ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

Ralf Krömer

The set of paths in a space and its algebraic structure. A historical account

Tome XXII, nº 5 (2013), p. 915-968.

<http://afst.cedram.org/item?id=AFST_2013_6_22_5_915_0>

Annales de la Faculté des Sciences de Toulouse Vol. XXII, n° 5, 2013
pp. 915–968

The set of paths in a space and its algebraic structure. A historical account

Ralf Krömer[(1)]

Abstract. — The present paper provides a test case for the significance of the historical category “structuralism” in the history of modern mathematics. We recapitulate the various approaches to the fundamental group present in Poincaré’s work and study how they were developed by the next generations in more “structuralist” manners. By contrasting this development with the late introduction and comparatively marginal use of the notion of fundamental groupoid and the even later consideration of equivalence relations finer than homotopy of paths (their implicit presence from the outset in the proof of the group property of the fundamental group notwithstanding), we encounter “delay” phenomena which are explained by focussing on the actual uses of a concept in mathematical discourse.

Résumé. — Le présent document fournit un cas d’école pour la signification de la catégorie historique « structuralisme » dans l’histoire des mathématiques modernes. Nous récapitulons les différentes approches pour le groupe fondamental présentes dans les travaux de Poincaré et étudions comment celles-ci ont été développées par les générations suivantes dans des directions plus « structuralistes ». En comparant cette évolution avec l’introduction tardive et l’utilisation relativement marginale de la notion de groupoïde fondamental et la prise en compte même plus tardive de relations d’équivalence plus fines que l’homotopie de chemins (nonobstant leur présence implicite dès le départ dans la preuve de la propriété de groupe du groupe fondamental), nous rencontrons des phénomènes « retard » qui sont expliqués en se concentrant sur les usages réels d’un concept dans le discours mathématique.

[(1)] Arbeitsgruppe Didaktik und Geschichte der Mathematik, Bergische Universität Wuppertal, Gaußstraße 20 D-42119 Wuppertal
rkroemer@uni-wuppertal.de

Contents

1 Introduction . . . 917
2 Poincaré's approaches to the fundamental group . . . 920
2.1. 1883 . . . 921
2.2. 1892 . . . 923
2.3. 1895 . . . 924
2.4. 1908 . . . 926
3 The fundamental group after Poincaré: coverings and "structuralist" outcomes . . . 928
3.1. Definitions of the fundamental group in the literature after Poincaré . . . 928
3.2. Weyl . . . 931
3.3. The fundamental group and decktransformations . . . 932
3.4. Schreier: coverings for groups . . . 934
3.5. Reidemeister's knot theory: Subgroups and coverings . . . 936
4 Composition of general paths and groupoids concept 939
4.1. Poincaré and partially defined group operations 939
4.2. Reidemeister . . . 939
4.3. Algebraic origin of the explicit groupoid concept . . . 940
4.4. Veblen and Whitehead . . . 942
4.5. Steenrod, local systems and axiomatic homotopy theory . . . 943
4.6. Groupoids in Bourbaki . . . 945
5 Equivalence relations finer than Homotopy . . . 947
5.1. The intuitive conception of continuous deformation . . . 947
5.2. From paths to general mappings . . . 950
5.3. Paths and changes of parameter . . . 953
5.4. Homotopic paths and the group(oid) property . 955
5.5. "Thin" homotopies . . . 959
6 Conclusions . . . 961
Bibliography . . . 963

1. Introduction

Considerations of algebraic structures on the set of paths in a topological space play an important role in topology and many other parts of modern mathematics where topological tools are used. Most important is perhaps the so-called fundamental group $\pi_1(X, x_0)$ of the space X with respect to a certain point $x_0 \in X$, given by homotopy classes of closed paths with x_0 as initial and endpoint. But there are more structures which can be defined in a similar way, including the fundamental groupoid (given by homotopy classes of not necessarily closed paths) or the "thin fundamental group" (given by classes of paths with respect to an equivalence relation finer than homotopy).

Now, all this obviously has a history. First of all, the very notions of path, topological space, set, group and structure underwent a development during the period relevant to this history; here, the notion of "structure" is much less of a precise mathematical notion than the other four,[1] and the whole talk about structures on the set of paths is retrospective in that it uses a vocabulary of a later period in the description of events of an earlier period. It is true, mathematics (or at least a large part of it) in the first half of the twentieth century underwent what has been called a "structural transition" by the advent of structural notions like group, topological space and so on[2]. But what is the significance of the historical category "structuralism"? The present paper provides a test case for this significance. Renaud Chorlay in a different context provided a sample of the kind of questions one encounters:

> The detailed study of the links between problem families and structural transition should help flesh out the very general description of the structural approach that can be found, for instance, in Bourbaki. In 1942, they described mathematics as a "a storehouse of abstract forms" [8, p. 231]; forms which are also tools, whose abstract (i.e. context-free), object-like definition warrant general applicability [...]. The arguments as to why it should be done, and what the epistemic gains result (economy of thought, insight into formal analogies, uniform treatment of seemingly different problems etc.) are quite clear; but no clue is given as to how it is done. In particular, Bourbaki stress the fact that the list of important structures is not

(1) See [54, p. 207ff]. Interestingly, it seems that in this case mathematical language even developed towards *less* precision, given the quite precise meaning of the term "structure" in Elie Cartan's dissertation [14]. This matter certainly would need further historical elucidation.

(2) See [23], a study which in an illuminating way extends the description of this structural transition provided by the now classical [24].

> closed, and that invention of structures accounts for numerous recent breakthroughs [...]. They also emphasise the fact that the axiomatic method is at its best when it succeeds in showing that some structure plays an important part in a field where, a priori, it seems it played none [...]. Yet, as to how these breakthroughs were made, these new structures invented and these unexpected structures identified in more classical settings, Bourbaki give no clue; or rather, they leave it to intuition: "more than ever does intuition dominate in the genesis of discoveries." [...]! [23, p. 67]

Invention of new structures, identification of unexpected structures in more classical settings, intuition guiding these discoveries: the present paper is a study of a case of it. We will focus on the following phenomena: The standard proof of the group property of the fundamental group (i.e., of the fact that homotopy classes of closed paths with respect to a point x_0 actually form a group) is valid for certain equivalence relations finer than homotopy as well (see section 5.4). This fact seems to have been "hidden" for a long time, and these other structures have not been used themselves until much more recently. Thus, while the rhetoric in various textbooks, as we will see below, wants the motive to consider homotopy of paths to be that one thus gets a group or groupoid structure, this cannot describe completely the historical motives since one could get such a structure in another way. Another "delay" can be observed with respect to the notion of fundamental groupoid. It is the aim of the present paper to explain these two "delay" phenomena.

