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BO BERNDTSSON
Convexity on the space of Kähler metrics

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Abstract. — These are the lecture notes of a minicourse given at a winter school in Marseille 2011. The aim of the course was to give an introduction to recent work on the geometry of the space of Kähler metrics associated to an ample line bundle. The emphasis of the course was the role of convexity, both as a motivating example and as a tool.

Résumé. — On présente ici les notes d'un mini-cours donné lors d'une école d'hiver à Marseille en 2011. Le but du cours était de fournir une introduction à des travaux récents sur la géométrie de l'espace des métriques kählériennes associées à un fibré en droites ample. Le cours a mis l'accent sur le rôle de la convexité, en tant qu'exemple motivant et en tant qu'outil.

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1. Introduction

Let X be a compact Kähler manifold and L a positive line bundle over X. Any positive(ly curved) metric, ϕ, on L, then defines a Kähler metric on X through

$$\omega = \omega^\phi = i\partial\bar\partial\phi.$$

We denote

$$\mathcal{H}_L = \{\phi, \text{ metric on } L, i\partial\bar\partial\phi > 0\}$$

and think of it as the class of potentials for Kähler metrics whose Kähler forms lie in the Chern class of L. We can think of $\mathcal{H}_L$ as a differentiable manifold of infinite dimension (it is an affine space modeled on $C^\infty(X)$), with tangent space at any point ϕ being equal to $C^\infty(X)$. Following Mabuchi, [18], Semmes [24] and Donaldson [13] we then introduce a (pre)Hilbert norm (depending on the point) on the tangent spaces, making $\mathcal{H}_L$ a Riemannian manifold (at least formally). Finally we study certain functions defined on this space, i.e. functions of metrics, and study their convexity properties (along geodesics).

One main motivation for this setup comes from the problems of existence and uniqueness of privileged Kähler metrics on X, notably Kähler-Einstein metrics and more generally metrics of constant scalar curvature. The equations that such privileged metrics have to satisfy are complicated nonlinear expressions of fourth order in the potential. A main idea is to reformulate these equations as the equation for a critical point of a certain function(al) defined on $\mathcal{H}_L$, a little bit like one can study the Dirichlet problem by looking at critical points of the energy functional. Convexity properties are then obviously relevant for the existence and uniqueness of critical points.

Interestingly, there are also other problems that lead up to the same structure. One is the homogenous complex Monge-Ampère equation (HCMA) on $\Omega \times X$ where Ω is a domain in the complex plane. One then studies 'curves' of metrics $\phi_t(x) = \phi(t,x)$ on L. The HCMA for such curves is then

$$(i\partial\bar\partial\phi)^{n+1} = 0,$$

where the $\partial\bar\partial$ is now taken w r t all the variables including t. The Dirichlet problem consists in solving this equation with given boundary values when t lies on the boundary of Ω. It turns out that the special case when Ω is a vertical strip and the metrics only depend on the real part of t is precisely the geodesic equation for the aforementioned Riemannian structure [24].

Another motivation comes from symplectic geometry, see [13]. A Kähler form ω on X is in particular a symplectic form on X. Fixing ω we can look at

S_ω, the group of diffeomorphisms of X that preserve ω. The tangent space to the full group of diffeomorphisms is the space of vector fields on X, and the tangent space of the symplectomorphisms is the space of Hamiltonian fields (at least if $H^1(X, \mathbb{R})$ vanishes). Each such field is the symplectic gradient of a Hamiltonian function, so up to constants the tangent space to S_ω is $C^\infty(X)$ – the 'same' tangent space as $\mathcal{H}_L$! There is a neat explanation of this.

Disregarding the slight problem with constants, the complexification of the tangent bundle of S_ω is the space of complex valued smooth functions on X. It turns out that there is a complex manifold, D_ω which has this space as its tangent space at any point. D_ω is then a sort of complexification of the group S_ω and it can be presented as the set of diffeomorphisms of X that map ω to a positive $(1, 1)$ form. Its complex structure is simply given by saying that a complex curve F_t in D_ω is holomorphic if $F_t(x)$ is a holomorphic curve in X for each x fixed. Such curves are called *holomorphic motions* on X. There is a natural map, or fibration, from D_ω to $\mathcal{H}_L$ which maps a diffemorphism F to $F^*(\omega)$. Its fiber over ω equals S_ω and its fibers over other points are conjugate to this group. At any point the tangent space of D_ω decomposes as a direct sum of the tangent space to the fiber (a group of symplectomorphisms) and the tangent space of the base $\mathcal{H}_L$, and the second of these summands is J times the first one, where J is the complex structure on D_ω. The relation to the Riemannian structure on $\mathcal{H}_L$ is that a curve in $\mathcal{H}_L$ is a geodesic if and only if it lifts to a holomorphic curve in D_ω. We will not go into any details of these constructions, but just point out that it means that convex functions on $\mathcal{H}_L$ lift to plurisubharmonic functions on this complex manifold, much like a convex function on $\mathbb{R}^n$ can be viewed as a plurisubharmonic function on $\mathbb{C}^n$, independent of the imaginary part.

In these notes from a series of lectures given at the CIRM, Luminy in February 2011 we will try to describe parts of the picture outlined above. Our point of view comes from a comparison with the analogous picture for convex functions on $\mathbb{R}^n$ [24]. We therefore start with a discussion of the real case emphasising the role of the Legendre transform and functional versions of the Brunn-Minkowski inequality. After that we discuss some theorems from [5] on the curvature of certain vector bundles associated to the space $\mathcal{H}_L$. From there we obtain variants of Donaldson's $\mathcal{L}$-functional, [12] as metrics on the determinants of these vector bundles. Using Bergman kernel asymptotics we then get two other important functionals, the Aubin-Yau energy and the Mabuchi K-energy, as limits of $\mathcal{L}$-functionals for kL when k tends to infinity. In the final section we argue that the vector bundle constructions can be seen as an analog of the Legendre transform.

These notes are in a very sketchy form and most detailed proofs are missing. The list of references to this very rapidly growing field is also far from exhaustive. The aim of the notes is to serve as an easy introduction to this beautiful field, all the time emphasizing the role of convexity. Almost all of the text is a survey of known material; the only part that has some claim to originality is the last section that proposes a notion of 'Legendre transform' for positively curved metrics on a line bundle. Finally, I would like to thank the referee for numerous comments leading to an improved presentation.

2. Real convexity

In this section we let ϕ, ψ etc denote convex functions on $\mathbb{R}^n$. We will also be interested in convex sets in $\mathbb{R}^n$ and will then use that the space of convex sets can be embedded in the space of convex functions. Actually, this can be done in two natural ways:

1. If K is a convex set in $\mathbb{R}^n$ we let

$$\phi_K(x) = 0, \infty$$

depending on whether x lies in K or not.

This function then takes on the value $+\infty$ but most of the theory of convex functions extends to such functions, since they can be written as limits of increasing sequences of classical convex functions.

2. If K is a convex set in $\mathbb{R}^n$ we let h_K be the supporting function of K

$$h_K(y) = \sup_{x \in K} y \cdot x.$$

It is easy to check that both these functions are convex; in fact the supporting function is convex even if K is not. The relation between the two is that h_K is the *Legendre transform* of ϕ_K, where the Legendre transform of a function in general is defined as

$$\hat{\phi}(y) = \sup_x (y \cdot x - \phi(x)).$$

Again, the Legendre transform is always convex (being the sup of a class of affine functions), and a basic result is that if ϕ is convex, then the Legendre transform of $\hat{\phi}$ equals ϕ. In general, the Legendre transform of $\hat{\phi}$ is the largest convex minorant of ϕ. We stress that one should think of the Legendre tranform as being defined on the dual space of $\mathbb{R}^n$.

The class of all convex functions has a natural affine structure since the sum of two convex functions is again convex. If we consider in particular convex functions of the form ϕ_K, then addition corresponds to taking the intersection of the corresponding convex sets. On the class of convex sets there is however a more interesting affine structure, given by the *Minkowski sum*

$$K + L = \{x + z; x \in K \text{ and } z \in L\}.$$

It is not hard to see that the supporting function of $K + L$ is the sum of the supporting functions of K and L

$$h_{K+L} = h_K + h_L.$$

Hence this more interesting affine structure is related to the first one by the Legendre transform.

In the same way we get two affine structures on the class of convex functions; one defined by ordinary addition, the other by taking the sum of the Legendre transforms and taking the Legendre transform back again. Explicitly, the second affine structure is given by the *infimal convolution*

$$\phi * \psi(z) = \inf_t (\phi(z - t) + \psi(t)).$$

Exercise 1. — Check that the Legendre transform of $\phi * \psi$ equals $\hat\phi + \hat\psi$.

