Tien Duc Luu

On some properties of three-dimensional minimal sets in $\mathbb{R}^4$


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Tien Duc LUU(1)

ABSTRACT. — We prove in this paper the Hölder regularity of Almgren minimal sets of dimension 3 in $\mathbb{R}^4$ around a $Y$-point and the existence of a point of particular type of a Mumford-Shah minimal set in $\mathbb{R}^4$, which is very close to a $T$. This will give a local description of minimal sets of dimension 3 in $\mathbb{R}^4$ around a singular point and a property of Mumford-Shah minimal sets in $\mathbb{R}^4$.

RÉSUMÉ. — On prouve dans cet article la régularité Höldérienne pour les ensembles minimaux au sens d’Almgren de dimension 3 dans $\mathbb{R}^4$ autour d’un point de type $Y$ et dans le cas d’un ensemble Mumford-Shah minimal dans $\mathbb{R}^4$ qui est très proche d’un $T$, l’existence d’un point avec une densité particulière. Cela donne une description locale des ensembles minimaux de dimension 3 dans $\mathbb{R}^4$ autour d’un point singulier et une propriété des ensembles Mumford-Shah minimaux dans $\mathbb{R}^4$.

1. Introduction

In this paper we will prove two theorems. The first theorem is about local Hölder regularity of three-dimensional minimal sets in $\mathbb{R}^4$ and the second theorem is about the existence of a point of a particular type of a Mumford-Shah minimal set, which is close enough to a cone of type $T$.

Let us give the list of notions that we will use in this paper.

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(1) Bâtiment 430, Département de Mathématique, Université Paris Sud XI, 91405 Orsay
luutienduc@gmail.com
Article proposé par Gilles Carron.
$H^d$ the $d$-dimensional Hausdorff measure.

$$\theta_A(x,r) = \frac{H^d(A \cap B(x,r))}{r^d},$$

where $A \subset \mathbb{R}^n$ is a set of dimension $d$ and $x \in A$.

$$\theta_A(x) = \lim_{r \to 0} \theta_A(x,r),$$
called the density of $A$ at $x$, if the limit exists.

Local Hausdorff distance $d_{x,r}(E,F)$. Let $E,F \subset \mathbb{R}^n$ be closed sets which meet the ball $B(x,r)$. We define

$$d_{x,r}(E,F) = \frac{1}{r} \left[ \sup \{ \text{dist}(z,F) ; x \in E \cap B(x,r) \} + \sup \{ \text{dist}(z,E) ; z \in F \cap B(x,r) \} \right].$$

Let $E,F \subset \mathbb{R}^n$ be closed sets and $H \subset \mathbb{R}^n$ be a compact set. We define

$$d_H(E,F) = \sup \{ \text{dist}(x,F) ; x \in E \cap H \} + \sup \{ \text{dist}(x,E) ; x \in F \cap H \}.$$

Convergence of a sequence of sets. Let $U \subset \mathbb{R}^n$ be an open set, $\{E_k\} \subset U$, $k \geq 1$, be a sequence of closed sets in $U$ and $E \subset U$. We say that $\{E_k\}$ converges to $E$ in $U$ and we write $\lim_{k \to \infty} E_k = E$, if for each compact $H \subset U$, we have

$$\lim_{k \to \infty} d_H(E_k,E) = 0.$$

Blow-up limit. Let $E \subset \mathbb{R}^n$ be a closed set and $x \in E$. A blow-up limit $F$ of $E$ at $x$ is defined as

$$F = \lim_{k \to \infty} \frac{E - x}{r_k},$$

where $\{r_k\}$ is any positive sequence such that $\lim_{k \to \infty} r_k = 0$ and the limit is taken in $\mathbb{R}^n$.

Now we give the definition of Almgren minimal sets of dimension $d$ in $\mathbb{R}^n$.

**Definition 1.1.** — Let $E$ be a closed set in $\mathbb{R}^n$ and $d \leq n - 1$ be an integer. An Almgren competitor (Al-competitor) of $E$ is a closed set $F \subset \mathbb{R}^n$ that can be written as $F = \varphi(E)$, where $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz mapping such that $W_\varphi = \{ x \in \mathbb{R}^n ; \varphi(x) \neq x \}$ is bounded.

An Al-minimal set of dimension $d$ in $\mathbb{R}^n$ is a closed set $E \subset \mathbb{R}^n$ such that $H^d(E \cap B(0,R)) < +\infty$ for every $R > 0$ and

$$H^d(E \setminus F) \leq H^d(F \setminus E)$$

for every Al-competitor $F$ of $E$. 

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Next, we give the definition of Mumford-Shah (MS) minimal sets in \( \mathbb{R}^n \).

**Definition 1.2.** — Let \( E \) be a closed set in \( \mathbb{R}^n \). A Mumford-Shah competitor (also called MS-competitor) of \( E \) is a closed set \( F \subset \mathbb{R}^n \) such that we can find \( R > 0 \) such that

\[
F \setminus B(0, R) = E \setminus B(0, R)
\]

(1.2.1)

and \( F \) separates \( y, z \in \mathbb{R}^n \setminus B(0, R) \) when \( y, z \) are separated by \( E \).

A Mumford-Shah minimal (MS-minimal) set in \( \mathbb{R}^n \) is a closed set \( E \subset \mathbb{R}^n \) such that

\[
H^{n-1}(E \setminus F) \leq H^{n-1}(F \setminus E)
\]

(1.2.2)

for any MS-competitor \( F \) of \( E \).

Here, \( E \) separates \( y, z \) means that \( y \) and \( z \) lie in different connected components of \( \mathbb{R}^n \setminus E \).

It is easy to show that any MS-minimal set in \( \mathbb{R}^n \) is also an Al-minimal set of dimension \( n - 1 \) in \( \mathbb{R}^n \). Next, if \( E \) is an MS-minimal set in \( \mathbb{R}^n \), then \( E \times \mathbb{R} \) is also an MS-minimal set in \( \mathbb{R}^n \times \mathbb{R} \), by exercice 16, p 537 of [5].

We give now the definition of minimal cones of type \( P \), \( Y \) and \( T \), of dimension 2 and 3 in \( \mathbb{R}^n \).

**Definition 1.3.** — A two-dimensional minimal cone of type \( Y \) is just a two-dimensional affine plane in \( \mathbb{R}^n \). A three-dimensional minimal cone of type \( P \) is a three-dimensional affine plane in \( \mathbb{R}^n \).

Let \( S \) be the union of three half-lines in \( \mathbb{R}^2 \subset \mathbb{R}^n \) that start from the origin 0 and make angles \( 120^\circ \) with each other at 0. A two-dimensional minimal cone of type \( Y \) is set of the form \( Y' = j(S \times L) \), where \( L \) is a line passing through 0 and orthogonal to \( \mathbb{R}^2 \) and \( j \) is an isometry of \( \mathbb{R}^n \). A three-dimensional minimal cone of type \( Y \) is a set of the form \( Y = j(S \times P) \), where \( P \) is a plane of dimension 2 passing through 0 and orthogonal to \( \mathbb{R}^2 \) and \( j \) is an isometry of \( \mathbb{R}^n \). We call \( j(L) \) the spine of \( Y' \) and \( j(P) \) the spine of \( Y \).

Take a regular tetrahedron \( R \subset \mathbb{R}^3 \subset \mathbb{R}^n \), centered at the origin 0, let \( K \) be the cone centered at 0 over the union of the 6 edges of \( R \). A two-dimensional minimal cone of type \( T \) is of the form \( j(K) \), a three-dimensional minimal cone of type \( T \) is a set of the form \( T = j(K \times L) \), where \( L \) is the line passing through 0 and orthogonal to \( \mathbb{R}^3 \) and \( j \) is an isometry of \( \mathbb{R}^n \). We call \( j(L) \) the spine of \( T \).
We denote by $d_P, d_Y, d_T$ the densities at the origin of the 3-dimensional minimal cones of type $\mathbb{P}$, $\mathbb{Y}$ and $\mathbb{T}$, respectively. It is clear that $d_P < d_Y < d_T$.

We can now define a Hölder ball for a set $E \subset \mathbb{R}^n$.

**Definition 1.4.** — Let $E$ be a closed set in $\mathbb{R}^n$. Suppose that $0 \in E$. We say that $B(0, r)$ is a Hölder ball of $E$, of type $\mathbb{P}, \mathbb{Y}$ or $\mathbb{T}$ with exponent $1 + \alpha$, if there exists a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ and a cone $Y$ of dimension 2 or 3, centered at the origin, of type $\mathbb{P}, \mathbb{Y}$ or $\mathbb{T}$, respectively, such that

$$|f(x) - x| \leq \alpha r \text{ for } x \in B(0, r) \quad (1.4.1)$$

$$\left(1 - \alpha\right) \left|\frac{x - y}{r}\right|^{(1 + \alpha)} \leq \left|\frac{f(x) - f(y)}{r}\right| \leq \left(1 + \alpha\right) \left|\frac{x - y}{r}\right|^{(1 - \alpha)} \text{ for } x, y \in B(0, r) \quad (1.4.2)$$

$$E \cap B(0, (1 - \alpha)r) \subset f(Y \cap B(0, r)) \subset E \cap B(0, (1 + \alpha)r). \quad (1.4.3)$$

For the sake of simplicity, we will say that $E$ is Bi-Hölder equivalent to $Y$ in $B(0, r)$, with exponent $1 + \alpha$.

If in addition, our function $f$ is of class $C^{1,\alpha}$, then we say that $E$ is $C^{1,\alpha}$ equivalent to $Y$ in the ball $B(0, r)$. Here, $f$ is said to be of class $C^{1,\alpha}$ if $f$ is differentiable and its differential is a Hölder continuous function, with exponent $\alpha$.

J. Taylor in [11] has obtained the following theorem about local $C^1$-regularity of two-dimensional minimal sets in $\mathbb{R}^3$.

**Theorem 1.5.** [11]. — Let $E$ be a two-dimensional minimal set in $\mathbb{R}^3$ and $x \in E$. Then there exists a radius $r > 0$ such that in the ball $B(x, r)$, $E$ is $C^{1,\alpha}$ equivalent to a minimal cone $Y(x, r)$ of dimension 2, of type $\mathbb{P}, \mathbb{Y}$ or $\mathbb{T}$. Here $\alpha$ is a universal positive constant.

As we know, any two-dimensional minimal cone in $\mathbb{R}^3$ is automatically of type $\mathbb{P}, \mathbb{Y}$ or $\mathbb{T}$. This is a great advantage when we study two-dimensional minimal sets of dimension 2 in $\mathbb{R}^3$, because each blow-up limit at some point of a two-dimensional minimal set is a minimal cone of the same dimension. So we can approximate our minimal set by cones which we know the structure of.

The problem of two-dimensional minimal sets in $\mathbb{R}^n$ with $n > 3$ is more difficult. Here we don’t know the list of two-dimensional minimal cones. But G. David gives in section 14 of [3] a description of two-dimensional minimal
On some properties of three-dimensional minimal sets in $\mathbb{R}^4$ cones in $\mathbb{R}^n$. Thanks to this, he can prove the local Hölder regularity of two-dimensional minimal sets in $\mathbb{R}^n$.

**Theorem 1.6.** [3].— *Let $E$ be a two-dimensional minimal set in $\mathbb{R}^n$ and $x \in E$. Then for each $\alpha > 0$, there exists a radius $r > 0$ such that in the ball $B(x, r)$, $E$ is Hölder equivalent to a two-dimensional minimal cone $Y(x, r)$, with exponent $\alpha$.*

The $C^1$ regularity of two-dimensional minimal sets in $\mathbb{R}^n$ needs more efforts. We have to prove that the local distance between $E$ and a two-dimensional minimal cone in $B(x, r)$ is of order $r^a$, where $a$ is a positive universal constant when $r$ tends to 0. G. David in [4] shows the $C^1$ regularity of $E$ locally around $x$, but he needs to add an additional condition, called ”full length” to some blow-up limit of $E$ in $x$.

