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## Twisted matings and equipotential gluings

XAVIER BUFF<sup>(1)</sup>, ADAM L. EPSTEIN<sup>(2)</sup>, SARAH KOCH<sup>(3)</sup>

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**ABSTRACT.** — One crucial tool for studying postcritically finite rational maps is Thurston’s topological characterization of rational maps. This theorem is proved by iterating a holomorphic endomorphism on a certain Teichmüller space. The graph of this endomorphism covers a correspondence on the level of moduli space. In favorable cases, this correspondence is the graph of a map, which can be used to study matings. We illustrate this by way of example: we study the mating of the basilica with itself.

**RÉSUMÉ.** — Un outil crucial pour l’étude des fractions rationnelles post-critiquement finies est la caractérisation topologique des fractions rationnelles due à Thurston. La démonstration de ce théorème repose sur l’itération d’un endomorphisme holomorphe d’un certain espace de Teichmüller. Le graphe de cet endomorphisme revêt une correspondance au niveau de l’espace des modules. Dans des cas favorables, cette correspondance est le graphe d’une application qui peut être utilisée pour étudier les accouplements. Nous illustrons ceci par un exemple : nous étudions l’auto-accouplement de la basilique.

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## Introduction

Our aim in this article is to present a worked example of a family of quadratic matings. We study the *twisted matings* of the *basilica* polynomial  $B : z \mapsto z^2 - 1$  with itself. We ultimately prove that for  $\theta = k/(2^n - 1) \notin \mathbb{Z}$ , the geometric twisted mating of angle  $-\theta$  of  $B^{\circ n}$  with  $B^{\circ n}$  exists and that the mating is otherwise obstructed (see Proposition 4.6). We also define *equipotential gluings*, a construction whose implementation is fundamental to the algorithms with which Arnaud Chéritat produces his superb still and moving images.

The organization is as follows. Section 1 recalls the standard definitions and constructions concerning moduli spaces, Teichmüller spaces, and universal curves over them. We review the formulation of Thurston's Theorem, and describe the mating operation in that context. In Section 2 we define twisted matings of polynomials, and in Section 3, we equipotential gluings. Following this general discussion, we consider the simplest nontrivial case in much greater detail: the basilica polynomial mated with itself in Section 4. We apply the Bartholdi-Nekrashevych recipe to obtain a skew-product dynamical system on the universal curve over moduli space. While this particular mating happens to be obstructed, equatorial twists of its iterates are realized. In Section 5 we consider a compactified universal curve, with attention to the dynamics at infinity, and finally in Section 6, we count the number of combinatorial classes of twisted basilica matings.

## 1. Preliminaries

### 1.1. Teichmüller theory

Let  $S$  be an oriented topological 2-sphere and  $\mathcal{P} \subset S$  be a finite set containing at least 3 points.

#### 1.1.1. The moduli space

The moduli space  $\mathcal{M}_{\mathcal{P}}$  is the set of equivalence classes of injective maps  $\phi : \mathcal{P} \rightarrow \mathbb{P}^1$  under the relation whereby maps  $\phi_1 : \mathcal{P} \rightarrow \mathbb{P}^1$  and  $\phi_2 : \mathcal{P} \rightarrow \mathbb{P}^1$  are equivalent when there is a Möbius transformation  $M : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  such that  $\phi_2 = M \circ \phi_1$ .

Choose three distinct points  $p_1, p_2$  and  $p_3$  in  $\mathcal{P}$ . In each equivalence class, there is a unique representative  $\phi : \mathcal{P} \rightarrow \mathbb{P}^1$  such that  $\phi(p_1) = 0$ ,  $\phi(p_2) = 1$  and  $\phi(p_3) = \infty$ . It follows that  $\mathcal{M}_{\mathcal{P}}$  may be identified with

$(\mathbb{C} - \{0, 1\})^{\mathcal{Q}} - \mathcal{N}_{\mathcal{Q}}$ , where  $\mathcal{Q} := \mathcal{P} - \{p_1, p_2, p_3\}$  and  $\mathcal{N}_{\mathcal{Q}} \subset \mathbb{C}^{\mathcal{Q}}$  consists of the non injective maps.

In this way, the set  $\mathcal{M}_{\mathcal{P}}$  acquires the structure of a smooth irreducible quasiprojective variety of dimension  $|\mathcal{P}| - 3$ . The structure is easily seen not to depend on the choice of distinct points  $p_1, p_2$  and  $p_3$  in  $\mathcal{P}$ .

*Example 1.1.* — If  $|\mathcal{P}| = 4$ , then  $\mathcal{M}_{\mathcal{P}}$  is identified with  $\mathbb{C} - \{0, 1\}$ . If  $|\mathcal{P}| = 5$ , then  $\mathcal{M}_{\mathcal{P}}$  is identified with  $\mathbb{C}^2$  minus the five lines defined by the equations  $x = 0, x = 1, y = 0, y = 1$  and  $x = y$ .

### 1.1.2. Teichmüller space

In our setting, the Teichmüller space  $\mathcal{T}_{\mathcal{P}}$  is the set of equivalence classes of orientation-preserving homeomorphisms  $\phi : S \rightarrow \mathbb{P}^1$  under the relation whereby homeomorphisms  $\phi_1 : S \rightarrow \mathbb{P}^1$  and  $\phi_2 : S \rightarrow \mathbb{P}^1$  are equivalent when there is a Möbius transformation  $M : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  such that

$$\phi_2|_{\mathcal{P}} = M \circ \phi_1|_{\mathcal{P}} \quad \text{and} \quad \phi_2 \text{ is isotopic to } M \circ \phi_1 \text{ relative to } \mathcal{P}.$$

The Teichmüller space  $\mathcal{T}_{\mathcal{P}}$  is canonically equipped with the structure of a  $\mathbb{C}$ -analytic manifold of dimension  $|\mathcal{P}| - 3$  so that the restriction  $\phi \mapsto \phi|_{\mathcal{P}}$  induces a universal covering map  $\pi : \mathcal{T}_{\mathcal{P}} \rightarrow \mathcal{M}_{\mathcal{P}}$ .

### 1.1.3. The universal curve over the moduli space

There are

- a  $\mathbb{C}$ -analytic manifold  $\mathfrak{M}_{\mathcal{P}}$ ,
- an analytic submersion  $\mathfrak{m} : \mathfrak{M}_{\mathcal{P}} \rightarrow \mathcal{M}_{\mathcal{P}}$  whose fibers are Riemann spheres,
- analytic sections  $(m_p : \mathcal{M}_{\mathcal{P}} \rightarrow \mathfrak{M}_{\mathcal{P}})_{p \in \mathcal{P}}$  with pairwise disjoint images,

so that for each  $x \in \mathcal{M}_{\mathcal{P}}$ , the injection

$$\mathcal{P} \ni p \mapsto m_p(x) \in \mathfrak{m}^{-1}(x)$$

represents  $x$ . The map  $\mathfrak{m} : \mathfrak{M}_{\mathcal{P}} \rightarrow \mathcal{M}_{\mathcal{P}}$  and sections  $(m_p : \mathcal{M}_{\mathcal{P}} \rightarrow \mathfrak{M}_{\mathcal{P}})_{p \in \mathcal{P}}$  are unique up to isomorphism. They are called a universal curve with universal sections over the moduli space. The word universal refers to the following property which in fact gives a characterization up to unique isomorphism: given a proper flat family of curves  $\mathfrak{s} : \mathfrak{U}_{\mathcal{P}, \mathcal{Q}} \rightarrow \Delta$  (see [6] for a definition), with distinct sections  $s_p : \Delta \rightarrow \mathfrak{U}_{\mathcal{P}, \mathcal{Q}}$  there exists a unique map

$\beta : \Delta \rightarrow \mathcal{M}_{\mathcal{P}}$ , such that the pullback by this map is canonically isomorphic to the given family. In particular, there is an analytic map  $\alpha : \mathfrak{U}_{\mathcal{P}, \mathcal{Q}} \rightarrow \mathfrak{M}_{\mathcal{P}}$  such that  $\mathfrak{m} \circ \alpha = \beta \circ s$ , the restriction  $\alpha : \mathfrak{s}^{-1}(x) \rightarrow \mathfrak{m}^{-1}(\beta(x))$  is an isomorphism for every  $x \in \Delta$  and  $m_p \circ \beta = \alpha \circ s_p$  for every  $p \in \mathcal{P}$ .

If  $\mathcal{M}_{\mathcal{P}}$  is identified with  $(\mathbb{C} - \{0, 1\})^{\mathcal{Q}} - \mathcal{N}_{\mathcal{Q}}$  via a choice of three distinct points  $p_1, p_2$  and  $p_3$  in  $\mathcal{P}$  as above, one may set  $\mathfrak{M}_{\mathcal{P}} := \mathcal{M}_{\mathcal{P}} \times \mathbb{P}^1$ . Let  $\mathfrak{m} : \mathfrak{M}_{\mathcal{P}} \rightarrow \mathcal{M}_{\mathcal{P}}$  be the projection to the first coordinate and consider the sections  $m_p : \mathcal{M}_{\mathcal{P}} \rightarrow \mathfrak{M}_{\mathcal{P}}$  defined at  $\phi \in \mathcal{M}_{\mathcal{P}}$  by

$$m_{p_1}(\phi) = (\phi, 0), \quad m_{p_2}(\phi) = (\phi, 1), \quad m_{p_3}(\phi) = (\phi, \infty)$$

and

$$m_q(\phi) = (\phi, \phi(q)) \quad \text{for } q \in \mathcal{Q}.$$

*Example 1.2.* — If  $\mathcal{P} = \{p_1, p_2, p_3, p_4\}$  and  $\mathcal{M}_{\mathcal{P}}$  is identified with  $\mathbb{C} - \{0, 1\}$ , the universal curve over moduli space is given by the projection

$$\mathfrak{m} : (\mathbb{C} - \{0, 1\}) \times \mathbb{P}^1 \ni (x, z) \mapsto x \in \mathbb{C} - \{0, 1\}$$

with sections

$$m_{p_1}(x) = (x, 0), \quad m_{p_2}(x) = (x, 1), \quad m_{p_3}(x) = (x, \infty) \quad \text{and} \quad m_{p_4}(x) = (x, x).$$

#### 1.1.4. The universal curve over the Teichmüller space

There is also a universal curve  $\mathfrak{t} : \mathfrak{T}_{\mathcal{P}} \rightarrow \mathcal{T}_{\mathcal{P}}$  over the Teichmüller space. This is obtained by considering the pullback of  $\pi : \mathcal{T}_{\mathcal{P}} \rightarrow \mathcal{M}_{\mathcal{P}}$  along  $\mathfrak{m} : \mathfrak{M}_{\mathcal{P}} \rightarrow \mathcal{M}_{\mathcal{P}}$ , that is

$$\mathfrak{T}_{\mathcal{P}} := \{(\tau, \mu) \in \mathcal{T}_{\mathcal{P}} \times \mathfrak{M}_{\mathcal{P}} \mid \pi(\tau) = \mathfrak{m}(\mu)\},$$

$\mathfrak{t} : \mathfrak{T}_{\mathcal{P}} \rightarrow \mathcal{T}_{\mathcal{P}}$  is the projection to the first coordinate and  $\varpi : \mathfrak{T}_{\mathcal{P}} \rightarrow \mathfrak{M}_{\mathcal{P}}$  is the projection to the second coordinate:

$$\begin{array}{ccc} \mathfrak{T}_{\mathcal{P}} & \xrightarrow{\mathfrak{t}} & \mathcal{T}_{\mathcal{P}} \\ \varpi \downarrow & & \downarrow \pi \\ \mathfrak{M}_{\mathcal{P}} & \xrightarrow{\mathfrak{m}} & \mathcal{M}_{\mathcal{P}}. \end{array}$$

The map  $\mathfrak{t} : \mathfrak{T}_{\mathcal{P}} \rightarrow \mathcal{T}_{\mathcal{P}}$  is a submersion with fiber over  $\tau \in \mathcal{T}_{\mathcal{P}}$  isomorphic to  $\mathbb{P}^1$ . For each  $p \in \mathcal{P}$ , there is a section  $t_p : \mathcal{T}_{\mathcal{P}} \rightarrow \mathfrak{T}_{\mathcal{P}}$  given by

$$t_p(\tau) = (\tau, m_p \circ \pi(\tau)).$$

The injection  $\mathcal{P} \ni p \mapsto t_p(\tau) \in \mathfrak{t}^{-1}(\tau)$  represents  $\pi(\tau)$  in  $\mathcal{M}_{\mathcal{P}}$ .

## 1.2. Thurston's theory

We remain in the setting where  $S$  is an oriented topological 2-sphere, and we let  $f : S \rightarrow S$  be an orientation-preserving ramified self-covering map.

### 1.2.1. Combinatorial equivalence

The map  $f$  is a *Thurston map* if it is postcritically finite, that is, if all the points in the critical set  $\Omega_f$  have finite forward orbits, or equivalently if the *postcritical set*

$$\mathcal{P}_f := \bigcup_{n>0} f^{\circ n}(\Omega_f)$$

is finite. Two Thurston maps  $f_1 : S_1 \rightarrow S_1$  and  $f_2 : S_2 \rightarrow S_2$  are *combinatorially equivalent* if there are orientation-preserving homeomorphisms  $\phi, \psi : S_1 \rightarrow S_2$  so that:

- the following diagram commutes,

$$\begin{array}{ccc} (S_1, \mathcal{P}_{f_1}) & \xrightarrow{\psi} & (S_2, \mathcal{P}_{f_2}) \\ f_1 \downarrow & & \downarrow f_2 \\ (S_1, \mathcal{P}_{f_1}) & \xrightarrow{\phi} & (S_2, \mathcal{P}_{f_2}) \end{array}$$

- $\phi$  and  $\psi$  agree on  $\mathcal{P}_{f_1}$ ,
- $\phi(\mathcal{P}_{f_1}) = \psi(\mathcal{P}_{f_1}) = \mathcal{P}_{f_2}$  and
- $\phi$  is isotopic to  $\psi$  relative to  $\mathcal{P}_{f_1}$ .

### 1.2.2. Thurston's Theorem

Thurston's topological characterization of rational maps provides a purely topological criterion under which a Thurston map  $f : S \rightarrow S$  is combinatorially equivalent to a critically finite rational map  $F : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ . If  $f$  is not combinatorially equivalent to a rational map, then  $f$  is *obstructed*. Thurston's criterion is formulated in terms of curve systems on  $S - \mathcal{P}_f$  which are "invariant" under pullback by  $f$ .

**THEOREM 1.3 (Thurston).** — *A Thurston map  $f$  with hyperbolic orbifold<sup>1</sup> is combinatorially equivalent to a rational map if and only if there are no obstructing multicurves. In that case, the rational map is unique up to conjugation by a Möbius transformation.*

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<sup>(1)</sup> For more information about the orbifold associated to a Thurston map, see [3].

A *multicurve*  $\Gamma := \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ , is a set of simple, closed, disjoint, nonisotopic, nonperipheral curves in  $S - \mathcal{P}_f$ . A multicurve is *f-stable* if for every  $\gamma \in \Gamma$ , every nonperipheral component of  $f^{-1}(\gamma)$  is isotopic in  $S - \mathcal{P}_f$  to a curve in  $\Gamma$ .