Here already a question arises: is $\phi * \psi$ convex? In general the suprema of families of convex functions are convex, but infima are not. That the infimal convolution of convex functions is nevertheless always convex follows from the next proposition, known as the minimum principle for convex functions.

PROPOSITION 2.1. — *Let $\phi(z,t)$ be convex on $\mathbb{R}^n_z \times \mathbb{R}^m_t$. Then*

$$f_\phi(z) := \inf_t \phi(z, t)$$

is convex.

This can be seen by considering epigraphs , i e the sets

$$E_\phi = \{(x, s); s > \phi(x)\}.$$

A function is convex if and only if its epigraph is a convex set. The epigraph of f_ϕ is the projection of the epigraph of ϕ. The proposition therefore follows from the geometrically obvious fact that projections of convex sets are convex.

Let us now take a different look at the two affine structures on the space of convex functions. We then shift focus slightly and consider the class of strictly convex and smooth functions. Then the gradient map

$$x \to d\phi(x)$$

is locally, and therefore globally, injective. We will use the following basic property of the Legendre transform.

PROPOSITION 2.2. — *The range of the gradient map of ϕ is the interior of the set where $\hat{\phi}$ is finite. The range is therefore always a convex open set. The inverse of the gradient map is the gradient map of $\hat{\phi}$ restricted to this open set.*

The proof uses the fact that if the supremum in the definition of $\hat{\phi}(y)$ is attained at a point x, then $y = d\phi(x)$.

Let us now, to fix ideas, look at the class of functions such that the range is all of $\mathbb{R}^n$. This means that the functions grow faster than linearly in all directions. We call this class Φ.

We now introduce a structure of infinite dimensional Riemannian manifold on this space of convex functions. We consider its tangent space at any point ϕ to be the set of smooth functions of say compact support and give it the trivial Riemannian norm

$$\|\chi\|^2 = \int_{\mathbb{R}^n} \chi^2 dx.$$

Remark 2.3. — Of course this is not completely correct. Allowing only functions with compact support as tangent vectors would only allow us to consider curves in the space that change the functions on compact subsets only. To get a really well defined Riemannian manifold we would need to specify more closely a set of convex functions with prescribed behaviour at infinity, and then define the tangent space accordingly. However, compactly supported functions would always be dense in the tangent space, so to simplify life we only consider such functions.

This Riemannian norm does not depend on the point ϕ so it is flat and geodesics are linear segments. As mentioned above we can consider the Legendre transform, $\mathcal{L}$, as a map from Φ to itself. The next proposition computes its derivative.

PROPOSITION 2.4. — *If χ is smooth, compactly supported,*

$$\mathcal{L}'_\phi(\chi)(d\phi(x)) = -\chi(x),$$

so, using Proposition 2.2,

$$\mathcal{L}'_\phi(\chi) = -\chi \circ d\hat{\phi}.$$

This means that the Legendre transform maps the trivial metric on Φ at a point ϕ to the nontrivial metric

$$\int \chi^2 d\mu$$

where

$$d\mu = d\psi^*(dx) = \det(\psi_{jk})dy := MA(\psi)$$

at the image point $\psi = \hat{\phi}$. Here MA stands for the Monge Ampère operator.

We can now write down the geodesic equation for this new metric. If ϕ_t is a curve it is the covariant derivative of the velocity vector $\dot{\phi}_t := d\phi/dt$, along the curve which we write as $D_{\dot{\phi}_t}\dot{\phi}_t$. It is not too hard to show that it equals

$$D_{\dot{\phi}_t}\dot{\phi}_t = d^2\phi/dt^2 - |d\dot{\phi}_t|^2_{(\phi_{jk})} = 0.$$

Here the norm in the right hand side $|v|_{(\phi_{jk})}$ is the norm of a one-form measured with respect to the metric

$$\frac{\partial^2 \phi_t}{\partial x_j \partial x_k}.$$

In general, the expression in the left hand side

$$c(\phi) := d^2\phi/dt^2 - |d\dot{\phi}_t|^2_{(\phi_{jk})}$$

is the geodesic curvature of the curve ϕ_t.

By a famous observation of Semmes [24], the geodesic curvature can be rewritten in yet another way

$$c(\phi)\det((\phi_t)_{jk}) = \det(\phi_{jk}) \tag{2.1}$$

where in the right hand side we take the determinant of the Hessian with respect to *all* of the variables, including t.

Exercise 2. — Check this for $n = 1$.

The conclusion is that a curve ϕ_t is a geodesic if and only if the function on $\mathbb{R}^{n+1}$

$$\phi(t, x) = \phi_t(x)$$

solves the homogenous Monge-Ampère equation. One interesting consequence of this is the following corollary.

COROLLARY 2.5. — *Let $\phi(t,x)$ be a solution of the homogenous Monge-Ampère equation for x in $\mathbb{R}^n$ and t in some interval. Let*

$$\psi_t(y) = \hat{\phi}_t$$

be the Legendre transform of ϕ with respect to x for t fixed. Then

$$\psi_t = \psi_a + (t - a)\chi$$

is an affine function in t.

This follows since the Legendre transform maps geodesics for the nontrivial Riemannian structure to geodesics for the trivial Riemannian structure, i e to linear segments.

More generally we say that a curve ϕ_t is a *subgeodesic* (for the nontrivial structure) if $c(\phi) \geqslant 0$. By (1.1) this is equivalent to saying that the product of all the eigenvalues of the full Hessian of $\phi(t,x)$ is nonnegative. Since the Hessian with respect to x is always nonnegative, this is equivalent to saying that $\phi(t,x)$ is convex with respect to all the variables.

Exercise 3. — Why?

Since the geodesic curvature for the nontrivial structure is mapped to the geodesic curvature for the trivial structure by the Legendre transform in x, we see that subgeodesics are mapped to curves ϕ_t that are convex in t (but perhaps not in all variables, cf tx).

3. Convex functions on the space of convex functions

We start with an almost trivial result.

PROPOSITION 3.1. — *Let $\phi(t,x)$ be convex in t. Then*

$$t \mapsto \log \int_{\mathbb{R}^n} e^{\phi(t,x)} dx$$

is convex (if the integral is convergent).

This follows easily from Hölder's inequality, and also by differentiating with respect to t.

Here is a somewhat less obvious consequence of this.

PROPOSITION 3.2. — *Let ϕ_t be a geodesic for the nontrivial structure. Then*

$$t \mapsto \log \int_{\mathbb{R}^n} e^{-\phi(t,x)} \det((\phi_t)_{jk}) dx$$

is convex.

Proof. — Let $\psi_t = \hat{\phi}_t$ and change variables by

$$x = d\psi_t(y),$$

where d denotes differentiation with respect to the y variable. Then

$$\int_{\mathbb{R}^n} e^{-\phi_t(x)} \det((\phi_t)_{jk}) = \int_{\mathbb{R}^n} e^{-\phi_t \circ d\psi_t(y)} dy.$$

But by the definition of the Legendre transform

$$-\phi_t \circ d\psi_t(y) = \psi_t(y) - d\psi_t \cdot y,$$

and we have seen that if ϕ_t is a geodesic then ψ_t and hence $d\psi_t$ are affine in t. The result then follows from the previous proposition. □

Notice that it is important here that ϕ_t is a geodesic, not just a subgeodesic. In fact,

$$\phi_t = t^2 + x^2$$

is a subgeodesic for which the result clearly does not hold.

Our final result is due to Prekopa [22].

THEOREM 3.3. — *Let ϕ_t be a subgeodesic. Then*

$$f(t) := \log \int_{\mathbb{R}^n} e^{-\phi_t(x)} dx$$

is concave.

This is deeper and more useful than the earlier propositions. Note that it does not follows in the same way as Proposition 2.1 since Hölder's inequality goes in the opposite direction. Before discussing its proof we mention some consequences.

If we take $\phi = \phi_K$ where K is a convex set in $\mathbb{R}^{n+1}$, we see that

$$\int_{\mathbb{R}^n} e^{-\phi_t(x)} dx = |K_t|$$

is the volume of the slices of K

$$K_t = \{x; (t, x) \in K\}.$$

Prekopa's theorem then says that $\log |K_t|$ is concave, or explicitly

$$|K_{(t+s)/2}|^2 \leqslant |K_t||K_s|$$

This is known as (the multiplicative form of) the Brunn-Minkowski theorem, which is one of the most important results in convex geometry.

We also mention that the minimum principle follows from Theorem 3.3. Applying Theorem 3.3 to the function

$$\psi_t^{(p)} = p\phi(t, x) + |x|^2$$

we get that

$$-(1/p) \log \int e^{-\psi_t^{(p)}} dx$$

is convex. When p tends to infinity this goes to $\inf_t \phi$, so the minimum principle follows. This is perhaps not so impressive since a direct proof of the minimum principle is much simpler, but it serves to highlight the relation between the two facts.