**Theorem 1.7.** [4].— *Let $E$ be a two-dimensional minimal set in the open set $U \subset \mathbb{R}^n$ and $x \in E$. We suppose that some blow-up limit of $E$ at $x$ is a full length minimal cone. Then there is a unique blow-up limit $X$ of $E$ at $x$, and $x + X$ is tangent to $E$ at $x$. In addition, there is a radius $r_0 > 0$ such that $E$ is $C^{1,\alpha}$ equivalent to $x + X$ in the ball $B(x, r_0)$, where $\alpha > 0$ is a universal constant.*

Let us say more about the “full length” condition for a two dimensional minimal cone $F$ centered at the origin in $\mathbb{R}^n$. As in [3, Sect 14], the set $K = F \cap \partial B(0, 1)$ is a finite union of great circles and arcs of great circles $C_j, j \in J$. The $C_j$ can only meet when they are arcs of great circles and only by sets of 3 and at a common endpoint. Now for each $C_j$ whose length is more than $\frac{9\pi}{10}$, we cut $C_j$ into 3 sub-arcs $C_{j,k}$ with the same length so that we have a decomposition of $K$ into disjoint arcs of circles $C_{j,k}, (j, k) \in \tilde{J}$ with the same length and for each $C_{j,k}$, we have $\text{length}(C_{j,k}) \leq \frac{9\pi}{10}$. The full length condition says that if we have another net of geodesics $K_1 = \bigcup_{(i,j) \in \tilde{J}} C^1_{j,k}$, for which the Hausdorff distance $d(C_{j,k}, C^1_{j,k}) \leq \eta$, where $\eta$ is a small constant which depends only on $n$, and if $H^1(K_1) > H^1(K)$, then we can find a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(x) = x$ out of the ball $B(0, 1)$ and $f(B(0, 1)) \subset B(0, 1)$ such that $H^2(f(F_1) \cap B(0, 1)) \leq H^2(F_1 \cap B(0, 1)) - C[H^1(K_1) - H^1(K)]$. Here $C > 0$ is a constant and $F_1$ is the cone over $K_1$. See [4, Sect 2] for more details.

It happens that all two-dimensional minimal cones in $\mathbb{R}^3$ satisfy the full length condition. So the theorem of G. David is a generalization of the theorem of J. Taylor.
For minimal sets of dimension $\geq 3$, little is known. Almgren in [1] showed that if $F$ is a three-dimensional minimal cone in $\mathbb{R}^4$, centered at the origin and over a smooth surface in $S^3$, the unit sphere of dimension 3, then $E$ must be a 3-plane. Then J. Simon in [10] showed that this is true for hyper minimal cones in $\mathbb{R}^n$ with $n < 7$. That is, if $F$ is a minimal cone of dimension $n - 1$ in $\mathbb{R}^n$, centered at the origin and over a smooth surface in $S^{n-1}$, then $F$ must be an $n - 1$ plane. There is no theorem yet about the regularity of minimal sets of dimension $\geq 3$ with singularities.

Our first theorem is to prove a local Hölder regularity of three-dimensional minimal sets in $\mathbb{R}^4$. But we don’t know the list of three-dimensional minimal cones in $\mathbb{R}^4$ and we don’t have a nice description of three-dimensional minimal cones as we have for two-dimensional minimal cones. So we shall restrict to some particular type of points, at which we can obtain some information about the blow-up limits.

Now let $E$ be a three-dimensional minimal set in $\mathbb{R}^4$ and $x \in E$. We want to show that $E$ is Bi-Hölder equivalent to a three-dimensional minimal cone of type $\mathbb{P}$ or $\mathbb{Y}$ in the ball $B(x, r)$, for some radius $r > 0$. If $\theta_E(x) = d_P$, then W. Allard in [2] showed that there exists a radius $r > 0$ such that in the ball $B(x, r)$, $E$ is $C^1$ equivalent to a 3-dimensional plane. We consider then the next possible density of $E$ at $x$, so we suppose that $\theta_E(x) = d_Y$. Since every blow-up limit of $E$ at $x$ is a 3-dimensional minimal cone of type $\mathbb{Y}$, then for each $\epsilon > 0$, there exists a radius $r > 0$ and a 3-dimensional minimal cone $Y(x, r)$ of type $\mathbb{Y}$ such that

$$d_{x,r}(E, Y(x, r)) \leq \epsilon. \tag{*}$$

By using (*) and the minimality of $E$, we shall be able to approximate $E$ by 3-dimensional minimal cones of type $\mathbb{P}$ or $\mathbb{Y}$ at every point in $E \cap B(x, r/2)$ and at every scale $t \leq r/2$. We shall then use Theorem 1.1 in [6] to conclude that $E$ is Bi-Hölder equivalent to a 3-dimensional minimal cone of type $\mathbb{Y}$ in the ball $B(x, r/2)$. Our first theorem is the following.

**Theorem 1.** — Let $E$ be a 3-dimensional minimal set in $\mathbb{R}^4$ and $x \in E$ such that $\theta_E(x) = d_Y$. Then for each $\alpha > 0$, we can find a radius $r > 0$, which depends also on $x$, such that $B(x, r)$ is a Hölder ball (see Def 1.4) of type $\mathbb{Y}$ of $E$, with exponent $1 + \alpha$.

Our second theorem concerns Mumford-Shah minimal sets in $\mathbb{R}^4$. In [3], G. David showed that there are only 3 types of Mumford-Shah minimal sets in $\mathbb{R}^3$, which are the cones of type $\mathbb{P}$, $\mathbb{Y}$ and $\mathbb{T}$. The most difficult part is to show that if $F$ is a Mumford-Shah minimal set in $\mathbb{R}^3$, which is close enough in $B(0, 2)$ to a $\mathbb{T}$ centered at 0, then there must be a $\mathbb{T}$-point of $F$ in $B(0, 1)$. To prove this proposition, G. David used very nice techniques which involve
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the list of connected components. We want to obtain a similar result for a Mumford-Shah minimal set in $\mathbb{R}^4$ which is close enough to a $T$ of dimension 3. But we cannot obtain a result which is as good as in [3, 18.1]. The reason is that we don’t know if there exists a minimal cone $C$ of dimension 3 in $\mathbb{R}^4$, centered at 0, which satisfies $d_\gamma < \theta_C(0) < d_T$. Our second theorem is the following.

**Theorem 2.** There exists an absolute constant $\epsilon > 0$ such that the following holds. Let $E$ be an MS-minimal set in $\mathbb{R}^4$, $r > 0$ be a radius, and $T$ be a 3-dimensional minimal cone of type $T$ centered at the origin such that

$$d_{0,r}(E, T) \leq \epsilon.$$

Then in the ball $B(0, r)$, there is a point of $E$ which is neither of type $P$ nor $\Upsilon$.

See Definition 2.5 for the definition of points of type $P$ and $\Upsilon$. We divide the paper into two parts. In the first part, we prove Theorem 1. In the second part, we prove Theorem 2.

I would like to thank Professor Guy David for many helpful discussions on this paper.

2. Hölder regularity near a point of type $\Upsilon$

for a 3-dimensional minimal set in $\mathbb{R}^4$

In this section we prove Theorem 1. We start with the following lemma.

**Lemma 2.1.** Let $F$ be a 3-dimensional minimal cone in $\mathbb{R}^4$, centered at the origin, and let $x \in F \cap \partial B(0,1)$. Then each blow-up limit $G$ of $F$ at $x$ is a 3-dimensional minimal cone $G$ of type $P$, $\Upsilon$ or $T$ and centered at 0. The type of $G$ depends only on $x$ and $\theta_E(x) = \theta_G(0)$.

We define the type of $x$ to be the type of $G$.

**Proof.** We denote by $0x$ the line passing by 0 and $x$. Suppose that $G$ is a blow-up limit of $F$ at $x$. Then $G = \lim_{k \to \infty} \frac{F-x}{r_k}$ with $\lim_{k \to \infty} r_k = 0$. Let $y \in G$, we want to show that $y + 0x \subset G$. Setting $F_k = \frac{F-x}{r_k}$, as $\{F_k\}$ converges to $G$, we can find a sequence $y_k \in F_k$ such that $\{y_k\}_{k=1}^\infty$ converges to $y$. Setting $z_k = r_k y_k + x$, then $z_k \in F$ by definition of $F_k$, and $z_k$ converges to $x$ because $r_k$ converges to 0. We fix $\lambda \in \mathbb{R}$ and we set $v_k = (1 + \lambda r_k)z_k$. Then $v_k \in F$ as $F$ is a cone centered at 0. We have next that $w_k = r_k^{-1}(v_k - x) \in F_k$. On the other hand,
\[
w_k = r^{-1}((1 + \lambda r_k)z_k - x)
\]
\[
= r^{-1}((1 + \lambda r)(rk y_k + x) - x)
\]
\[
= r^{-1}(rk y_k + \lambda r^2 k y_k + \lambda rk x)
\]
\[
= y_k + \ldots (0)\left((r + 1)^3/r^3\right) \geq \theta_F(x)
\]
for each \(r > 0\). We let \(r \to +\infty\) and we obtain then \(\theta_F(0) \geq \theta_F(x)\), which is (2.2.2).

We see from this lemma that for each \(x \in F \setminus \{0\}\), where \(F\) is a 3-dimensional minimal cone in \(\mathbb{R}^4\) centered at the origin,
\[
\theta_F(x)\]
can take only one of the three values \(d_P, d_Y, d_T\). \hfill (1)

But we do not know the list of possible values of \(\theta_F(0)\). However, the following lemma says that for this cone \(F\), it is not possible that \(d_P < \theta_F(0) < d_Y\).

**Lemma 2.2.** — There does not exist a 3-dimensional minimal cone \(F\) in \(\mathbb{R}^4\), centered at the origin such that \(d_P < \theta_F(0) < d_Y\).

**Proof.** — Suppose that there is a cone \(F\) as in the hypothesis and
\[
d_P < \theta_F(0) < d_Y. \hfill (2.2.1)
\]

We first show that
\[
\text{for each } x \in F \cap \partial B(0, 1), \text{ we have } \theta_F(0) \geq \theta_F(x). \hfill (2.2.2)
\]

Indeed, since \(F\) is a minimal cone, for each \(z \in F\), the function \(\theta_F(z, t)\) is nondecreasing. So for \(r > 0\), we have \(\theta_F(x, r) \geq \theta_F(x)\), which means that \(H^3(F \cap B(x, r))/r^3 \geq \theta_F(x)\). Since \(B(x, r) \subset B(0, r + 1)\), we obtain \(H^3(F \cap B(x, r)) \leq H^3(F \cap B(0, r + 1))\) and thus \(H^3(F \cap B(0, r + 1))/r^3 \geq \theta_F(x)\). We deduce that \((H^3(F \cap B(0, r + 1))/(r + 1)^3)((r + 1)^3/r^3) \geq \theta_F(x)\).