Here, the talk about "delays" and their "explanation" asks for methodological discussion. I think that delay phenomena can be sensibly explained by focussing on the actual uses of a concept in mathematical discourse. In my opinion, the conceptual history of mathematics cannot be simply described as being composed of transitions from vague original formulations into later formal precision (and it is thus not the only task of the historian of mathematics to analyze carefully these transitions). I rather think (and stressed repeatedly before) that even when the definition of a concept has reached a state of consolidation, its use in mathematical discourse and research is by far not exclusively determined by its formal definition; there are many other aspects relevant to this usage. Mathematicians when using concepts often play "language games" in Wittgenstein's sense; for instance, only part of the actual models of the formal definition are considered as "relevant". They have learned to play these games through training, and this is the source of what Bourbaki labelled "intuition" in the quote above. They operate with the concept in accordance not only with the formal definition, but also with the "intended model" (what they have in mind about the "right" usage of the concept through their training). In this situation, some usage can perfectly be in the scope of the formally defined concept, but pathological with respect to that intended model (and thus delayed until the intended model changes or is enlarged).[3]

The present paper's originality does reside in this way of interpreting the historiographical information, while the major part of this information in itself actually is already contained in an important corpus of secondary literature.[4] In addition to the use of this secondary literature, the building of the corpus of primary sources at least for the period up to 1942 has been backed up by automatic query in the *Jahrbuch* data.[5] Of course, not every hit of these queries has been included in the list of references, but the usefulness of the procedure will be pointed out at several places in the paper. One might hope that such automatic procedures allow to continue Volkert's short and "hand-made" account of the reception of Poincaré's topological work up to 1908 [106, p. 177ff].

We will first recapitulate the various approaches to the fundamental group present in Poincaré's work (section 2) and then see how in the work of the next generation some moves which might be described as "structuralist" made increase the usefulness of the fundamental group (section 3). The introduction of the fundamental groupoid is studied in section 4. In section 5, we approach the case of equivalence relations finer than homotopy by analysing the role played by the intuitive content of continuous deformation and by the definition of paths as mappings.

---

(3) This focus on language use and its training has been applied in the analysis of the historical development of category theory in [54]. In that case like in the present one, this focus in itself is thought of as a epistemological method for understanding historical events, especially by showing that there are such things like "intended models" in mathematics.

(4) The development of the concept of homotopy of paths up to the first decades of the twentieth century has been studied by Ria Van den Eynde [102, 103], and very few is to be added concerning this period in the present paper. Van den Eynde's second paper being essentially a shortened version of the first, I will use the first paper here (and the corresponding pagination even for passages also present in the second paper).

For the later period, too, it would be wrong to say that the history of the concept still has to be written. But the information used in the present work is scattered in studies written under different perspectives: we have historical accounts of algebraic topology as a whole [29], accounts of the work of Poincaré [34], [90], [80], histories of particular concepts (like the concept of group [112] or of manifold [92], the local-global distinction [22] etc.) or of particular theories in which homotopy played some role at some stage (like combinatorial group theory [20], the theory of Lie groups [36], [37], knot theory [31], the theory of 3-manifolds [106], etc.). But we do not have a study which takes these threads together under the perspective to highlight the role of homotopy in these histories, and the interaction of the concept's development with them. The present paper cannot fill this gap either, but I hope it can go some steps further towards this bold aim.

(5) See http://www.emis.de/MATH/JFM/JFM.html. The queries made were on "deform*", "Fundamentalgruppe", "Gruppoid", "homotop", "Knotengruppe", "Poincaré group", and "Wegegruppe".

2. Poincaré's approaches to the fundamental group

A nice introduction to Poincaré's topological work is given by a paper by K.S.Sarkaria contained in James' Volume on the history of topology [90]. Concerning Poincaré's 1895 definition of the fundamental group, Sarkaria notes:[6]

> Poincaré gives four approaches to his groups g and G. Firstly, as all *deck transformations* of a covering space over M, viz., that whose projection map is the inverse of the multiple valued function F_α [...]. Secondly, his differential equations definition [...] gives g as the *holonomy group* of a "curvature zero" or integrable connection on a vector bundle over M [...]. Thirdly, his definition using "loops", "equivalences" and "lacets" amounts to that which one usually finds in most text books. Lastly, [...], for any M obtained from a polytope by facet conjugations, Poincaré defines $\pi_1(M)$ via some simple and elegant (yet intriguing) cyclic relations. [90, p. 144]

In the present section, I intend to pursue these different approaches, to locate them in Poincaré's overall work and its historical contexts.[7] First of all, Poincaré does not in all cases explicitly state that the groups defined according to the various approaches are in fact the same (abstract) group, and in no case he gives a proof of such a statement. In a more traditional vocabulary of historiography of mathematics, we would presumably say in this situation that he only "implicitly" defines the fundamental group in these various ways; and in a more recent setting of this historiography, we would say that such a statement about what Poincaré did is worthless for it is of a retrospective nature. My line of argument will however be a bit different: I intend to show that the various approaches of Poincaré have been pursued partly independently of each other, and that the result of this pursuit was not only that a "common content" has been eventually discovered, but also that the interpretation of one determination has been made stable by the existence of the others.[8]

---

(6) The meaning of the letters g and G used in this quote will be explained in section 2.3. In the present paper, emphasis in quotes is always original.

(7) The focus will be mostly on the first, third and fourth approach in Sarkaria's list and their interaction; we will later have occasion to focus on holonomy groups, too, but there does not seem to be a historical connection to Poincaré's work in this case. Moreover, my presentation of Poincaré's work will be very short and concentrated on quotes from Poincaré's original research papers (pagination in references to which will always be with respect to the *Œuvres*) giving some idea about the way in which the approaches mentioned by Sarkaria appear there. The reader finds more extensive descriptive and historical accounts of this work, along with Sarkaria's paper, in [92], [102] and [106].

(8) This is a variant of a Wimsattian robustness argument; see [56] for a similar consideration.