An obstructing multicurve is a special kind of *f-stable* multicurve; we will restrict our attention to obstructing multicurves which are *Levy cycles*. Let  $\Gamma$  be an *f-stable* multicurve, and let  $\Lambda := \{\gamma_0, \dots, \gamma_k = \gamma_0\} \subseteq \Gamma$  be such that for each  $i = 0, \dots, k - 1$ ,  $\gamma_i$  is isotopic relative to  $\mathcal{P}_f$  to at least one component  $\gamma'$  of  $f^{-1}(\gamma_{i+1})$  and  $f : \gamma' \rightarrow \gamma_{i+1}$  has degree 1. Then  $\Lambda$  is called a *Levy cycle*. A Thurston map with a *Levy cycle* is necessarily obstructed, see chapter 9 of [5].

### 1.2.3. The Thurston endomorphism

A Thurston map  $f : S \rightarrow S$  with  $|\mathcal{P}_f| \geq 3$  induces a holomorphic endomorphism  $\sigma_f : \mathcal{T}_{\mathcal{P}_f} \rightarrow \mathcal{T}_{\mathcal{P}_f}$  as follows. Choose three distinct points  $p_1, p_2$  and  $p_3$  in  $\mathcal{P}_f$ . Let  $\tau \in \mathcal{T}_{\mathcal{P}_f}$  be represented by a homeomorphism  $\phi : S \rightarrow \mathbb{P}^1$  sending  $p_1$  to 0,  $p_2$  to 1 and  $p_3$  to  $\infty$ . There is a homeomorphism  $\psi : S \rightarrow \mathbb{P}^1$  sending  $p_1$  to 0,  $p_2$  to 1 and  $p_3$  to  $\infty$  and a rational map  $F : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  such that the following diagram commutes:

$$\begin{array}{ccc} (S, \mathcal{P}_f) & \xrightarrow{\psi} & (\mathbb{P}^1, \psi(\mathcal{P}_f)) \\ f \downarrow & & \downarrow F \\ (S, \mathcal{P}_f) & \xrightarrow{\phi} & (\mathbb{P}^1, \phi(\mathcal{P}_f)). \end{array} \tag{1.1}$$

The homeomorphism  $\psi$  and rational map  $F$  are uniquely determined by  $\phi$ . Moreover, since  $\mathcal{P}_f$  contains the critical value set of  $f$ , if  $\phi_1 : S \rightarrow \mathbb{P}^1$  is isotopic to  $\phi_2 : S \rightarrow \mathbb{P}^1$  relative to  $\mathcal{P}_f$ , then  $\psi_1$  is isotopic to  $\psi_2$  relative to  $f^{-1}(\mathcal{P}_f) \supset \mathcal{P}_f$  and  $F_1 = F_2$ . As a consequence,

- the class of  $\psi$  in  $\mathcal{T}_{\mathcal{P}_f}$  is uniquely determined by  $\tau \in \mathcal{T}_{\mathcal{P}_f}$  (it does not even depend on the choice of points  $p_1, p_2, p_3$ ) ; we denote by  $\sigma_f(\tau)$  the class of  $\psi$  in  $\mathcal{T}_{\mathcal{P}_f}$ ;
- the rational map  $F_\tau := F$  only depends on  $\tau$  and the choice of points  $p_1, p_2, p_3$ .

The map

$$\mathcal{T}_{\mathcal{P}_f} \ni \tau \mapsto \sigma_f(\tau) \in \mathcal{T}_{\mathcal{P}_f}$$

is called the *Thurston endomorphism* associated to  $f$ . This map is holomorphic on  $\mathcal{T}_{\mathcal{P}_f}$ , and weakly-contracting for the Teichmüller metric (see [3] for details). In the context of Theorem 1.3, a Thurston map  $f : S \rightarrow S$  is

combinatorially equivalent to a rational map  $F : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  if and only if  $\sigma_f$  has a fixed point in  $\mathcal{T}_{\mathcal{P}_f}$ .

### 1.2.4. A map between curves over Teichmüller space

Recall that  $\mathfrak{t} : \mathfrak{T}_{\mathcal{P}_f} \rightarrow \mathcal{T}_{\mathcal{P}_f}$  is the universal curve over Teichmüller space. According to the previous discussion, the construction of the Thurston endomorphism  $\sigma_f : \mathcal{T}_{\mathcal{P}_f} \rightarrow \mathcal{T}_{\mathcal{P}_f}$  comes with a family of rational maps  $F_\tau : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ , whose domains are marked with  $\psi : S \rightarrow \mathbb{P}^1$ , and whose ranges are marked with  $\phi : S \rightarrow \mathbb{P}^1$ . So, each  $F_\tau : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  induces a holomorphic map

$$\mathfrak{F}_\tau : \mathfrak{S}_v \rightarrow \mathfrak{S}_\tau \quad \text{with} \quad v := \sigma_f(\tau), \quad \mathfrak{S}_\tau := \mathfrak{t}^{-1}(\tau) \quad \text{and} \quad \mathfrak{S}_v := \mathfrak{t}^{-1}(v).$$

Such a family does not necessarily arise from an endomorphism of  $\mathfrak{T}_{\mathcal{P}_f}$  because  $\mathfrak{S}_\tau$  is the range, not the domain: if  $v = \sigma_f(\tau_1) = \sigma_f(\tau_2)$  with  $\tau_1 \neq \tau_2$ , there are distinct maps  $\mathfrak{F}_{\tau_1} : \mathfrak{S}_v \rightarrow \mathfrak{S}_{\tau_1}$  and  $\mathfrak{F}_{\tau_2} : \mathfrak{S}_v \rightarrow \mathfrak{S}_{\tau_2}$ . However there is a map  $\mathfrak{F} : \mathfrak{U}_{\mathcal{P}_f} \rightarrow \mathfrak{T}_{\mathcal{P}_f}$  where

$$\begin{aligned} \mathfrak{U}_{\mathcal{P}_f} &:= \{(\tau, \xi) \in \mathcal{T}_{\mathcal{P}_f} \times \mathfrak{T}_{\mathcal{P}_f} \mid \mathfrak{t}(\xi) = \sigma_f(\tau)\} \\ &\approx \{(\tau, v, \mu) \in \mathcal{T}_{\mathcal{P}_f} \times \mathcal{T}_{\mathcal{P}_f} \times \mathfrak{M}_{\mathcal{P}_f} \mid v = \sigma_f(\tau) \text{ and } \mathfrak{m}(\mu) = \pi(v)\} \end{aligned}$$

and  $\mathfrak{F}(\tau, \xi) := \mathfrak{F}_\tau(\xi)$ .

By construction, we have the following commutative diagram:

$$\begin{array}{ccccccccc} \mathfrak{M}_{\mathcal{P}_f} & \xleftarrow{\varpi} & \mathfrak{T}_{\mathcal{P}_f} & \xleftarrow{\mathfrak{F}} & \mathfrak{U}_{\mathcal{P}_f} & \xrightarrow{\mathfrak{p}_2} & \mathfrak{T}_{\mathcal{P}_f} & \xrightarrow{\varpi} & \mathfrak{M}_{\mathcal{P}_f} \\ \mathfrak{m} \downarrow & & \mathfrak{t} \downarrow & & \mathfrak{p}_1 \downarrow & & \mathfrak{t} \downarrow & & \downarrow \mathfrak{m} \\ \mathcal{M}_{\mathcal{P}_f} & \xleftarrow{\pi} & \mathcal{T}_{\mathcal{P}_f} & \xleftarrow{\text{id}} & \mathcal{T}_{\mathcal{P}_f} & \xrightarrow{\sigma_f} & \mathcal{T}_{\mathcal{P}_f} & \xrightarrow{\pi} & \mathcal{M}_{\mathcal{P}_f} \end{array} \quad (1.2)$$

where  $\mathfrak{p}_1 : \mathfrak{U}_{\mathcal{P}_f} \rightarrow \mathcal{T}_{\mathcal{P}_f}$  is the projection  $(\tau, \xi) \mapsto \tau$  and  $\mathfrak{p}_2 : \mathfrak{U}_{\mathcal{P}_f} \rightarrow \mathfrak{T}_{\mathcal{P}_f}$  is the projection  $(\tau, \xi) \mapsto \xi$ .

In Section 4, we shall further study this diagram in the context of the mating of the basilica polynomial  $z \mapsto z^2 - 1$  with itself.

## 1.3. Matings

### 1.3.1. Polynomials

All polynomials  $P : \mathbb{C} \rightarrow \mathbb{C}$  considered in this article will be monic of degree  $d \geq 2$ . They will be postcritically finite and hyperbolic, that is, the orbit of any critical point eventually lands on a superattracting cycle.



The filled-in Julia set of  $P$  is the set

$$K(P) := \{z \in \mathbb{C} \mid n \mapsto P^{\circ n}(z) \text{ is bounded}\}.$$

The Julia set  $J(P)$  is the boundary of  $K(P)$ .

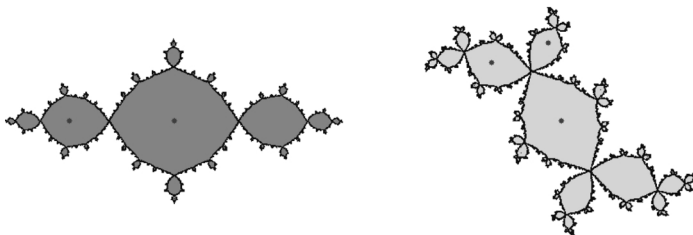


Figure 1. — Left: the filled-in Julia set of the *basilica* polynomial  $z \mapsto z^2 - 1$ . The critical point is periodic of period 2. Right: the filled-in Julia set of the *rabbit* polynomial  $z \mapsto z^2 + c_0$ , where  $c_0$  is chosen so that  $\text{Im}(c_0) > 0$  and the critical point is periodic of period 3. The forward orbit of the critical point is marked in both pictures.

When  $P$  is postcritically finite,  $K(P)$  and  $J(P)$  are connected.<sup>2</sup> In this situation, the complement of  $K(P)$  is isomorphic to  $\mathbb{C} - \overline{\mathbb{D}}$ , and there is an isomorphism  $\mathbb{C} - \overline{\mathbb{D}} \rightarrow \mathbb{C} - K(P)$  conjugating  $z \mapsto z^d$  to  $P$ . Since  $P$  is monic, there is a unique such *Böttcher coordinate*  $\text{böt} : \mathbb{C} - \overline{\mathbb{D}} \rightarrow \mathbb{C} - K(P)$  tangent to the identity at infinity.

If  $\theta \in \mathbb{R}/\mathbb{Z}$ , the external ray  $\mathcal{R}_P(\theta)$  of angle  $\theta$  is the set of points of the form  $\text{böt}(\rho e^{2i\pi\theta})$  with  $\rho > 1$ . The polynomial  $P$  sends the external ray of angle  $\theta$  to the external ray of angle  $d \cdot \theta$ .

### 1.3.2. Formal mating

We add to the complex plane  $\mathbb{C}$  the *circle at infinity* which is symbolically denoted  $\{\infty \cdot e^{2i\pi\theta} \mid \theta \in \mathbb{R}/\mathbb{Z}\}$ . We define

$$\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty \cdot e^{2i\pi\theta} \mid \theta \in \mathbb{R}/\mathbb{Z}\},$$

which is homeomorphic to a closed disk with the standard topology. A monic polynomial  $P : \mathbb{C} \rightarrow \mathbb{C}$  of degree  $d$  extends to  $\tilde{\mathbb{C}}$  by setting

$$P(\infty \cdot e^{2i\pi\theta}) = \infty \cdot e^{2i\pi d\theta}.$$

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<sup>(2)</sup> The filled-in Julia set of a polynomial is connected if and only if all the critical points have bounded orbits.

Let  $\tilde{\mathbb{C}}_P$  and  $\tilde{\mathbb{C}}_Q$  be two copies of  $\tilde{\mathbb{C}}$  and assume  $P : \tilde{\mathbb{C}}_P \rightarrow \tilde{\mathbb{C}}_P$  and  $Q : \tilde{\mathbb{C}}_Q \rightarrow \tilde{\mathbb{C}}_Q$  are monic postcritically finite polynomials with common degree  $d \geq 2$ . Let  $\sim$  be the equivalence relation in  $\tilde{\mathbb{C}}_P \sqcup \tilde{\mathbb{C}}_Q$  (disjoint union) which, for  $\theta \in \mathbb{R}/\mathbb{Z}$ , identifies  $\infty \cdot e^{2i\pi\theta}$  in  $\tilde{\mathbb{C}}_P$  to  $\infty \cdot e^{-2i\pi\theta}$  in  $\tilde{\mathbb{C}}_Q$ . The quotient by  $\sim$  is topologically an oriented 2-sphere. The formal mating  $P \uplus Q$  of  $P$  and  $Q$  is the orientation-preserving branched self-covering which is given by  $P$  on the image of  $\tilde{\mathbb{C}}_P$  and by  $Q$  on the image of  $\tilde{\mathbb{C}}_Q$ .

One can give an equivalent definition as follows (see [12]). Let  $S$  be the unit sphere in  $\mathbb{C} \times \mathbb{R}$ . If  $P : \mathbb{C} \rightarrow \mathbb{C}$  and  $Q : \mathbb{C} \rightarrow \mathbb{C}$  are two monic polynomials of the same degree  $d \geq 2$ , the formal mating of  $P$  and  $Q$  is the ramified covering  $f = P \uplus Q : S \rightarrow S$  obtained as follows.

We identify the dynamical plane of  $P$  to the upper hemisphere  $H^+$  of  $S$  and the dynamical plane of  $Q$  to the lower hemisphere  $H^-$  of  $S$  via the gnomonic projections:

$$\nu_P : \mathbb{C} \rightarrow H^+ \quad \text{and} \quad \nu_Q : \mathbb{C} \rightarrow H^-$$

given by

$$\nu_P(z) = \frac{(z, 1)}{\|(z, 1)\|} = \frac{(z, 1)}{\sqrt{|z|^2 + 1}} \quad \text{and} \quad \nu_Q(z) = \frac{(\bar{z}, -1)}{\|(\bar{z}, -1)\|} = \frac{(\bar{z}, -1)}{\sqrt{|z|^2 + 1}}.$$

Since  $P$  and  $Q$  are monic polynomials of degree  $d$ , the map  $\nu_P \circ P \circ \nu_P^{-1}$  defined on the upper hemisphere and  $\nu_Q \circ Q \circ \nu_Q^{-1}$  defined in the lower hemisphere extend continuously to the equator of  $S$  by

$$(e^{2i\pi\theta}, 0) \mapsto (e^{2i\pi d\theta}, 0).$$

The two maps fit together so as to yield a ramified covering map  $f : S \rightarrow S$ , which is called the formal mating  $P \uplus Q$  of  $P$  and  $Q$ .

### 1.3.3. Geometric mating

Let us now consider the smallest equivalence relation  $\sim_{\text{ray}}$  on  $S$  such that for all  $\theta \in \mathbb{R}/\mathbb{Z}$ ,

- points in the closure of  $\nu_P(\mathcal{R}_P(\theta))$  are in the same equivalence class, and
- points in the closure of  $\nu_Q(\mathcal{R}_Q(\theta))$  are in the same equivalence class.

In particular, for all  $\theta \in \mathbb{R}/\mathbb{Z}$ ,  $\nu_P(\mathcal{R}_P(\theta))$  and  $\nu_Q(\mathcal{R}_Q(-\theta))$  are in the same equivalence class since the closures of these sets intersect at the point  $(e^{2i\pi\theta}, 0)$  on the equator of  $S$ .