Since \(F\) is a cone centered at 0, \(H^3(F \cap B(0, r + 1))/(r + 1)^3 = \theta_F(0)\) for each \(r > 0\). We deduce then \(\theta_F(0)((r + 1)^3/r^3) \geq \theta_F(x)\) for each \(r > 0\). We let \(r \to +\infty\) and we obtain then \(\theta_F(0) \geq \theta_F(x)\), which is (2.2.2).
Now (2.2.1) and (2.2.2) give us that $\theta_F(x) < d_Y$ for each $x \in F \cap \partial B(0, 1)$. By (1), we have $\theta_F(x) = d_P$ for $x \in F \cap \partial B(0, 1)$. So by [2, 8.1], there exists a neighborhood $U_x$ of $x$ in $\mathbb{R}^4$ such that $F \cap U_x$ is a 3-dimensional smooth manifold. We deduce that $F \cap \partial B(0, 1)$ is a 2-dimensional smooth sub-manifold of $\partial B(0, 1)$. By [1, Lemma 1], $F$ is a 3-plane passing through 0. But this implies that $\theta_F(0) = d_P$, we obtain then a contradiction, Lemma 2.2 follows.

**Lemma 2.3.** — Let $F$ be a 3-dimensional minimal cone in $\mathbb{R}^4$, centered at the origin 0. If $\theta_F(0) = d_Y$, then $F$ is a 3-dimensional cone of type $\mathbb{Y}$.

**Proof.** — As in the argument for (2.2.2), we have that for each $x \in F \cap \partial B(0, 1)$, $\theta_F(x) \leq \theta_F(0) = d_Y$. So $\theta_F(x)$ can only take one of the two values $d_P$ or $d_Y$. If all $x \in F \cap \partial B(0, 1)$ are of type $\mathbb{P}$, then by the same argument as above, $F$ will be a 3-plane, and then $\theta_F(0) = d_P$, a contradiction. So there must be a point $y \in F \cap \partial B(0, 1)$, such that $\theta_F(y) = d_Y$. By the same argument like above, $\theta_F(0)(r + 1)^3/r^3 \geq \theta_F(y, r)$ for each $r > 0$. Letting $r \to \infty$ and noting that $\theta_F(y, r)$ is non-decreasing in $r$, we have $d_Y \geq \lim_{r \to \infty} \theta_F(y, r)$. But $\theta_F(y, r) \geq \theta_F(y) = d_Y$ for each $r > 0$, so we must have $\theta_F(y, r) = d_Y$ for $r > 0$. By [3, 6.2], $F$ must be a cone centered at $y$. But we have also that $F$ is a cone centered at 0. So $F$ is of the form $F = F' \times 0y$, where $F'$ is a cone in a 3-plane $H$ passing through 0 and orthogonal to $0y$. Since $F$ is a minimal cone, by [3, 8.3], $F'$ is also a 2-dimensional minimal cone in $H$ and centered at 0. So $F'$ must be a cone of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$. Since $\theta_F(0) = d_Y$, we must have that $F'$ is a 2-dimensional minimal cone of type $\mathbb{Y}$ and we deduce that $F$ is a 3-dimensional minimal cone of type $\mathbb{Y}$. □

We can now consider 3-dimensional minimal sets in $\mathbb{R}^4$. We start with the following lemma.

**Lemma 2.4.** — Let $E$ be a 3-dimensional minimal set in $\mathbb{R}^4$. Then

(i) There does not exist a point $z \in E$ such that $d_P < \theta_E(z) < d_Y$.

(ii) If $x \in E$ such that $\theta_E(x) = d_P$, then each blow-up limit of $E$ at $x$ is a 3-dimensional plane.

(iii) If $\theta_E(x) = d_Y$, then each blow-up limit of $E$ at $x$ is a 3-dimensional minimal cone of type $\mathbb{Y}$.

**Proof.** — The proof uses Lemmas 2.2 and 2.3. Take any point $z \in E$, let $F$ be a blow-up limit of $E$ at $z$. Then by [3, 7.31], $F$ is a cone and $\theta_F(0) = \theta_E(x)$. By Lemma 2.2, it is not possible that $d_P < \theta_F(0) < d_Y$, which means that it is also not possible that $d_P < \theta_E(x) < d_P$, (i) follows.
If \( x \in E \) such that \( \theta_E(x) = d_P \), then any blow-up limit \( F \) of \( E \) at \( x \) satisfies \( \theta_F(0) = \theta_E(x) = d_P \). By the same arguments as in Lemma 2.2, for each \( y \in F \cap \partial B(0,1) \), \( \theta_F(y) \leq \theta_F(0) = d_P \). We deduce that \( \theta_F(y) = d_P \) for each \( y \in F \cap \partial B(0,1) \), and then \( F \) will be a 3-dimensional minimal cone over a smooth sub-manifold of \( \partial B(0,1) \). By \([1, \text{Lemma 1}]\), \( F \) must be a 3-dimensional plane, (ii) follows.

If \( x \in E \) such that \( \theta_E(x) = d_Y \), then any blow-up limit \( F \) of \( E \) at \( x \) satisfies \( \theta_F(0) = \theta_E(x) = d_Y \). By Lemma 2.3, \( F \) must be a 3-dimensional minimal cone of type \( Y \), (iii) follows. \( \square \)

Lemma 2.4 allows us to define the points of type \( P \) and \( Y \) of a 3-dimensional minimal set in \( \mathbb{R}^4 \).

**Definition 2.5.** — Let \( E \) be a 3-dimensional minimal set in \( \mathbb{R}^4 \) and \( x \in E \). We call \( x \) a point of type \( P \) if \( \theta_E(x) = d_P \). We call \( x \) a point of type \( Y \) if \( \theta_E(x) = d_Y \).

The following proposition says that if a 3-dimensional minimal set \( E \) is close enough to a 3-dimensional plane \( P \) in the ball \( B(x,2r) \), then \( E \) is Bi-Hölder equivalent to \( P \) in \( B(x,r) \).

**Proposition 2.6.** — For each \( \alpha > 0 \), we can find \( \epsilon > 0 \) such that the following holds.

Let \( E \) be a 3-dimensional minimal set in \( \mathbb{R}^4 \) and \( x \in E \). Let \( P \) be a 3-dimensional plane such that

\[
d_{x,2^3r}(E,P) \leq \epsilon. \tag{2.6.1}
\]

Then \( E \) is Bi-Hölder equivalent to \( P \) in the ball \( B(x,r) \), with Hölder exponent \( 1 + \alpha \).

**Proof.** — Take any point \( y \in B(x,r) \). Since \( B(y,2^4r) \subset B(x,2^5r) \), we have

\[
d_{y,2^4r}(E,P) \leq 2d_{x,2^5r}(E,P) \leq 2\epsilon. \tag{2.6.2}
\]

By \([3, 16.43]\), for each \( \epsilon_1 > 0 \), we can find \( \epsilon > 0 \) such that if (2.6.2) holds, then

\[
H^3(E \cap B(y,2^3r)) \leq H^3(P \cap B(y,(1+\epsilon_1)2^4r)) + \epsilon_1r^3 \\
\leq d_P(2^3r)^3 + C\epsilon_1r^3. \tag{2.6.3}
\]

Now (2.6.3) implies that \( \theta_E(y,2^3r) \leq d_P + C\epsilon_1 \). If \( \epsilon_1 \) is small enough, then \( \theta_E(y) \leq \theta_E(y,2^3r) < d_Y \). We deduce that \( \theta_E(y) = d_P \) and \( y \) is a \( P \) point.
Since $\theta_E(y,t)$ is a non-decreasing function in $t$, we have
\[ 0 \leq \theta_E(y,t) - \theta_E(y) \leq C\epsilon_1 \text{ for } 0 < t \leq 2^3 r. \] (2.6.4)
By [3, 7.24], for each $\epsilon_2 > 0$, we can find $\epsilon_1 > 0$ such that if (2.6.4) holds, then there exists a 3-dimensional minimal cone $F$, centered at $y$, such that
\[ d_{y,t/2}(E,F) \leq \epsilon_2 \text{ for } 0 < t \leq 2^3 r, \] (2.6.5)
and
\[ |\theta_E(y,2^2 r) - \theta_F(y,2^2 r)| \leq \epsilon_2. \] (2.6.6)
Since $d_P \leq \theta_E(y,2^2 r) \leq d_P + C\epsilon_1$, we deduce from (2.6.6) that $\theta_F(y,2^2 r) \leq d_P + C\epsilon_1 + \epsilon_2$. So if $\epsilon_1$ and $\epsilon_2$ are small enough, then $\theta_F(y,2^2 r) < d_{y}$. Which implies $\theta_F(y) < d_{y}$. Since $F$ is a minimal cone centered at $y$, we deduce that $F$ must be a 3-dimensional plane, by the same arguments as in second part of Lemma 2.4.

Now we can conclude that for each $y \in E \cap B(x,r)$ and each $t \leq r$, there exists a 3-dimensional plane $P(y,t)$, which is $F$ in (2.6.5), such that
\[ d_{y,t}(E,P(y,t)) \leq \epsilon_2. \] By [6, 2.2], for each $\alpha > 0$, we can find $\epsilon_2 > 0$, and then $\epsilon > 0$, such that $E$ is Bi-Hölder equivalent to a $P$ in the ball $B(x,r)$.

PROPOSITION 2.7. — For each $\eta > 0$, we can find $\epsilon > 0$ with the following properties. Let $E$ be a minimal set of dimension 3 in $\mathbb{R}^4$ and $Y$ be a 3-dimensional minimal cone of type $Y$, centered at the origin. Suppose that $d_{0,1}(E,Y) \leq \epsilon$. Then in the ball $B(0,\eta)$, there must be a point $y \in E$, which is not of type $P$.

Proof. — Suppose that the lemma fails. Then each $z \in B(0,\eta)$ is of type $P$. We note $F_1, F_2, F_3$ the three half-plane of dimension 3 which form $Y$ and $L$ the spine of $Y$, which is a plane of dimension 2. Then $F_i, 1 \leq i \leq 3$ have common boundary $L$. Take $w_i \in F_i \cap \partial B(0,\eta/4), 1 \leq i \leq 3$, such that the distance $\text{dist}(w_i,L) = \eta/4$. We see that the $w_i$ lie in a 2-dimensional plane orthogonal to $L$. Since $d_{0,1}(E,Y) \leq \epsilon$, we have that for each $1 \leq i \leq 3$, there exists $z_i \in E$ such that $d(z_i,w_i) \leq \epsilon$. Now $d(z_i,0) \leq d(w_i,0) + \epsilon = \eta/4 + \epsilon < 3\eta/8$ and $\text{dist}(z_i,L) \geq \text{dist}(w_i,L) - \epsilon = \eta/4 - \epsilon > 3\eta/16$. So if $\epsilon$ is small enough, we have that for each $1 \leq i \leq 3$, the ball $B(z_i,\eta/8)$ does not meet $L$. As a consequence, $Y$ coincide with $F_i$ in the ball $B(z_i,\eta/8)$ for $1 \leq i \leq 3$. We have next
\[ d_{z_i,\eta/8}(E,F_i) = d_{z_i,\eta/8}(E,Y) \]
\[ \leq \frac{8}{\eta} d_{0,1}(E,Y) \]
\[ \leq \frac{8\epsilon}{\eta}. \] (2.7.1)
Take a very small constant $\alpha > 0$, say, $10^{-15}$. Then by Proposition 2.6, we can find $\epsilon > 0$ such that if (2.7.1) holds, then

$$E \text{ is Bi-Hölder equivalent to } F_i \text{ in the ball } B(z_i, \eta/2^8) \text{ for each } 1 \leq i \leq 3 \text{ with Hölder exponent } 1 + \alpha.$$  

(2.7.2)

Next, since we suppose that each $z \in B(0, \eta)$ is of type $\mathbb{P}$, we have that there exists a radius $r_z > 0$, such that

$$E \text{ is Bi-Hölder equivalent to a 3-dimensional plane in the ball } B(z, r_z), \text{ with exponent } 1 + \alpha.$$  

(2.7.3)

In the ball $B(0, \eta)$, we have

$$d_{0,\eta}(E, Y) \leq \frac{1}{\eta}d_{0,1}(E, Y) \leq \frac{\epsilon}{\eta}.$$  

(2.7.4)

We can adapt the arguments in [3], section 17 to obtain that there does not exist a set $E$, which satisfies the conditions (2.7.2), (2.7.3) and (2.7.4). The idea is as follows, we construct a sequence of simple and closed curves $\gamma_0, \gamma_1, \ldots, \gamma_k$ such that $\gamma_k \cap E = \emptyset$ and $\gamma_0$ intersects $E$ transversally at exactly 3 points in the ball $B(z_i, \eta/2^8)$. For each $0 \leq i \leq k-1$, $\gamma_i$ intersects $E$ transversally at a finite number of points and $|\gamma_i \cap E| - |\gamma_{i+1} \cap E|$ is even, here $|\gamma_i \cap E|$ denotes the number of intersections of $\gamma_i$ with $E$. This is impossible since $|\gamma_0 \cap E| = 3$ and $|\gamma_k \cap E| = 0$. We obtain then a contradiction. Proposition 2.7 follows. □

**Lemma 2.8.** — For each $\delta > 0$, we can find $\epsilon > 0$ such that the following holds.

Let $F$ be a 3-dimensional minimal cone in $\mathbb{R}^4$, centered at the origin. Suppose that $d_Y < \theta_F(0) < d_Y + \epsilon$. Then there exists a 3-dimensional minimal cone $Y_F$, of type $\mathbb{Y}$, centered at 0 such that $d_{0,1}(F, Y_F) \leq \delta$.