For the proof of Prekopa's theorem, we first notice that by Fubini's theorem, it is enough to prove it for $n = 1$.

Exercise 4. — Why?

Assuming things are nice and differentiable, we get after some computation that the theorem amounts to the inequality

$$\int_{\mathbb{R}} (\ddot{\phi} - (\dot{\phi}_t - \tilde{\dot{\phi}}_t)^2) e^{-\phi_t(x)} dx \geqslant 0 \tag{3.1}$$

where $\tilde{u}$ is the average of a function. The main point in the argument is the following lemma, due to Brascamp and Lieb [8].

LEMMA 3.4. — *Let ϕ be a convex function on $\mathbb{R}$ and let u be a function on $\mathbb{R}$ such that*

$$\int_{\mathbb{R}} u e^{-\phi} = 0.$$

Then

$$\int_{\mathbb{R}} u^2 e^{-\phi} \leqslant \int_{\mathbb{R}} |du|^2 / \phi'' e^{-\phi}.$$

This inequality is the real variable analog of the Hörmander L^2-estimate for the $\bar{\partial}$-equation, but it is of course much easier to prove in this very simple situation. We do not prove it here, but just mention that this is the main reason why one can develop an analogous theory in the complex setting.

Inserting the Brascamp-Lieb inequality in (3.1), with $u = \dot{\phi}_t - \dot{\tilde{\phi}}_t$ we obtain

$$\int_{\mathbb{R}} (\ddot{\phi} - (\dot{\phi}_t - \dot{\tilde{\phi}}_t)^2)\, e^{-\phi_t(x)} dx \geqslant \int_{\mathbb{R}} (\ddot{\phi} - |du|^2/\phi''_{xx})\, e^{-\phi_t(x)} dx =$$

$$= \int_{\mathbb{R}} c(\phi)\, e^{-\phi_t(x)} dx \geqslant 0$$

and we are done.

4. The space of Kähler potentials

In this section (and in the subsequent sections), we consider a compact Kähler manifold X with a positive line bundle L over it. We let

$$\mathcal{H}_L = \{\phi;\, \text{metric on } L,\, i\partial\bar{\partial}\phi > 0\}$$

This is the class of potentials for Kähler metrics ω on X that belong to a fixed cohomology class, determined by the Chern class of L. It will play the role of the space of convex functions; fixing the cohomology class is analogous to fixing the behaviour of the convex functions at infinity.

Toric varieties

A particularily simple class of examples where the analogy with convex functions is the clearest is the class of toric varieties. These are varieties that contain complex tori $\mathbb{C}^n_*$ as open and dense subsets, and for which the natural action of $\mathbb{C}^n_*$ on itself by multiplication extends to the whole (compact) manifold. Such manifolds can be obtained in the following way. Let K be a convex subset of $\mathbb{R}^n$ and let $P = K \cap \mathbb{Z}^n$. We assume moreover that K is the convex hull of P. Then let

$$Z_P = \{z^\alpha;\, \alpha \in P\}.$$

We are going to construct a compactification of $\mathbb{C}^n_*$ over which there is a line bundle L, trivial over $\mathbb{C}^n_*$, such that all elements in Z_P extend as holomorphic sections of L. For this we first assume that P is sufficiently large so that the map from $\mathbb{C}^n_*$

$$\iota : z \to [z^\alpha]_\alpha$$

to $\mathbb{P}^N$ ($N = \#Z_P - 1$) separates points on $\mathbb{C}_*^n$. We moreover assume that the closure, X, of the image of this map in $\mathbb{P}^N$ is a smooth manifold. This is then our toric manifold.

If we use $[w_\alpha]$ as homogenous coordinates on $\mathbb{P}^N$, the embedding of $\mathbb{C}_*^n$ is given by $w_\alpha = z^\alpha$. The sections of $\mathcal{O}(1)$, the hyperplane section bundle over $\mathbb{P}^N$ are linear forms

$$\sum a_\alpha w_\alpha.$$

The pullback of such a section under ι is therefore a linear combination

$$\sum a_\alpha z^\alpha.$$

The restriction of $\mathcal{O}(1)$ to X is thus a line bundle L such that z^α form a basis for $H^0(X, L)$. A metric ϕ on L restricts to a metric on the trivial line bundle over $\mathbb{C}_*^n$, i.e. a function which has to be plurisubharmonic if the curvature of the metric is nonnegative. We call such a metric *toric* if this plurisubharmonic function is invariant under the action of the real torus $(S^1)^n$ on $\mathbb{C}_*^n$. This means that the plurisubharmonic function can be written

$$\phi(z) = \varphi(\log|z_1|^2, \dots \log|z_n|^2)$$

where $\varphi(x_1, \dots x_n)$ is convex. That ϕ extends to a metric on L over all of X implies in particular that the norm of any section is bounded so that

$$|z^\alpha|^2 e^{-\phi(z)}$$

stays bounded on the complex torus. Hence

$$\alpha \cdot x \leqslant \varphi(x) + C$$

if α lies in P. Taking the maximum over all α we conclude that the supporting function of K satisfies

$$h_K(x) \leqslant \varphi(x) + C.$$

If we moreover want a nonsingular metric on L at infinity we need an opposite inequality to be satisfied so that

$$|\varphi - h_K| \leqslant C,$$

otherwise the norm of all sections of L would need to vanish at some point at infinity. Thus toric metrics on such line bundle correspond to convex functions on $\mathbb{R}^n$ that behave like h_K at infinity.

Now we proceed in the same way as in the real case. The tangent space to $\mathcal{H}_L$ at any point is the space of smooth functions on X. This is still

perhaps not quite satisfactory since smooth functions do not form a Banach space, but it is more correct than what we did in the real setting since the condition of compact support disappears. We define a (pre)Hilbert norm on the tangent space by putting

$$\|\chi\|_\phi^2 = \int_X |\chi|^2 (i\partial\bar{\partial}\phi)^n.$$

This is the metric introduced by Mabuchi in [18], and then rediscovered and further studied by Semmes [24] and Donaldson [13]. It gives $\mathcal{H}_L$ the structure of a Riemannian manifold of infinite dimension that corresponds to what we called the 'nontrivial' Riemannian structure for the space of convex functions. Note that there is nothing obvious that corresponds to the 'trivial' structure since we don't have any canonical substitute for Lebesgue measure on X. Still, we have of course a trivial affine structure on $\mathcal{H}_L$.

One very important difference between the real and the complex settings is that the Riemannian structure on $\mathcal{H}_L$ turns out to have negative, in particular nonzero, curvature, see [24]. In the real setting the Legendre transform gave an isometry onto a flat space, so the nontrivial structure is also flat. In the complex setting, Legendre transformation corresponds to a version of quantization, that we shall discuss in section 7.

First however we mention that the geodesic curvature for a curve ϕ_t is given by a formula very similar to the one in the real setting

$$c(\phi_t) = \ddot{\phi}_t - |\bar{\partial}\dot{\phi}_t|^2_{i\partial\bar{\partial}\phi_t}.$$

This formula makes sense even if we let t be a complex parameter and consider

$$\dot{\phi}_t = \frac{\partial\phi_t}{\partial t}$$

and

$$\ddot{\phi}_t = \frac{\partial^2\phi_t}{\partial t \partial\bar{t}}.$$

This function is related to the complex Monge-Ampère operator in the same way as before

$$nc(\phi)(i\partial\bar{\partial}_X\phi)^n \wedge idt \wedge d\bar{t} = (i\partial\bar{\partial}\phi)^{n+1}.$$

We identify such complex curves with real curves if they are independent of the imaginary part of t, so that in particular a geodesic corresponds to a solution of the homogenous complex Monge-Ampère equation that only depends on he real part of t. By a theorem of Chen [9], any two metrics

ϕ_0 and ϕ_1 can be connected by a generalized geodesic, in the sense that the Monge-Ampère equation holds in the generalized sense, and that ϕ is not known to be smooth. In a recent paper by Lempert and Vivas [16], it is shown that in general two positively curved metrics in $\mathcal{H}_L$ cannot be connected by a smooth geodesic in $\mathcal{H}_L$. The best regularity known is that $\partial\bar{\partial}\phi$ is bounded. Therefore the curve is not properly speaking a curve in $\mathcal{H}_L$, and we also do not know that $i\partial\bar{\partial}\phi_t > 0$. As before we say that ϕ_t is a subgeodesic if it is psh on $X \times U$ where U is some open set in $\mathbb{C}$.

Of course for a general metric on a line bundle over X the integral of $e^{-\phi}$ over X has no meaning (with one notable exception that as shall see later on). What corresponds to such integrals are instead L^2-norms of holomorphic sections. The analogy perhaps becomes clearer if we think of

$$\int_{\mathbb{R}^n} e^{-\phi}$$

as the weighted L^2-norm of the constant function 1, which forms a basis for the space $\text{Ker}(d)$, which is the real variable analog of $\text{Ker}(\bar{\partial})$. Variations of such norms on the space of holomorphic sections can be interpreted as hermitian metrics on certain vector bundles.