2.1. 1883

Central contributions to uniformization theory have been made by Schwarz and later his student Koebe, and by Poincaré who (inspired by Schwarz) presented a paper in 1883 [72][9]. His discussion quite explicitly resides on analytic continuation of functions along a path; see also [36, p. 189]. And concerning these paths, homotopy is the relevant equivalence relation, as Van den Eynde points out: "The fact that Poincaré considers the behaviour of many-valued functions on the surface [...] and not the integrals of such functions, forces him to use continuous deformation, and not homology" [102, p. 158]. Moreover,

> it seems likely that Poincaré was inspired by his work on automorphic functions. [...] Poincaré knew that to a given Riemann surface corresponds a group of linear transformations in the plane. If the Riemann surface has genus $p > 1$ these automorphisms can be interpreted as isometries of the non-euclidean plane. The collection of the replicas of a fundamental domain of the group can be seen as the surface S he defines in his paper of 1883. [102, p. 159]

The following quote shows that the surface S mentioned by van den Eynde is what latter became called the universal covering.

> We consider m analytic functions of x,
>
> $$y_1, y_2, \dots, y_m$$
>
> which are in general not uniform. [...] We [...] consider the variable x as moving [...] on a Riemann surface S. [...] We trace in the plane an arbitrary closed contour C which begins and ends at the same point x. The surface will be completely defined if we state the conditions under which the initial and final point of this contour must be regarded as belonging to the same sheet or to different sheets.
>
> Now there are two sorts of contours C:
>
> 1^o Those which are such that at least one of the m functions y does not return to its initial value when the variable describes the contour C;
>
> 2^o Those which are such that the m functions y return to their initial values when the variable describes the contour C.

---

(9) See [35] and [69] for more details. The contributions to our topic contained in a second paper from 1908 will be discussed below.

> Among the contours of the second sort, I will distinguish two species:
>
> 1^o C will be of the first species, if, by deforming this contour in a continuous manner, one can pass to an infinitesimal contour so that the contour never ceases to be of the second sort.
>
> 2^o C will be of the second species in the contrary case.
>
> Well, the initial and the final point of C will belong to different sheets if this contour is of the first sort or of the second species of the second sort. They belong to the same sheet if C is of the first species of the second sort. [...]
>
> The Riemann surface is then defined completely. It is simply connected [...]. [77, p. 58f][10]

Thus, the definition of S is such that the lifting of homotopically trivial loops ("second species") is still closed ("second sort"). In other words, S is the universal covering. Klaus Volkert describes the content of this passage thus: "Poincaré already in 1883 in a paper on algebraic curves [...] studied universal coverings and recognized the connection between decktransformations and the fundamental group."[11] [106, p. 111]. Volkert points to [29, p. 295] and [92, p. 202f] for this reading. But I think that one would read too much into this passage when claiming that Poincaré already discusses the connection between decktransformations and the fundamental group here.

---

(10) Translation quoted from [36, p. 179]. Original passage: "Considérons m fonctions de x, $y_1, y_2, \ldots, y_m$, analytiques, non uniformes en général. [...] Nous considérons la variable x se mouvant [...] sur une surface de Riemann S. [...] Traçons dans le plan un contour fermé quelconque C partant d'un point initial x et revenant finit à ce même point x. La surface S sera complètement définie, si nous disons à quelles conditions le point initial et le point final de ce contour devront être regardés comme appartenant à un même feuillet ou à des feuillets différents.

Or il y a deux sortes de contours C :

1^o Ceux qui sont tels que l'une au moins des m fonctions y ne revient pas à sa valeur initiale quand la variable x décrit le contour C ;

2^o Ceux qui sont tels que les m fonctions y reviennent à leurs valeurs initiales quand la variable x décrit le contour C.

Parmi les contours de la deuxième sorte, je distinguerai deux espèces :

1^o C sera de la première espèce, si l'on peut, en déformant ce contour d'une façon continue, passer à un contour infinitésimal de telle façon que le contour ne cesse jamais d'être de la seconde sorte ;

2^o C sera de la seconde espèce dans le cas contraire.

Eh bien, le point initial et le point final de C appartiendront à des feuillets différents si ce contour est de la première sorte, ou de la seconde espèce de la seconde sorte. Ils appartiendront au même feuillet si C est de la première espèce de la seconde sorte. [...]

La surface de Riemann est alors complètement définie. Elle est simplement connexe [...]."

(11) "Poincaré hatte bereits 1883 in einer Arbeit über algebraische Kurven [...] universelle überlagerungen studiert und den Zusammenhang zwischen Decktransformationen und Fundamentalgruppe erkannt."

2.2. 1892

Poincaré's bold aim in 1892 is the classification of closed (hyper-)surfaces in n-space up to homeomorphism. In the case of $\mathbb{R}^3$, the Betti numbers are topological invariants; in $\mathbb{R}^n$ for $n > 3$, finer invariants than Betti numbers are needed. Poincaré's proof of this fact consists of an ingenious construction of an example; see the end of section 2.3. Given this situation, Poincaré is lead to introduce a certain group which seems to be appropriate to characterize a hypersurface; at least, this would be in agreement with his example since the latter consists of two manifolds with same Betti numbers but different group. Here is the definition (where "surface" means "hypersurface", actually):

> Let $x_1, x_2, \ldots, x_{n+1}$ be the coordinates of a point on the surface. These $n+1$ quantities are connected by the equation of the surface. Now let
>
> $$F_1, F_2, \ldots, F_p$$
>
> be any p functions of the $n+1$ coordinates x (which I always suppose to be connected by the equation of the surface, and which I suppose to take only real values).
>
> [...] suppose that our point now describes a *finite* closed contour on the surface. It may then happen that the p functions do not return to their initial values, but instead become
>
> $$F_1', F_2', \ldots, F_p'$$
>
> In other words, they undergo the substitution
>
> $$(F_1, F_2, \ldots, F_p; F_1', F_2', \ldots, F_p').$$
>
> All the substitutions corresponding to the different closed contours that we can trace on the surface form a group which is discontinuous [78, p. 190].[12]

---

(12) Quoted after [80, p. 1]. Original quote: "[...] Soient $x_1, x_2, \ldots, x_{n+1}$ les coordonnées d'un point de la surface ; ces $n+1$ quantités sont liées entre elles par l'équation de la surface. Soient maintenant

$$F_1, F_2, \ldots, F_p$$

p fonctions quelconques de ces $n+1$ coordonnées x (coordonnées que je suppose toujours liées par l'équation de la surface et auxquelles je conviens de ne donner que des valeurs réelles).

