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Introduction

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Introduction

At the Bourbaki seminar of Fall 1982, Adrien Douady sketched the theory of the iteration of rational functions. He writes [1]:

"L'étude de l'itération des polynômes et des fractions rationnelles complexes a été inaugurée en 1918/1919 par Julia et Fatou. Mais, à part le Théorème de Siegel en 1942 et ses perfectionnements ultérieurs et l'article de Brohlin en 1966 (...) le sujet a pratiquement dormi jusqu'à ces dernières années (...) C'est la possibilité de faire des expériences numériques sur ordinateur, voire sur un microordinateur, qui l'a réveillé, avec les observations de Feigenbaum pour le cas réel (...) et pour le cas complexe celles de B.Mandelbrot pour l'ensemble que nous appelons pour cela M, (...)"[1]

This seminar occured just "two years after this awakening" (as said A. Douady, so an awakening around 1980). In this paper, Adrien Douady already introduces the notion of mating ([1]§3). Two marvelous drawing of the Julia sets of rational maps illustrate this notion. The picture below (Figure 1) reproduces one of them (originally drawn by J.H. Hubbard) to give a rough idea. It represents the mating of two polynomials of the form $z^2 + c$ with $c \sim -1,3 + 0,74i$ (usually called "rabbits").

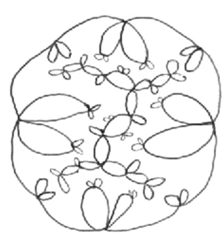

Figure 1. — Mating of two rabbits

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(1) "The study of iteration of complex polynomials and rational maps was inaugurated in 1918/1919 by Julia and Fatou. But, apart from Siegel's Theorem in 1942 and its later improvements and the article of Brohlin in 1966 (...) the subjet has pratically slept until these last years (...). It is the possibility to make numerical experiments on computer and even on microcomputer that revived it, with the observations by Feigenbaum in the real case (...) and in the complex case those by B. Mandelbrot for the set that we call for this reason M, (...)"

Benoît Mandelbrot relates in his book [2] :

"Later I investigated the map $g(\lambda, z) = \lambda(z + 1/z)$. *(...). I described this paper to Adrien Douady on a day when I drove him from NewYork City to catch a plane at JFK airport. (...) Be that as it may, Douady went on to describe the interlacing that characterizes Figure 2 as a "marriage". Choosing a different* λ*, he mated a "rabbit" and a "cathedral", (...) ."*

Here is a picture similar to the one presenting the interlacing mentioned by B. Mandelbrot. It represents the matings of polynomials still called rabbits but the values are a little different.

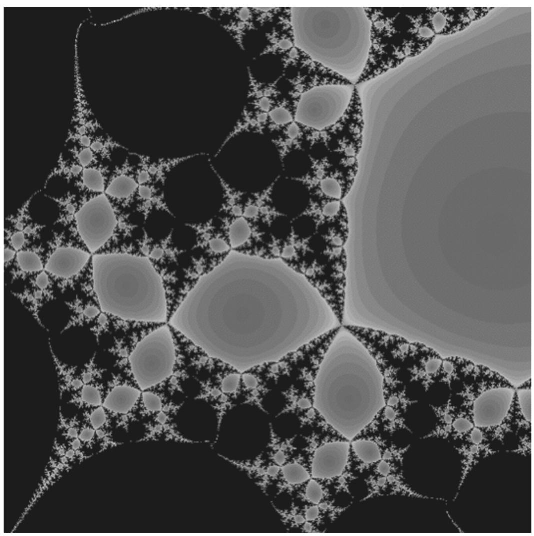


Figure 1. — Mating of other kinds of rabbits

The birth of the notion of matings (and all the theory that came with) was certainly impulsed by the production of new computer pictures.

Let us now give the first precise definition extracted from [1].