**Proof.** — Suppose that the lemma fails. Then there exists $\delta > 0$, such that we can find 3-dimensional minimal cones $F_1, \ldots, F_k, \ldots$ centered at 0, satisfying $d_Y \leq \theta_F \leq d_Y + 1/2^i$, and for any 3-dimensional minimal cone $Y$ of type $\mathbb{Y}$, centered at 0, we have $d_{0,1}(Y, F_i) > \delta$.

Now we can find a sub-sequence $\{F_{j_k}\}_{k=1}^\infty$ of $\{F_i\}_{i=1}^\infty$ such that this sub-sequence converges to a closed set $G \subset \mathbb{R}^4$. By [3, 3.3], $G$ is also a minimal set. Since each $F_{i_k}$ is a cone centered at 0, $G$ is also a cone centered at 0. So $G$ is a 3-dimensional minimal cone centered at 0. By [3, 3.3], we have

$$H^3(G \cap B(0, 1)) \leq \liminf_{k \to \infty} H^3(F_{j_k} \cap B(0, 1)),$$  

(2.8.1)

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which implies that

$$\theta_G(0) \leq \liminf_{k \to \infty} (d_Y + 1/2^{j_k}) = d_Y.$$  (2.8.2)

By [3, 3.12], we have

$$H^3(G \cap \overline{B}(0, 1)) \geq \limsup_{k \to \infty} H^3(F_{j_k} \cap \overline{B}(0, 1)),$$  (2.8.3)

which implies that

$$\theta_G(0) \geq \limsup_{k \to \infty} (d_Y + 1/2^{j_k}) = d_Y.$$  (2.8.4)

From (2.8.2) and (2.8.4), we have that $\theta_G(0) = d_Y$. Then by Lemma 2.3, $G$ must be a 3-dimensional minimal cone of type $\mathbb{Y}$, centered at 0. Since $\lim_{k \to \infty} F_{j_k} = G$, there is $k > 0$ such that $d_{0,1}(F_{j_k}, G) \leq \delta/2$, which is a contradiction. The lemma follows. $\square$

The following lemma is similar to Lemma 2.8, but we consider minimal sets in general.

**Lemma 2.9.** — For each $\delta > 0$, we can find $\epsilon > 0$ such that the following holds.

Suppose that $E$ is a 3-dimensional minimal set in $\mathbb{R}^4$ and $0 \in E$. Suppose that

$$d_Y \leq \theta_E(0) \leq d_Y + \epsilon,$$  (2.9.1)

and

$$\theta_E(0, 4) - \theta_E(0) \leq \epsilon.$$  (2.9.2)

Then there exists a 3-dimensional minimal cone $Y_E$, of type $\mathbb{Y}$, centered at 0 such that

$$d_{0,1}(E, Y_E) \leq \delta.$$  (2.9.3)

**Proof.** — By [3, 7.24], for each $\epsilon_1 > 0$, we can find $\epsilon > 0$ such that if (2.9.2) holds, then there is a 3-dimensional minimal cone $F$ centered at the origin, such that

$$d_{0,2}(F, E) \leq \epsilon_1.$$  (2.9.3)

and

$$|\theta_F(0, 2) - \theta_E(0, 2)| \leq \epsilon_1.$$  (2.9.4)

Since $E$ is minimal, $\theta_E(0, 4) \geq \theta_E(0, 2) \geq \theta_E(0)$. So from (2.9.1) and (2.9.2), we have that $d_Y \leq \theta_E(0, 2) \leq d_Y + 2\epsilon$. With (2.9.4), we have

$$d_Y - \epsilon_1 \leq \theta_F(0, 2) \leq d_Y + 2\epsilon + \epsilon_1.$$  (2.9.5)
Now if we choose $\epsilon_1$ small enough, then $\theta_F(0) = \theta_F(0, 2) \geq d_Y - \epsilon_1 > d_P$, so by Lemma 2.2, we have $\theta_F(0) \geq d_Y$. Thus
\[ d_Y \leq \theta_F(0) \leq d_Y + 2\epsilon + \epsilon_1. \quad (2.9.6) \]

By Lemma 2.8, for each $\epsilon_3 > 0$, we can find $\epsilon_1 > 0$, and then $\epsilon > 0$, such that if (2.9.6) holds, then there is a 3-dimensional minimal cone $Y_F$ of type $\mathbb{Y}$, centered at 0 such that
\[ d_{0,2}(F,Y_F) \leq \epsilon_3. \quad (2.9.7) \]

From (2.9.3) and (2.9.7) we have
\[ d_{0,1}(E,Y_F) \leq 2(d_{0,2}(E,F) + d_{0,2}(F,Y_F)) \leq 2(\epsilon_1 + \epsilon_3). \quad (2.9.8) \]

Now for each $\delta > 0$, we choose $\epsilon > 0$ such that $2(\epsilon_1 + \epsilon_3) < \delta$, we set then $Y_E = Y_F$ and the lemma follows. \(\square\)

We are ready to prove Theorem 1.

**Theorem 2.10.** — For each $\alpha > 0$, we can find $\epsilon > 0$ such that the following holds.

Let $E$ be a 3-dimensional minimal set in $\mathbb{R}^4$, which contains the origin 0. Suppose that there exists a radius $r > 0$ such that
\[ d_Y \leq \theta_E(0) \leq d_Y + \epsilon, \quad (2.10.1) \]

and
\[ \theta_E(0,2^{11}r) - \theta_E(0) \leq \epsilon. \quad (2.10.2) \]

Then $E$ is Bi-Hölder equivalent to a 3-dimensional minimal cone $Y$ of type $\mathbb{Y}$ and centered at 0 in the ball $B(0,r)$, with Hölder exponent $1 + \alpha$.

**Proof.** — By Lemma 2.9, for each $\epsilon_1 > 0$, we can find $\epsilon > 0$ such that if (2.10.1) and (2.10.2) hold, then there exists a 3-dimensional minimal cone $Y$, of type $\mathbb{Y}$, centered at 0 such that
\[ d_{0,2^{11}r}(E,Y) \leq \epsilon_1. \quad (2.10.3) \]

We consider a point $y \in E \cap B(0,r)$. We set
\[ E_Y = \{ z \in E \cap \overline{B}(0,4r) \} \] is not a $\mathbb{P}$-point. \( (2.10.4) \)

We note that $E_Y$ is closed. Indeed, if $z$ is an accumulation point of $E_Y$, then if $z$ is a $\mathbb{P}$-point, then there exists a neighborhood $V_z$ of $z$ in $E$ such
that $V_z$ has only points of type $P$, as in the proof of Proposition 2.6, which is not possible. So $z$ cannot be a $P$-point and as a consequence, $z \in E_Y$.

**Case 1,** $y \in E_Y$.

Since $y$ is not a $P$-point, $\theta_E(x) \neq d_P$, then by Lemma 2.4, we have

$$\theta_E(y) \geq d_Y; \quad (2.10.5)$$

Next, $B(y, 2^8r) \subset B(0, 2^9r)$, by (2.10.3), we have

$$d_{y,2^8r}(E,Y) \leq 2d_{0,2^9r}(E,Y) \leq 2\epsilon_1. \quad (2.10.6)$$

By [3, 16.43], for each $\epsilon_2 > 0$, we can find $\epsilon_1 > 0$ such that if (2.10.6) holds, then

$$H^3(E \cap B(y, 2^7r)) \leq H^3(Y \cap B(y, (1 + \epsilon_2)2^7r)) + \epsilon_2r^3, \quad (2.10.7)$$

which, together with (2.10.5), imply

$$d_Y \leq \theta_E(y, 2^7r) \leq d_Y + C\epsilon_2. \quad (2.10.8)$$

But $E$ is a minimal set, so the function $\theta_E(y, \cdot)$ is non-decreasing. So we have

$$d_Y \leq \theta_E(y, t) \leq d_Y + C\epsilon_2 \text{ for } 0 < t \leq 2^7r. \quad (2.10.9)$$

By Lemma 2.8, for each $\epsilon_3 > 0$, we can find $\epsilon_2, \epsilon_1 > 0$, and then $\epsilon > 0$, such that if (2.10.5) and (2.10.8) hold, then there exists a 3-dimensional minimal cone $Y(y, t)$ of type $Y$, centered at $y$, such that

$$d_{y,t}(E,Y(y,t)) \leq \epsilon_3 \text{ for } 0 < t \leq 2^5r. \quad (2.10.10)$$

We note as above, for $y \in B(0, r)$ and $t \leq 2^5r$, $Y(y, t)$ the cone of type $Y$ that satisfies (2.10.10).

**Case 2,** $y$ is a $P$ point.

Let $d = \text{dist}(y, E_Y) > 0$. Take a point $u \in E_Y$ such that $d(y, u) = d$. Since $z \in B(0, r)$ and $0 \in E_Y$, we have $d \leq d(0, y) \leq r$. We take the cone $Y(u, 2d)$ as in (2.10.10), then

$$d_{u,2d}(E,Y(u,2d)) \leq \epsilon_3. \quad (2.10.11)$$

Call $L$ the spine of $Y(u, 2d)$, then $L$ is a 2-dimensional plane passing through $u$. We want to show that

$$\text{dist}(y, L) \geq d/2. \quad (2.10.12)$$
Indeed, if (2.10.12) fails, then there exists $u' \in L$ such that $d(y, u') = \text{dist}(y, L) < d/2$. So $d(u', u) \leq d(u', y) + d(y, u) \leq 3d/2$. As a consequence, $B(u', d/2) \subset B(u, 2d)$. We have next

$$d_{u', d/2}(E, Y(u, 2d)) \leq 4d_{u, 2d}(E, Y(u, 2d)) \leq 4\epsilon_3. \quad (2.10.13)$$

By Proposition 2.7, we can choose $\epsilon_3 > 0$ such that if (2.10.13) holds, then there is a point $u_1 \in E \cap B(u', d/1000)$, which is not of type $\mathbb{P}$. Next, $d(y, u_1) \leq d(y, u') + d(u', u_1) \leq d/2 + d/1000 < 3d/4$ and since $y \in B(0, r)$, $u' \in B(0, r + 3d/4) \subset B(0, 4r)$. As $u'$ is not a $\mathbb{P}$-point, we have that $u' \in E_Y$. So we can find a point $u' \in E_Y$ for which $d(y, u') < d$, a contradiction. We have then (2.10.12).