We say that a rational map  $F : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  is a geometric mating of  $P$  and  $Q$  if

- (1) the quotient space  $S/\sim_{\text{ray}}$  is homeomorphic to  $S$  (which will have a natural orientation), and
- (2) the formal mating  $P \uplus Q$  induces a map  $S/\sim_{\text{ray}} \rightarrow S/\sim_{\text{ray}}$  which is topologically conjugate to  $F : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  via an orientation preserving homeomorphism.

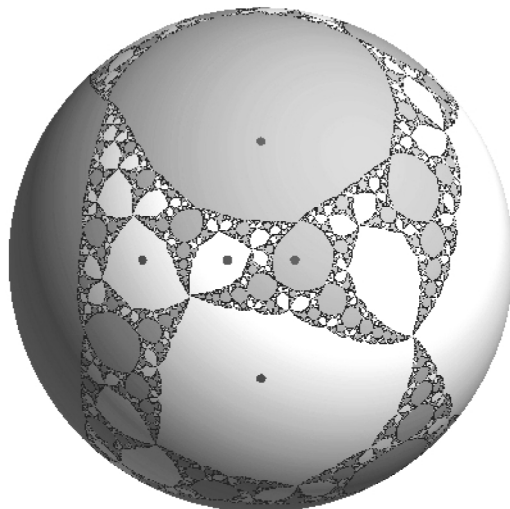


Figure 2. — The rational map  $F : z \mapsto (z^2 - e^{-2i\pi/3})/(z^2 - 1)$  is a geometric mating of the basilica polynomial and the rabbit polynomial (see Figure 1). The rational map  $F$  has a superattracting cycle of period 3 (the basin of which is colored in light grey), and a superattracting cycle of period 2 (the basin of which is colored in dark grey).

#### 1.3.4. The Rees Theorem

The following theorem of Rees in [13] relates the two mating criteria above to Theorem 1.3.

**THEOREM 1.4 (Rees).** — *Assume  $P : \mathbb{C} \rightarrow \mathbb{C}$  and  $Q : \mathbb{C} \rightarrow \mathbb{C}$  are two postcritically finite hyperbolic polynomials and  $F : \mathbb{P} \rightarrow \mathbb{P}$  is a rational map. The formal mating  $P \uplus Q$  is combinatorially equivalent to  $F$  if and only if  $F$  is a geometric mating of  $P$  and  $Q$ .*

*Remark 1.5.* — A similar result also holds in the case  $P$  and  $Q$  are post-critically finite polynomials, not necessarily hyperbolic; see [15] for an appropriate modification of the formal mating.

## 2. Twisted matings

### 2.1. Definition and first property

If  $P$  is a monic polynomial of degree  $d \geq 2$ , then the complex polynomial  $T(P) : \mathbb{C} \rightarrow \mathbb{C}$  defined by

$$T(P)(z) = e^{-2i\pi/(d-1)}P(e^{2i\pi/(d-1)}z)$$

is also monic. The filled Julia set of  $T(P)$  is the image of the Julia set of  $P$  by the rotation of angle  $-1/(d-1)$  turns, centered at 0. The external ray of argument  $\theta$  of  $T(P)$  is the image of the external ray of argument  $\theta + 1/(d-1)$  of  $P$ .

**DEFINITION 2.1.** — *If  $P$  and  $Q$  are monic polynomials of degree  $d$ , the twisted mating of angle  $k/(d-1)$  of  $P$  and  $Q$  is the mating (formal or geometric depending on the case) of  $P$  with  $T^{\circ k}(Q)$ .*

This corresponds to constructing the formal mating as follows: consider

$$P : \tilde{\mathbb{C}}_P \rightarrow \tilde{\mathbb{C}}_P \quad \text{and} \quad Q : \tilde{\mathbb{C}}_Q \rightarrow \tilde{\mathbb{C}}_Q,$$

and glue  $\tilde{\mathbb{C}}_P$  to  $\tilde{\mathbb{C}}_Q$  by identifying  $\infty \cdot e^{2i\pi\theta_1}$  in  $\tilde{\mathbb{C}}_P$  to  $\infty \cdot e^{2i\pi\theta_2}$  in  $\tilde{\mathbb{C}}_Q$  if and only if  $\theta_1 + \theta_2 = -k/(d-1)$ .

**PROPOSITION 2.2.** — *Assume  $P$  and  $Q$  are two monic polynomials of degree  $d \geq 2$ , which are critically finite. Then,  $P \uplus T(Q)$  is combinatorially equivalent to  $D \circ (P \uplus Q)$ , where  $D : S \rightarrow S$  is a positive Dehn twist around the equator of  $S$  relative to the postcritical set of  $P \uplus Q$ .*

*Proof.* — Set  $f := P \uplus Q$  and  $g := P \uplus T(Q)$ . Choose  $1 < r < \infty$ , and define  $A \subset \tilde{\mathbb{C}}_P - K(P)$  to be the image of the annulus  $\{z \in \mathbb{C} ; |z| > r\}$  by the Böttcher coordinate  $\text{böt}$  of  $P$  and  $A' \subset A$  to be the image of the annulus  $\{z \in \mathbb{C} ; |z| > r^d\}$ . Let

$$h : [r^d, +\infty] \rightarrow [0, 1]$$

be a continuous increasing function such that  $h(r^d) = 0$  and  $h(+\infty) = 1$ . Then, define  $\phi_P : A' \rightarrow A'$  and  $\psi_P : A \rightarrow A$  by

$$\phi_P \circ \text{böt}(\rho e^{2i\pi\theta}) = \text{böt}\left(\rho e^{2i\pi(\theta+h(\rho)/(d-1))}\right)$$

for  $\rho > r^d$  and  $\theta \in \mathbb{R}/\mathbb{Z}$  and by

$$\psi_P \circ \text{böt}(\rho e^{2i\pi\theta}) = \text{böt}\left(\rho e^{2i\pi(\theta+h(\rho^d)/(d-1))}\right)$$

for  $\rho > r$  and  $\theta \in \mathbb{R}/\mathbb{Z}$ . The maps  $\phi_P$  and  $\psi_P$  extend to self-homeomorphisms of  $\widetilde{\mathbb{C}}_P$  by the identity on  $\mathbb{C}_P - A'$  and  $\mathbb{C}_P - A$  respectively and by

$$\phi_P(\infty \cdot e^{2i\pi\theta}) = \infty \cdot e^{2i\pi(\theta+1/(d-1))}$$

on the circle at infinity. They induce self-homeomorphisms of the upper hemisphere of  $S$  via the gnomonic projection  $\nu_P$ , which themselves extend to self-homeomorphisms of  $S$  by

$$\phi \circ \nu_Q(z) = \psi \circ \nu_Q(z) = \nu_Q(e^{-2i\pi/(d-1)}z)$$

on the lower hemisphere of  $S$ . The homeomorphisms  $\phi$  and  $\psi$  are isotopic relative to  $\mathcal{P}_f$ .

Let us now consider the Dehn twist  $D : S - \mathcal{P}_f \rightarrow S - \mathcal{P}_f$  defined on  $\nu_P(A')$  by

$$D \circ \nu_P \circ \text{böt}(\rho e^{2i\pi\theta}) = \nu_P \circ \text{böt}\left(\rho e^{2i\pi(\theta+h(\rho))}\right)$$

for  $\rho > r^d$  and  $\theta \in \mathbb{R}/\mathbb{Z}$ , extended by the identity outside  $\nu_P(A')$ .

For  $z \in \nu_P(A)$ , we have  $\phi \circ D \circ f = g \circ \psi$ . Indeed, a point  $z \in \nu_P(A)$  is of the form  $z = \nu_P \circ \text{böt}(\rho e^{2i\pi\theta})$ . We say that  $z$  has Böttcher coordinates  $(\rho, \theta)$ . Then,  $f(z)$  has Böttcher coordinates  $(\rho^d, d\theta)$ . Thus,  $D \circ f(z)$  has Böttcher coordinates  $(\rho^d, d\theta + h(\rho^d))$ . Finally,  $\phi \circ D \circ f(z)$  has Böttcher coordinates

$$\left(\rho^d, d\theta + h(\rho^d) + \frac{h(\rho^d)}{d-1}\right) = \left(\rho^d, d\theta + \frac{d}{d-1}h(\rho^d)\right).$$

Now, the point  $\psi(z)$  has Böttcher coordinates  $(\rho, \theta + h(\rho^d)/(d-1))$ . Thus, the point  $g \circ \psi(z)$  has Böttcher coordinates

$$\left(\rho^d, d \cdot \left(\theta + \frac{h(\rho^d)}{d-1}\right)\right) = \left(\rho^d, d\theta + \frac{d}{d-1}h(\rho^d)\right).$$

A similar (and easier) proof shows that the same equality holds outside  $\nu_P(A')$ : we have the following commutative diagram:

$$\begin{array}{ccccc} (S, \mathcal{P}_f) & \xrightarrow{f} & (S, \mathcal{P}_f) & \xrightarrow{D} & (S, \mathcal{P}_f) \\ \psi \downarrow & & & & \downarrow \phi \\ (S, \mathcal{P}_g) & \xrightarrow{g} & & \xrightarrow{g} & (S, \mathcal{P}_g). \end{array}$$

The homeomorphisms  $\phi$  and  $\psi$  give a combinatorial equivalence between  $g$  and  $D \circ f$ .  $\square$

### 3. Equipotential gluing

Let  $P : \mathbb{C} \rightarrow \mathbb{C}$  and  $Q : \mathbb{C} \rightarrow \mathbb{C}$  be monic polynomials of degree  $d \geq 2$  with connected Julia sets (not necessarily postcritically finite). Let  $h_P : \mathbb{C} \rightarrow [0, +\infty)$  be the Green's function associated to  $P$ :

$$h_P(z) := \lim_{n \rightarrow +\infty} \frac{1}{d^n} \max(0, \log |P^{\circ n}(z)|),$$

and  $h_Q : \mathbb{C} \rightarrow [0, +\infty]$  the Green's function associated to  $Q$ . Let  $U_P$  and  $U_Q$  be the subsets of  $\mathbb{D} \times \mathbb{C}$  defined by

$$U_P := \{(t, z) \mid h_P(z) < -\log |t|\} \quad \text{and} \quad U_Q := \{(t, w) \mid h_Q(w) < -\log |t|\}.$$

Let  $V \subset \mathbb{C}^3$  be the complex surface defined by

$$V := \{(t, z, w) \in \mathbb{C}^3 \mid zw = t, |t| < 1, |z| < 1, |w| < 1\}.$$

Let  $\mathfrak{U}_{P,Q}$  be the analytic manifold obtained by gluing  $V$ ,  $U_P$  and  $U_Q$ , in the following way (see Figure 3):

- a point  $(t, z, w)$  in  $V$  with  $z \neq 0$  is identified with the point  $(t, \text{böt}_P(1/z))$  in  $U_P$  and
- a point  $(t, z, w)$  in  $V$  with  $w \neq 0$  is identified with the point  $(t, \text{böt}_Q(1/w))$  in  $U_Q$ .

Note that automatically, if  $t \neq 0$  and  $(t, z, w) \in V$ , then the points

$$(t, \text{böt}_P(1/z)) \in U_P \quad \text{and} \quad (t, \text{böt}_Q(1/w)) \in U_Q$$

are identified.

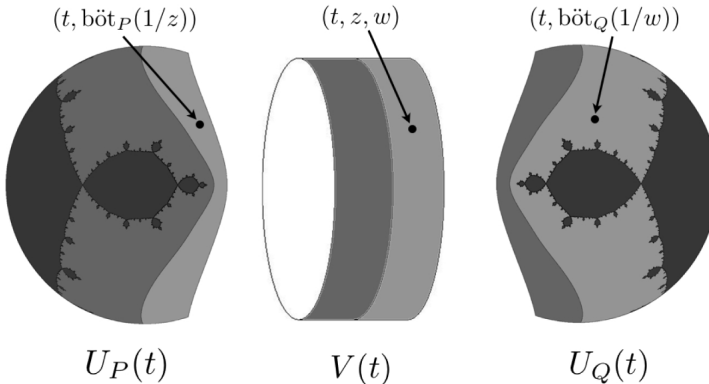


Figure 3. — We obtain  $\Sigma_t$  by gluing  $U_P(t) := U_P \cap (\{t\} \times \mathbb{C})$  and  $U_Q(t) := U_Q \cap (\{t\} \times \mathbb{C})$  along the annulus  $V(t) := V \cap (\{t\} \times \mathbb{D} \times \mathbb{D})$ .

Let  $\mathfrak{s} : \mathfrak{U}_{P,Q} \rightarrow \mathbb{D}$  be the map given by  $\mathfrak{s}(t, z) := t$  on  $U_P$ ,  $\mathfrak{s}(t, w) := t$  on  $U_Q$  and  $\mathfrak{s}(t, z, w) := t$  on  $V$ . According to the characterization given in [6] Definition 3.1, this is a proper flat family of curves. The fiber  $\Sigma_t$  of this map over a point  $t \in \mathbb{D} - \{0\}$  is a Riemann sphere, whereas the fiber  $\Sigma_0$  over 0 is singular: it is the union of two Riemann spheres intersecting at an ordinary double point.

It is important to observe that there is a skew-product  $S_{P,Q} : \mathfrak{U}_{P,Q} \rightarrow \mathfrak{U}_{P,Q}$  which is defined by

- $(t, z) \mapsto (t^d, P(z))$  on  $U_P$ ,
- $(t, w) \mapsto (t^d, Q(w))$  on  $U_Q$  and
- $(t, z, w) \mapsto (t^d, z^d, w^d)$  on  $V$ .

and satisfies the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{U}_{P,Q} & \xrightarrow{S_{P,Q}} & \mathfrak{U}_{P,Q} \\ \mathfrak{s} \downarrow & & \downarrow \mathfrak{s} \\ \mathbb{D} & \xrightarrow{t \mapsto t^d} & \mathbb{D} \end{array}$$

Each component of  $\Sigma_0$  is fixed by  $S_{P,Q}$ . The restriction of  $S_{P,Q}$  to one of them is conjugate to  $P$  and the restriction to the other is conjugate to  $Q$ . Under iteration of  $S_{P,Q}$ , the orbit of every point in  $\mathfrak{U}_{P,Q}$  converges to  $\Sigma_0$ . The fiber  $\Sigma_t$  lying above  $t \in \mathbb{D} - \{0\}$  contains an isomorphic copy of  $K(P)$  and an isomorphic copy of  $K(Q)$ , separated by an annulus of modulus  $-\log |t|$ . In  $\Sigma_t$ , points in the external ray  $\mathcal{R}_P(\theta_1)$  are glued to points in the external ray  $\mathcal{R}_Q(\theta_2)$  if and only if

$$\theta_1 + \theta_2 = -\frac{1}{2\pi} \arg(t).$$

The skew-product  $S_{P,Q}$  sends the fiber  $\Sigma_t$  to the fiber  $\Sigma_{t^d}$ . The copy of  $K(P)$  in  $\Sigma_t$  is mapped to the copy of  $K(P)$  in  $\Sigma_{t^d}$  and the copy of  $K(Q)$  in  $\Sigma_t$  is mapped to the copy of  $K(Q)$  in  $\Sigma_{t^d}$ .