Let $f : \Sigma \to \Sigma$ (where Σ is the Riemann sphere) a rational map and P a polynomial such that its filled Julia set K_P is connected. One says that P *figures* in f if there exist a continuous map $\varphi : K_p \to \Sigma$ such that :

- (i)$f \circ \varphi = \varphi \circ P$;
- (ii) φ is analytic on K_P ;
- (iii) φ is a uniform limit of continuous injections from K_P to Σ.

Let P_1 and P_2 be two polynomials of degree d such that the Julia sets K_{P_1} and K_{P_2} are connected and locally connected. One says that a rational map f of degree d is a *mating* of P_1 and P_2 if there exist maps ϕ_1 and ϕ_2 letting P_1 and P_2 figure in f, and such that the diagram

$$\begin{array}{ccccc} & & K_{P_1} & & \\ & \gamma_1 \nearrow & & \searrow \varphi_1 & \\ \mathbf{T} & & & & \Sigma \\ & \tilde{\gamma}_2 \searrow & & \nearrow \varphi_2 & \\ & & K_{P_2} & & \end{array}$$

where $\tilde{\gamma}_2(t) = \gamma_2(-t)$, is cocartésien in the category of Hausdorff spaces. Here $\gamma_s : \mathbf{T} \to K_{P_s}$ is the Carathéodory curve parameterizing the boundary of K_{P_s} by the unit circle $\mathbf{T}$.

In this definition, one recognizes in a rational map the Julia sets of two polynomials glued along their boundaries. The mating appears as a procedure, as well as the surgery, to describe the dynamics of a rational map starting with two polynomials.

Ten years later, J. Milnor proposed a new vision of the matings in his article [3]. This new definition was motivated by the attempt to generalize the result of Tan Lei to the whole Mandelbrot set (in particular to polynomials having non locally connected Julia sets). This generalization can be stated as follows: if f_a and f_b do not belong to limbs of the Mandelbrot set which are complex conjugated, it is possible to define a mating $f_a \sqcup f_b$ (this is part of the "Quadratic Mating Conjecture").

The idea of this new definition is to glue neighborhoods of the Julia sets of the two polynomials along an equipotential. One gets on that way a sphere and also a dynamics by juxtaposing both dynamics. The range is no more the same sphere as the one of departure, but rather another sphere obtained by gluing further equipotentials (the neighborhoods of the Julia sets are larger). Normalizing this procedure, one gets a sequence of rational maps defined on the Riemann sphere. If this sequence converges, the limit will be called *the mating of the two polynomials*. This notion is nowadays called *slow-mating* (see also the articles of Meyer-Petersen, of Chéritat and of Buff-Epstein-Koch in this volume). One does not need for this construction to have locally connected Julia sets.

This definition of slow matings was inspiring for numerous works and in particular allowed X. Buff and A. Chéritat to develop computer programs. Improving those programs, A.Chéritat produces now films showing the procedure of matings steps by steps, namely one can see the rays shrinking to points. This new visualization is a succession of pictures such as the one at the beginning of this volume (due to A. Chéritat) showing the mating of the polynomial $z^2 + c$ with $c = -0.126520 + 1.032247i$ (usually called "Kokopeli") with the polynomial $z^2 + c'$ with $c' = -1.754877666$ (usually called the "airplane"). Thanks to those kind of movies one can finally "see" and better understand the mating procedure. The article of X. Buff, S. Koch and A. Epstein gives the basis for the theory underlying these numerical experiments.

It is not possible to describe here all the developments of the theory of matings during the last thirty years. Renewed interest was shown recently, maybe due to the novel way to visualize them.

This volume naturally appeared after the conference "Polynomial matings", held in Toulouse during June 2001, funded by the ANR grant ANR-08-JCJC-002 of the Agence National de la Recherche, the Institute of Mathematics of Toulouse and the Institut Universitaire de France. The purpose of this conference was to combine introductory lectures to this notion together with the presentation of recent developments.

