).

[17] Tian (G.). — On a set of polarized Kähler metrics on algebraic manifolds. J. Differential Geom., 32(1), p. 99-130 (1990).

[18] Varolin (D.). — Geometry of Hermitian algebraic functions. Quotients of squared norms. Amer. J. Math., 130(2), p. 291-315 (2008).

[19] Zelditch (S.). — Szegő kernels and a theorem of Tian. Internat. Math. Res. Notices, 6, p. 317-331 (1998).