[...] supposons que notre point décrive sur la surface un contour fermé *fini*, il pourra se faire que nos p fonctions ne reviennent pas à leurs valeurs initiales, mais deviennent

$$F_1', F_2', \ldots, F_p'$$

Even if Poincaré does not give a reference to his work on the monodromy group of linear differential equations here, there is obviously a great similarity between the two conceptions; see [71] and [73]. There, in turn, he makes clear from the outset that this work continues his earlier work on fuchsian functions. The same group had been studied earlier by Camille Jordan [50], [49]. In particular, Jordan there makes some rather informal allusions which might be read as describing an enveloping of the singular points of a surface by continuous deformation of paths. We will come back on this issue in section 5 in order to show that Jordan's treatment represents a certain way of interpreting the conception of continuous deformation.[13]

2.3. 1895

The treatment in the 1895 paper contains a novel element. Poincaré first defines a group g in a way very similar to what he did in 1892. He then enters a study of an equivalence relation and a composition of closed paths, eventually yielding the definition of a second group G. It is worth to be noted that Poincaré does not speak about a group with respect to the homotopy classes themselves. To the contrary, he will make a (substitution) group correspond to the set of closed paths with this relation and composition:[14]

> This being given, it is clear that we can envisage a group G satisfying the following conditions:
>
> 1^o Each closed contour M_0BM_0 corresponds to a substitution S of the group,
>
> 2^o The necessary and sufficient condition for S to reduce to the identity substitution is that
>
> $$M_0BM_0 \equiv 0;$$

---

ou, en d'autres termes, qu'elles subissent la substitution

$$(F_1, F_2, \ldots, F_p; F'_1, F'_2, \ldots, F'_p).$$

Toutes les substitutions correspondant aux divers contours fermés que l'on peut tracer sur la surface forment un groupe qui est discontinu [...]."

(13) The reader can find a short presentation of the theory of the monodromy group of a linear differential equation in an *Enzyklopädie* article by Emil Hilb [38]. Hilb on p. 499 n.110 cites Poincaré's and Jordan's papers as the first studies using this group; one might however argue that it is implicitly present in Riemann's work on the hypergeometric equation [68, p. 76]. While Jordan and Poincaré spoke only about the "groupe de l'équation", Hilb gives credit to Klein [51, p. 134] for the terminology "monodromy group". A short discussion of Jordan's work on monodromy groups of linear differential equations in relation to earlier work by Puiseux and Jordan's work on substitution groups can be found in [10] p. 346; on this aspect of Puiseux' work, see also [102, p. 140].

(14) Here, $M_0BM_0 \equiv 0$ means the closed path with initial point and endpoint M_0 and passing through B is nullhomotopic.

> 3^o If S and S' correspond to contours C and C' and if
>
> $$C'' \equiv C + C',$$
>
> then the substitution corresponding to C'' will be SS'.
>
> [78, p. 242][15]

It is to be assumed that Poincaré proceeds in this manner because a group for him is still a group of substitutions. (There is a passage in the 5th complement very clearly confirming this hypothesis; see [78, p. 450] or [102, p. 161]). Next, Poincaré compares the groups g and G:

> We compare [the group G] to the group g of substitutions undergone by the functions F.
>
> The group g will be isomorphic to G.
>
> The isomorphism can be holoedric, but it will not be if there is a closed contour M_0BM_0 indecomposable into hairpin bends on which the functions F return to their original values [78, p. 242][16]

Poincaré's terminology is still that of [48] where isomorphisms can be "mériédrique" (i.e., surjective, but not necessarily injective homomorphisms) or "holoédrique" (isomorphisms in the modern sense of the term). Poincaré's last remark is crucial: a function continued analytically along a homotopically nontrivial path can nevertheless be single-valued. This constitutes a difference (expressed in the noninjectiveness of the homomorphism) of the two groups. A proof of the claimed "isomorphy" of the two groups has been

---

(15) Quoted after [80, p. 40]. Original quote:
"Cela posé, il est clair que l'on peut imaginer un groupe G satisfaisant aux conditions suivantes :

1^o A chaque contour fermé M_0BM_0 correspondra une substitution S du groupe ;

2^o La condition nécessaire et suffisante pour que S se réduise à la substitution identique, c'est que

$$M_0BM_0 \equiv 0 \;;$$

3^o Si S et S' correspondent aux contours C et C' et si

$$C'' \equiv C + C',$$

la substitution correspondant à C'' sera SS'."

(16) Quoted after [80, p. 40]. Original quote: "Comparons [le groupe G] au groupe g des substitutions subies par les fonctions F.
Le groupe g sera isomorphe à G.
L'isomorphisme pourra être holoédrique.
Il pourra être mériédrique si un contour fermé M_0BM_0 non décomposable en lacets ramène les fonctions F à leurs valeurs primitives."

worked out by Volkert; in this proof, the universal covering is used [106, p. 121f]. Poincaré later on (p. 247) stresses that in some cases, the fuchsian group is the fundamental group. In the fifth complement, there is a longer passage where this relation is studied [78, p. 448-459]. Therefore, Volkert conjectures that Poincaré was inspired by the terminology "domaine fondamental" from the context of fuchsian functions when choosing the term "groupe fondamental" [106, p. 117, 130].

We should ask which role the approach by generators and relations (Sarkaria's fourth approach) actually played in Poincaré's topological work. This approach to groups in general certainly had some influence in the field of discontinuous groups (see the remarks in [20, p. 11] concerning the work of Fricke and Klein, for instance). It was also used before Poincaré in the field of classification of manifolds: Camille Jordan achieves a classification of compact orientable surfaces with a boundary and determines what we would call now generators and relations of the fundamental group of these surfaces [47], [46].[17] Chandler and Magnus also point to the crucial role played by this approach for one of the central results of [75]:

> The construction of a three-dimensional space for which the Betti number and the torsion coefficients [...] are the same as for the closed three-dimensional spherical space but which has a fundamental group Γ a perfect group which, in turn, has the group of the icosahedron [...] as a quotient group. [...] this group is given by a finite presentation. But apart from this example, one cannot say that Poincaré used group-theoretical methods in any decisive manner. [20, p. 14f]

The example in question is obtained by identification of faces, and the group is calculated from the gluing data; see [78, p. 493-496]. While Chandler and Magnus suggest that the influence of the approach was limited to this example (see the last sentence of the quotation), Poincaré at least treated other examples in this way; detailed descriptions can be found in [90] and [106].