This volume contains 11 carefully refereed articles written by participants of the Conference. We give now a quick overview of them.

– The article "On the notions of mating" by D. Meyer and C. Petersen presents all the background necessary to understand the construction of matings. The relations between the various definitions of matings is the subject of this paper. They describe also the obstructions to the existence of the mating of two polynomials. These obstructions can be just topological when the quotient space where the mating should occur is not a sphere. When it is a sphere, and in the particular case when the critical points have finite orbits (post-critically finite case), an other type of obstructions comes from the Thurston Theorem which characterizes rational maps among branched coverings of the sphere.

– Recall that the original purpose of the theory of matings was to describe the dynamics of rational maps (as a combination of the dynamics of two polynomials), with the motivation of classifying rational maps (dynamically) and of describing their parameter spaces. The articles of I. Mashanova and V. Timorin as well as the one of A. Epstein and T. Sharland are in this spririt.

- In their article "Captures, matings and regluing", I. Mashanova and V. Timorin work in the parameter space of quadratic rational maps. They give topological models for these rational maps in the boundary of some hyperbolic components within certain slices. As a result of this, many such boundary points can be represented as matings. In particular, a regluing of a capture which represents a boundary point of the hyperbolic component is a mating, under certain conditions.

- The article "A classification of bicritical rational maps with a pair of period two super-attracting cycles" by A. Epstein and T. Sharland considers those rational maps of degree $d \geqslant 3$ with exactly two critical points, both of which are periodic of period two. The dynamics is described (combinatorially) by proving that any such maps is the mating of two unicritical degree d polynomials.

– The mating of two polynomials is not always neither Thurston equivalent nor topologically conjugated to a rational map : there can be topological or analytic obstructions.

- In his article "Tan Lei and Shishikura's example of non matable degree 3 polynomials without a *Levy cycle*", Arnaud Chéritat describes an example of obstructed mating. Since the obstruction in this case is not the classical one encountered in degree 2, namely is not a Levy cycle, the goal here is to get a clue on this obstruction. In order to visualize where the obstruction comes from, A. Chéritat uses the construction of slow matings, procedure that he describes in details. He gives conformally correct computer generated pictures and a conjectural interpretation of the obstruction given by these pictures. Finally, the section "zooming in a tunnel" presents some heuristically discovered phenomena concerning the rates of growth of some moduli.

- In the article "On a theorem of Rees-Shishikura", G. Cui, W. Peng and L. Tan simplify the original proof the theorem saying that if the formal mating of two post-critically finite polynomials is Thurston equivalent to a rational map, then the topological mating is conjugate to the rational map. Recall that saying that post-critically finite maps F, G are Thurston equivalent means that there exist orientation preserving homeomorphisms ϕ_0, ϕ_1 of $\widehat{\mathbf{C}}$ such that ϕ_1 is isotopic to ϕ_0 relatively to the post-critical set such that $\phi_0 \circ F = G \circ \phi_1$.

  This simplification allows them to extend the result to post-critically finite branched covering. They obtain that if F is a post-critically finite branched covering, which is holomorphic in a neighborhood of the critical cycles, and which is Thurston equivalent to a rational map f, then the pull-back sequence ϕ_n (satisfying $\phi_n \circ F = f \circ \phi_{n+1}$) converges uniformly to a semi-conjugacy satisfying several good properties.

- Recall that Thurston's Theorem gives a criterium for a map to be Thurston equivalent to a rational map. The map which Thurston introduced in this characterization is studied by X. Buff, A. Epstein and S. Koch in their article "Twisted matings and equipotential glu-ings". This Thurston map encodes for instance the pullback sequence as explained above, and any of its fixed points provides a conjugacy to a rational map. In their article, they obtain that the slow-mating of the basilica with the basilica, which is obstructed, is in fact realized by a rational map when twisted along the equator by some specific angle, during the slow mating.