Since $B(y, d/2) \subset B(u, 2d)$, we have

$$d_{y, d/2}(E, Y(u, 2d)) \leq 4d_{u, 2d}(E, Y(u, 2d)) \leq 4\epsilon_3. \quad (2.10.14)$$

By [3, 16.43], for each $\epsilon_4 > 0$, we can find $\epsilon_3 > 0$ such that if (2.10.14) holds, then

$$H^3(E \cap B(y, d/4)) \leq H^3(Y(u, 2d) \cap B(y, (1 + \epsilon_4)d/4) + \epsilon_4 d^3. \quad (2.10.15)$$

Now as $\text{dist}(y, L) \geq d/2$, we see that $Y(u, 2d)$ coincide with a 3-dimensional plane in the ball $B(y, (1 + \epsilon_4)d/4)$. So $H^3(Y(u, 2d) \cap B(y, (1 + \epsilon_4)d/4) \leq d_P((1 + \epsilon_4)d/4)^3$, together with (2.10.15), we obtain

$$\theta_E(y, d/4) \leq d_P + C\epsilon_4. \quad (2.10.16)$$

By the proof of Proposition 2.6, we have that for each $\epsilon_5 > 0$, we can find $\epsilon_4 > 0$ such that for each $t \leq d/8$, there exists a plane $P(y, t)$ of dimension 3 passing by $y$, such that

$$d_{y, t}(E, P(y, t)) \leq \epsilon_5. \quad (2.10.17)$$

For the case $d/8 \leq t \leq r$, we take the cone $Y(u, t + d)$ as in 2.10.10 which is possible since $t + d < 8r$. Since $B(y, t) \subset B(u, t + d)$, we have

$$d_{y, t}(E, Y(u, t + d)) \leq \frac{t + d}{t}d_{u, t + d}(E, Y(u, t + d)) \leq 10\epsilon_3. \quad (2.10.18)$$

From (2.10.10), (2.10.17) and (2.10.18) we conclude that, for each $y \in E \cap B(0, r)$ and $t \leq r$, there exists a 3-dimensional minimal cone $Z(y, t)$ of type $\mathbb{P}$ or $\mathbb{Y}$, such that $d_{y, t}(E, Z(y, t)) \leq \epsilon_6$, where $\epsilon_6 = \max\{\epsilon_5, 10\epsilon_3\}$. By [6,2.2], we conclude that for each $\alpha > 0$, we can find $\epsilon > 0$ such that if (2.10.1) and (2.10.2) hold, then $E$ is Bi-Hölder equivalent to a 3-dimensional minimal
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cone of type \( Y \), centered at 0 in the ball \( B(x, r) \), with Hölder exponent \( 1 + \alpha \).

Now we see that Theorem 1 is a consequence of Theorem 2.10, since \( \theta_E(x) = d_Y \) which lies between \( d_Y \) and \( d_Y + \epsilon \) for any \( \epsilon > 0 \). Next, for each \( \epsilon > 0 \), since \( \lim_{r \to 0} \theta_E(x, r) = \theta_E(x) \), so we can find \( r > 0 \) such that \( \theta_E(x, 2^{11}r) \leq \theta_E(x) + \epsilon = d_Y + \epsilon \). We conclude that \( E \) is Bi-Hölder equivalent to a cone of type \( Y \) in the ball \( B(x, r) \).

**Corollary 2.11.** — For each \( \alpha > 0 \), we can find \( \epsilon > 0 \) such that the following holds. Let \( E \) be a 3-dimensional minimal set in \( \mathbb{R}^4 \), \( x \in E \), \( r \) be a radius \( > 0 \) and \( Y \) be a 3-dimensional minimal cone of type \( Y \), centered at \( x \) such that

\[
d_{x,2^{14}r}(E,Y) \leq \epsilon. \tag{2.11.1}
\]

Then \( E \) is Bi-Hölder equivalent to \( Y \) in the ball \( B(x, r) \), with Hölder exponent \( 1 + \alpha \).

**Proof.** — By Proposition 2.7, we can find \( \epsilon \) small enough such that there exists a point \( y \in B(x, r/1000) \) which is not of type \( \mathbb{P} \). So \( \theta_E(y) \geq d_Y \). Since \( B(y, 2^{12}r) \subset B(x, 2^{13}r) \), we have

\[
d_{y,2^{13}r}(E,Y) \leq 2d_{x,2^{14}r}(E,Y) \leq 2\epsilon. \tag{2.11.2}
\]

By [3, 16.43], for each \( \epsilon_1 > 0 \), we can find \( \epsilon > 0 \) such that if (2.11.2) holds, then

\[
H^3(E \cap B(y, 2^{12}r)) \leq H^3(Y \cap B(y, (1 + \epsilon_1)2^{12}r)) + \epsilon_1 r^3, \tag{2.11.3}
\]

which implies that

\[
\theta_E(y, 2^{12}r) \leq d_Y + C\epsilon_1. \tag{2.11.4}
\]

Now (2.11.4) together with the fact that \( \theta_E(y) \geq d_Y \) are the conditions in the hypothesis of Theorem 2.10 with the couple \((x, 2r)\). Following the proof of the theorem, for each \( \epsilon_2 > 0 \), we can find \( \epsilon_1 > 0 \) such that for each \( z \in B(y, 2r) \) and for each \( t \leq 2r \), there is a 3-dimensional minimal cone \( Z(z, t) \) of type \( \mathbb{P} \) or \( Y \) such that \( d_{z,t}(Z(z, t), E) \leq \epsilon_2 \). Since \( B(x, r) \subset B(y, 2r) \), the above holds for any \( z \in B(x, r) \) and \( t \leq r \). Now since \( d_{x,r}(E,Y) \leq 2^{14} \epsilon \leq \epsilon_2 \), we can apply [DDT,2.2] to conclude that for each \( \alpha > 0 \), we can find \( \epsilon > 0 \) such that if (2.11.1) holds, then \( E \) is Hölder equivalent to \( Y \) in \( B(x, r) \), with Hölder exponent \( 1 + \alpha \). □

By construction of the Bi-Hölder function in [6], we see that if \( E \) is Bi-Hölder equivalent to a \( Y \) of type \( Y \) in \( B(x, r) \) by a function \( f \), then \( f \) is a bijection of the spine of \( Y \) in \( B(x, r/2) \) to the points of type non-\( \mathbb{P} \) of \( E \) in a neighborhood of \( x \). We have the remark.
Remark 2.12. — Let $E$ be a 3-dimensional minimal set in $\mathbb{R}^4$, $x \in E$ and $r > 0$. Suppose that $E$ is Bi-Hölder equivalent to a 3-dimensional minimal cone $Y$ of type $\mathbb{Y}$ and centered at $x$ in the ball $B(x, r)$. Note $E_Y$ the set of the points of type non-$\mathbb{Y}$ of $E$ in $B(x, r)$ and $L$ the spine of $Y$. Then
\[ E_Y \cap B(x, r/8) \subset f(L \cap B(x, r/4)) \subset E_Y \cap B(x, r/2). \tag{2.12.1} \]

3. Existence of a point of type non-$\mathbb{P}$ and non-$\mathbb{Y}$ for a Mumford-Shah minimal set in $\mathbb{R}^4$ which is near a $T$

Let us restate Theorem 2.

Theorem 2. — There exists an absolute constant $\epsilon > 0$ such that the following holds. Let $E$ be an MS-minimal set in $\mathbb{R}^4$, $r > 0$ be a radius and $T$ be a 3-dimensional minimal cone of type $\mathbb{T}$ centered at the origin such that
\[ d_{0,r}(E, T) \leq \epsilon. \tag{2.1} \]
Then in the ball $B(0, r)$, there is a point which is neither of type $\mathbb{P}$ nor $\mathbb{Y}$ of $E$.

We will prove Theorem 2 by contradiction. By homothety, we may assume that $r = 2^{10}$. Suppose that (2.1) fails, that is
\[ \text{there are only points of type $\mathbb{P}$ and $\mathbb{Y}$ in $E \cap B(0, 2^{10})$.} \tag{2.2} \]

We fix a coordinate $(x_1, x_2, x_3, x_4)$ of $\mathbb{R}^4$. Without loss of generality, we suppose that $T$ is of the form $T = T' \times l$, where $T'$ is a 2-dimensional minimal cone of type $\mathbb{T}$ which belong to a 3-dimensional plane $P$ of equation $P = \{x_1, x_2, x_3, x_4\} : x_4 = 0$ and $l$ the line of equation $x_1 = x_2 = x_3 = 0$. We call $l$ the spine of $T$, which is also the set of $\mathbb{T}$-points of $T$. Let $l_1, l_2, l_3, l_4$ be the four axes of $T'$; then $L_i = l_i \times l, i = 1, \ldots, 4$ are the 2-faces of $T$. We see that $\bigcup_{i=1}^4 L_i \setminus l$ is the set of $\mathbb{Y}$-points of $T$. Finally, let $F_j, 1 \leq j \leq 6$ the faces of $T'$ in $P$. Then $F_j \times l, 1 \leq j \leq 6$ are the 3-faces of $T$ and $\bigcup_{j=1}^6 F_j$ minus the set of $\mathbb{Y}$-points and the set of $\mathbb{T}$-points of $T$ is the set of $\mathbb{P}$-points of $T$. The proof of Theorem 2 requires several lemmas. We begin with a lemma about the connected components of $B(0, 2) \setminus E$.

Lemma 3.1. — Let $a_i, 1 \leq i \leq 4$ be the four points in $\partial B(0, 2^9) \cap P$ whose distances to $T'$ are maximal. Set $V_i, 1 \leq i \leq 4$ the connected component of $B(0, 2^{10}) \setminus E$ which contains $a_i$. Then we have $V_i \neq V_j$ for $1 \leq i \neq j \leq 4$.

Proof. — Suppose that the lemma fails. Then there are $i \neq j$ such that $V_i = V_j$. Without loss of generality, we may assume that $V_1 = V_2 = V$. Now
the point \( a = (a_1 + a_2)/2 \) belongs to a 3-face \( P_{12} \) of \( T \) and \( T \) coincide with \( P_{12} \) in \( B(a, 2^8) \).

Since \( d_{0,2^5}(E, T) \leq \epsilon \), we have

\[
d_{a,2^8}(E, T) = d_{a,2^8}(E, P_{12}) \leq 4\epsilon. \tag{3.1.1}
\]

By Proposition 2.6, for a constant \( r \) very small, say, \( 10^{-25} \), we can find \( \epsilon > 0 \) such that \( E \) is Bi-Hölder equivalent to \( P_{12} \) in the ball \( B(a, 2^3) \), with Hölder exponent \( 1 + r \). We note \( f \) this Hölder function; then \( f \) is a homeomorphism and

\[
E \cap B(a, 4) \subset f(P_{12} \cap B(a, 8)) \subset E \cap B(a, 16), \tag{3.1.2}
\]

and

\[
|f(x) - x| \leq r \text{ for } x \in B(a, 16). \tag{3.1.3}
\]

We want to show that

\[
\text{if } z \in \partial B(a, 4) \setminus E, \text{ then } z \in V. \tag{3.1.4}
\]

Indeed, set \( z' = f^{-1}(z) \), then \( z' \in B(a, 8) \) and as \( z \notin E \), we have \( z' \notin P_{12} \). Now the 3-plane \( P_{12} \) separate \( \mathbb{R}^4 \) into two half-spaces \( H_1 \) and \( H_2 \) which contain \( a_1 \) and \( a_2 \), respectively. Let \( z_1 \in H_1 \) and \( z_2 \in H_2 \) be two points in \( \partial B(a, 4) \) whose distances to \( P_{12} \) are maximal. We see that \( a \) is the mid-point of the segment \([z_1, z_2]\) and this segment is orthogonal to \( P_{12} \). Since \( z_1 \) and \( z_2 \) lie in two different half-spaces of \( \mathbb{R}^4 \) separated by \( P_{12} \), one of the two segment \([z', z_1]\) and \([z', z_2]\) doesn’t meet \( P_{12} \). We suppose that is the case of \([z', z_1]\); then the curve \( \gamma = f([z', z_1]) \) doesn’t meet \( E \).