2.4. 1908

In 1908, Poincaré publishes a second paper on uniformization in which the concept of covering is discussed in greater generality.[18] We find the

(17) See [102, p. 148] and [92] for details.
(18) I should justify my dating, for throughout the secondary literature, this paper is cited as published in 1907. This is the year given in the *Œuvres* [77]; maybe most authors cite the *Œuvres* rather than the *Acta* themselves. Now, the volume 31 of the *Acta* bears

following definition of the universal covering in [76]:[19]

> Now let D be an arbitrary domain. I claim that we can find a domain Δ which is a regular multiple of D and simply connected. [...] One can go from M_0 to M on D by many paths. Consider two of these paths. They could be *equivalent*, that is they could bound a continuous area situated on D; but they may not be, at least if D is not simply connected [77, p. 91].[20]

Thus, like in 1895, "equivalent" means homotopic, but here for not necessarily closed paths. (We will see in section 5.1 that the usage in [75] is similar). Poincaré continues:

> That given, let us define the domain Δ. A point of this domain will be characterized by the point M of D to which it corresponds and by the path by which one proceeds from M_0. In order that two points so characterized be identical it is necessary and sufficient that one has come from M_0 by equivalent paths. It is clear that Δ is simply connected [77, p. 91].[21]

In the considerations to follow, Poincaré uses neighborhoods on Δ without having defined them explicitly in the construction of Δ quoted above. Maybe this is implicit in his notion of "domain". As we will see below when discussing Weyl's work, composition of paths plays a role in the definition of these neighborhoods.

---

actually 1908 as year of publication, and this is a sufficient reason to adopt this date, given that readers might wish to look up the paper in the *Acta* themselves. Things get more complicated when on the first page of Poincaré's paper we read "printed March 19, 1907" ("imprimé le 19 mars 1907"). It is clear from the correspondence with Mittag-Leffler that Poincaré had finished the paper even well before, since he sent it to Mittag-Leffler on June 13, 1906 (see [79] letters 232, 233). It was not unusual that sheets of the volumes of journals were printed as the papers came along, whence the discrepancy between the printing date of the paper and the publication date of the volume.

(19) See [36, p. 181] for further discussion of the following passage.

(20) Translation partly taken from the quotation given in [36, p. 181] ; original passage : "Soit maintenant D un domaine quelconque ; je dis que nous pourrons trouver un domaine Δ, régulièrement multiple de D et simplement connexe. [...] On peut aller sur le domaine D de M_0 en M par plusieurs chemins; envisageons deux de ces chemins ; ils pourront être *équivalents*, c'est-à-dire qu'ils pourront limiter une aire continue située sur D ; mais ils pourront aussi ne pas l'être, à moins que D ne soit simplement connexe."

(21) Translation quoted from [36, p. 181] ; original passage : "Cela posé, définissons le domaine Δ ; un point de ce domaine sera caractérisé par le point M de D qui lui correspond, et par le chemin par lequel on est venu de M_0 en M ; pour que deux points de Δ ainsi caractérisés soient identiques, il faudra et il suffira qu'ils correspondent à un même point M de D et qu'on soit venu de M_0 en M par deux chemins équivalents. Il est clair que Δ est simplement connexe [...]."

3. The fundamental group after Poincaré: coverings and "structuralist" outcomes

In this section, we try to show that the notion of fundamental group reached a certain "stability" in the work of the generation after Poincaré thanks to several reasons: the simultaneous pursuit of the various approaches to the group, notably including its fruitful interplay with the concept of covering,[22] and a focus on "structural" aspects like the study of subgroups and quotients.

3.1. Definitions of the fundamental group in the literature after Poincaré

First of all, we should note that the early literature shows a great diversity in the treatment of the fundamental group, not only with respect to the definition used, but as well with respect to the name given. We encounter not only other terms for that group, but also another usage of the term "fundamental group". The *Jahrbuch* has 174 entries containing the term "Fundamentalgruppe" and 25 entries containing "Wegegruppe". Seven of them actually contain both terms; thus, we have 192 entries in all. Among those containing "Fundamentalgruppe", 131 concern the fundamental group in one of Poincaré's senses, while the 43 remaining concern other usages, 35 of them in a sense promoted by Elie Cartan and ultimately relying on the role groups play in Klein's *Erlanger Programm*.

In fact, Cartan uses the term "groupe fondamental" for the group operating on a space. This usage is applied throughout his work,[23] and actually by several other authors relying, like Cartan, on the work of Klein and Lie.[24] Cartan expresses the idea very clearly:

> I developed, in these last years, a general theory of spaces comprising the classical theory of Riemannian spaces as well as the more recent theory of Weyl spaces. [...] I tried to extend that fruitful principle of Klein's according to which every geometry is the study of the properties of a group of transformations G:

(22) From the presentation of the uniformization problem above, the reader might have got the impression that the concept of covering exclusively developed from this context. This impression would however be misleading; other relevant contexts are the question of orientation in the work of Möbius and Klein and the "Raumproblem" of Clifford-Klein (see [32] here). As to the history of the concept of covering in general, these few remarks should suffice, since we are not trying to write such a history here, but rather study the role of coverings for the conceptual development of the fundamental group.

(23) See for instance [16], [15], [17]. In [19, p. 14], the space is homogeneous.

(24) Fano seems to use the terminology like Cartan [33, p. 367f], and it is extensively used by Koebe in various publications.

> the *continuum* in which the figures of this geometry are located and the only essential properties of which are those which are stable under an arbitrary transformation of G is called a *space with fundamental group G*.[25]

A group isomorphic with the fundamental group in our sense is called "groupe de connexion au sens de l'Analysis situs" on p. 28 of [19].

Some authors simply speak about the "Poincaré group".[26] In 1929, Hellmuth Kneser suggests to replace Poincaré's terminology of fundamental group by "Wegegruppe" (path group; [52, p. 256]). He argues that this term cannot be misunderstood as easily as Poincaré's. However, he does not point to the competing usage of Cartan; rather, he finds that homology groups are more or less "fundamental" as well — a statement which reflects well the situation in 1929 but would not make sense in Poincaré's situation. This proposal has been pursued for a while, as we have seen: there are 25 entries in the *Jahrbuch*.

Now to definitions; some definitions to be found in the early literature after Poincaré are presented here while others will be presented later together with their applications (because the conceptual outcome of these applications will prove important for my line of argument).

As far as the wording is concerned, Dehn and Heegaard in their 1907 *Enzyklopädie* article give the now standard definition of the fundamental group, based on the totality of closed oriented curves through a given point and identified if homotopic. However, their notion of homotopic curves is combinatorial (defined with respect to the corresponding "Streckenkomplexe"; p. 164); they postulate the equivalence of homotopy so defined and continuous deformability in the "Deformationsaxiom" (p. 169). As to the proof of the group property, they merely say that the curves running through a point of an M_n yield a discontinuous group "by their composition properties" [28, p. 207]. Dehn further stresses in [26] that the question whether two based paths are homotopic is a group theoretical question (the "word problem"); see [20, p. 17] and [102, p. 173].