#### 4. Twisted matings of basilicas

From now on, we will restrict to a particular case:  $P = Q = B$ , where the polynomial  $B$  is given by  $z \mapsto z^2 - 1$ ; the critical point 0 is periodic of period 2, and  $f = B \uplus B$  is the formal mating of this polynomial with itself. The postcritical set  $\mathcal{P}_f$  has cardinality 4: there are two critical points  $p_0$  and  $q_0$  (each critical point corresponds to one of the basilica polynomials  $B$ ), and two associated critical values  $p_1 = f(p_0)$  and  $q_1 = f(q_0)$ ;  $f$  has the following ramification portrait.

$$\begin{array}{ccc}
 p_0 & \xrightarrow{2} & p_1 \\
 & \searrow & \swarrow \\
 & & 
 \end{array}
 \quad
 \begin{array}{ccc}
 q_0 & \xrightarrow{2} & q_1 \\
 & \searrow & \swarrow \\
 & & 
 \end{array}$$

#### 4.1. Obstructed mating

PROPOSITION 4.1. — *No rational map is a geometric mating of  $B$  and  $B$ .*

*Proof.* — We propose two independent proofs of this result. The first is algebraic: if there were such a geometric mating, it would be a rational map of degree 2 with two superattracting cycles of period 2, one from each basilica. But a rational map of degree 2 is entitled to at most one periodic cycle of period 2, superattracting or otherwise.

The second relies on Theorem 1.4. If there were a rational map  $F$  which was a geometric mating of  $B$  with  $B$ , then  $F$  would be combinatorially equivalent to  $f$ . Consider the curve  $\gamma$  in  $S - \mathcal{P}_f$  composed of the arcs

$$\nu_P(\mathcal{R}_P(1/3)) \cup \nu_P(\mathcal{R}_P(2/3)) \text{ in } \tilde{\mathbb{C}}_P \text{ and } \nu_Q(\mathcal{R}_Q(2/3)) \cup \nu_Q(\mathcal{R}_Q(1/3)) \text{ in } \tilde{\mathbb{C}}_Q$$

with  $P = B$  in  $\mathbb{C}_P$  and  $Q = B$  in  $\mathbb{C}_Q$ . It is easy to show that  $\Gamma := \{\gamma\}$  is a Levy cycle for  $f$ ; therefore,  $f$  is obstructed and no such  $F$  exists (see Figure 4).

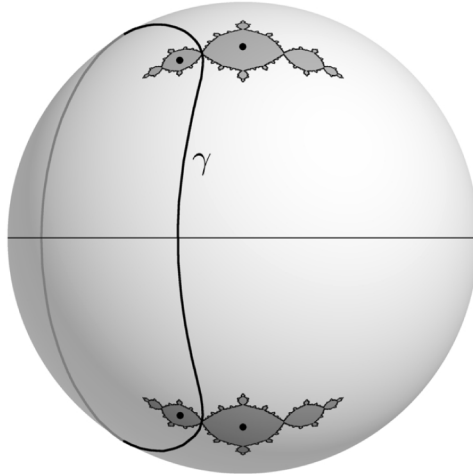


Figure 4. — The Levy cycle  $\Gamma := \{\gamma\}$  is an obstructing multcurve for  $f$ , so  $f$  is not combinatorially equivalent to a rational map  $F : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ .



## 4.2. Maps between universal curves

We now analyze Diagram (1.2) in the case where  $f = B \uplus B$ .

For the remainder of Section 4, we assume that the homeomorphisms  $\phi : S \rightarrow \mathbb{P}^1$  representing points in  $\mathcal{T}_{\mathcal{P}_f}$  and the injective maps  $\phi : \mathcal{P}_f \rightarrow \mathbb{P}^1$  representing points in  $\mathcal{M}_{\mathcal{P}_f}$  satisfy the condition

$$\phi(p_0) = 0, \quad \phi(q_0) = \infty, \quad \text{and} \quad \phi(q_1) = 1.$$

We will refer to such maps as *normalized* maps.

In this way, the moduli space  $\mathcal{M}_{\mathcal{P}_f}$  is identified with  $\mathbb{C} - \{0, 1\}$  via  $\phi \mapsto \phi(p_1)$ . The Teichmüller space  $\mathcal{T}_{\mathcal{P}_f}$  is isomorphic to the unit disc  $\mathbb{D}$  and  $\pi : \mathcal{T}_{\mathcal{P}_f} \rightarrow \mathbb{C} - \{0, 1\}$  is a universal cover. The universal curve over moduli space is given by the projection

$$m : (\mathbb{C} - \{0, 1\}) \times \mathbb{P}^1 \ni (x, z) \mapsto x \in \mathbb{C} - \{0, 1\}$$

with sections

$$m_{p_0} : x \mapsto (x, 0), \quad m_{q_0} : x \mapsto (x, \infty), \quad m_{q_1} : x \mapsto (x, 1) \quad \text{and} \quad m_{p_1} : x \mapsto (x, x).$$

- A point in  $\mathfrak{M}_{\mathcal{P}_f}$  is identified with a pair  $(x, z) \in (\mathbb{C} - \{0, 1\}) \times \mathbb{P}^1$ .
- A point in  $\mathfrak{T}_{\mathcal{P}_f}$  is identified with a triple  $(\tau, x, z) \in \mathcal{T}_{\mathcal{P}_f} \times (\mathbb{C} - \{0, 1\}) \times \mathbb{P}^1$  satisfying  $x = \pi(\tau)$ , so that  $(x, z) = \varpi(\tau, x, z)$ .
- A point in  $\mathfrak{U}_{\mathcal{P}_f}$  is identified with a quadruple  $(\tau, v, x, z)$  such that  $\tau \in \mathcal{T}_{\mathcal{P}_f}$ ,  $v = \sigma_f(\tau) \in \mathcal{T}_{\mathcal{P}_f}$ ,  $x = \pi(v) \in \mathbb{C} - \{0, 1\}$  and  $z \in \mathbb{P}^1$ .

Let  $(\tau, v, x, z)$  belong to  $\mathfrak{U}_{\mathcal{P}_f}$  and let  $(\tau, y, w)$  be its image by the map  $\mathfrak{F} : \mathfrak{U}_{\mathcal{P}_f} \rightarrow \mathfrak{T}_{\mathcal{P}_f}$ . Then, there are a normalized homeomorphism  $\phi : S \rightarrow \mathbb{P}^1$  representing  $\tau \in \mathcal{T}_{\mathcal{P}_f}$ , a normalized homeomorphism  $\psi : S \rightarrow \mathbb{P}^1$  representing  $v = \sigma_f(\tau) \in \mathcal{T}_{\mathcal{P}_f}$  and a rational map  $F_\tau : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  such that the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\psi} & \mathbb{P}^1 \\ f \downarrow & & \downarrow F_\tau \\ S & \xrightarrow{\phi} & \mathbb{P}^1. \end{array} \quad (4.1)$$

In addition,

$$\phi(p_1) = y, \quad \psi(p_1) = x \quad \text{and} \quad F_\tau(z) = w.$$

In particular,  $F_\tau : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  is a rational map of degree 2 with critical points at 0 and  $\infty$  such that

$$F_\tau(\infty) = 1, \quad F_\tau(1) = \infty, \quad F_\tau(x) = 0, \quad \text{and} \quad F_\tau(0) = y.$$

Imposing the first three conditions, we see that

$$w := F_\tau(z) = \frac{z^2 - x^2}{z^2 - 1},$$

and imposing the condition  $F_\tau(0) = y$  gives the relation  $y = x^2$ .

We have just shown that

$$\mathfrak{F}(\tau, v, x, z) = (\tau, y, w) \quad \text{with} \quad y = x^2 \quad \text{and} \quad w = \frac{z^2 - x^2}{z^2 - 1}.$$

In other words, there are maps  $G_f : \mathfrak{M}_{\mathcal{P}_f} \dashrightarrow \mathfrak{M}_{\mathcal{P}_f}$  and  $g_f : \mathcal{M}_{\mathcal{P}_f} \dashrightarrow \mathcal{M}_{\mathcal{P}_f}$  so that the following diagram commutes.

$$\begin{array}{ccccccc}
 & & & G_f & & & \\
 & & & \text{---} & & & \\
 \mathfrak{M}_{\mathcal{P}_f} & \xleftarrow{\varpi} & \mathfrak{T}_{\mathcal{P}_f} & \xleftarrow{\mathfrak{F}} & \mathfrak{M}_{\mathcal{P}_f} & \xrightarrow{\mathfrak{p}_2} & \mathfrak{T}_{\mathcal{P}_f} & \xrightarrow{\varpi} & \mathfrak{M}_{\mathcal{P}_f} \\
 \downarrow \mathfrak{m} & & \downarrow \mathfrak{t} & & \downarrow \mathfrak{p}_1 & & \downarrow \mathfrak{t} & & \downarrow \mathfrak{m} \\
 \mathcal{M}_{\mathcal{P}_f} & \xleftarrow{\pi} & \mathcal{T}_{\mathcal{P}_f} & \xleftarrow{\text{id}} & \mathcal{T}_{\mathcal{P}_f} & \xrightarrow{\sigma_f} & \mathcal{T}_{\mathcal{P}_f} & \xleftarrow{\pi} & \mathcal{M}_{\mathcal{P}_f} \\
 & & & & & & & & \\
 & & & g_f & & & & & 
 \end{array}$$

The map  $g_f : \mathcal{M}_{\mathcal{P}_f} \dashrightarrow \mathcal{M}_{\mathcal{P}_f}$  is defined on  $\mathcal{M}_{\mathcal{P}_f} - \{-1\} = \mathbb{C} - \{0, 1, -1\}$  by

$$g_f(x) = x^2.$$

The map  $G_f : \mathfrak{M}_{\mathcal{P}_f} \dashrightarrow \mathfrak{M}_{\mathcal{P}_f}$  is defined on  $\mathfrak{M}_{\mathcal{P}_f} - \mathfrak{m}^{-1}(-1)$  by

$$G_f(x, z) = \left( x^2, \frac{z^2 - x^2}{z^2 - 1} \right).$$

We have the following commutative diagram:

$$\begin{array}{ccc}
 \mathfrak{M}_{\mathcal{P}_f} & \xrightarrow{G_f} & \mathfrak{M}_{\mathcal{P}_f} \\
 \mathfrak{m} \downarrow & & \downarrow \mathfrak{m} \\
 \mathcal{M}_{\mathcal{P}_f} & \xrightarrow{g_f} & \mathcal{M}_{\mathcal{P}_f}
 \end{array} \tag{4.2}$$

For  $x \in \mathbb{C} - \{0, 1, -1\}$ , the restriction of the map  $G_f$  to the fiber  $\mathfrak{S}_x := \mathfrak{m}^{-1}(x)$  takes its values in the fiber  $\mathfrak{S}_{x^2} := \mathfrak{m}^{-1}(x^2)$  and is therefore not a dynamical system in general. However, when  $x$  is periodic of period  $n$  for  $g_f$ , the  $n$ -th iterate of the skew-product  $G_f$  fixes the fiber  $\mathfrak{S}_x := \mathfrak{m}^{-1}(x) \subset \mathcal{M}_{\mathcal{P}_f}$ . In that case, it makes sense to consider the rational map  $G_f^{\circ n} : \mathfrak{S}_x \rightarrow$

$\mathfrak{S}_x$  as a dynamical system. We show that it is the geometric mating of two polynomials.

Our result is the following.

PROPOSITION 4.2. — *If  $x = e^{2\pi ik/(2^n-1)}$  with  $k/(2^n-1) \notin \mathbb{Z}$ , then  $G_f^{\text{on}} : \mathfrak{S}_x \rightarrow \mathfrak{S}_x$  is the geometric twisted mating of angle  $-k/(2^n-1)$  of  $B^{\text{on}}$  with  $B^{\text{on}}$ .*

### 4.3. An intermediate covering space

Before proceeding to the proof of Proposition 4.2, we prove that the Thurston endomorphism  $\sigma_f : \mathcal{T}_{\mathcal{P}_f} \rightarrow \mathcal{T}_{\mathcal{P}_f}$  descends to a self-map of an intermediate covering space (see Diagram (4.3) below). We first need to define the maps involved in this diagram.

Recall that  $f = B \uplus B$ , and that with our chosen normalization,  $\mathcal{M}_{\mathcal{P}_f}$  is identified with  $\mathbb{C} - \{0, 1\}$ . Let  $\alpha : \mathbb{C} - \mathbb{Z} \rightarrow \mathbb{C} - \{0, 1\}$  and  $L : \mathbb{C} - \mathbb{Z} \rightarrow \mathbb{C} - \frac{1}{2}\mathbb{Z}$  be the covering maps defined by

$$\alpha(X) := e^{2\pi i X} \quad \text{and} \quad L(X) := X/2.$$

We have the following commutative diagram of covering spaces.

$$\begin{array}{ccc} \mathbb{C} - \mathbb{Z} & \xrightarrow{L} & \mathbb{C} - \frac{1}{2}\mathbb{Z} \\ \alpha \downarrow & & \downarrow \alpha \\ \mathbb{C} - \{0, 1\} & \xleftarrow{g_f} & \mathbb{C} - \{0, 1, -1\} \end{array}$$

The universal cover  $\pi : \mathcal{T}_{\mathcal{P}_f} \rightarrow \mathbb{C} - \{0, 1\}$  factors as  $\pi = \alpha \circ \beta$  for some universal cover  $\beta : \mathcal{T}_{\mathcal{P}_f} \rightarrow \mathbb{C} - \mathbb{Z}$ . We wish to choose a particular  $\beta$  in a way that we now describe. Let  $\Gamma$  be the topological circle in  $S - \mathcal{P}_f$  obtained as the closure of  $\nu_P(\mathbb{R}) \cup \nu_Q(\mathbb{R})$ , where the gnomonic projections  $\nu_P$  and  $\nu_Q$  satisfy

$$\nu_P(\mathbb{R}) \subset H^+ \quad \text{and} \quad \nu_Q(\mathbb{R}) \subset H^-,$$

and  $\mathbb{R}$  is the real axis in the respective dynamical planes of the basilicas  $P = B$  and  $Q = B$  (see Figure 5). Note that  $p_0, p_1 \in \nu_P(\mathbb{R})$ , and  $q_0, q_1 \in \nu_Q(\mathbb{R})$ . Let  $\Gamma' \subset \Gamma$  be the subarc joining the two critical values  $p_1$  and  $q_1$ , avoiding the critical points  $p_0$  and  $q_0$ , oriented from  $q_1$  to  $p_1$ . If  $\phi : S \rightarrow \mathbb{P}^1$  is a normalized homeomorphism representing  $\tau \in \mathcal{T}_{\mathcal{P}_f}$ , then  $\phi(\Gamma')$  is a curve joining  $\phi(q_1) = 1$  to  $\phi(p_1) = \pi(\tau)$  in  $\mathbb{C} - \{0\}$  and we set

$$\beta(\tau) := \frac{1}{2\pi i} \int_{\phi(\Gamma')} \frac{dz}{z}.$$

Note that  $\beta(\tau) \notin \mathbb{Z}$  because if this were not the case, then  $\phi(\Gamma')$  would be a closed loop, and we would have  $\phi(p_1) = 1$ , but  $\phi(q_1) = 1$ .

By Cauchy's Theorem, the result does not depend on the choice of normalized representative.