There seems to be two natural ways to do this.

4.1. A negatively curved vector bundle

Let $F = H^0(X, L)$ be the space of global holomorphic sections of L. For u in F we define

$$\|u\|_\phi^2 := \int |u|^2 e^{-\phi} (\omega^\phi)^n,$$

where $\omega^\phi = i\partial\bar{\partial}\phi$.

THEOREM 4.1. — *Let Ω be a domain in $\mathbb{C}$ and define a trivial vector bundle over Ω as*

$$F \times \Omega.$$

Let $\underline{F}$ denote this vector bundle. Let ϕ_t for t in Ω be a complex curve in $\mathcal{H}_L$ and let this curve define a hermitian metric on $\underline{F}$ by

$$\|u\|_t = \|u\|_{\phi_t}^2.$$

Assume ϕ_t is a (complex) geodesic. Then the curvature of this metric, $\Theta^{\underline{F}}$, is seminegative.

To understand this we recall the elementary formula (see e g [15])

$$i\partial\bar{\partial}\|u_t\|^2 = - <\Theta u, u> +\|D'u\|^2, \tag{4.1}$$

where D' is the $(0,1)$-part of the Chern connection, if u is a holomorphic section. From this we first see that a vector bundle is seminegative, i e $<\Theta u, u >\leqslant 0$ for any u, if and only if $\|u_t\|$ is subharmonic for any holomorphic section u_t. One direction of this is clear, and the converse follows since given any point u in a fiber over some point, we can always extend it holomorphically to a neighbourhood in such a way that $D'u = 0$ at the given point.

This is in turn equivalent to the seemingly stronger statement that $\log\|u_t\|$ is subharmonic for any holomorphic section u_t, simply since we can replace u_t by $e^{at}u_t$ and $\log g$ is subharmonic if and only if $e^{at}g$ is subharmonic for any a.

Hence we get that Theorem 4.1 is a complex variant of Proposition 3.2. As in that case, it is important here that we really are dealing with a geodesic; subgeodesics will not do. We can also use (4.1) to compute the curvature. To compute $i\partial\bar{\partial}\|u_t\|^2$ we write the norm as a pushforward

$$\|u_t\|^2 = p_*(|u|^2 e^{-\phi}(i\partial\bar{\partial}\phi)^n)$$

under the projection map from $X \times \Omega$ to Ω, and then use that pushforwards commute with differentiation. When taking the pushforward here we have two choices how to interpret $i\partial\bar{\partial}\phi$: Either we take the $\partial\bar{\partial}$-operator on X or $\partial\bar{\partial}$ on $X \times \Omega$. The first alternative is what we really want, but the second is better in computations since $i\partial\bar{\partial}\phi$ then is a closed form. Fortunately, both alternatives give the same result since terms containg differentials with respect to t give no contribution to the pushforward.

4.2. A positively curved vector bundle

Let $E = H^0(X, K_X + L)$, where K_X is the canonical line bundle of X, i e the bundle of holomorphic $(n,0)$-forms. Here and in the sequel we use additive notation for line bundles so that $K_X + L$ is the tensor product of the canonical bundle with L. E can therefore be viewed as the space of global holomorphic $(n,0)$-forms with values in L. For u in E we define

$$\|u\|_\phi^2 := \int |u|^2 e^{-\phi}.$$

Here we think of $|u|^2$ as $c_n u \wedge \bar{u}$, an $L \otimes \bar{L}$-valued (n, n)-form that can be integrated directly over X. More precisely, we write locally $u = a \otimes b$ where

a is an $(n,0)$-form and b is a section of L and define

$$|u|^2 e^{-\phi} := c_n u \wedge \bar{u} e^{-\phi} := c_n a \wedge \bar{a} |b|^2 e^{-\phi}.$$

We then define the hermitian vector bundle $\underline{E}$ in a way similar to $\underline{F}$, but using our new definition of $\|u\|_\phi^2$ instead,

$$\|u\|_t^2 = \int_X |u|^2 e^{-\phi_t}.$$

THEOREM 4.2. — *Let ϕ_t be a subgeodesic and use it to define a hermitian metric on the vector bundle $\underline{E}$. Then the curvature $\Theta^{\underline{E}}$ is semipositive.*

Clearly, this corresponds in the same way to Prekopa's theorem. It can be proved in several ways; see [5], [6] and [7]. The most elementary way mimics the proof of Prèkopa's theorem, see [7]. It consists in computing the second derivative

$$\partial \bar{\partial}_t \|u_t\|_t^2 = - < \Theta u_t, u_t > + \|D' u_t\|_t^2$$

of the norm squared of a holomorphic section of $\underline{E}$, cf formula (4.1). First

$$\partial_t \|u_t\|_t^2 = \int_X (\partial_t^\phi u) \bar{u} e^{-\phi},$$

where

$$\partial_t^\phi u = e^\phi \partial_t e^{-\phi} u_t = dt \wedge \dot{u} - \dot{\phi}_t dt \wedge u_t.$$

By the definition of Chern connection, this shows that the $(1,0)$-part of the Chern connection on $\underline{E}$ is

$$D' u = \pi(\dot{u} - \dot{\phi}_t u),$$

where π is the orthogonal projection onto the space of holomorphic sections. Then differentiate once more, with respect to $\bar{t}$, to get

$$\frac{\partial^2}{\partial t \partial \bar{t}} \|u_t\|_t^2 = - \int \phi_{t\bar{t}} |u|^2 e^{-\phi} + \|(\dot{u} - \dot{\phi}_t u)\|_t^2 =$$

$$- \int \phi_{t\bar{t}} |u|^2 e^{-\phi} + \|D' u_t\|_t^2 + \|\pi_\perp(\dot{u} - \dot{\phi}_t u)\|_t^2,$$

where $\pi_\perp$ is the orthogonal projection to the orthogonal complement of holomorphic forms. Comparing with (4.1) we see that

$$< \Theta u, u > = \int \phi_{t\bar{t}} |u|^2 e^{-\phi} - \|\pi_\perp(\dot{u} - \dot{\phi}_t u)\|_t^2.$$

This is where Hörmander's estimate enters the picture. The form $w :=$ $\pi_\perp(\dot{u} - \dot{\phi}_t u)$ is orthogonal to holomorphic forms, so it is the minimal solution to the $\bar{\partial}$-equation

$$\bar{\partial}w = \bar{\partial}\pi_\perp(\dot{u} - \dot{\phi}_t u) = -\bar{\partial}\dot{\phi}_t \wedge u.$$

By Hörmander's inequality

$$\|w\|^2 \leqslant \int_X |\bar{\partial}\dot{\phi}_t|^2_{\partial\bar{\partial}\phi_t} |u|^2 e^{-\phi}.$$

Therefore Theorem 4.2 follows from the inequality

$$|\bar{\partial}\dot{\phi}_t|^2_{\partial\bar{\partial}_X \phi_t} \leqslant \phi_{t\bar{t}},$$

i e that $c(\phi) \geqslant 0$, which as we have seen above means that ϕ is plurisubharmonic with respect to all the variables. In the end we then even get a lower bound for the curvature

$$< \Theta u, u >> \geqslant \int_X c(\phi)|u|^2 e^{-\phi}.$$

There are however some drawbacks with this proof. First, it presupposes that ϕ_t is strictly plurisubharmonic for each t. Second it does not give an explicit formula for the curvature, but just an inequality. Finally, there is a more general version of Theorem 4.2, dealing with nontrivial fibrations, i e situations when not only the metric, but also the manifold depends on t, which seems hard to prove in this way.