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(25) My translation. Original quote: "J'ai développé, dans ces dernières années, une théorie générale des espaces englobant la théorie classique des espaces de Riemann et celle plus récente des espaces de Weyl. [...] j'ai cherché à étendre le principe si fécond de Klein, d'après lequel toute Géométrie est l'étude des propriétés d'un groupe de transformations G: le *continuum* dans lequel sont localisées les figures dont s'occupe cette Géométrie, et dont les seules propriétés jugées essentielles sont celles qui se conservent par une transformation arbitraire de G, s'appelle un *espace à groupe fondamental G*" [18, p. 1].

(26) The *Jahrbuch* has only five relevant entries, four by Zariski and one by Flexner. But the terminology is also used in [57, p. 82f].

Tietze defines the fundamental group of a manifold by generators and relations obtained from the "scheme" ("Schema") of the manifold (the combinatorial encoding of the manifold he is working with) [101, p. 65ff]. He notes that there is a connection of the fundamental group so defined with "the multivalued but unbranched functions conceived of as extended on the manifold"[27] but this connection is, in Tietze's words, only allusively touched upon in the paper. Tietze thinks here of his n.6 on p. 67 where he explains in detail how the values of such functions determine substitutions corresponding to paths ("Wegstücke"). On p. 69, Tietze proceeds to the proof that his fundamental group is a topological invariant (which means, in his case, that it is invariant by transition to a homeomorphic scheme), and notes that in Poincaré's presentation, this emanates from the "significance of the fundamental group for the unbranched functions extended on the manifold".[28] Hence, in Tietze's view, Poincaré did not need a proof of this invariance thanks to the connection to those functions — while Tietze needs such a proof, given his strictly combinatorial approach. Tietze does not prove the group property of his group, which is clear since he explicitly gives generators and relations for it.

Veblen's treatment in [104, p. 132f] is again strictly combinatorial and very close to Dehn-Heegaard. The role of paths is played by oriented closed 1-cells called "equivalent" if homotopic in the combinatorial sense. Veblen explicitly notes the group axioms, and says it is "clear" that they are fulfilled here.

Solomon Lefschetz in [57, p. 82] defines the fundamental group ("Poincaré group") by homotopy classes of loops; he notes how the composition, the identity and inverses are defined, but otherwise does not bother about the proof. He refers to [74] and [75]. Later on, he says:

> Recently the Poincaré group has been the object of highly interesting investigations, notably in connection with the theory of knots by Alexander and Reidemeister, and with the isolation of fixed points of certain surface transformations by Nielsen and other manifolds by Hopf. The group appears also in disguise in Morse's investigations on geodesics of open and closed surfaces.[29]

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(27) "den auf der Mannigfaltigkeit ausgebreitet gedachten mehrdeutigen, jedoch unverzweigten Funktionen" (p. 65). He refers to [74, p. 60f] for this viewpoint; this is [78, p. 239f].

(28) "Bedeutung der Fundamentalgruppe für die in der Mannigfaltigkeit ausgebreiteten unverzweigten Funktionen".

(29) The references given by Lefschetz are [1], [4], [3], [40], [61], [62], [64], [65], [66], [85] and [86]. Among these 11 references, only five (the first, the sixth and the last three, actually) have been found by our *Jahrbuch* query; this shows the limitations of the automatic search procedure.

The "disguise" in Morse's case is the group of decktransformations of the universal covering—at least in the reading of the *Jahrbuch* reviewer. Morse himself does not use this terminology, but rather goes directly back to Poincaré's seminal paper on the theory of fuchsian groups [70].

Threlfall and Seifert define the fundamental group (or "Wegegruppe") $\mathfrak{F}$ as the set of "Wegeklassen" [100, p. 44]; these are classes of closed oriented curves through a given point identified if continuously deformable one in another. They indicate how to form the product of two such classes, and how the unit element is defined, but give no further details of the proof that there is indeed such a group.

3.2. Weyl

One prominent place where the theory of coverings is developed is *Die Idee der Riemannschen Fläche* [107].[30] Weyl defines paths explicitly as continuous mappings of $[0, 1]$ into the space; he however calls them curves ("Kurven").[31] Weyl's construction of the universal covering ("Universelle Überlagerungsfläche") in [107, p. 51] is the following. As expected, Weyl introduces the construction only for surfaces $\mathfrak{F}$ with a base point $\mathfrak{p}_0$. A point $\tilde{\mathfrak{p}}$ of $\tilde{\mathfrak{F}}$ over a point $\mathfrak{p}$ of $\mathfrak{F}$ is given by a path γ from $\mathfrak{p}_0$ to $\mathfrak{p}$, and two such paths γ, γ' define the same point $\tilde{\mathfrak{p}}$ if in any covering of $\mathfrak{F}$, two paths starting in the same point and mapped to γ, γ' in $\mathfrak{F}$ always end in the same point. (We have seen in section 2.4 that Poincaré expressed this condition in a way more close to the modern standpoint according to which γ, γ' are homotopic.)

In the definition of the topology on $\tilde{\mathfrak{F}}$, composition of paths (written additively by Weyl) plays some role: given a path γ_0 from $\mathfrak{p}_0$ to $\mathfrak{p}$ and defining the point $\tilde{\mathfrak{p}}$ in $\tilde{\mathfrak{F}}$, a neighbourhood of $\tilde{\mathfrak{p}}$ consists of the points defined by all the paths $\gamma_0 + \gamma$ where γ has $\mathfrak{p}$ as a starting point and is completely contained in a neighbourhood of $\mathfrak{p}$.

The fact that Weyl does not use the notions of homotopy of paths, and of fundamental group, has already been stressed in the literature.[32] Actually, instead of using the fundamental group, Weyl exclusively considers the group of automorphisms ("Gruppe der Decktransformationen") of the covering. When using the composition of paths again later (p. 68ff), it is in order to speak about "lineare Kurvenfunktionen" and "Integralfunktionen"

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(30) For further discussion, see in particular [36, p. 182ff] and [22, p. 506]. [23, p. 52] stresses that there are important changes in later editions.

(31) There is also a usage of the term "Weg" (which is the German equivalent of "path") in a different sense; this usage will be discussed in section 5.

(32) Weyl managed "to eschew the use of homotopy" [102, p. 154]. See also [36, p. 183].