  As a by product, they settle the basis of the theory lying under the computer programs done by A. Chéritat, programs which produce the movies of matings.

– Thurston's Theorem of characterization of rational functions is a fundamental tool in order to know if a mating can be realized as a rational map. Like in the previous article of X. Buff, A. Epstein and S. Koch, the articles of K. Pilgrim and also of S. Godillon study from an algebraic point of view this theory.

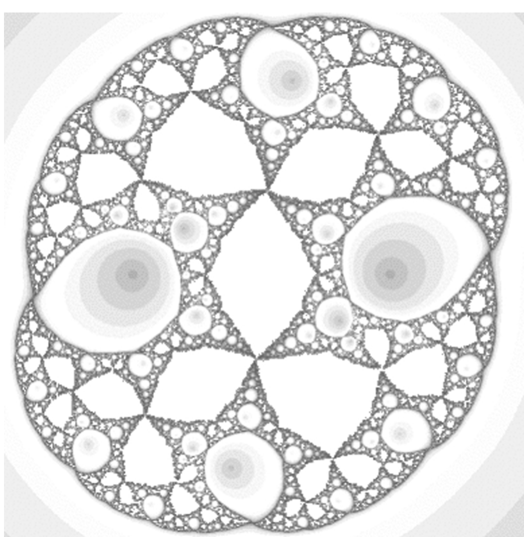
- The paper "An algebraic formulation of Thurston's characterization of rational functions" of K. Pilgrim studies pull-backs of (multi)curves and Dehn twists by post-critically finite branched coverings of the sphere, and formulates Thurston's characterization of rational functions in terms of the virtual endomorphism of the mapping class group naturally associated with the branched covering. it is also shown when the pull-back map acting on multi-curves has a finite global attractor.
- " Introduction to Iterated Monodromy Groups" by S. Godillon is an exposition of new algebraic techniques. In particular, it is explained how to see that a rational map f is not combinatorially equivalent to a formal mating. Namely, it suffices to check some algebraic conditions on the Iterated Monodromy Group, group which is presented in details. This will be probably a powerful tool.

– The mating of polynomials can be seen as particular case of the combination of two dynamical systems on the sphere, which includes also the combination of the action of a Fuchsian group on one hemisphere and a polynomial on the other hemisphere or the combination of two Fuchsian groups. This is the subject of the two next articles.

- In "A holomorphic correspondence at the boundary of the Klein combination locus", S. Bullet and W. Curtis focus on a family called "matings of the modular group with quadratic polynomials". The parameter space of this family contains a compact subset which is conjecturally homeomorphic to the Mandelbrot set and whose parameters represent matings with quadratic polynomials with a connected Julia set. Bullett and Curtis study particular families of correspondences lying on the boundary of this set. The dynamics of those parameters is supposed to represent the mating of the modular group with a quadratic polynomial with disconnected Julia set.

- The article "Matings and the other side of the dictionary" by J. H. Hubbard, provides a new light on the subject. Sullivan's dictionary shows precise and deep analogies between Complex dynamics and Kleinian groups. Here, J. Hubbard develops a new line of this dictionary explaining the analogy between the mating of polynomials and the "Double Limit Theorem" of Thurston.

– The last article is in fact a short survey, the goal of which being to present a list of questions and problems concerning the mating of polynomials. It was written by X. Buff, A. Epstein, S. Koch, D. Meyer, K. Pilgrim, M. Rees and Tan Lei. Most of the questions were posed during the question session of the conference but some arose also later. It should certainly serve as a point of departure for future work and even maybe on the other side of the dictionary.



This last picture represents the Julia set of the rational map $\frac{z^2+c}{z^2-1}$ with $c = \frac{1+\sqrt{-3}}{2}$ which can be described as the mating of the rabbit and the basilica. It appears as a logo on the first page of the preprints of Stony Brook. A mark of the importance of this notion? of its beauty? certainly a mark that it will last through times.

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