We shall therefore also indicate an alternative route that avoids these problems. It also circumvents the use of Hörmander's theorem, but rather proves a statement of that kind along the way. Again we want to compute the second derivative of the norm squared of a holomorphic section. We write the norm squared as a push forward

$$\|u_t\|^2_t = p_*(c_n u_t \wedge \bar{u}_t e^{-\phi})$$

where p_* is the natural projection from $X \times \mathbb{C}$ to $\mathbb{C}$, and $c_n = i^{n^2}$ is a unimodular constant chosen to make the form positive. (Note that this projection is defined for general fibrations.) The point is that we can here replace u in the right hand side by

$$\hat{u} := u_t + dt \wedge v$$

where v is an arbitrary form of bidegree $(n-1, 0)$, since the second term that contains a factor dt gives no contribution to the push forward. Differenting twice one arrives at the formula (see [6])

$$i\partial\bar{\partial}\|u_t\|^2_t = -p_*(c_n i\partial\bar{\partial}\phi \wedge \hat{u} \wedge \bar{\hat{u}} e^{-\phi})+$$

$$p_*(c_{n+1}\partial^\phi \hat{u} \wedge \overline{\partial^\phi \hat{u}}e^{-\phi}) + p_*(c_n \bar{\partial}v \wedge \overline{\bar{\partial}v}e^{-\phi})idt \wedge d\bar{t}.$$

Here $\partial^\phi = e^\phi \partial e^{-\phi}$. We next need to choose v. This can be done in different ways, each leading in principle to a formula for the curvature. One choice is to take v to solve the equation

$$\partial_X^\phi v = \pi_\perp(\dot{u} - \dot{\phi}_t u). \tag{4.2}$$

Then

$$\partial^\phi \hat{u} = dt \wedge \pi(\dot{u} - \dot{\phi}_t u) = dt \wedge D'u,$$

as we have seen above. We then get that

$$p_*(c_{n+1}\partial^\phi \hat{u} \wedge \overline{\partial^\phi \hat{u}}e^{-\phi}) = \|D'u\|_t^2 idt \wedge d\bar{t}.$$

The reason we can solve (4.2) is that ∂_X^ϕ is basically the adjoint of $\bar{\partial}$ under the pairing

$$< v, \alpha >= \int c_n v \wedge \bar{\alpha}e^{-\phi}$$

between $(n-1,0)$-forms and $(n,1)$-forms, so the range of ∂_X^ϕ is the orthogonal complement of the kernel of $\bar{\partial}$. One can also show that we may in fact choose v to satisfy the additional requirement $\bar{\partial}v \wedge \omega = 0$ where ω is a fixed Kähler form on X. Then $\bar{\partial}v$ is a *primitive form* which implies that

$$p_*(\bar{\partial}v \wedge \overline{\bar{\partial}v}e^{-\phi}) = -\|\bar{\partial}v\|^2.$$

All in all we find that

$$i\partial\bar{\partial}p_*(c_n u \wedge \bar{u}e^{-\phi}) = -p_*(c_n i\partial\bar{\partial}\phi \wedge \hat{u} \wedge \overline{\hat{u}}e^{-\phi}) + (\|D'u\|_t^2 - \|\bar{\partial}v\|^2)idt \wedge d\bar{t}.$$

Comparing with formula (4.1) we then finally arrive at a semiexplicit formula for the curvature

$$\langle \Theta u, u \rangle_t = \int_{X_t} c_n i\partial\bar{\partial}\phi \wedge \hat{u} \wedge \overline{\hat{u}}e^{-\phi} + \|\bar{\partial}v\|_t^2. \tag{4.3}$$

Without entering into details we mention that one can get an almost explicit formula for the curvature if $i\partial\bar{\partial}_X\phi > 0$. We then choose v in a different way, as $v = V \rfloor u$, the interior multiplication of u with V, the *complex gradient* (see below) of $\dot{\phi}_t$. This leads to the formula

$$\langle \Theta u, u \rangle_t = \int_{X_t} c(\phi)|u|^2 e^{-\phi} + \langle (1+\Box)^{-1}\bar{\partial}v, \bar{\partial}v \rangle_t, \tag{4.4}$$

where $\Box$ is the $\bar{\partial}$-Laplacian, see [6].

Here $c(\phi)$ is the geodesic curvature so the first term is completely explicit; the second term is perhaps a bit less so since it involves the solution of a certain equation. At any rate, the second term is nonnegative, so, as we have already seen, as an operator, the curvature is greater than the operator defined by the first term in the right hand side

$$\int_{X_t} c(\phi)|u|^2 e^{-\phi}.$$

Recall that if χ is any real valued function, the *Toeplitz operator with symbol* χ is the operator T_χ defined by

$$< T_\chi u, u >= \int_{X_t} \chi |u|^2 e^{-\phi}.$$

It is clearly an hermitean operator if χ is real valued, and it is sometime interpreted as the quantization of the function χ. Thinking also of the passage from metrics on L to the induced metrics on $\underline{E}$ as a sort of quantization, we arrive at the slogan *the curvature of the quantization is greater than the quantization of curvature*. In particular, if the curvature of the vector bundle E is zero, then the curve ϕ_t must be a geodesic. Actually much more can be said.

In the last of the three proofs that we gave for Theorem 4.2 we used the complex gradient of $\dot\phi_t$. In general, the complex gradient of a function χ, with respect to a given Kähler metric ω, is a $(1,0)$ vector field defined by

$$V \rfloor \omega = i\bar\partial \chi.$$

Let us now go back to the second proof of Theorem 4.2, where v was chosen to solve $\partial_X^\phi v = \pi_\perp(\dot\phi_t u)$. Assume the curvature is zero; then it follows from formula (4.3) that $\bar\partial v = 0$. Using the commutator relation

$$\bar\partial\partial^\phi + \partial^\phi\bar\partial = \partial\bar\partial\phi,$$

we get

$$\partial\bar\partial\phi \wedge v = \bar\partial\pi_\perp(\dot\phi_t u) = -\bar\partial(\dot\phi_t u) = -\bar\partial\dot\phi_t \wedge u.$$

If V is the complex gradient of $\dot\phi_t$, with respect to $\omega = i\partial\bar\partial\phi$, the right hand side here is

$$\partial\bar\partial_X\phi \wedge V \rfloor u.$$

Hence $v = V \rfloor u$, i e the same v that we used in the last proof! If v is holomorphic on X, then V must be holomorphic too. In particular, if X has no nontrivial holomorphic vector fields, $\dot\phi_t$ must be a constant (it is a real valued holomorphic function). In general an elaboration of this argument

shows that if the curvature vanishes, then the family of Kähler metrics $i\partial\bar{\partial}_X\phi_t$ moves by the flow of a holomorphic vector fields, and the proof can actually be made to work, even without the strong regularity assumptions we have used here, see [4].

In the last section we will argue that the "quantization", i e the passage from metrics on L to metrics on $\underline{E}$ can be seen as a counterpart of the Legendre transform. The interplay between the curvature of $\underline{E}$ and the geodesic curvature of ϕ_t is then analogous to how solutions of the real homogenous Monge-Ampère equation are linearized by the Legendre transform.

5. Functions on the space of Kähler metrics

In this section we shall see how one arrives at convex functions on $\mathcal{H}_L$ from the vector bundles in the previous section. We shall focus on a construction that leads to functions that have played an important role in the study of special Kähler metrics but we stress that in passing from the vector bundles to these functions one loses a lot of information – the positivity of the vector bundles is in general a much stronger statement than the convexity of these particular functions. The idea in the construction is to consider the determinants of the vector bundles $\underline{E}$ and $\underline{F}$, which are line bundles with curvature equal to the trace of the curvature of the vector bundles, see Donaldson [12].

We can formulate this in a less technical way as follows. Fixing a metric ϕ in $\mathcal{H}_L$ we get induced hermitian metrics, $h_\phi^{F,E}$ on F and E as described earlier. Choosing a basis these metrics are given by matrices and we can look at their determinants. They depend on the choice of basis, but the quotient of two such determinants does not. We define the $\mathcal{L}$-functional as

$$\mathcal{L}(\phi,\psi) = \log \det h_\phi - \log \det h_\psi$$

and use superscripts like $\mathcal{L}^F$ do indicate which of the vector spaces we are dealing with.

Let ϕ_t be a curve in $\mathcal{H}_L$ and consider the functions

$$\mathcal{L}(t) := \mathcal{L}(\phi_t, \psi)$$

for some arbitrary fixed ψ. Then

$$i\partial\bar{\partial}\mathcal{L}^{F,E}(t) = \text{trace } \Theta^{F,E},$$

since the curvature of the determinant line bundle of a vector bundle is the trace of the curvature of the vector bundle. From this and Theorems 4.1 and 4.2 we immediately get the next proposition.

Proposition 5.1. — $\mathcal{L}^F$ *is convex (subharmonic) along (complex) geo-*
desics. $\mathcal{L}^E$ *is concave (superharmonic) along (complex) subgeodesics.*

Let us now look at the first order derivative of $\mathcal{L}$. We concentrate on
$\mathcal{L}^E$; there is a similar formula for $\mathcal{L}^F$ but it is a litle bit more complicated.
The formula for the first order derivative uses only the standard fact that
the derivative of $\log \det A(t)$ equals the trace of $\dot{A}A^{-1}$. We apply this to
$A(t) = h_{\phi_t}$. We may assume that we have chosen the basis of E to be
orthonormal for the scalar product induced by ϕ_{t_0} when we compute the
derivative at $t = t_0$, so that A is the identity at that point. Then

$$\frac{\partial \mathcal{L}^E}{\partial t}\Big|_{t_0} = \sum \int (-\dot{\phi}_t)|u_j|^2 e^{-\phi_{t_0}}.$$

The important point is that

$$\sum_j |u_j|^2 =: K_{\phi_{t_0}}$$

is the *Bergman kernel* for the metric induced by ϕ_{t_0}. Recall that in a Hilbert
space of holomorphic functions, the Bergman kernel is defined as

$$K = \sum_j |u_j|^2$$

if u_j is any orthonormal basis for the Hilbert space; it does not depend on
the choice of orthonormal basis. The same definition can be used for a space
of holomorphic sections of a line bundle L, with the understanding that

$$|u_j|^2$$

should then be understood as defining a metric on L, since

$$|u|^2/|u_j|^2$$

is a well defined function if u is an arbitrary section of L. Hence, if ϕ is
another metric on L, $Ke^{-\phi}$ is globally well defined function. In our present
case, the Bergman kernel is built from sections of $K_X + L$ instead of L, and
one sees directly that $Ke^{-\phi}$ is an (n,n)-form on X.