(locally null-homologous linear “Kurvenfunktionen”), these notions giving rise to the notion of genus of a Riemann surface (half the maximal number of linearly independent Integralfunktionen) and later the Riemann-Roch theorem (p. 122). With the exception of §19 and §20, Weyl’s book is very much geared towards integration, so that it is not surprising that Weyl focuses on homology rather than homotopy. This is a first reason why the absence of homotopy is not so astonishing; another could be that Weyl was giving a lecture course on function theory, after all, and certainly was not interested in multiplying more than necessary the uses of abstract notions his students were not acquainted with.

Another point is that Weyl is concerned with surfaces, where homology is enough to obtain a topological classification. That is, he might have been aware that Poincaré had used a certain group to bring forward the classification of higher-dimensional manifolds, but thought that this is irrelevant to his interest in surfaces. Moreover, given his own background, he might have found the presentation of the surface by functions and the definition of the group by linear transformations more natural than the definition by composition and deformation of paths.

Weyl might also have disliked Poincaré’s terminology “fundamental group” (and therefore have avoided its use), for the following reason: he certainly was aware of the usage of the term by Fano, Koebe and others in reference to the ideas of Klein’s *Erlanger Programm*. Incidentally, this terminology is adopted in the *Jahrbuch* review of [108]; even if Weyl does not use this terminology himself in that paper, this shows that this usage was both widespread and related to Weyl’s preferred fields of work.

3.3. The fundamental group and decktransformations

The connection skipped by Weyl between the fundamental group of a space and the group of decktransformations of the universal covering has been explicitly stressed by other authors of that generation. In the 1920s, Jakob Nielsen made important applications of the fundamental group concerned with “the isolation of fixed points of certain surface transformations” [57, p. 82]. On the group-theoretical aspects of Nielsen’s work, the reader can find a wealth of informations in [20, p. 81ff]. We will pick out here just one article by Nielsen, [67], since his reading of Poincaré’s achievements presented there is relevant to my line of argument.

Nielsen on p. 204 defines the fundamental group F of a closed two-sided surface as a group of substitutions (shown to be hyperbolic on p. 208). He then constructs the universal covering Φ of a surface ϕ with respect to the fundamental group F of ϕ as group of decktransformations (he does not use this terminology, and does not give a reference to Weyl here; such a reference can be found only on p. 286). On p. 206, Nielsen credits Poincaré to have disclosed completely, in the fifth complement, the connection between curves on a surface ϕ and curves on the universal covering Φ using the fundamental group F of ϕ. This connection Nielsen explains thus (p. 207): he proves that two curves on ϕ corresponding to the same element of F are homotopic and vice versa, and the proof actually runs through a parametrized "Schar" of curves yielding the deformation.

Now, an inspection of the fifth complement shows that Nielsen refers here to [78, p. 465f], but that most of the matters mentioned by Nielsen are far less explicitly said there. Nielsen is presenting a "synthetic" reading of Poincaré here, thus testifying the claimed features of stabilization.

We have already seen how the definition of the fundamental group is discussed in [100]. Their overall goal is related to Nielsen's work, but phrased already in more "structural" terms: it is the theorem that the fundamental group $\mathfrak{F}$ of the domain of discontinuity of a finite group of motions ("Bewegungsgruppe") $\mathfrak{G}$ of the hypersphere is the quotient of $\mathfrak{G}$ and the subgroup $\mathfrak{N}$ of $\mathfrak{G}$ generated by all motions admitting a fixed point ("fixpunkthaltigen Bewegungen") of $\mathfrak{G}$. We might conjecture that the letter $\mathfrak{N}$ has been chosen for "Nielsen".

This "quotient" approach is treated in a more abstract manner in Seifert and Threlfall's monograph on topology. A notion of monodromy group (without mention of multivalued functions) is defined and its relation to the fundamental group is made explicit in terms of coverings [95, p. 199f]. They are looking for all coverings of a finite complex $\mathfrak{K}$ with a finite number g of sheets. To a closed path W from a point O in $\mathfrak{K}$ belong g covering paths with initial points $\tilde{O}_1, \tilde{O}_2, \ldots, \tilde{O}_g$ and endpoints a permutation of these $\tilde{O}_{k_1}, \tilde{O}_{k_2}, \ldots, \tilde{O}_{k_g}$. The monodromy group is simply the group $\mathfrak{M}$ of permutations

$$\begin{pmatrix} 1 & 2 & \ldots & g \\ k_1 & k_2 & \ldots & k_g \end{pmatrix}.$$

Since homotopic paths correspond to the same permutation, and a composed path to the composition of the permutations, there is a homomorphism of the fundamental group $\mathfrak{F}$ onto $\mathfrak{M}$. The result is that $\mathfrak{M}$ is isomorphic to the quotient group $\mathfrak{F}/\mathfrak{T}$ where $\mathfrak{T}$ is obtained from the conjugate subgroups of $\mathfrak{F}$ corresponding to the covering.

Reidemeister in [89] studies the relation between two ways of defining the fundamental group: "as the path group of the complex on the one hand, as the group of automorphisms of the corresponding universal covering complex on the other hand".[33] He shows the equivalence of the two definitions in his strictly combinatorial setting.

3.4. Schreier: coverings for groups

Hermann Weyl, despite making path-breaking contributions to the theory of Lie groups in the 1920s, still avoided using the fundamental group in these years, as far as I can see.[34] It was Otto Schreier who transferred the interplay of coverings and the fundamental group to the theory of topological groups, and thereby contributed to important progress in the treatment of Hilbert's fifth problem. In the case of interest for us, Schreier investigated the relation between topological groups with identical "group germ" (i.e., there is a neighborhood of unity identical in both groups). Schreier's main result is the following:

> All continuous groups locally having the same structure can be considered as factor groups of a particular group among them, the covering group. The normal subgroups occuring are abelian and isomorphic to the fundamental group of the corresponding factor group, considered as a space.[35]

Schreier gives two accounts of this, one with proofs couched in terms of abstract group theory (generators and relations) [93], another with new proofs stressing, as Schreier puts it, "the topological point of view" (which means: how the covering group ("überlagerungsgruppe") can directly be defined globally) [94]. Schreier gives credit to Artin for the terminology "überlagerungsgruppe" [93, p. 24 n.6] and also for a certain example [93, p. 32]. This is but one of several benefits of a "Vienna-Hamburg-exchange" taking place in these years; see below.

The explicit constructions are the following. Schreier starts with $\mathfrak{R}$ a "Konvergenzraum" (some kind of topological space). He then defines the notion of path ("Weg") as follows: "By a path w on $\mathfrak{R}$, we mean a univalent function $F(t)$ continuous in the interval $0 \leqslant t \leqslant 1$ and whose values are

(33) "als die Wegegruppe des Komplexes einerseits, als die Automorphismengruppe des zugehörigen universellen überlagerungskomplexes andererseits".