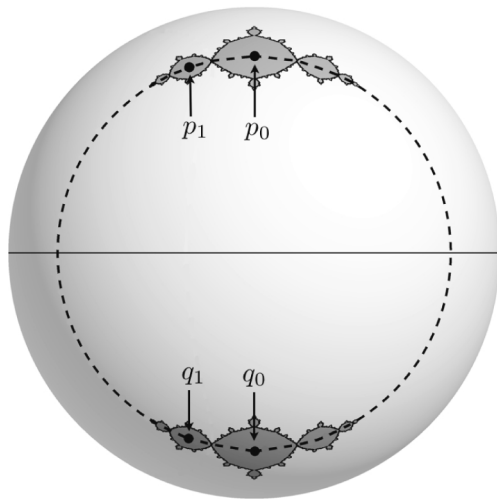


Figure 5. — The curve drawn with the dashes is  $\Gamma$ ; it is a topological circle, which can be decomposed into two subarcs  $\Gamma = \Gamma' \cup \Gamma''$ , where  $\Gamma'$  is the subarc connecting  $p_1$  and  $q_1$ , which does not contain  $p_0$  and  $q_0$ , and  $\Gamma''$  is the subarc connecting  $p_1$  and  $q_1$  which does contain  $p_0$  and  $q_0$ .

PROPOSITION 4.3 *We have the following commutative diagram of covering maps:*

$$\begin{array}{ccccc}
 \mathcal{T}_{\mathcal{P}_f} & \xrightarrow{\sigma_f} & \mathcal{T}_{\mathcal{P}_f} - \pi^{-1}(-1) & & \\
 \searrow \beta & & \searrow \beta & & \\
 \mathbb{C} - \mathbb{Z} & \xrightarrow{L} & \mathbb{C} - \frac{1}{2}\mathbb{Z} & & \\
 \swarrow \alpha & & \swarrow \alpha & & \\
 \mathbb{C} - \{0, 1\} & \xleftarrow{g_f} & \mathbb{C} - \{0, 1-1\} & & \\
 \uparrow \pi & & \uparrow \pi & & \\
 \mathcal{T}_{\mathcal{P}_f} & & \mathcal{T}_{\mathcal{P}_f} - \pi^{-1}(-1) & & 
 \end{array}
 \tag{4.3}$$

*Proof.* — What remains is to show that  $\sigma_f : \mathcal{T}_{\mathcal{P}_f} \rightarrow \mathcal{T}_{\mathcal{P}_f} - \pi^{-1}(-1)$  is a covering map and that  $\beta \circ \sigma_f = L \circ \beta$  on  $\mathcal{T}_{\mathcal{P}_f}$ .

Since  $L \circ \beta : \mathcal{T}_{\mathcal{P}_f} \rightarrow \mathbb{C} - \frac{1}{2}\mathbb{Z}$  is a universal cover and  $\beta : \mathcal{T}_{\mathcal{P}_f} - \pi^{-1}(-1) \rightarrow \mathbb{C} - \frac{1}{2}\mathbb{Z}$  is a covering map, if  $\tau \in \mathcal{T}_{\mathcal{P}_f}$  and

$v \in \mathcal{T}_{\mathcal{P}_f} - \pi^{-1}(-1)$  satisfy  $\beta(v) = L \circ \beta(\tau)$ , there is a unique covering map  $\sigma : \mathcal{T}_{\mathcal{P}_f} \rightarrow \mathcal{T}_{\mathcal{P}_f} - \pi^{-1}(-1)$  such that

$$L \circ \beta = \beta \circ \sigma \quad \text{and} \quad \sigma(\tau) = v.$$

We will show that there is such a pair  $(\tau, v)$  with  $v = \sigma_f(\tau)$ . Both  $\sigma$  and  $\sigma_f$  are lifts of  $\pi : \mathcal{T}_{\mathcal{P}_f} \rightarrow \mathcal{M}_{\mathcal{P}_f}$  through  $g_f \circ \pi : \mathcal{T}_{\mathcal{P}_f} - \pi^{-1}(-1) \rightarrow \mathcal{M}_{\mathcal{P}_f}$  and they coincide at  $\tau$ . By uniqueness of the lift,  $\sigma$  and  $\sigma_f$  coincide everywhere.

So, let  $\tau \in \mathcal{T}_{\mathcal{P}_f}$  be represented by a normalized homeomorphism  $\phi : S \rightarrow \mathbb{P}^1$ , such that  $\phi(\Gamma) = \mathbb{R} \cup \{\infty\}$ . Set  $v := \sigma_f(\tau)$ . We need to prove that  $\beta(v) = L \circ \beta(\tau)$ . Let  $\psi : S \rightarrow \mathbb{P}^1$  be the normalized representative of  $v$  such that

$$F := \phi \circ f \circ \psi^{-1} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$$

is rational. Set  $y := \phi(p_1)$  and  $x := \psi(p_1)$ .

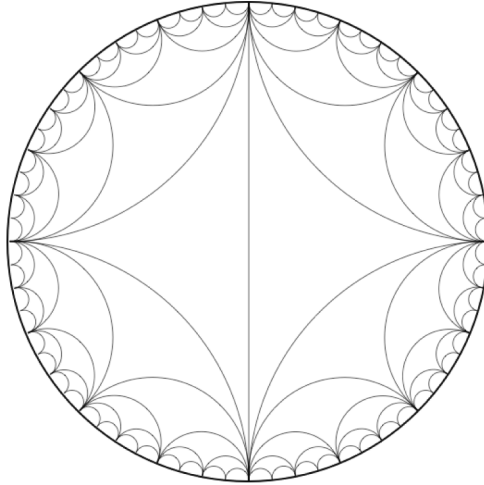


Figure 6. — The Julia set  $J_{g_f}$  of the map  $g_f : x \mapsto x^2$  is the unit circle. We can lift  $J_{g_f} \cap \mathcal{M}_{\mathcal{P}_f}$  through the map  $\pi$  to the Teichmüller space  $\mathcal{T}_{\mathcal{P}_f}$ , which is isomorphic to  $\mathbb{D}$ . This is a picture of  $\pi^{-1}(J_{g_f} \cap \mathcal{M}_{\mathcal{P}_f})$ . The structure of the lifted Julia set can be used to understand the map  $\sigma_f$ .

LEMMA 4.4. — *We have that  $\psi(\Gamma) = \mathbb{R} \cup \{\infty\}$ .*

*Proof.* — As previously discussed in Section 4.2:  $y = x^2$  and

$$F : z \mapsto \frac{z^2 - x^2}{z^2 - 1}$$

with critical values  $x^2$  and 1. Since  $\phi(\Gamma) = \mathbb{R} \cup \{\infty\}$ , we have that  $y \in (0, 1)$ , so  $0 < x^2 < 1$ . Note that

$$\phi \circ f(\Gamma) = \phi(\Gamma'') = \mathbb{R} \cup \{\infty\} - (x^2, 1) = F \circ \psi(\Gamma).$$

Since  $\Gamma = f^{-1}(\Gamma'')$ , we have  $\psi(\Gamma) = F^{-1}(\mathbb{R} \cup \{\infty\} - (x^2, 1))$ . The map  $F$  can be decomposed as  $F = M \circ s$ , where  $s : z \mapsto z^2$ , and  $M : z \rightarrow (z - x^2)/(z - 1)$ . Note that for the Möbius transformation  $M$ , we have

$$M(x^2) = 0, \quad M(0) = x^2, \quad M(\infty) = 1 \quad \text{and} \quad M(1) = \infty.$$

Furthermore,

$$M^{-1}(\mathbb{R} \cup \{\infty\} - (x^2, 1)) = [0, \infty] \quad \text{and} \quad s^{-1}([0, \infty]) = \mathbb{R} \cup \{\infty\}.$$

It follows that  $\psi(\Gamma) = \mathbb{R} \cup \{\infty\}$ .  $\square$

We conclude the proof of the proposition as required:

$$\beta(v) = \frac{1}{2\pi i} \ln y = L \left( \frac{1}{2\pi i} \ln x \right) = L(\beta(\tau)).$$

$\square$

**PROPOSITION 4.5.** — *Let  $D : S \rightarrow S$  be a positive Dehn twist around the equator of  $S$  relative to  $\mathcal{P}_f$ . For any normalized homeomorphism  $\phi : S \rightarrow \mathbb{P}^1$ , we have*

$$\beta([\phi \circ D]) = \beta([\phi]) - 1.$$

*Proof.* — Recall that  $\Gamma'$  is oriented from  $q_1$  to  $p_1$ . We orient the equator  $\Gamma_{eq}$  of  $S$  so that  $q_1$  is on the left of  $\Gamma_{eq}$  with respect to this orientation. The orientations of  $\Gamma'$  and  $\Gamma_{eq}$  induce orientations of the curves  $\phi(\Gamma')$  and  $\phi(\Gamma_{eq})$ :

- if  $y := \phi(p_1)$ , then the initial point of  $\phi(\Gamma')$  is 1, and its terminal point is  $y$ , and
- $\phi(\Gamma_{eq}) \subset \mathbb{P}^1$  is a simple closed curve positively-oriented around  $\infty$ ; that is,  $\infty$  is on the left of  $\phi(\Gamma_{eq})$ , and 0 is on the right with respect to this orientation.

The chains  $D(\Gamma')$  and  $\Gamma' + \Gamma_{eq}$  are homologous in  $S - \mathcal{P}_f$ . By Cauchy's theorem,

$$\int_{\phi(D(\Gamma'))} \frac{dz}{z} = \int_{\phi(\Gamma')} \frac{dz}{z} + \int_{\phi(\Gamma_{eq})} \frac{dz}{z}.$$

We ultimately have

$$\beta([\phi \circ D]) = \frac{1}{2\pi i} \int_{\phi(D(\Gamma'))} \frac{dz}{z} = \frac{1}{2\pi i} \left( \int_{\phi(\Gamma')} \frac{dz}{z} - 2\pi i \right) = \beta([\phi]) - 1.$$

□

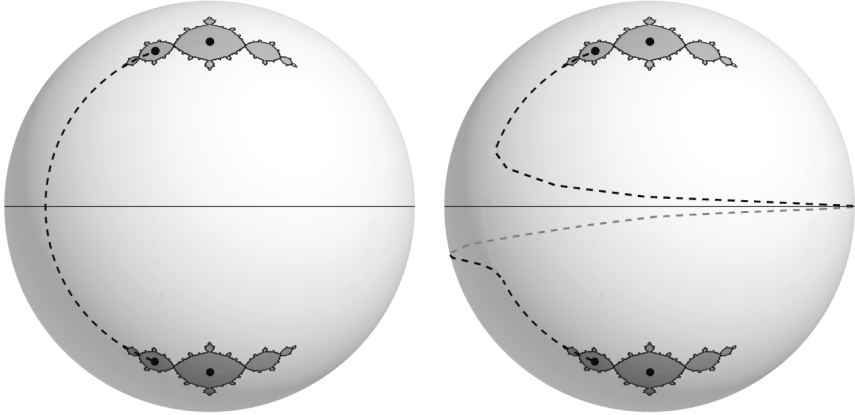


Figure 7. — The curve  $\Gamma'$  is drawn in dashes on the left sphere, and its image under the Dehn twist  $D$  is drawn on the right sphere; its image is wrapped once around the equator,  $\Gamma_{eq}$ .

#### 4.4. Proof of Proposition 4.2

For  $n \geq 0$ , we define  $F_x^n : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  by

$$G_f^{\circ n}(x, z) = (g_f^{\circ n}(x), F_x^n(z)).$$

We must show that if  $x$  is a periodic point of  $g_f$  of period  $n$ , that is, if

$$x = e^{2\pi i k / (2^n - 1)} \quad \text{for } k / (2^n - 1) \notin \mathbb{Z},$$

then the  $F_x^n$  is the geometric twisted mating of angle  $-k / (2^n - 1)$  of  $B^{\circ n}$  with  $B^{\circ n}$ . This is a direct consequence of the more precise following result.

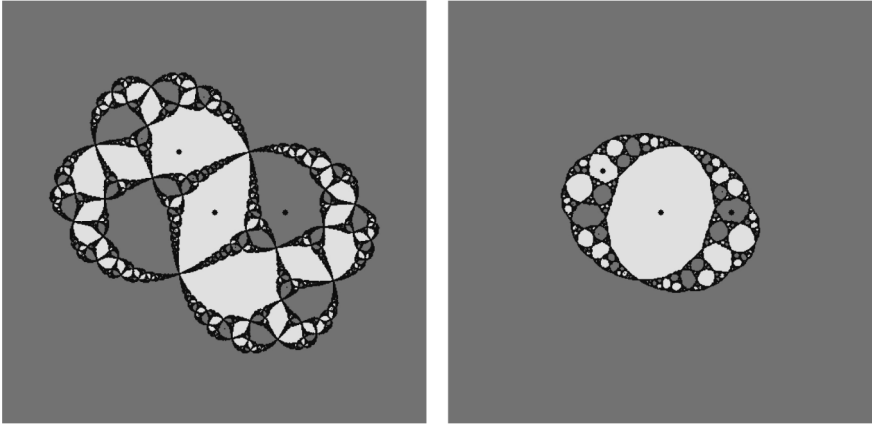


Figure 8. — Left: the Julia set and Fatou set of  $F_y^2$  for  $y = e^{2i\pi/3}$ . Right: the Julia set and Fatou set of  $F_y^4$  for  $y = e^{2i\pi/5}$ . Points are colored darker grey if they are in the basin of  $\infty$  and lighter grey if they are in the basin of 0. The Julia set is colored black. The points 0, 1 and  $y$  are shown.

PROPOSITION 4.6. — Fix  $n \in \mathbb{Z}_+$  and  $k \in \mathbb{Z}$ , and let  $f_{n,k} : S \rightarrow S$  be the twisted mating of angle  $-k/(2^n - 1)$  of  $B^{\circ n}$  with  $B^{\circ n}$ . Then  $f_{n,k}$  is combinatorially equivalent to a rational map if and only if

$$x := e^{2\pi ik/(2^n - 1)} \neq 1.$$

In this case,  $f_{n,k}$  is combinatorially equivalent to  $F_x^n$ .

*Proof.* — By definition,  $f_{n,k} = B^{\circ n} \uplus T^{\circ k}(B^{\circ n})$ . According to Proposition 2.2,  $f_{n,k}$  is combinatorially equivalent to

$$D^{\circ k} \circ (B^{\circ n} \uplus B^{\circ n}) = D^{\circ k} \circ (B \uplus B)^{\circ n} = D^{\circ k} \circ f^{\circ n}.$$

The Thurston endomorphism  $\sigma_{f_{n,k}} : \mathcal{T}_{\mathcal{P}_f} \rightarrow \mathcal{T}_{\mathcal{P}_f}$  therefore satisfies

$$\sigma_{f_{n,k}} = \sigma_{D^{\circ k} \circ f^{\circ n}} = \sigma_{f^{\circ n}} \circ \sigma_{D^{\circ k}} = \sigma_f^{\circ n} \circ \sigma_D^{\circ k}.$$

Define  $L_{n,k} : \mathbb{C} \rightarrow \mathbb{C}$  by

$$L_{n,k}(X) := (X - k)/2^n.$$

According to Proposition 4.5,

$$\beta \circ \sigma_{f_{n,k}} = L_{n,k} \circ \beta.$$

Fix  $\tau_0 \in \mathcal{T}_{\mathcal{P}_f}$ , and define the sequence of iterates

$$\tau_m := \sigma_{f_{n,k}}^{\circ m}(\tau_0) \quad \text{for } m \geq 0.$$



- Either  $\sigma_{f_{n,k}}$  has a fixed point, in which case the sequence  $m \mapsto \tau_m$  converges to it and  $f_{n,k}$  is equivalent to a rational map.
- Or  $\sigma_{f_{n,k}}$  has no fixed point, in which case  $m \mapsto \tau_m$  diverges and  $f_{n,k}$  is obstructed.