Denoting by

$$B_\phi := K_\phi e^{-\phi}$$

we thus see that

$$\frac{\partial \mathcal{L}^E}{\partial t}\Big|_{t_0} = \int -\dot{\phi}_t B_\phi.$$

Next we will replace L by kL where k is a (large) positive integer. We therefore pause a moment to discuss the asymptotics of the Bergman kernels for kL.

5.1. Bergman kernel asymptotics

The following theorem, due to Bouche-Tian-Zelditch-Catlin is crucial.

THEOREM 5.2. — *There is an asymptotic expansion such that when k goes to infinity, we have for any m*

$$B_\phi k^{-n} = C_L(1 + b_1 k^{-1} + ...b_m k^{-m})(\omega^\phi)^n + O(k^{-(m+1)})$$

where b_j are certain smooth functions on X.

(Notice that in our setting B_ϕ is an (n, n)-form.) We will only be interested in the first two terms of the expansion and it will be convenient to rewrite the formula a little bit. Let N_k be the dimension of $H^0(X, K_X + kL)$. Then

$$N_k = C_L k^n + O(k^{n-1})$$

and we write

$$B_\phi / N_k = (1/V + \hat{b}_1 k^{-1})(\omega^\phi)^n + O(k^{-2}).$$

Since the left hand side here has integral 1 over X it follows that

$$V = \int (\omega^\phi)^n$$

and that $\hat{b}_1$ has integral zero over X with respect to the measure $(\omega^\phi)^n$. By a theorem of Lu, [17],

$$\hat{b}_1 = a(S_\phi - \tilde{S}_\phi)$$

where $a > 0$ and S_ϕ is the *scalar curvature* of the metric ω^ϕ, and $\tilde{S}_\phi$ its average over X.(See the next section for the definition of scalar curvature.)

Let us take a closer look at the function $\mathcal{L}_k^E$, i e the function $\mathcal{L}^E$ defined with L replaced by kL . Putting together the formula for the derivative of $\mathcal{L}^E$ with the Bergman kernel asymptotics we get that

$$-N_k^{-1} \frac{\partial \mathcal{L}_k^E}{\partial t}|_{t_0} = V^{-1} \int \dot{\phi}_t (\omega_t^\phi)^n + k^{-1} \int \dot{\phi}_t a(S_{\phi_t} - \tilde{S}_{\phi_t})(\omega^\phi)^n + O(k^{-2}).$$
$$(5.1)$$

The left hand side here is by definition the derivative of a certain function on $\mathcal{H}_L$, i e its integral along any curve ϕ_t depends only on the endpoints of

the curve. Therefore both of the terms in the right hand side must have the same property and thus correspond to functions defined on $\mathcal{H}_L$. The first of these functions is the Monge-Ampère energy, which is defined by

$$\frac{\partial \mathcal{E}(\phi_t, \psi)}{\partial t} = V^{-1} \int \dot{\phi}_t (\omega_t^\phi)^n \tag{5.2}$$

and $\mathcal{E}(\psi, \psi) = 0$. Explicitly

$$\mathcal{E}(\phi, \psi) = ((n+1)V)^{-1} \int (\phi - \psi) \sum_0^n (\omega^\phi)^k \wedge (\omega^\psi)^{n-k}.$$

Since $\mathcal{L}^E$ is concave along subgeodesics we see by taking limits that $\mathcal{E}$ is convex along subgeodesics, and it is not hard to verify that

$$\frac{\partial^2 \mathcal{E}(\phi_t, \psi)}{\partial t \partial \bar{t}} = V^{-1} \int c(\phi_t)(\omega_t^\phi)^n.$$

Thus $\mathcal{E}$ is also linear (or harmonic) on (complex) geodesics.

The second function is the Mabuchi K-energy; it is defined by

$$\frac{\partial \mathcal{M}(\phi_t, \psi)}{\partial t} = \int \dot{\phi}_t (S_{\phi_t} - \tilde{S}_{\phi_t})(\omega_t^\phi)^n \tag{5.3}$$

and $\mathcal{M}(\psi, \psi) = 0$.

Since $\mathcal{L}^E$ is concave and $\mathcal{E}$ is linear along geodesics it follows that $\mathcal{M}$ is convex along (smooth) geodesics. A not so easy computation shows that

$$\frac{\partial^2 \mathcal{M}(\phi_t, \psi)}{\partial t \partial \bar{t}} = \int c(\phi_t)(S_{\phi_t} - \tilde{S}_{\phi_t})(\omega_t^\phi)^n + c_{>0} \int |\bar{\partial} V_\phi|^2 (\omega_t^\phi)^n,$$

where V_ϕ is the complex gradient of $\dot{\phi}_t$, the complex vector field defined by

$$V_\phi \rfloor \omega^\phi = \bar{\partial} \dot{\phi}_t$$

along any smooth curve. An important property – or the *raison d'être* – of the Mabuchi K-energy is that by (5.3) its critical points are exactly the (potentials of) metrics of constant scalar curvature.

The importance of the convexity, and more precisely the formula for the second derivative of, the Mabuchi K-energy, is that it implies *formally* uniqueness properties of metrics of constant scalar curvature: if ϕ_0 and ϕ_1 are two such metrics they are both critical points of $\mathcal{M}$. If, and this is

an important proviso, we can join them with a smooth geodesic it follows that $\mathcal{M}$ must be constant along this geodesic, and therefore V_ϕ must be a holomorphic vector field. If we moreover assume that X carries no nontrivial holomorphic vector fields $\dot{\phi}_t$ must be constant on X, and this easily implies that ω^{ϕ_t} is constant. More generally, if there are nontrivial holomorphic vector fields it follows that ω^{ϕ_t} must move by the flow of such fields.

In the next section we will give an alternative argument in a simpler special case, the Kähler-Einstein metrics.

6. Kähler-Einstein metrics and the Bando-Mabuchi theorem

Let ω be a Kähler form on X and let

$$\omega^n/n! = \det(\omega_{j\bar{k}})c_n dz \wedge d\bar{z}$$

be its volume form. Then

$$\mathrm{Ric}(\omega) = i\partial\bar{\partial}\log\det(\omega_{j\bar{k}})$$

is the Ricci form of the metric; it does not depend on the choice of local coordinates. The trace of $\mathrm{Ric}(\omega)$

$$\Delta\log\det(\omega_{j\bar{k}})$$

is the scalar curvature. One says that ω is a Kähler-Einstein metric if

$$\mathrm{Ric}(\omega) = a\omega,$$

with a constant. Notice that this is only possible if ω lies in a multiple of the class of $\mathrm{Ric}(\omega)$ which is always $c[-K_X]$, the Chern class of the anticanonical bundle, i e the inverse of the canonical bundle.

By multiplying ω by a positive constant (which leaves $\mathrm{Ric}(\omega)$ intact) we may take this multiple to be -1, 0 or 1. The Kähler-Einstein problem thus divides into three cases, depending on whether K_X is positive, flat or negative. (In case it is neither the problem is not solvable). In these cases ω solves the Kähler-Einstein equation if and only if it has constant scalar curvature, i e is a CSC metric.

The Kähler-Einstein problem is a special case of the CSC problem – to find metrics of constant scalar curvature – when the bundle L is a multiple of the canonical bundle. To see this, assume ω is a CSC metric and that $[\omega] = ac[K_X]$. Then ω^n is a volume form and so determines a metric ψ on the canonical bundle. If $\omega = i\partial\bar{\partial}\phi$ where ϕ is a metric on aK_X then

$$\Delta(\phi - a\psi) = n - a\Delta\psi$$

is a constant that integrates to zero, hence zero. Therefore the function $\phi - a\psi$ is constant so

$$\omega = i\partial\bar{\partial}\phi = ai\partial\bar{\partial}\psi = a\mathrm{Ric}(\omega)$$

and ω has constant Ricci curvature.