Next, it is clear that \( \text{dist}(u, T) \geq 2 \) for \( u \in [a_1, f(z_1)] \) as \( |f(z_1) - z_1| \leq r \). Since \( d_{0,2^5}(E, T) \leq \epsilon \), the segment \([a_1, f(z_1)]\) doesn’t meet \( E \). Now the curve \( \gamma' \) which goes first from \( a_1 \) to \( f(z_1) \) by the segment \([a_1, f(z_1)]\) and then from \( f(z_1) \) to \( f(z') = z \) by the curve \( \gamma \) is a curve in \( B(0, 2^9) \) which joint \( a_1 \) to \( z \) and doesn’t meet \( E \). We deduce that \( z \in V_1 = V \), which is (3.1.4).

Now we want to obtain a contradiction. We will construct an MS-competitor \( F \) for \( E \) whose Hausdorff measure in \( B(0, 2^{10}) \) is smaller than that of \( E \) in the same ball. We set

\[
F = E \setminus B(a, 4). \tag{3.1.5}
\]

It is clear that \( F \setminus \overline{B}(0, 2^{10}) = E \setminus \overline{B}(0, 2^{10}) \). We want to show that \( F \) is an MS-competitor for \( E \). For this, we suppose that \( x_1, x_2 \in \mathbb{R}^4 \setminus (\overline{B}(0, 2^{10}) \cup E) \) such that \( x_1, x_2 \) are separated by \( E \). We want to show that they are also separated by \( F \).
We proceed by contradiction. Suppose that
there is a curve $\Gamma \subset \mathbb{R}^4$ connecting $x_1$ and $x_2$ which doesn’t meet $F$.

(3.1.6)

Now if $\Gamma \cap \overline{B}(a, 4) = \emptyset$, then $\Gamma$ doesn’t meet $E$. Next, as $F = E \setminus B(a, 4)$, we have that $x_1, x_2$ are not separated by $E$, a contradiction. So we must have that $\Gamma$ meets $\overline{B}(a, 4)$. Let $x_1'$ be the first point at which $\Gamma$ meets $\overline{B}(a, 4)$ and $x_2'$ be the last point at which $\Gamma$ meets $\overline{B}(a, 4)$. Then it is clear that $x_1', x_2' \in \partial B(a, 4)$. We note $\Gamma_1$ the sub-curve of $\Gamma$ from $x_1$ to $x_1'$ and $\Gamma_2$ the sub-curve of $\Gamma$ from $x_2'$ to $x_2$. Since $\Gamma_1$ and $\Gamma_2$ belong to the same connected component of $F$ and $\Gamma_1, \Gamma_2$ don’t meet $B(a, 4)$ and $F = E \setminus B(a, 4)$, we deduce that $\Gamma_1$ and $\Gamma_2$ belong to the same connected component of $\mathbb{R}^4 \setminus E$.

In addition, since $x_1', x_2' \in \partial B(a, 4) \setminus E$, so by (3.1.4), they both belong to $V$ and then we can connect $x_1'$ and $x_2'$ by a curve $\Gamma_3$ which doesn’t meet $E$.

Now the curve $\Gamma_4$ which is the union of $\Gamma_1, \Gamma_2$ and $\Gamma_3$ is a curve that connects $x_1$ and $x_2$ and doesn’t meet $E$. This is a contradiction, as we suppose that $x_1$ and $x_2$ are separated by $E$.

Now since $\text{dist}(a, E) \leq 2^{10}\epsilon$, there is a point $a' \in E$ such that $d(a, a') \leq 2^{10}\epsilon$ and by consequence $B(a', 2) \subset B(a, 4)$. Next

\[
H^3(F \cap B(0, 2^{10})) = H^3(E \cap B(0, 2^{10}) \setminus B(a, 4)) \\
\leq H^3(E \cap B(0, 2^{10}) \setminus B(a', 2)) \\
= H^3(E \cap B(0, 2^{10})) - H^3(E \cap B(a', 2)) \\
\leq H^3(E \cap B(0, 2^{10})) - C2^3 < H^3(E \cap B(0, 2^{10})).
\]

(3.1.7)

Where the last line is obtained from the fact that $E$ is Alhfors-regular (see [7]). Now (3.1.7) contradicts the hypothesis that $E$ is MS-minimal, we thus obtain the lemma. □

If $x$ is a point of type $\mathbb{P}$ or $\mathbb{Y}$ of $E$, then by Proposition 2.6 and Theorem 1, for $\tau = 10^{-25}$, for example, we can find a radius $r > 0$ and a Bi-Hölder mapping $\psi_x : B(x, 2r) \to \mathbb{R}^4$, and a 3-dimensional minimal cone $Y$ of type $\mathbb{P}$ or $\mathbb{Y}$, respectively, centered at $x$, such that

$|\psi_x(z) - z| \leq \tau r$ for $z \in B(x, 2r)$  

(2)

$E \cap B(x, r) \subset \psi_x(Y \cap B(x, 3r/2)) \subset E \cap B(x, 2r)$.

(3)
By (2.2), there are only points of type $P$ or $Y$ of $E \cap \overline{B}(0, 2^{10})$. We set then

$$E_Y \text{ the set of } Y\text{-points of } E \cap \overline{B}(0, 2^{10}).$$

(4)

It is clear that $E_Y$ is closed by the proof of Theorem 2.10. If $x \in E_Y \cap B(0, 2^{10})$, then there exists $r_x > 0$ such that $B(x, r_x) \subset B(0, 2^{10})$ and a minimal cone $Y_x$ of type $Y$, centered at $x$, and a Hölder mapping $\psi_x : B(x, 2r_x) \to \mathbb{R}^4$ such that (2) and (3) hold for $\psi_x$ and $Y_x$. Let $L_x$ be the spine of $Y_x$, then $L_x$ is a 2-plane passing through $x$. By Remark 2.12, there is a neighborhood $U_x$ of $x$ such that

$$E_Y \cap U_x = \psi_x(B(x, r_x) \cap L_x).$$

(5)

Now we take four points $d_i, 1 \leq i \leq 4$ such that $0$ is the mid-point of the segments $[a_i, d_i], 1 \leq i \leq 4$, here $a_i$ is as in Lemma 3.1. It is clear that $d_i \in T' \subset T$. In addition, $d_i \in L_i, 1 \leq i \leq 4$, where $L_i$ are described just after the second statement of Theorem 2. Next, for $1 \leq i \leq 4$, we have $d_{d_i, 4}(E, T) \leq 2^8 d_{0, 2^{10}}(E, T) \leq 2^8 \epsilon$. But in the ball $B(d_i, 4), T$ coincide with a cone $Y_i$ of type $Y$ whose spine is $L_i$. So $d_{d_i, 4}(E, Y_i) \leq 2^8 \epsilon$. By Corollary 2.11, for $\tau = 10^{-25}$, we can find $\epsilon > 0$ such that $E$ is Bi-Hölder equivalent to $Y_i$ in the ball $B(d_i, 2)$, with Hölder exponent $1 + \tau$. Call $\psi_i$ this Hölder mapping, then by Remark 2.12

$$E_Y \cap B(d_i, 1) \subset \psi_i(L_i \cap B(d_i, 3/2)) \subset E_Y \cap B(d_i, 2)$$

(6)

and

$$|\psi_i(z) - z| \leq \tau \text{ for } z \in B(d_i, 2).$$

(7)

Setting

$$b_i = \psi_i(d_i), 1 \leq i \leq 4.$$  

(8)

By (7), we have $d(d_i, b_i) \leq \tau$. We want to prove the following lemma.

**Lemma 3.2.** — The point $b_1 \in E_Y$ can be connected to another point $b_i \in E_Y, i \neq 1$ by a curve $\gamma \subset E_Y \cap B(0, 3 \cdot 2^8)$.

**Proof.** — Recall that $\psi_i, b_i, d_i$ are the same as (6),(7),(8) above. In addition, for each $x \in E_Y \cap B(0, 2^{10})$, there are a radius $r_x$ and a Bi-Hölder mapping $\psi_x$, a minimal cone $Y_x$ of type $Y$, centered at $x$ such that (2),(3), and (5) hold.

We proceed by contradiction. We denote by $E_Y^1$ the connected component of $E_Y \cap B(0, 2^{10})$ which contains $b_1$. Since in each ball $B(b_i, 2), E_Y$ is Hölder equivalent to a 2-plane, by (6), we deduce that each $z \in E_Y \cap B(b_i, 1)$
can be connected to \( b_i \) by a curve in \( E_Y \). So if the lemma fails, that is \( E_Y^{1} \) doesn’t contain any \( b_i, i \neq 1 \), we must have
\[
E_Y^{1} \cap B(b_i, 1) = \emptyset \text{ for } i \neq 1. \tag{3.2.1}
\]

Recall next that \( T = T' \times l \), where \( T' \) is a 2-dimensional minimal cone of type \( \mathbb{T} \) in the 3-plane \( P \) of equation \( x_4 = 0 \) and \( l \) is the line of equation \( x_1 = x_2 = x_3 = 0 \).

Now we construct a family of functions \( f_t, 0 \leq t \leq 1 \) from \( \mathbb{R}^4 \) to \( \mathbb{R}^2 \) by the formula
\[
f_t(x) = (x_4, |x - td_2|^2 - ((1-t)2^9)^2), \tag{3.2.2}
\]
where \( x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \) and \( 0 \leq t \leq 1 \). If \( x \in E_Y^{1} \), then
\[
|f_1(x)| \geq |x - d_2| \geq 1/2, \tag{3.2.3}
\]
by (3.2.1) and the fact that \( |d_2 - b_2| \leq \tau \). We will construct a finite number of functions to go from \( f_0 \) to \( f_1 \). First, let \( K = E_Y^{1} \cap \overline{B}(0, 3 \cdot 2^8) \). Then for each \( z \in K \), there is a radius \( r_z \) such that \( E_Y^{1} \) is Bi-Hölder equivalent to a 2-plane \( P_z \), with Hölder exponent \( 1 + \tau \). Since \( K \) is compact, we can cover \( K \) by a finite number of balls \( B(z_i, r_{z_i}), 1 \leq i \leq N \). Finally, we choose \( \eta > 0 \) which is smaller than \( \frac{1}{10} \min \{r_{z_i}\}, 1 \leq i \leq N \).