Let  $\gamma_0 : [0, 1] \rightarrow \mathcal{T}_{\mathcal{P}_f}$  be a path connecting  $\gamma_0(0) = \tau_0$  and  $\gamma_0(1) = \tau_1$ . Let  $\kappa_0 : [0, 1] \rightarrow \mathbb{C} - \mathbb{Z}$  be the path  $\kappa_0 = \beta \circ \gamma_0$ .

Set  $\kappa_m := L_{n,k}^{\circ m} \circ \kappa_0$ . Define  $X_m := \beta(\tau_m)$ ; the sequence  $m \mapsto X_m$  converges to  $X_\star := -k/(2^n - 1)$ , the unique fixed point of  $L_{n,k}$ . Concatenating the  $\kappa_m$  we obtain a path  $\kappa$  that joins  $X_0$  to  $X_\star$ . The sequence  $m \mapsto \kappa_m$  will converge to  $X_\star$ . Either  $X_\star \in \mathbb{Z}$ , or it is not. We discuss each case.

If  $X_\star \in \mathbb{C} - \mathbb{Z}$ , then the path  $\kappa$  lifts uniquely to a path  $\gamma$  connecting  $\tau_0$  to a point  $\tau_\star \in \beta^{-1}(X_\star) \subset \mathcal{T}_{\mathcal{P}_f}$ . By induction on  $m$ , we see that the lift of  $\kappa_m$  joins  $\tau_{m-1}$  to  $\tau_m$  and therefore coincides with  $\gamma_m := \sigma_{f_{n,k}}^{\circ m}(\gamma_0)$ . In particular,  $\tau_m \rightarrow \tau_\star$  as  $m \rightarrow \infty$ , so  $\tau_\star$  is a fixed point of the Thurston endomorphism:

$$\sigma_{f_{n,k}}(\tau_\star) = \tau_\star, \quad \beta(\tau_\star) = X_\star.$$

Set  $x_\star := \alpha(X_\star) = e^{2\pi i X_\star} \in \mathbb{C} - \{0, 1\}$ ; in particular,  $x_\star \neq 1$ , and the following diagram commutes

$$\begin{array}{ccc} S & \xrightarrow{\psi} & \mathbb{P}^1 \\ f_{n,k} \downarrow & & \downarrow F_{x_\star}^n \\ S & \xrightarrow{\phi} & \mathbb{P}^1 \end{array}$$

where  $\phi$  and  $\psi$  are normalized homeomorphisms representing  $\tau_\star$ . Hence  $f_{n,k}$  and  $F_{x_\star}^n$  are combinatorially equivalent.

If  $X_\star \in \mathbb{Z}$ , then  $\sigma_{f_{n,k}} : \mathcal{T}_{\mathcal{P}_f} \rightarrow \mathcal{T}_{\mathcal{P}_f}$  cannot have a fixed point. Indeed, if  $\sigma_{f_{n,k}}$  fixes  $\tau_\star$  then  $L$  fixes  $\beta(\tau_\star)$ . However,  $X_\star \in \mathbb{Z}$  is the unique fixed point of  $L$ , and  $\beta(\tau_\star) \in \mathbb{C} - \mathbb{Z}$ .  $\square$

The following corollary follows immediately.

**COROLLARY 4.7.** — *If  $\theta = k/(2^n - 1) \notin \mathbb{Z}$ , the geometric twisted mating of angle  $-\theta$  of  $B^{\text{on}}$  with  $B^{\text{on}}$  exists.*

## 5. Compactifying the universal curve

We continue in the setting where  $f = B \uplus B$ , and  $g_f : \mathcal{M}_{\mathcal{P}_f} \dashrightarrow \mathcal{M}_{\mathcal{P}_f}$ , and  $G_f : \mathfrak{M}_{\mathcal{P}_f} \dashrightarrow \mathfrak{M}_{\mathcal{P}_f}$  are the maps defined in Section 4.2.

If  $(x, z) \in \mathfrak{M}_{\mathcal{P}_f}$  with  $|x| < 1$  or  $|x| > 1$ , then  $G_f^{on}(x, z)$  eventually leaves every compact set of  $\mathfrak{M}_{\mathcal{P}_f}$  as  $n \rightarrow \infty$ . Indeed,  $g_f^{on}(x) \rightarrow 0$  if  $|x| < 1$  and  $g_f^{on}(x) \rightarrow \infty$  if  $|x| > 1$ . In order to study the asymptotic behaviors, we will compactify  $\mathfrak{M}_{\mathcal{P}_f}$  and study how the map  $G_f$  extends to the compactified space. Work of Nikita Selinger [14] suggests that when studying Thurston endomorphisms, it might be interesting to consider the Deligne-Mumford compactification of the moduli space and its universal curve.

### 5.1. The Deligne-Mumford compactification

Let  $\mathcal{P}$  be a finite set containing at least three points. A *curve of genus zero marked by  $\mathcal{P}$*  is an injection  $\phi : \mathcal{P} \rightarrow \mathcal{C}$  where

- $\mathcal{C}$  is a connected algebraic curve whose only singularities are ordinary double points (called *nodes*), such that
- each irreducible component is isomorphic to  $\mathbb{P}^1$
- the graph, whose vertices are the irreducible components and whose edges connect components intersecting at a node, is a tree
- each  $\phi(p)$  is a smooth point of  $\mathcal{C}$ .

The curve is *stable* if the number of marked points and nodes on each irreducible component is at least 3. Two stable marked curves  $\phi_1 : \mathcal{P} \rightarrow \mathcal{C}_1$  and  $\phi_2 : \mathcal{P} \rightarrow \mathcal{C}_2$  are isomorphic if there is an isomorphism  $\alpha : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  such that  $\phi_2 = \alpha \circ \phi_1$ .

The set  $\overline{\mathcal{M}}_{\mathcal{P}}$  of stable curves of genus zero marked by  $\mathcal{P}$  modulo isomorphism may be naturally regarded as a smooth compact projective variety (see Knudsen [10], and Knudsen and Mumford [9]). It contains  $\mathcal{M}_{\mathcal{P}}$  as an open dense Zariski subset, and it is called the Deligne-Mumford compactification of  $\mathcal{M}_{\mathcal{P}}$ .

If  $\mathcal{P} \subset \mathcal{Q}$ , there is a map  $\overline{\mathcal{M}}_{\mathcal{Q}} \rightarrow \overline{\mathcal{M}}_{\mathcal{P}}$  which consists of forgetting the points in  $\mathcal{Q} - \mathcal{P}$ , collapsing only the irreducible components which then contain fewer than three nodes and marked points.

When  $\mathcal{Q} = \mathcal{P} \cup \{q\}$  with  $q \notin \mathcal{P}$ , there are sections  $\overline{m}_p : \overline{\mathcal{M}}_{\mathcal{P}} \rightarrow \overline{\mathcal{M}}_{\mathcal{Q}}$  which may be defined as follows: given a stable curve  $\phi : \mathcal{P} \rightarrow \mathcal{C}$ , one may form a new stable curve  $\psi : \mathcal{Q} \rightarrow \mathcal{C}'$  where

- $\mathcal{C}'$  is obtained by adjoining to  $\mathcal{C}$  a Riemann sphere intersecting  $\mathcal{C}$  at  $\phi(p)$ ,
- $\psi$  agrees with  $\phi$  on  $\mathcal{P} - \{p\}$ , and
- $\psi(p)$  and  $\psi(q)$  are distinct points in  $\mathcal{C}' - \mathcal{C}$ .

We define  $\overline{m}_p([\phi]) := [\psi]$ . In fact, the forgetful map  $\overline{m} : \overline{\mathcal{M}}_{\mathcal{Q}} \rightarrow \overline{\mathcal{M}}_{\mathcal{P}}$  with the sections  $(\overline{m}_p : \overline{\mathcal{M}}_{\mathcal{P}} \rightarrow \overline{\mathcal{M}}_{\mathcal{Q}})_{p \in \mathcal{P}}$  is the universal curve over  $\overline{\mathcal{M}}_{\mathcal{P}}$ : for each  $x \in \overline{\mathcal{M}}_{\mathcal{P}}$ , the injection

$$\mathcal{P} \ni p \mapsto \overline{m}_p(x) \in \overline{m}^{-1}(x)$$

represents  $x$ . We use the notation  $\overline{m} : \overline{\mathfrak{M}}_{\mathcal{P}} \rightarrow \overline{\mathcal{M}}_{\mathcal{P}}$  and  $\overline{m}_p : \overline{\mathcal{M}}_{\mathcal{P}} \rightarrow \overline{\mathfrak{M}}_{\mathcal{P}}$ .

## 5.2. Basilica matings

In our setting, the Deligne-Mumford compactification  $\overline{\mathcal{M}}_{\mathcal{P}_f}$  is identified with  $\mathbb{P}^1$ . The three added points  $x = 0$ ,  $x = 1$  and  $x = \infty$  correspond to marked stable curves  $\phi : \mathcal{P}_f \rightarrow \mathcal{C}$  where  $\mathcal{C}$  is the union of two Riemann spheres intersecting at a single node:

- for  $x = 0$ , the injection  $\phi$  sends  $p_0$  and  $p_1$  to the same component and  $q_0$  and  $q_1$  to the other component.
- for  $x = 1$ , the injection  $\phi$  sends  $p_0$  and  $q_0$  to the same component and  $p_1$  and  $q_1$  to the other component.
- for  $x = \infty$ , the injection  $\phi$  sends  $p_0$  and  $q_1$  to the same component and  $p_1$  and  $q_0$  to the other component.

Describing the universal curve  $\overline{m} : \overline{\mathfrak{M}}_{\mathcal{P}_f} \rightarrow \overline{\mathcal{M}}_{\mathcal{P}_f}$  is slightly more difficult. We first describe  $\overline{\mathfrak{M}}_{\mathcal{P}_f}$ . Recall that  $\mathfrak{M}_{\mathcal{P}_f}$  is identified with  $(\mathbb{C} - \{0, 1\}) \times \mathbb{P}^1$  and one may be tempted to deduce that  $\overline{\mathfrak{M}}_{\mathcal{P}_f}$  is identified with  $\mathbb{P}^1 \times \mathbb{P}^1$ . But this is not the case. In fact,  $\overline{\mathfrak{M}}_{\mathcal{P}_f}$  is identified with  $\mathbb{P}^1 \times \mathbb{P}^1$  blown up at the three points  $(0, 0)$ ,  $(1, 1)$  and  $(\infty, \infty)$  (see [8] for example). The map  $\overline{m} : \overline{\mathfrak{M}}_{\mathcal{P}_f} \rightarrow \overline{\mathcal{M}}_{\mathcal{P}_f}$  consists of blowing down  $\overline{\mathfrak{M}}_{\mathcal{P}_f}$  to  $\mathbb{P}^1 \times \mathbb{P}^1$  and then projecting to the first coordinate. As expected, the fibers above  $\mathbb{C} - \{0, 1\}$  are Riemann spheres, and each of the fibers above 0, 1 and  $\infty$  consists two Riemann spheres touching at a point, where one of the spheres is a component of the exceptional divisor produced by the blow up. The sections  $\overline{m}_p : \overline{\mathcal{M}}_{\mathcal{P}_f} \rightarrow \overline{\mathfrak{M}}_{\mathcal{P}_f}$  are the continuous extensions of the sections  $m_p : \mathcal{M}_{\mathcal{P}_f} \rightarrow \mathfrak{M}_{\mathcal{P}_f}$  (see Figure 9). We will use the notation

$$\mathfrak{S}_x := \overline{m}^{-1}(x) \quad \text{for } x \in \overline{\mathcal{M}}_{\mathcal{P}_f}.$$

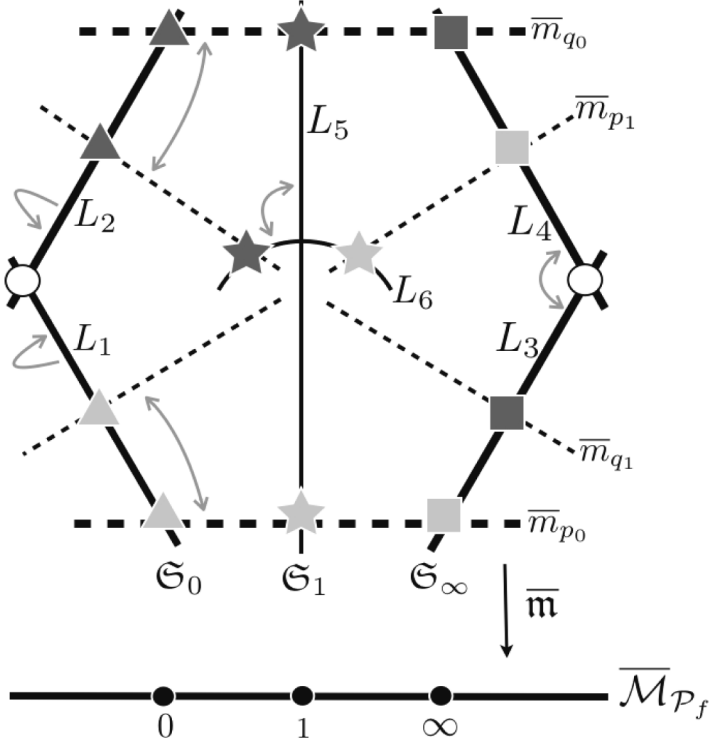


Figure 9. — The universal curve  $\bar{m}: \bar{\mathcal{M}}_{\mathcal{P}_f} \rightarrow \bar{\mathcal{M}}_{\mathcal{P}_f}$ . The fibers  $\mathfrak{S}_0, \mathfrak{S}_\infty$  and  $\mathfrak{S}_1$  are each stable curves consisting of two spheres (or projective lines) intersecting at a node:  $\mathfrak{S}_0$  is the pair of intersecting lines  $L_1$  and  $L_2$ ,  $\mathfrak{S}_\infty$  is the pair of intersecting lines  $L_3$  and  $L_4$ , and  $\mathfrak{S}_1$  is the pair of intersecting lines  $L_5$  and  $L_6$ . The dashed lines indicate the images of the sections  $\bar{m}_{p_0}, \bar{m}_{q_0}, \bar{m}_{p_1}, \bar{m}_{q_1}: \bar{\mathcal{M}}_{\mathcal{P}_f} \rightarrow \bar{\mathcal{M}}_{\mathcal{P}_f}$ . The thick lines (continuous and dashed) form the critical set of the rational map  $G_f: \bar{\mathcal{M}}_{\mathcal{P}_f} \rightarrow \bar{\mathcal{M}}_{\mathcal{P}_f}$ . The white circles are superattracting fixed points of  $G_f$ . The light and dark grey triangles and squares form superattracting 2-cycles of  $G_f$ . The light and dark grey stars form 2-cycles of  $G_f$  with eigenvalues 0 and 4.