It was shown by Aubin and Yau that the Kähler-Einstein equation is always solvable when $c[K_X]$ is negative, and by Yau's solution of the Calabi problem this also holds when $c[K_X]$ is zero. When the curvature is positive the equation is not always solvable and a lot of deep recent work has been done to characterize when it is solvable and to what extent the solution is unique. A complete solution to the problem of characterizing the Fano manifolds (i e the manifolds that posess a metric of positive Ricci curvature) that have a Kähler-Einstein metrics has very recently been given by Chen-Donaldson-Sun, [14] and subsequent work, see also [26].

Since the Kähler-Einstein equation is equivalent to the CSC equation when $L = -K_X$, its solutions are critical points of the Mabuchi K-energy. In this particular case there is however a simpler functional with the same property. If L is equal to $-K_X$, $K_X + L$ trivial so $E = H^0(X, K_X + L) = \mathbb{C}$. Consider in particular the section $u = 1$ of the trivial bundle and think of it as an $(n, 0)$-form with values in $-K_X$. Let ϕ be an element in $\mathcal{H}_L = \mathcal{H}_{-K_X}$. By the notational conventions we have used earlier the integrals

$$\int_X |u|^2 e^{-\phi} = \int_X e^{-\phi}$$

are then well defined.

Consider the function

$$\mathcal{L}(\phi) = \log \int_X e^{-\phi}.$$

This is precisely the $\mathcal{L}^E$-function introduced in section 5, and in this case we do not need to use any auxiliary metric ψ in the definition since we have a privileged section $u = 1$ to choose as a frame for E. By Proposition 5.1, $\mathcal{L}$ is concave along real subgeodesics and superharmonic along complex subgeodesics. Next we define the *Ding functional* by

$$\mathcal{F}(\phi) = \mathcal{L}(\phi) + \mathcal{E}(\phi, \phi_0)$$

for some arbitrary choice of ϕ_0 this function is still concave along geodesics since $\mathcal{E}$ gives a linear contribution. We claim that the critical points of $\mathcal{F}$

are the Kähler-Einstein metrics. To see this, note first that by definition, ϕ is critical if and only if

$$e^{-\phi} / \int e^{-\phi} = (1/V)(\omega^\phi)^n. \tag{6.1}$$

Taking first logarithms and then $i\partial\bar{\partial}$ this is equivalent to saying that

$$\omega^\phi = \mathrm{Ric}(\omega^\phi)$$

which is the (normalized) Kähler-Einstein condition.

One therefore expects that the existence of Kähler-Einstein metrics should be related to some 'properness' condition on the functional $\mathcal{F}$; that it goes to infinity at infinity. The classical Moser-Trudinger inequality is perhaps the most basic manifestation of this; it says that if $\mathcal{H}_L$ contains some Kähler-Einstein metric then $\mathcal{F}$ is at least bounded from below and moreover in a very precise way.

THEOREM 6.1. — *Assume $L = -K_X$ is positive and that ϕ_0 in $\mathcal{H}_L$ is a Kähler-Einstein metric. Then for any other ϕ in $\mathcal{H}_L$*

$$\mathcal{F}(\phi) \leqslant \mathcal{F}(\phi_0)$$

or more explicitly

$$\log \int e^{-\phi} \leqslant \log \int e^{-\phi_0} + \mathcal{E}(\phi_0, \phi).$$

As pointed out by Berman in [3], this is in principle clear since ϕ_0 is a critical point of the concave function $\mathcal{F}$. For the proof one needs to consider geodesics between ϕ and ϕ_0. These are not necessarily smooth but they can be approximated by smooth subgeodesics so Proposition 5.1 still applies and says that $\mathcal{L}$ is concave along generalized (e g bounded) geodesics. Since $\mathcal{E}$ is always linear along generalized geodesics, it follows that $\mathcal{F}$ is concave which immediately gives the theorem.

From this we also see that properness of $\mathcal{F}$ is related to strict concavity. See [10] and [21] for generalized Moser-Trudinger inequalities, including an extra term that 'goes to infinity at infinity'.

Another reason to be interested in strict concavity of $\mathcal{L}$ is uniqueness of Kähler-Einstein metrics. A famous theorem of Bando and Mabuchi says that the potentials of two Kähler-Einsten metrics ϕ_0 to ϕ_1 must be given by the flow of a holomorphic vector field.

THEOREM 6.2 *(Bando-Mabuchi, [1]).* — *Let $K_X < 0$ and let ω^{ϕ_0} and ω^{ϕ_1} be two Kähler-Einstein metrics. Then there is an automorphism of X, F homotopic to the identity, such that*

$$\omega^{\phi_0} = F^*(\omega^{\phi_1}).$$

By (6.1) and the computations immediately afterwards, in the case of negative canonical bundle ϕ defines a Kähler-Einstein metric if and only if

$$e^{-\phi} = c(\omega^\phi)^n,$$

$c > 0$. In the case of positive canonical bundle the equation is

$$e^\phi = c(\omega^\phi)^n.$$

In this latter case it is easy to see that uniqueness holds, unconditionally: Look at the difference between two Kähler-Einstein potentials

$$\phi_1 - \phi_0.$$

This is a function and it must have a maximum somewhere. At that point its complex Hessian is seminegative, and from the Kähler-Einstein equation it follows that the function is nonpositive at its maximum. Similarily, it is nonnegative at its minimum, so it must be identically zero.

Now, it is clear from the equation $\mathrm{Ric}(\omega) = a\omega$ that it is preserved under biholomorphic transformations. Therefore any biholomorphic map must be an isomorphism for the Kähler-Einstein metric if $K_X > 0$. This does not hold in the case of negative canonical bundle. When X is the Riemann sphere

$$\frac{|a|^2 i dz \wedge d\bar{z}}{(1 + |a|^2 |z|^2)^2}$$

are Kähler-Einstein metrics for any choice of $a \neq 0$, and they are of course related by the automorphism that sends z to az – which is homotopic to the identity. The Bando-Mabuchi theorem says that this is all that can happen. In particular, it follows from this remarkable theorem that if the automorphism group is discrete, then all automorphisms *are* isomorphisms for the Kähler-Einstein metric – if there is one.

Theorem 6.2 (and an extension of it) can be proved using the remarks at the end of subsection 4.2. We have seen that potentials of Kähler-Einstein metrics are critical points for the Ding functional. Since the Ding functional is concave it must therefore be an affine function of t along the (generalized!) geodesic connecting two Kähler-Einstein metrics. By subsection 4.2, the metrics along the geodesic therefore move by the flow of a holomorphic vector field – which is precisely the conclusion of the Bando-Mabuchi theorem.

7. A Legendre tranform of metrics on a line bundle?

Recall that for a function on $\mathbb{R}^n$ its Legendre transform is defined by

$$\hat{\phi}(y) = \sup_x (x \cdot y - \phi(x)). \tag{7.1}$$

One reason that this notion is so useful is that one can recover ϕ from its Legendre transform, provided ϕ is convex. This in turn comes from the fact that a convex function is equal to the supremum of all affine functions below it. If we look for a similar construction for plurisubharmonic functions, or like we do here positively curved metrics on a line bundle L, the first approximation is to try to write a metric ϕ in $\mathcal{H}_L$ as the supremum of all expressions

$$\log |h|^2$$

where h runs over elements in $H^0(X, L)$ with $\log |h|^2 \leqslant \phi$. This is however in general not possible. A better attempt is to try

$$\sup_h (1/k) \log |h|^2$$

where h runs over $H^0(X, kL)$ for a large k. This will still not work, but if ϕ is regular, it can be written as the limit of a sequence of such suprema.

Inspired by this we (preliminarily!) define, for each k and h in $H^0(X, kL)$

$$\hat{\phi}_k(h) = \sup_X ((1/k) \log |h(x)|^2 - \phi(x)).$$

This is equal to

$$(1/k) \log \|h\|_{k\phi, \infty}^2$$

where

$$\|h\|_{k\phi, \infty}^2 := \sup_X |h|^2 e^{-k\phi}$$

is the weighted L^∞-norm of h. Somewhat like the usual Legendre transform is defined on the dual of $\mathbb{R}^n$, i.e. the space of linear functions on $\mathbb{R}^n$, we then get functions defined on 'the holomorphic dual of X', i e spaces of holomorphic sections on X.

The L^∞-norms here are somewhat complicated to work with, so we replace them by L^2-norms

$$\|h\|_{k\phi, 2}^2 := \int_X |h|^2 e^{-k\phi} d\mu$$

where μ is some suitable measure. If μ has the *Bernstein-Markov property*, then

$$|(1/k)\log\|h\|^2_{k\phi,\infty} - (1/k)\log\|h\|^2_{k\phi,2}|$$

goes to zero so we will not lose much. We therefore *define* the Legendre transform of ϕ to be the sequence

$$(1/k)\log\|h\|^2_{k\phi,2}$$

of logarithms of norms on $H^0(X, kL)$. From now on we decide on using L^2-norms and therefore write $\|h\|^2_{k\phi}$ instead of $\|h\|^2_{k\phi,2}$.