Next, let \( \{x_i\}, 1 \leq i \leq l \) be a maximal collection of points in \( K \) such that \( |x_i - x_j| \geq \eta \) for \( i \neq j \). Set \( \varphi_j \) a bump function with support in \( B(x_j, 2\eta) \) and such that \( \tilde{\varphi}_j(x) = 1 \) for \( x \in \overline{B}(x_j, \eta) \) and \( 0 \leq \tilde{\varphi}_j(x) \leq 1 \) everywhere. We note that \( \sum_j \tilde{\varphi}_j(x) \geq 1 \) for \( x \in E_Y^{1} \cap \overline{B}(0, 3 \cdot 2^8) \) since \( x \) must lie in one of the ball \( B(x_j, \eta) \) by the maximality of the family \( \{x_i\} \). Set \( \varphi_0 \) a \( C^\infty \) function in \( \mathbb{R}^4 \) such that \( \tilde{\varphi}_0(x) = 0 \) for \( |x| \leq 3 \cdot 2^8 - \eta \) and \( \tilde{\varphi}_0(x) = 1 \) for \( |x| \geq 3 \cdot 2^8 \) and \( 0 \leq \tilde{\varphi}_0(x) \leq 1 \) everywhere. We have then \( \sum_{j=0}^l \tilde{\varphi}_j(x) \geq 1 \) on \( E_Y^{1} \) and we set
\[
\varphi_j(x) = \varphi_j(x)\left\{ \sum_{j=0}^l \tilde{\varphi}_j(x) \right\}^{-1} \text{ for } x \in E_Y^{1} \text{ and } 0 \leq j \leq l. \tag{3.2.4}
\]

The functions \( \varphi_j, 0 \leq j \leq l \) have the following properties.
\[
\varphi_j \text{ has support in } B(x_j, 2\eta) \text{ for } j \geq 1, \tag{3.2.5}
\]
\[
\sum_{j=0}^l \varphi_j(x) = 1 \text{ for } x \in E_Y^{1},
\]
\[
\sum_{j=1}^l \varphi_j(x) = 1 \text{ for } x \in E_Y^{1} \cap B(0, 3 \cdot 2^8 - \eta), \tag{3.2.6}
\]
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since \( \varphi_0(x) = 0 \) on \( B(0, 3 \cdot 2^8 - \eta) \). Our first approximation is a sequence of functions given by

\[
g_k = f_0 + \sum_{0 < j < k} \varphi_j(f_1 - f_0),
\]

with \( 0 \leq k \leq l \). Then \( g_0 = f_0 \) and

\[
g_l(x) = f_1(x) \text{ for } x \in E \cap B(0, 3 \cdot 2^8 - \eta).
\]

We note that for \( k \geq 1 \)

\[
g_k(x) - g_{k-1}(x) = \varphi_k(x)(f_1(x) - f_0(x)) \text{ is supported in } B(x_k, 2\eta).
\]

We compute the number of solutions in \( E_Y^1 \) of the equations \( g_k(x) = 0 \). We will modify \( f_0 \) and the \( g_k \) such that they have only a finite number of zeroes. We modify first \( f_0 \).

**Sub-lemma 3.2.1.** — There exists a continuous function \( h_0 \) on \( E_Y^1 \) such that

\[
|h_0(x) - f_0(x)| \leq 10^{-6} \text{ for } x \in E_Y^1,
\]

\( h_0 \) has exactly one zero \( b_1 \) in \( E_Y^1 \), and \( b_1 \) is a simple, non-degenerate zero of \( h_0 \).

Here, we say that \( \xi \in E_Y^1 \) is a non-degenerate, simple zero of a continuous function \( h \) on \( E_Y^1 \) if \( h(\xi) = 0 \) and there is a ball \( B(\xi, \rho) \) and a Bi-Hölder function \( \gamma \) with Hölder exponent \( 1 + \tau \) which maps \( E_Y^1 \cap B(\xi, \rho) \) to an open set \( V \) of a 2-plane, such that \( h \circ \gamma^{-1} \) is of class \( C^1 \) on \( V \) and the differential \( D(h \circ \gamma^{-1}) \) at the point \( \gamma(\xi) \) is of rank 2.

**Proof.** — We modify \( f_0 \) in a neighborhood of \( d_1 \). We have already our Bi-Hölder homeomorphism \( \psi \) which satisfies (6), (7) and (8). Next, since \( E_Y^1 \) is the connected component of \( E_Y \) which contains \( b_1 \), we have

\[
E_Y \cap B(d_1, 1) = E_Y^1 \cap B(d_1, 1),
\]

thus

\[
E_Y^1 \cap B(d_1, 1/3) \subset \psi_1(B(L_1 \cap B(d_1, 1/2))) \subset E_Y^1 \cap B(d_1, 1),
\]

(3.2.10)

\( \psi_1 \) is the 2-face of \( T \) that contains \( d_1 \), which is Bi-Hölder equivalent to \( E_Y^1 \) in the ball \( B(d_1, 1) \).

Set \( h_0 = f_0 \) outside the ball \( B(d_1, 1/2) \). In \( B(d_1, 1/4) \), we set \( h_0 = f_0 \circ \psi^{-1} \). In the region between the two balls \( R = \overline{B(d_1, 1/2)} \setminus B(d_1, 1/4) \), we set

\[
h_0(x) = \alpha(x)f_0(x) + (1 - \alpha(x))f_0 \circ \psi^{-1}(x),
\]

(3.2.11)
where \( \alpha(x) = 4|x - d_1| - 1 \). We have then \( |h_0(x) - f_0(x)| \leq |f_0(x) - f_0 \circ \psi_1^{-1}(x)| \leq C \tau \) for \( x \in B(d_1, 1/2) \) since \( |\psi_1(x) - x| \leq \tau \) and the differential of \( f_0 \) is bounded in this ball. We have then (3.2.9).

Since \( f_0(x) = (x_4, |x|^2 - 4^9) \), so \( |f_0(x)| \geq 1/500 \) for \( x \in E_1 \setminus B(d_1, 10^{-2}) \).

By consequence, all the zeroes of \( h_0 \) must lie in the ball \( B(d_1, 1/4) \).

We verify next that \( h_0 \) has exactly one zero in \( B(d_1, 1/4) \), which is simple and non-degenerate. Set \( \gamma_1(x) = \psi_1^{-1}(x) \) for \( x \in E_1 \cap B(d_1, 1/4) \). Then \( \gamma_1 \) is a homeomorphism from \( E_1 \cap B(d_1, 1/4) \) onto its image, which is an open set in \( L_1 \).

Since \( h_0 = f_0 \circ \psi_1^{-1} = f_0 \circ \gamma_1 \) on \( E_1 \cap B(d_1, 1/4) \), we have that \( h_0(\xi) = 0 \) for \( \xi \in E_1 \cap B(d_1, 1/4) \) if and only if \( \gamma_1(\xi) \) is a zero of \( f_0(\xi) = (x_4, |x|^2 - 4^9) \) in \( L_1 \cap B(d_1, 1/2) \), which can only be \( d_1 \). The verification that \( Df_0 \) is of maximal rank at \( d_1 \) is clear. The sub-lemma follows.

We need another sub-lemma which allows us to go from \( h_{k-1} \) to \( h_k \).

**Sub-Lemma 3.2.2.** — *We can find continuous functions \( \theta_k, 1 \leq k \leq l \), such that* \( \theta_k \) is supported in \( B(x_k, 3\eta) \), \( (3.2.12) \) and \( ||\theta_k||_{\infty} \leq 2^{-k}10^{-6} \), \( (3.2.13) \) and if we set \( h_k = h_{k-1} + \varphi_k(f_1 - f_0) + \theta_k \), \( (3.2.14) \) for \( 1 \leq k \leq l \), then \( (3.2.15) \)

*each \( h_k \) has a finite number of zeroes in \( E_1 \), which are all simple and non-degenerate.*

**Proof.** — We will construct \( h_k \) by induction. For \( k = 0 \), the function \( h_0 \) satisfy clearly (3.2.15). Let \( k \geq 1 \), and we suppose that we have already constructed \( h_{k-1} \) such that (3.2.15) holds.

We note that \( h_{k-1} + \varphi_k(f_1 - f_0) \) coincide with \( h_{k-1} \) outside the ball \( B(x_k, 2\eta) \), by (3.2.5). We take a thin annulus

\[
A = \overline{B}(x_k, \rho_2) \setminus B(x_k, \rho_1), 2\eta < \rho_1 < \rho_2 < 3\eta, \tag{3.2.16}
\]

which doesn’t meet the finite set of zeroes of \( h_{k-1} \). Recall that there is a Bi-Hölder function \( \psi_k : B(x_k, 20\eta) \to \mathbb{R}^4 \) and a 2-plane \( P_k \) passing through
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$x_k$ such that $|\psi_k(x) - x| \leq 10\eta \tau$ for $x \in B(x_k, 20\eta)$ and

$$E^1_Y \cap B(x_k, 19\eta) \subset \psi_k(P_k \cap B(x_k, 20\eta)) \subset E^1_Y.$$  \hspace{1cm} (3.2.17)

We choose $\theta_k$ such that $\theta_k$ is supported in $B(x_k, \rho_2)$ and $||\theta_k||_{\infty} < \min\{2^k 10^{-6}, \inf_{x \in A} |h_{k-1}(x)|\}$, of course $\inf_{x \in A} |h_{k-1}(x)| > 0$ since $A$ doesn’t meet the set of zeroes of $h_{k-1}$. Then $h_k = h_{k-1}$ outside the ball $B(x_k, \rho_2)$.

We will control $h_k$ in the ball $B(x_k, \rho_1)$. Set $\gamma(x) = \psi^{-1}_k(x)$ for $x \in E^1_Y \cap B(x_k, \rho_1)$. By (3.2.17) and since $\psi_k$ is Bi-Hölder on $B(x_k, 20\eta)$, $\gamma$ is a Bi-Hölder homeomorphism from $E^1_Y \cap B(x_k, \rho_1)$ onto an open set $V$ of the 2-plane $P_k$.

By the density of $C^1$ function in the space of bounded continuous functions on $V$ with the sup norm, we can choose $\theta_k$ with the above properties and such that

$$h_k \circ \theta_k \text{ is of class } C^1 \text{ on } V.$$

We can also add a very small constant $w \in \mathbb{R}^2$ to $\theta_k$ on $E^1_Y \cap B(x_k, \rho_1)$, and then interpolate continuously on $A$. We verify that for almost every choice of $w$,

$$h_k \text{ has a finite number of zeroes in } E^1_Y \cap B(x_k, \rho_1).$$  \hspace{1cm} (3.2.19)

For this, we set $Z_y = \{z \in V; h_k \circ \psi_k(z) = y\}$. By (3.2.18), we can apply the co-area formula ([9, 3.2.22]) for $h_k \circ \psi_k$ on $V$, and we obtain

$$\int_V J(z) dH^2(z) = \int_{y \in \mathbb{R}^2} H^0(Z_y) dH^2(y),$$  \hspace{1cm} (3.2.20)

here, $J(z)$ denote the Jacobian of $h_k \circ \psi_k$ at $z$, which is clearly bounded. We deduce that $Z_y$ is finite for almost-every $y \in \mathbb{R}^2$. If we choose $w$ such that $Z_w$ is finite and then add $-w$ to $\theta_k$ in $E^1_Y \cap B(x_k, \rho_1)$, then the new $Z_0$ will be finite, and we have (3.2.19).

We consider now the rank of the differential. By Sard’s theorem, the set of critical values of $h_k \circ \psi_k$ has measure 0 in $\mathbb{R}^2$. So if we choose $w \in \mathbb{R}^2$ which is not a critical value, and add $-w$ to $\theta_k$ in $E^1_Y \cap B(x_k, \rho_1)$, then the differential of the new function $h_k \circ \psi_k$ at each zero of $h_k \circ \psi_k$ is of rank 2.

So we take $w$ very small with the above properties, and add $-w$ to $\theta_k$ in $B(x_k, \rho_1)$; next, we interpolate in the region $A$, we obtain a function $h_k$ having a finite number of zeroes in $E^1_Y \cap B(x_k, \rho_1)$ which are all simple and non-degenerate. The sub-lemma follows.