Now consider the maps

$$g_f: x \mapsto x^2 \quad \text{and} \quad G_f: (x, z) \mapsto \left( x^2, \frac{z^2 - x^2}{z^2 - 1} \right).$$

The first trivial observation is that the map  $g_f$  extends to an endomorphism of  $\bar{\mathcal{M}}_{\mathcal{P}_f} = \mathbb{P}^1$ . On  $\mathbb{P}^1 \times \mathbb{P}^1$ , the map  $G_f$  has five points of indeterminacy:  $(\infty, \infty), (1, 1), (-1, 1), (-1, -1)$  and  $(1, -1)$ . The critical locus of  $G_f$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  consists of the lines of equation  $x = 0, z = 0, x = \infty$  and  $z = \infty$ . The map  $G_f$  has a superattracting fixed point at  $(0, 0)$ .

We blow up  $\mathbb{P}^1 \times \mathbb{P}^1$  at the indeterminacy points  $(\infty, \infty)$  and  $(1, 1)$ , and at the superattracting fixed point  $(0, 0)$  to obtain  $\overline{\mathfrak{M}}_{\mathcal{P}_f}$ . The map  $G_f$  may now be regarded as a rational map  $G_f : \overline{\mathfrak{M}}_{\mathcal{P}_f} \rightarrow \overline{\mathfrak{M}}_{\mathcal{P}_f}$  (see Figure 9).

The map has two symmetries we wish to emphasize, arising from permutations of the marked points  $p_0, p_1, q_0$ , and  $q_1$ . The first symmetry comes from exchanging the pair  $p_0$  and  $q_0$ , and the pair  $p_1$  and  $q_1$ ; this permutation induces the involution  $\iota_1 : \overline{\mathfrak{M}}_{\mathcal{P}_f} \rightarrow \overline{\mathfrak{M}}_{\mathcal{P}_f}$  given by

$$\iota_1(x, z) := \left(x, \frac{x}{z}\right).$$

Then,

$$\iota_1 \circ G_f(x, z) = \left(x^2, \frac{x^2 z^2 - x^2}{z^2 - 1}\right) = \left(x^2, \frac{x^2 - x^2/z^2}{1 - x^2/z^2}\right) = G_f \circ \iota_1(x, z).$$

The second symmetry comes from the permutation exchanging  $p_0$  and  $q_0$ , and keeping  $p_1$  and  $q_1$  fixed. This permutation induces the involution  $\iota_2 : \overline{\mathfrak{M}}_{\mathcal{P}_f} \rightarrow \overline{\mathfrak{M}}_{\mathcal{P}_f}$  given by

$$\iota_2(x, z) := \left(\frac{1}{x}, \frac{1}{z}\right)$$

which exchanges  $\mathfrak{m}^{-1}(\mathbb{D})$  and  $\mathfrak{m}^{-1}(\mathbb{P}^1 - \overline{\mathbb{D}})$ . Then,

$$\iota_2 \circ \iota_1(x, z) = \left(\frac{1}{x}, \frac{z}{x}\right) = \iota_1 \circ \iota_2(x, z)$$

and

$$\iota_2 \circ G_f(x, z) = \left(\frac{1}{x^2}, \frac{z^2 - 1}{z^2 - x^2}\right) = \left(\frac{1}{x^2}, \frac{z^2/x^2 - 1/x^2}{z^2/x^2 - 1}\right) = G_f \circ \iota_2 \circ \iota_1(x, z).$$

As a consequence,

$$\begin{aligned} \iota_2 \circ G_f^{\circ 2} &= (G_f \circ \iota_2 \circ \iota_1) \circ G_f = G_f \circ \iota_2 \circ G_f \circ \iota_1 \\ &= G_f \circ (G_f \circ \iota_2 \circ \iota_1) \circ \iota_1 = G_f^{\circ 2} \circ \iota_2. \end{aligned}$$

### 5.3. The fiber over 0

To analyze the dynamics near the fiber  $\mathfrak{S}_0$ , we work in the coordinates  $(t, z)$  with  $x = tz$ . The expression of the map  $G_f$  becomes:

$$(t, z) \mapsto \left(-t^2 \frac{1 - z^2}{1 - t^2}, -z^2 \frac{1 - t^2}{1 - z^2}\right).$$

The fiber  $\mathfrak{S}_0$  is the union of the line  $L_1$  given by  $z = 0$  and the line  $L_2$  given by  $t = 0$ . Those lines intersect at the node  $(t = 0, z = 0)$ . This node, represented by a white circle, is a superattracting fixed point. The map  $G_f$  fixes both of the lines  $L_1$  and  $L_2$ . In each line, the map restricts to a rational map of degree 2 which has a superattracting fixed point at the node and a superattracting cycle of period 2:

- the light grey triangles  $\overline{m}_{p_0}(0)$  and  $\overline{m}_{p_1}(0)$  are exchanged in  $L_1$  and
- the dark grey triangles  $\overline{m}_{q_0}(0)$  and  $\overline{m}_{q_1}(0)$  are exchanged in  $L_2$ .

The restriction to the line  $L_1$ , parametrized by  $t$ , is  $t \mapsto t^2/(t^2 - 1)$  and the restriction to the line  $L_2$ , parametrized by  $z$ , is  $z \mapsto z^2/(z^2 - 1)$ . The restrictions  $G_f : L_1 \rightarrow L_1$  and  $G_f : L_2 \rightarrow L_2$  are conjugate to the basilica polynomial  $B$ .

#### 5.4. The fiber over $\infty$

The dynamics near the fiber  $\mathfrak{S}_\infty$  may be analyzed using the symmetry  $\iota_2$ . The fiber  $\mathfrak{S}_\infty$  is the union of two lines  $L_3$  and  $L_4$  which intersect at a node. This node, represented by a white circle, is a superattracting fixed point. The lines  $L_3$  and  $L_4$  are exchanged by  $G_f$ :

- the light grey squares  $\overline{m}_{q_1}(\infty)$  in  $L_3$  and  $\overline{m}_{q_0}(\infty)$  in  $L_4$  are exchanged and form a superattracting 2-cycle and
- the dark grey squares  $\overline{m}_{p_0}(\infty)$  in  $L_3$  and  $\overline{m}_{p_1}(\infty)$  in  $L_4$  are exchanged and form a superattracting 2-cycle.

The second iterate of the map  $G_f$  commutes with  $\iota_2$  which sends  $L_3$  to  $L_2$  and  $L_4$  to  $L_1$ . So, it fixes  $L_3$  and  $L_4$  and the restrictions  $G_f^{\circ 2} : L_3 \rightarrow L_3$  and  $G_f^{\circ 2} : L_4 \rightarrow L_4$  are conjugate to the second iterate  $B^{\circ 2}$  of the basilica polynomial.

#### 5.5. The fiber over 1

To analyze the dynamics near the fiber  $\mathfrak{S}_1$ , we work in the coordinates  $(t, w)$  with  $x - 1 = tw$  and  $z - 1 = w$ . The expression of the map  $G_f$  becomes:

$$(t, w) \mapsto \left( -w(2 + w), -t \frac{2 + tw}{2 + w} \right).$$

The fiber  $\mathfrak{S}_1$  is the union of two lines  $L_5$  and  $L_6$  which intersect at the node  $(t = 0, w = 0)$ . The line  $L_5$  is given by  $t = 0$ , parameterized by  $w$ ,

and the line  $L_6$  is given by  $w = 0$ , parameterized by  $t$ . The derivative of  $G_f$  at the node is the linear map  $(t, w) \mapsto (-2w, -t)$  with eigenvalues  $\pm\sqrt{2}$ . The derivative of  $G_f^{\circ 2}$  at the node is multiplication by 2. The two lines are exchanged by  $G_f$ :

- the line  $L_5$ , which intersects the critical locus at  $m_{p_0}(1)$  and  $m_{q_0}(1)$  is mapped with degree 2 to the line  $L_6$ ,
- the line  $L_6$  is mapped isomorphically to  $L_5$ , and
- the points  $\overline{m}_{p_0}(1) \in L_5$  and  $\overline{m}_{p_1}(1) \in L_6$  form a cycle of period 2 with eigenvalues 0 and 4. Indeed, the line  $L_5$  (given by  $t = 0$ ), and the line given by  $w = -1$  intersect at  $m_{p_0}(1)$  which has coordinates  $(0, -1)$ ; each of the lines is periodic of period 2. The restriction of  $G_f^{\circ 2}$  to the line given by  $w = -1$  is  $t \mapsto t(2-t)(2-t(2-t))$  which fixes the point  $t = 0$  with derivative 4, and the restriction of  $G_f^{\circ 2}$  to  $L_5$  is  $w \mapsto w(2+w)$  which fixes the point  $w = -1$  with derivative 0.
- A similar calculation shows that the points  $\overline{m}_{q_0}(1) \in L_5$  and  $\overline{m}_{q_1}(1) \in L_6$  form a cycle of period 2 with eigenvalues 0 and 4.

The restriction of  $G_f^{\circ 2}$  to each line  $L_5$  and  $L_6$  is conjugate to  $z \mapsto z^2$  with a repelling fixed point at the node and superattracting fixed points at the marked points in  $\mathfrak{S}_1$  represented by the grey stars.

### 5.6. The fibers over $\mathbb{D} - \{0\}$

If  $x \in \mathbb{D} - \{0\}$ , the fiber  $\mathfrak{S}_x$  is attracted to  $\mathfrak{S}_0$  under iteration of  $G_f$ . We may partition  $\mathfrak{S}_x$  according to whether the orbit of a point is attracted by

- the superattracting fixed point in  $\mathfrak{S}_0$ ,
- the superattracting 2-cycle  $\{\overline{m}_{p_0}(0), \overline{m}_{p_1}(0)\}$ ,
- the superattracting 2-cycle  $\{\overline{m}_{q_0}(0), \overline{m}_{q_1}(0)\}$  or
- none of the above.

Figure 10 shows such a partition in different fibers. The color corresponds to the color of the cycle the orbit is attracted to and is black otherwise.

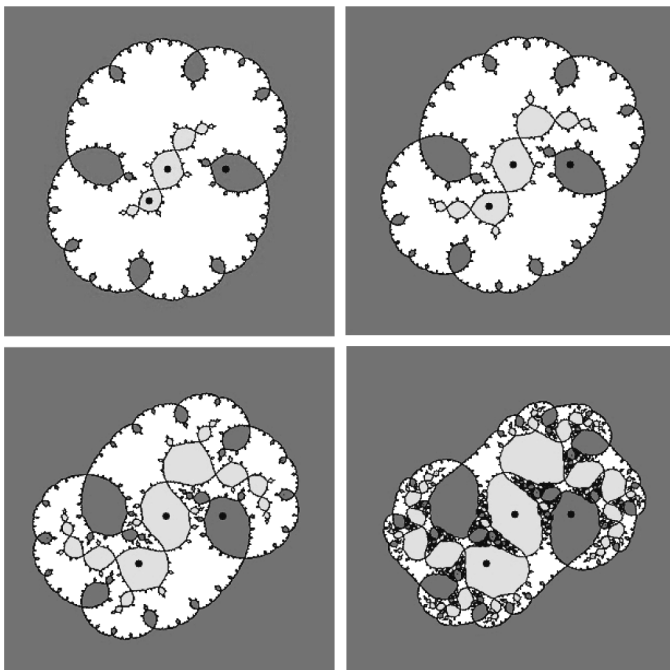


Figure 10. — The partition in four fibers  $\mathfrak{S}_{x_1}, \mathfrak{S}_{x_2}, \mathfrak{S}_{x_3}, \mathfrak{S}_{x_4}$  with  $\arg(x_j) = 4\pi/3$  and  $0 < |x_1| < |x_2| < |x_3| < |x_4| < 1$ . The rational map  $G_f^{\circ 2} : \mathfrak{S}_{e^{4\pi i/3}} \rightarrow \mathfrak{S}_{e^{4\pi i/3}}$  is the geometric twisted mating of angle  $1/3$  of  $B^{\circ 2}$  with itself. Points colored white are in the basin of the superattracting fixed point in  $\mathfrak{S}_0$ , those colored light grey are in the basin of the light grey 2-cycle in  $L_1$ , and those colored dark grey are in the basin of the dark grey 2-cycle in  $L_2$ .

### 5.7. The fibers over $\mathbb{C} - \overline{\mathbb{D}}$

If  $x \in \mathbb{C} - \overline{\mathbb{D}}$ , the fiber  $\mathfrak{S}_x$  is attracted to  $\mathfrak{S}_\infty$  under iteration of  $G_f$ . We may partition  $\mathfrak{S}_x$  according to whether the orbit of a point is attracted by

- the superattracting fixed point in  $\mathfrak{S}_\infty$ ,
- the superattracting 2-cycle  $\{\overline{m}_{p_0}(\infty), \overline{m}_{p_1}(\infty)\}$ ,
- the superattracting 2-cycle  $\{\overline{m}_{q_0}(\infty), \overline{m}_{q_1}(\infty)\}$  or
- none of the above.



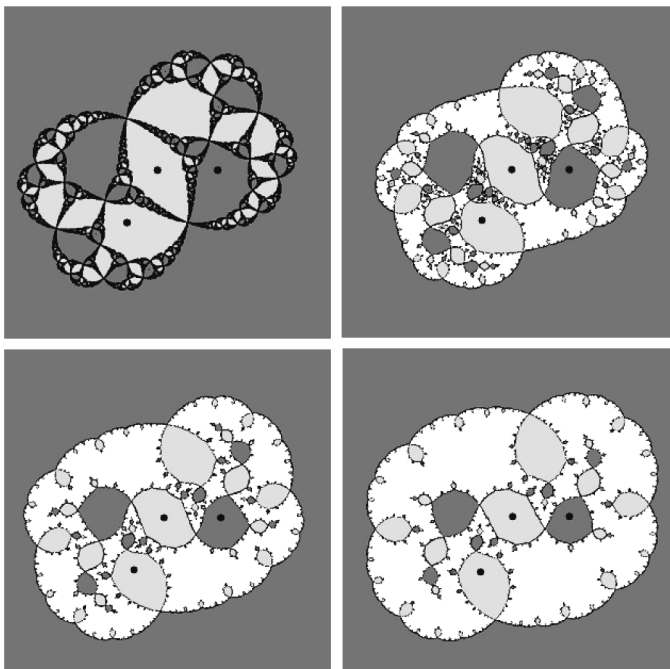


Figure 11. — The partition in four fibers  $\mathfrak{S}_{x_5}, \mathfrak{S}_{x_6}, \mathfrak{S}_{x_7}, \mathfrak{S}_{x_8}$  with  $\arg(x_j) = 4\pi/3$  and  $1 = |x_5| < |x_6| < |x_7| < |x_8|$ . In the upper-left,  $G_f^{o2} : \mathfrak{S}_{x_5} \rightarrow \mathfrak{S}_{x_5}$  is the geometric twisted mating of angle  $1/3$  of  $P^{o2}$  with itself. In the other fibers, points colored white are in the basin of the superattracting fixed point in  $\mathfrak{S}_{\infty}$ , those colored light grey are in the basin of the light grey 2-cycle in  $\mathfrak{S}_{\infty}$ , and those colored dark grey are in the basin of the dark grey 2-cycle in  $\mathfrak{S}_{\infty}$ .