We now turn to the inverse Legendre transform, and therefore consider a sequence

$$N_k(h) = (1/k)\log\|h\|^2_k$$

of logarithms of norms on $H^0(X, kL)$. In analogy with the real setting we then let

$$\hat{N}_k(x) = \sup_h((1/k)\log|h(x)|^2 - N_k(h)),$$

the sup taken over all h in $H^0(X, kL)$. For each k this is a metric on L and

$$e^{k\hat{N}_k(x)} = \sup_h \frac{|h(x)|^2}{\|h\|^2_k}.$$

But this is precisely the Bergman kernels for the norms $\|\cdot\|_k$.

To sum up the Legendre transform of a metric ϕ on L is a sequence of norms on $H^0(X, kL)$, and the 'inverse' Legendre tranform of such a sequence of norms is $(1/k$ times the log of) the sequence of Bergman kernels. If we start with a metric ϕ and take the Legendre transform twice, we therefore end up with the sequence

$$(1/k)\log K_{k\phi}$$

where K_ϕ is the Bergman kernel on the diagonal for the L^2-metric defined by $e^{-k\phi}$. It follows easily from the Bergman kernel asymptotics that

$$|(1/k)\log K_{k\phi} - \phi|$$

goes to zero uniformly on X at the rate $k/\log k$. This is what corresponds to the fact that the iterated Legendre transform of a convex function is equal to the function itself.

Associated to the real Legendre transform is the important *gradient map*

$$x \to y = d\phi(x)$$

whose inverse is the gradient map of $\hat{\phi}$. For ϕ smooth and strictly convex this map can alternatively be defined as the map that associates to x the unique y that realizes the sup in

$$\sup_{y}(x \cdot y - \hat{\phi}(y)).$$

In analogy with this we consider the iterated Legendre transform of a metric ϕ in $\mathcal{H}_L$

$$\sup_{h}((1/k)\log|h|^2 - (1/k)\log\|h\|_{k\phi}^2)).$$

For a given x in X, the sup is the same as the sup of

$$\sup \frac{|h(x)|^2}{\|h\|_{k\phi}^2},$$

which is attained for

$$h = K_{k\phi}(\cdot, x)$$

the (offdiagonal) Bergman kernel at x, or for any multiple of that section. The analogy of the gradient map is therefore the map from X to $H^0(X, kL)$

$$x \rightarrow K_{k\phi}(\cdot, x)$$

but to get rid of the ambiguity of chosing the right multiple it is natural to compose it with the natural map to the projectivization of $H^0(X, kL)$. All in all we thus get a map

$$\kappa_k(x) = [K_{k\phi}(\cdot, x)] = [\bar{u}_1(x), ...\bar{u}_N(x)].$$

In the last equality here we have chosen an orthonormal basis u_j for $H^0(X, kL)$ and expanded $K_{k\phi}$ in that basis

$$K_{k\phi}(\cdot, x) = \sum u_j \bar{u}_j(x).$$

This map depends heavily on the choice of ϕ, as it should, but if we postcompose it with the natural map from $H^0(X, kL)$ to its dual given by the L^2-norm $\|h\|_{k\phi}^2$ we instead get the map

$$x \rightarrow [\mathrm{ev}_x]$$

mapping a point x to the element 'evaluation at x'. This is the Kodaira map, and it does not depend on the choice of ϕ. In this way we may say that the analogy of the gradient map in the complex setting is the Kodaira map, but with the understanding that we mean the Kodaira map composed with the identification of $H^0(X, kL)$ and its dual, coming from ϕ.

As we have seen in Section 2 the pullback of Lebesgue measure under the gradient map is the (real) Monge-Ampère measure of ϕ. Again we have something similar in the complex setting. The counterpart of Lebesgue measure is the Fubini-Study volume form on projective space, $dV_{\text{F-S}}$. Then

$$\kappa_k^*(dV_{\text{F-S}}) = (i\partial\bar{\partial} \log \sum |u_j|^2)^n = (i\partial\bar{\partial} \log K_{k\phi}(x,x))^n.$$

Again it follows from a version of Bergman kernel asymptotics that suitably normalized this converges to

$$(\omega^\phi)^n$$

the Monge-Ampère-measure of ϕ, see [25] and [23] .

In section 2 we also noted that the Legendre transform linearized the real homogenous Monge-Ampère equation. This meant that geodesics ϕ_t map to functions affine in t when we take the Legendre transform in the x-variables. The ideal counterpart of this would be that a complex geodesic maps to a curve of norms $\| \cdot \|_{k\phi}$ on $H^0(X, kL)$, giving a vector bundle metric on $H^0(X, kL) \times \Omega$ of zero curvature. As usual, we can only expect this in an asymptotic sense. Let us look at the trace of Θ^E, which is equal to $\partial\bar{\partial}_t$ of $\mathcal{L}^E$. By formula (5.1), the leading order term here is the $i\partial\bar{\partial}$ of $\mathcal{E}$, which vanishes if ϕ_t is a geodesic. In this sense, the complex Legendre transform maps geodesics to a sequence of vector bundle metrics that are asymptotically flat. In [20], [5] , [6] a converse of this is given: Starting from a sequence of flat metrics on $H^0(X, kL) \times \Omega$, we get a geodesic in $\mathcal{H}_L$ by taking inverse Legendre transforms, i e the logarithms of the corresponding Bergman kernels.

We end this section with a toy example in which the correspondence between real and complex Legendre transforms is quite transparent.

7.1. A toy example

Let ϕ be a convex function on $\mathbb{R}^n$ and consider the weighted L^2-space of entire functions h on $\mathbb{C}^n$ such that

$$\|h\|_\phi^2 := \int |h(x+iy)|^2 e^{-\phi(x)} < \infty.$$

Such functions can be written as Fourier-Laplace transforms

$$h(x+iy) = \int_{\mathbb{R}^n} e^{t(x+iy)} \tilde{h}(t) dt$$

Computing the L^2-norm with the aid of Plancherel's formula (and skipping a few constants) we arrive at the formula

$$\|h\|^2_\phi = \int |\tilde{h}(t)|^2 e^{\tilde{\phi}(t)} dt$$

where

$$e^{\tilde{\phi}(t)} := \int e^{2x \cdot t - \phi(x)} dx.$$

From the latter formula we see that $\tilde{\phi}$ is a sort of smeared out Legendre transform; instead of taking the sup of

$$g = 2x \cdot t - \phi(x)$$

we take the integral of e^g. We can think of $e^{\tilde{\phi}}$ as giving the norm $\|h\|^2_\phi$ in the basis consisting of exponential functions, and in this way the 'smeared out' Legendre transform is similar to our definition of complex Legendre transform as the logarithm of the norm of holomorphic objects. Moreover, we can replace ϕ by $k\phi$ and recover the classical Legendre transform in the limit as k goes to infinity, if we renormalize the Fourier-Laplace transform by

$$h(x + iy) = \int_{\mathbb{R}^n} e^{kt(x+iy)} \tilde{h}(t) dt.$$

Conversely, let us start with some weighted L^2-norm on $\mathbb{R}^n$

$$\|f\|_{A_\psi} := \int |f|^2 e^{\psi(t)}$$

and look at the space A_ψ of entire functions

$$\hat{f}(x + iy) := \int e^{t(x+iy)} f(t) dt$$

with $\|f\|_{A_\psi} < \infty$. We then define the norm of $\hat{f}$ to be equal to the norm of f. In accordance with our earlier discussion the complex Legendre transform should then be the (log of the) Bergman kernel on the diagonal for A_ψ. To compute the Bergman kernel $K_\psi(\zeta, z)$ fix $z = x + iy$. Then there is a unique function f_z such that

$$K_\psi(\zeta, z) = \int e^{t\zeta} f_z(t) dt$$

and by the reproducing property of Bergman kernels we should have, if

$$h(z) = \hat{f}(z) = \int e^{t(x+iy)} f(t) dt,$$

that

$$h(z) = <h, K_\psi(\cdot, z)> = \int f\bar{f}_z e^\psi dt.$$

Comparing we get

$$f_z(t) = e^{t\bar{z}} e^{-\psi(t)},$$

hence

$$K_\psi(\zeta, z) = \int e^{t\zeta} f_z(t) dt = \int e^{t(\zeta+\bar{z}) - \psi(t)} dt.$$

Restricting to the diagonal

$$K_\psi(z, z) = \int e^{2t \cdot x - \psi(t)} dt$$

so

$$\log K_\psi(z, z) = \tilde{\psi}(x),$$

i.e. exactly the same transform as before. This is the main point. In this way the two complex Legendre transforms – the one mapping a weight to an L^2-norm and the one mapping a norm to its Bergman kernel – coincide, as one would have hoped.

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