Now let $N(k)$ be the number of zeroes of $h_k$ in $E^1_Y$. Then $N(0) = 1$ since the only zero of $h_0$ in $E^1_Y$ is $b_1$. Let us check that for the last index $l$,
\[ N(l) = 0. \text{ First we have} \]
\[ h_l - h_0 = \sum_{1 \leq k \leq l} (h_k - h_{k-1}) = \sum_{1 \leq k \leq l} \varphi_k(f_1 - f_0) + \sum_{1 \leq k \leq l} \theta_k. \]

If \( x \in E_Y^1 \cap B(0, 3 \cdot 2^8 - \eta) \), then \( \sum_{1 \leq k \leq l} \varphi_k(x) = 1 \), thus
\[ h_l(x) = h_0(x) + f_1(x) - f_0(x) + \sum_{1 \leq k \leq l} \theta_k(x) \]
so that
\[ |h_l(x)| \geq |f_1(x)| - |h_0(x) - f_0(x)| - \sum_{1 \leq k \leq l} |\theta_k(x)| \]
\[ \geq 1/4 - 10^{-6} - \sum_{1 \leq k \leq l} 2^{-k} 10^{-6} > 0 \]
by (3.2.3), (3.2.6) and (3.2.13).

If \( x \in E_Y^1 \cap B(0, 2^{10}) \setminus B(0, 3 \cdot 2^8 - \eta) \), then \( \sum_{1 \leq k \leq l} \varphi_k(x) = 1 - \varphi_0(x) \),
so
\[ h_l(x) = h_0(x) + (1 - \varphi_0(x))(f_1(x) - f_0(x)) + \sum_{1 \leq k \leq l} \theta_k(x) \]
which implies
\[ |h_l(x) - f_0(x) - (1 - \varphi_0(x))(f_1(x) - f_0(x))| \]
\[ \leq |h_0(x) - f_0(x)| + \sum_{1 \leq k \leq l} |\theta_k(x)| \leq 2.10^{-6}. \]

But the second coordinate of \( f_0(x) + (1 - \varphi_0(x))(f_1(x) - f_0(x)) \) is
\[ |x|^2 - 4^9 + (1 - \varphi_0(x))(|x - d_2|^2 - |x|^2 + 4^9) \]
\[ = \varphi_0(x)|x|^2 - 4^9 + (1 - \varphi_0(x))|x - d_2|^2 \geq 1/4, \]
by (3.2.2) and because \(|x| \geq 3 \cdot 2^8 - \eta\). Thus \( h_l(x) \neq 0 \) in this case also. We deduce that \( h_l \) has no zero in \( E_Y^1 \), and \( N(l) = 0 \).

**Sub-Lemma 3.2.3.** — \( N(k) - N(k - 1) \) is even for \( 1 \leq k \leq l \).

**Proof.** — We observe that \( h_{k-1} \) don’t vanish on \( A \), where \( A \) is the annulus defined in (3.2.16), and we took \( \|\theta_k\|_{\infty} \) very small so that \( h_k \) does not vanish on \( A \) as well. Next, by definition of \( \varphi_k, \varphi_k = 0 \) on \( A \). Setting
\[ m_t(x) = h_{k-1}(x) + t[h_k(x) - h_{k-1}(x)] = h_{k-1}(x) + \theta_k(x), \quad (3.2.21) \]
for \( x \in E_Y^1 \cap \overline{B}(x_k, \rho_2) \) and \( 0 \leq t \leq 1 \). Then \( m_0 = h_{k-1} \) and \( m_1 = h_k \) on \( E_Y^1 \cap \overline{B}(x_k, \rho_2) \). Since \( m_t(x) = h_{k-1}(x) + t \theta(x) \) for \( x \in E_Y^1 \cap A \) and \( 0 \leq t \leq 1 \), so \( m_t(x) \neq 0 \) if we take \( \theta \) small enough. Let \( \beta_k > 0 \) such that \( |m_t(x)| \geq \beta_k \) for \( x \in E_Y^1 \cap A \). Set \( S_\infty = \mathbb{R}^2 \cup \{\infty\} \), so that \( S_\infty \) can be stereographically identified with a sphere of dimension 2, we define \( \pi : \mathbb{R}^2 \to S_\infty \) by

\[
\pi(x) = \infty \text{ if } |x| \geq \beta_k \text{ and } \pi(x) = \frac{x}{\beta_k - |x|} \text{ otherwise.} \tag{3.2.22}
\]

Next, we set

\[
p_t(x) = \pi(m_t(x)) \text{ for } x \in E_Y^1 \cap \overline{B}(x_k, \rho_2) \text{ and } 0 \leq t \leq 1. \tag{3.2.23}
\]

Then \( p_t(x) \) is a continuous function of \( x \) and \( t \), which takes values in \( S_\infty \).

By the definition of \( \beta_k \),

\[
p_t(x) = \infty \text{ for } x \in E_Y^1 \cap A \text{ and } 0 \leq t \leq 1. \tag{3.2.24}
\]

We want to replace the domain \( E_Y^1 \cap \overline{B}(x_k, \rho_2) \) by an open set in a 2-plane \( P_k \). We keep our Bi-Hölder function \( \psi_k \) as above, which maps an open set \( V \) of a 2-plane \( P_k \) onto \( E_Y^1 \cap B(x_k, \rho_2) \) and its inverse \( \gamma \) which is also Bi-Hölder and maps \( E_Y^1 \cap B(x_k, \rho_2) \) onto \( V \). For \( 0 \leq t \leq 1 \), we set

\[
q_t(x) = p_t(\psi_k(x)) \text{ for } x \in V \text{ and } q_t(x) = \infty \text{ for } x \in P_k \setminus V. \tag{3.2.25}
\]

We check that \( q_t \) is continuous in \( P_k \times [0, 1] \). It is continuous in \( V \times [0, 1] \), since \( p_t \) is continuous in \( [E_Y^1 \cap B(x_k, \rho_2)] \times [0, 1] \). It is also continuous in \( [P_k \setminus V] \times [0, 1] \), because it is \( \infty \) here. Now if \( x \in \partial V \), then \( \psi_k(x) \in E_Y^1 \cap \partial B(x_k, \rho_2) \), so there is a neighborhood of \( \psi_k(x) \) in \( \overline{B}(x_k, \rho_2) \) which is contained in \( A \), and we have \( p_t(\psi_k) = \infty \) on this neighborhood, so \( q_t = \infty \) near \( x \).

We set \( q_t(\infty) = \infty \), so \( q_t \) is well defined on \( S' = P_k \cup \{\infty\} \) and it is clear that each \( q_t \) is continuous for \( 0 \leq t \leq 1 \).

Now since \( q_0 \) and \( q_1 \) are two continuous functions from the 2-sphere \( S' \) to the 2-sphere \( S_\infty \), we can compute their degrees. First, as \( q_0 \) and \( q_1 \) are homotopic, they have the same degrees. We compute the degree of \( q_0 \), for example. Let

\[
q_0^{-1}(\{0\}) = \{y_1, y_2, ..., y_m\}, \tag{3.2.26}
\]

the set of zeroes of \( q_0 \). This is a finite set since \( q_t \) has only finite number of zeroes for \( t \leq 1 \). Since each zero of \( q_0 \) is simple and non-degenerate, for each \( 1 \leq k \leq m \), there exists a neighborhood \( W_k \) of \( y_k \) such that

\[
q_0 \text{ is a homeomorphism from } W_k \text{ to } q_0(W_k), \tag{3.2.27}
\]
and
\[ W_k \cap W_l = \varnothing \text{ if } k \neq l. \]  \hspace{1cm} (3.2.28)
So the degree of \( q_0 \) is computed as follows. We begin by 0, next, for \( 1 \leq k \leq m \), if \( q_0 \) preserve the orientation of \( W_k \), we add 1, if \( q_0 \) doesn’t preserve the orientation of \( W_k \), we add -1. Then it is clear that
\[ d(q_0) \text{ is of the same parity as } m. \] \hspace{1cm} (3.2.29)
Here \( d(q) \) denote the degree of the function \( q \). By the same arguments, we have
\[ d(q_1) \text{ is of the same parity as the number of zeroes of } q_1. \] \hspace{1cm} (3.2.30)
But \( d(q_0) = d(q_1) \) as above, we obtain
\[ \text{the number of zeroes of } q_0 \text{ is of the same parity as the number of zeroes of } q_1. \] \hspace{1cm} (3.2.31)

We want to prove next that the number of zeroes of \( h_{k-1} \) is of the same parity as the number of zeroes of \( h_k \). Since \( h_{k-1} = h_k \) outside the ball \( B(x_k, \rho_2) \) and they both don’t vanish on \( E_Y^1 \cap A \), we need only to consider their number of zeroes in \( E_Y^1 \cap B(x_k, \rho_1) \). We verify that
\[ \text{the number of zeroes of } h_{k-1+s} \text{ in } E_Y^1 \cap B(x_k, \rho_1) \text{ is equal to the number of zeroes of } q_s \text{ in } S' \text{ for } s = 0, 1. \] \hspace{1cm} (3.2.32)

We verify for \( s = 0 \). If \( q_0(x) = 0 \), then \( x \in V \) (otherwise \( q_0(x) = \infty \)), so \( q_0(x) = p_0(\psi_k(x)) \) and then \( p_0(\psi_k(x)) = 0 \). Since \( m_0(\psi_k(x)) = 0 \), we have \( h_{k-1}(\psi_k(x)) = 0 \). Because \( x \in V \), we have \( \psi_k(x) \in B(x_k, \rho_1) \). So if \( q_0(x) = 0 \), then \( \psi_k(x) \in B(x_k, \rho_1) \) and is a zero of \( h_{k-1} \).

Conversely, if \( y \in B(x_k, \rho_1) \) is such that \( h_{k-1}(y) = 0 \), then \( p_0(y) = 0 \) and then there exists \( y' \in V \) such that \( \psi_k(y') = y \) because \( \psi_k \) is a homeomorphism from \( V \) to \( B(x_k, \rho_1) \). Now \( q_0(y') = p_0(\psi_k(y')) = 0 \) and thus \( y' \) is a zero of \( q_0 \).

So we have (3.2.32) for \( s = 0 \). The case \( s = 1 \) is the same, and we have then (3.2.32). By (3.2.31), we obtain that the number of zeroes of \( h_{k-1} \) is of the same parity as the number of zeroes of \( h_k \), which means that \( N(k) - N(k-1) \) is even. The sub-lemma follows.

Now by sub-lemma 3.2.3, we know that \( N(0) - N(1) \) is even, but it is 1, so we obtain a contradiction, and we finish the proof of Lemma 3.2. \( \square \)
3.3. Proof of Theorem 2

Let $U(y), y \in E_Y \cap B(0, 3 \cdot 2^{8})$ be the set of connected components $V$ of $B(0, 2^{10}) \setminus E$ such that $y \in V$. Since for each $y \in E_Y$, there is a neighborhood $W$ of $y$ on which $E$ is Bi-Hölder equivalent to a $Y$, we see that $U(y)$ is locally constant. By Lemma 3.2, we can connect $b_1$ to another point $b_i, i \neq 1$, by a curve in $E_Y$, and we can suppose that $i = 2$. Because $b_1, b_2 \in E_Y$ and $U(y)$ is locally constant on $E_Y$, we have $U(b_1) = U(b_2)$. By Lemma 3.1, and the fact that $E$ is Bi-Hölder equivalent to a $Y$ near each point of type $Y$, we have

$$\{V_2, V_3, V_4\} = U(b_1)$$

and

$$\{V_1, V_3, V_4\} = U(b_2),$$

where $V_i, 1 \leq i \leq 4$ is as in Lemma 3.1. So we see that $U(b_1) \neq U(b_2)$, which is a contradiction. We finish the proof of Theorem 2. \qed

Bibliography