### 5.8. The dynamics of $G_f$ over $\mathbb{D}$

As mentioned previously, the fiber  $\mathfrak{S}_0$  has a basin of attraction

$$\mathfrak{B}_f := \overline{m}^{-1}(\mathbb{D}).$$

Under iteration of  $G_f$ , the orbit of every point in  $\mathfrak{B}_f$  converges to  $\mathfrak{S}_0$ . We understand the dynamics of  $G_f$  in  $\mathfrak{B}_f$  in terms of the skew product  $S_{P,Q} : \mathfrak{U}_{P,Q} \rightarrow \mathfrak{U}_{P,Q}$  defined in Section 3 with  $P = Q = B$ . To lighten the notation, we will denote this skew product by  $S : \mathfrak{U} \rightarrow \mathfrak{U}$ . We shall use the notation  $U_P$  and  $U_Q$  to avoid the confusion between the two copies of  $U_B$  that are glued in order to form the analytic manifold  $\mathfrak{U}$ .

There are sections  $s_{p_0}, s_{q_0}, s_{p_1}, s_{q_1} : \mathbb{D} \rightarrow \mathfrak{U}$  such that for each  $t \in \mathbb{D}$ ,

- $s_{p_0}(t) \in \Sigma_t$  corresponds to  $0 \in U_P$ ,
- $s_{q_0}(t) \in \Sigma_t$  corresponds to  $0 \in U_Q$ ,
- $s_{p_1}(t) \in \Sigma_t$  corresponds to  $-1 \in U_P$  and
- $s_{q_1}(t) \in \Sigma_t$  corresponds to  $-1 \in U_Q$ .

PROPOSITION 5.1. — *There is an isomorphism  $\alpha : \mathfrak{U} \rightarrow \mathfrak{B}_f$  which conjugates the map  $S : \mathfrak{U} \rightarrow \mathfrak{U}$  to  $G_f : \mathfrak{B}_f \rightarrow \mathfrak{B}_f$ , that is  $\alpha \circ S = G_f \circ \alpha$  on  $\mathfrak{U}$ .*

*Proof.* — Note that  $\mathfrak{s} : \mathfrak{U} \rightarrow \mathbb{D}$  is a proper flat family of stable curves of genus zero with distinct sections  $s_{p_0}, s_{q_0}, s_{p_1}, s_{q_1} : \mathbb{D} \rightarrow \mathfrak{U}$ . By universality of  $\overline{\mathfrak{M}}_{\mathcal{P}_f}$ , there are analytic maps  $\alpha : \mathfrak{U} \rightarrow \overline{\mathfrak{M}}_{\mathcal{P}_f}$  and  $\beta : \mathbb{D} \rightarrow \mathbb{P}^1$  such that  $\mathfrak{m} \circ \alpha = \beta \circ \mathfrak{s}$ , the restriction  $\alpha : \Sigma_t \rightarrow \mathfrak{S}_{\beta(t)}$  is an isomorphism for every  $t \in \mathbb{D}$  and  $m_p \circ \beta = \alpha \circ s_p$  for every  $p \in \mathcal{P}_f$ .

Fix  $t \in \mathbb{D}$ . Set  $x := \beta(t)$  and  $y := \beta(t^2)$ . Let

- $\alpha_x : \Sigma_t \rightarrow \mathbb{P}^1$  be the normalized isomorphism which sends  $s_{p_0}(t)$  to 0,  $s_{q_0}(t)$  to  $\infty$  and  $s_{q_1}(t)$  to 1, and
- $\alpha_y : \Sigma_{t^2} \rightarrow \mathbb{P}^1$  be the normalized isomorphism which sends  $s_{p_0}(t^2)$  to 0,  $s_{q_0}(t^2)$  to  $\infty$  and  $s_{q_1}(t^2)$  to 1,

so that  $\alpha_x$  sends  $s_{p_1}(t)$  to  $x$  and  $\alpha_y$  sends  $s_{p_1}(t^2)$  to  $y$ . Then,

$$F := \alpha_y \circ S \circ \alpha_x^{-1} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$$

is a rational map of degree 2 with critical points at 0 and  $\infty$  such that

$$F(\infty) = 1, \quad F(1) = \infty, \quad F(x) = 0 \quad \text{and} \quad F(0) = y.$$

According to Section 4.2,  $y = x^2$  and the following diagram commutes:

$$\begin{array}{ccc} \Sigma_t & \xrightarrow{\alpha} & \mathfrak{S}_x \\ s \downarrow & & \downarrow G_f \\ \Sigma_{t^2} & \xrightarrow{\alpha} & \mathfrak{S}_{x^2}. \end{array}$$

In particular,  $\beta(t^2) = (\beta(t))^2$  and so,  $\beta$  is a power map:  $x = t^k$  for some  $k \geq 1$ .

LEMMA 5.2. — *We have that  $k = 1$ .*

*Proof.* — Since  $U_P = U_Q$ , there is an obvious involution  $\iota : \mathfrak{U} \rightarrow \mathfrak{U}$  exchanging the images of  $U_P$  and  $U_Q$  in  $\mathfrak{U}$ . For  $t \in \mathbb{D}$ , this involution restricts to an isomorphism  $\iota_t : \Sigma_t \rightarrow \Sigma_t$  which exchanges

- $s_{p_0}(t)$  with  $s_{q_0}(x)$ ,
- $s_{p_1}(t)$  with  $s_{q_1}(t)$  and
- the copy of  $K(P)$  in  $\Sigma_t$  with the copy of  $K(Q)$  in  $\Sigma_t$ .

In  $\mathfrak{S}_x$  the points 0 and  $x$  may be separated from the points  $\infty$  and 1 by a Euclidean annulus of modulus  $-\frac{1}{2\pi} \log |x|$ . Thus, in  $\Sigma_t$  the points  $s_{p_0}(t)$  and  $s_{p_1}(t)$  may be separated from the points  $s_{q_0}(t)$  and  $s_{q_1}(t)$  by an annulus of the same modulus  $-\frac{1}{2\pi} \log |x| = -\frac{k}{2\pi} \log |t|$ .

Let  $A(t) \subset \Sigma_t$  be the annulus of maximal modulus separating  $s_{p_0}(t)$  and  $s_{p_1}(t)$  from  $s_{q_0}(t)$  and  $s_{q_1}(t)$  in  $\Sigma_t$ :

$$\text{modulus}(A(t)) \geq -\frac{k}{2\pi} \log |t|.$$

The annuli  $A(t)$  and  $V(t)$  are invariant by the involution  $\iota_x : \Sigma_t \rightarrow \Sigma_t$ . So, they share the same equatorial curve : the image of the equipotential of level  $-\frac{1}{2} \log |t|$ . Cutting  $\Sigma_t$  along this equatorial curve, we deduce that  $U_P$  contains an annulus of modulus  $\frac{1}{2} \text{modulus}(A(t))$  separating 0 and  $x$  from the equipotential of level  $-\frac{1}{2} \log |t|$ . As  $t \rightarrow 0$ , this equipotential is asymptotic to the circle centered at 0 with radius  $\sqrt{|t|}$ . We deduce that

$$\text{modulus}(A(t)) \leq -\frac{1}{2\pi} \log |t| + \mathcal{O}(1).$$

Therefore, as  $t \rightarrow 0$ ,

$$-k \log |t| \leq -\log |t| + \mathcal{O}(1)$$

and  $k = 1$  as required.  $\square$

So,  $x = t$  and  $\beta : \mathbb{D} \rightarrow \mathbb{D}$  is an isomorphism. Thus,  $\alpha : \mathfrak{U} \rightarrow \mathfrak{B}_f$  is an isomorphism. It conjugates  $S : \mathfrak{U} \rightarrow \mathfrak{U}$  to  $G_f : \mathfrak{B}_f \rightarrow \mathfrak{B}_f$  according to the preceding diagram.  $\square$

## 6. Counting combinatorial equivalence classes of twisted basilica matings

For a given integer  $n > 1$ , Proposition 4.6 asserts that the periodic  $n$ -cycles of  $g_f : x \mapsto x^2$  correspond to rational maps which are combinatorially equivalent to twisted matings of  $B^{\text{on}}$  with itself; that is, if  $x \neq 0, 1$  satisfies

$$x^{2^n} - x = 0, \tag{6.1}$$

then  $F_x^n$  is a twisted mating of  $B^{\circ n}$  with  $B^{\circ n}$ . Excluding the values  $x = 0, 1$  there are precisely  $2^n - 2$  distinct solutions of (6.1). This provides an upper bound on the number of combinatorial equivalence classes of rational maps that are twisted matings of  $B^{\circ n}$  with itself.

**PROPOSITION 6.1.** — *Let  $n$  be an integer greater than 1. If  $n$  is even, there are exactly  $2^{n-1} - 1$  combinatorial equivalence classes of rational maps which are twisted matings of  $B^{\circ n}$  with  $B^{\circ n}$ . If  $n$  is odd, there are exactly  $2^n - 2$  combinatorial equivalence classes of rational maps which are twisted matings of  $B^{\circ n}$  with  $B^{\circ n}$ .*

*Proof.* — For an integer  $k \in [1, 2^n - 2]$ , set

$$x_k := e^{2\pi ik/(2^n-1)} \quad \text{and} \quad F_k := F_{x_k}^n,$$

then  $x_k \neq 1$  is a solution of (6.1), and  $F_k$  is a twisted mating of  $B^{\circ n}$  with  $B^{\circ n}$  of twist angle  $-k/(2^n - 1)$  by Proposition 4.6.

Each  $F_k$  is postcritically finite with postcritical set  $\mathcal{P}_{F_k} = \{0, 1, \infty, x_k\}$ .

For  $i \neq j \in [1, 2^n - 2]$ , the rational map  $F_i$  is combinatorially equivalent to  $F_j$  if and only if there is a Möbius transformation  $\mu$  so that  $\mu \circ F_i = F_j \circ \mu$ . If such a  $\mu$  exists, it must map the postcritical set of  $F_i$  to the postcritical set of  $F_j$ . Moreover, it must do this in a way that is compatible with the dynamics of the maps. This means that for such a Möbius transformation  $\mu$ , we necessarily have

- $\mu : 0 \mapsto 0$  and  $\mu : \infty \mapsto \infty$ , or
- $\mu : 0 \mapsto \infty$  and  $\mu : \infty \mapsto 0$ .

In the first case,  $\mu$  must have the form  $\mu : z \mapsto \alpha z$  for some  $\alpha$ . If  $\mu \neq \text{id}$ , then  $\mu : 1 \mapsto x_j$  and  $\mu : x_i \mapsto 1$ , and we see that necessarily  $x_j x_i = 1$ , and  $\mu$  must be of the form  $\mu : z \mapsto x_j z$ .

In the second case,  $\mu$  must have the form  $\mu : z \mapsto \alpha/z$  for some  $\alpha$ . There are two possible subcases:

- $\mu : 1 \mapsto 1$  and  $\mu : x_i \mapsto x_j$ , or
- $\mu : 1 \mapsto x_j$  and  $\mu : x_i \mapsto 1$ .

In the first subcase, we see that  $\alpha = 1$ , and  $1/x_i = x_j$ , or  $x_i x_j = 1$ , and  $\mu$  has the form  $\mu : z \mapsto 1/z$ . In the second subcase, we see that  $\alpha = x_j$ , and then  $\mu : x_i \mapsto 1$  implies that  $x_i = x_j$ , which is impossible since  $i \neq j$ .

In summary, if there is a Möbius transformation  $\mu$  such that  $\mu \circ F_i = F_j \circ \mu$ , then necessarily  $x_i x_j = 1$ , and  $\mu$  must be of the form  $\mu : z \mapsto x_j z$  or  $\mu : z \mapsto 1/z$ .

- If  $n$  is even, then for any  $k \in [1, 2^n - 2]$ ,

$$F_k : 0 \mapsto 0, \quad F_k : \infty \mapsto \infty, \quad F_k : 1 \mapsto 1, \quad \text{and} \quad F_k : x_k \mapsto x_k.$$

In this case,  $F_i$  and  $F_j$  are combinatorially equivalent only if  $x_i x_j = 1$ . We now show that the Möbius transformation  $\mu : z \mapsto 1/z$  provides a conjugacy between  $F_i$  and  $F_j$  if  $x_i x_j = 1$ . To this end, recall the maps

$$G_f : (x, z) \mapsto \left( x^2, \frac{z^2 - x^2}{z^2 - 1} \right) \quad \text{and} \quad \iota_2 : (x, z) \mapsto \left( \frac{1}{x}, \frac{1}{z} \right).$$

from Section 5.2, and recall that  $\iota_2$  is an automorphism of  $G_f^{\circ 2}$ ; that is,  $\iota_2 \circ G_f^{\circ 2} = G_f^{\circ 2} \circ \iota_2$ . Since  $n$  is even,

$$\iota_2 \circ G_f^{\circ n} = G_f^{\circ n} \circ \iota_2.$$

To lighten notation, set  $H := G_f^{\circ n}$ , and let  $\text{pr}_2 : \mathbb{C}^2 \rightarrow \mathbb{C}$  be given by  $\text{pr}_2 : (x, z) \mapsto z$ . We have

$$\iota_2 \circ H = H \circ \iota_2 \text{pr}_2 \circ \iota_2 \circ H = \text{pr}_2 \circ H \circ \iota_2$$

$$\frac{1}{\text{pr}_2(H(x, z))} = \text{pr}_2 \left( H \left( \frac{1}{x}, \frac{1}{z} \right) \right),$$

and in particular,

$$\frac{1}{F_{x_i}(z)} = F_{1/x_i} \left( \frac{1}{z} \right), \quad \text{or} \quad \frac{1}{F_{x_i}(z)} = F_{x_j} \left( \frac{1}{z} \right)$$

since  $x_i x_j = 1$ ; therefore  $F_i$  and  $F_j$  are conjugate via the Möbius transformation  $\mu : z \mapsto 1/z$ . So the condition  $x_i x_j = 1$  is necessary and sufficient to guarantee that  $F_i$  and  $F_j$  are combinatorially equivalent. Hence there are exactly  $(2^n - 2)/2 = 2^{n-1} - 1$  combinatorial equivalence classes of rational maps which are twisted matings of  $B^{\circ n}$  with itself.

- If  $n$  is odd, then for any  $k \in [1, 2^n - 2]$ ,

$$F_k : 0 \mapsto x_k, \quad F_k : x_k \mapsto 0, \quad F_k : \infty \mapsto 1, \quad \text{and} \quad F_k : 1 \mapsto \infty.$$

In this case, neither Möbius transformation is compatible with the dynamics of  $F_i$  and  $F_j$ . Indeed, if  $x_i x_j = 1$ , and  $\mu : z \mapsto x_j z$ , then

$\mu(F_i(0)) = 1$ , but  $F_j(\mu(0)) = x_j \neq 1$ , so this  $\mu$  cannot conjugate the map  $F_i$  to the map  $F_j$ . Now consider the map  $\mu : z \mapsto 1/z$ :  $\mu(F_i(0)) = x_j$ , but  $F_j(\mu(0)) = 1 \neq x_j$ . Thus for all  $i, j \in [1, 2^n - 2]$   $i \neq j$ ,  $F_i$  and  $F_j$  cannot be Möbius conjugate, so they are necessarily in distinct combinatorial equivalence classes, and there are  $2^n - 2$  combinatorial equivalence classes of rational maps which are twisted matings of  $B^{\text{on}}$  with itself.  $\square$

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