(34) See [36] and [37] for detailed accounts on Weyl's work in this direction.

(35) "alle kontinuierlichen Gruppen, die im kleinen dieselbe Struktur haben, [können] als Faktorgruppen einer bestimmten unter ihnen, der überlagerungsgruppe, aufgefaßt werden [...]. Die dabei auftretenden Normalteiler sind abelsch und mit der Fundamentalgruppe der — als Raum aufgefaßten — zugehörigen Faktorgruppe isomorph" [94, p. 233].

points of $\mathfrak{R}$"[36] (p. 233f). Schreier then defines the notions of closed path, contractible closed path (defined by continuation of a function on the border of a circle to the entire circle), inverse path, and product path, and from now on considers two paths as equal if $w_1^{-1}w_2$ is contractible (this being equivalent to w_1 being homotopic to w_2). He then says: "Thus, the same path can now be determined by various functions".[37] This remark incidentally underlines that by "Weg" he does mean neither just a point set nor a class of point sets related by change of parameter[38]—for in both cases, it would have been the case before that different functions can determine the same path.

Schreier then imposes the supplementary condition (α) which we could express by saying that $\mathfrak{R}$ is path-wise connected. According to Schreier, (α) implies that the paths form a groupoid;[39] Schreier cites [9]. This is presumably the first use of Brandt's concept in topology, and the first explicit appearance of the fundamental groupoid in the literature. We will discuss Brandt's work and the further history of the concept of fundamental groupoid in section 4.

On p. 236, Schreier passes to the consideration of a group $\mathfrak{G}$ which is supposed to be an "L-Gruppe" (which means that its elements form a "Konvergenzraum" and that product and passage to the inverse are continuous) fulfilling condition (α) and another condition. For A in $\mathfrak{G}$, the paths determined by $F(t)$ and by $AF(t)$ are considered as equal. Schreier then says:

> By an appropriate choice of A, we can transfer the initial point of a given path to an arbitrary element of $\mathfrak{G}$ or put differently, we can start the run through S the path from an arbitrary point. It is thus possible to compose any two given paths. It follows from **1.** that with this composition, the set of paths on $\mathfrak{G}$ is turned into a group $\mathfrak{g}$.[40]

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(36) "Unter einem Weg w auf $\mathfrak{R}$ verstehen wir eine im Intervall $0 \leqslant t \leqslant 1$ (eindeutige und) stetige Funktion $F(t)$, deren Werte Punkte von $\mathfrak{R}$ sind".

(37) "derselbe Weg kann also jetzt durch verschiedene Funktionen bestimmt werden".

(38) See section 5.3 below.

(39) They actually form a groupoid in the modern sense for any $\mathfrak{R}$; however, the groupoid is transitive, and hence a Brandt groupoid (see section 4.3 below), only if $\mathfrak{R}$ is path-wise connected.

(40) "Durch passende Wahl von A können wir den Anfangspunkt eines gegebenen Weges nach einem beliebigen Element von $\mathfrak{G}$ verlegen, den Weg also von einem beliebigen Punkt aus durchlaufen. Wir sind daher auch in der Lage, irgend zwei Wege miteinander zu multiplizieren. Aus **1.** folgt, daßwir so die Menge der Wege auf $\mathfrak{G}$ zu einer Gruppe $\mathfrak{g}$ machen" [94, p. 236].

By "**1.**", Schreier refers to the preceding paragraph where he showed that homotopy classes of paths form a groupoid. Thus, Schreier overcomes the fact that composition of these classes is only partially defined by passing to equivalence classes of these classes for left translation. $\mathfrak{g}$ is shown to be itself an "L-Gruppe"; in fact, topologically, $\mathfrak{g}$ is just the universal covering: Schreier could have worked on the topological level perfectly well with a base point instead of classes for left translation, and every path with the base point as the initial point represents an element of $\mathfrak{g}$. In this sense, he would have obtained the same group.

In the next step, Schreier reintroduces a base point (the unit element E of $\mathfrak{G}$), in order to get a well-defined homomorphism from $\mathfrak{g}$ to $\mathfrak{G}$ which maps any path with initial point E to its endpoint. The fundamental group $\mathfrak{f}$ of $\mathfrak{G}$ (which is the group of paths with initial point and end point the unit element E of $\mathfrak{G}$) is shown to be an invariant subgroup of $\mathfrak{g}$, and $\mathfrak{g}/\mathfrak{f}$ is isomorphic with $\mathfrak{G}$. From *Satz 8* of the 1925 paper, it follows that $\mathfrak{f}$ is abelian.

We cannot pursue here the far-reaching applications these results had in the context of Hilbert's fifth problem in work by Pontrjagin and van Dantzig.[41] At present, we should note that, as [36, p. 192] points out, Cartan arrived independently at a similar result. As we have stressed already, Cartan's wording for the fundamental group in [19] is "groupe de connexion au sens de l'Analysis situs" (p. 28). In fact, Cartan considers a connected topological group G operating on a homogeneous Lie space E and the largest subgroup g of G leaving unchanged a certain point O of the space. This g can itself be connected or "mixed" ("mixte") (this terminology goes back to Klein and Lie; see [33, p. 295], for instance). If g is mixed, to each of its connected components g_ι corresponds in E a homotopy class of closed paths (in our present terminology), constructed by joining the unit element of G to an element of g_ι, and applying the elements of the path in G so obtained to the point O in E yields a closed path in E. Thus, the number of homotopy classes is the number of connected components. This construction of the fundamental group of a Lie group is very similar to the one given by Schreier.

3.5. Reidemeister's knot theory: Subgroups and coverings

It is well known that great advances in the theory of knots have been made by the introduction of the "knot group" (the fundamental group of the complement of the knot), most notably in [27] and in the papers by

(41) See [81] and [25]. I plan to discuss these issues in more detail in a paper on the history of direct and inverse limits under preparation.

Alexander on our list of references.[42] Another crucial step has been made by Kurt Reidemeister in [84] by extending the theory of coverings by the study of the subgroups of the fundamental group of the covered space. The discovery of this tool[43] lead to important progress not only in knot theory, but also in the theory of Lie groups. As explained in [20, p. 91ff], Reidemeister's method to calculate a presentation for a subgroup H of G, given a presentation for G, has been extended and simplified by Schreier and is therefore duly called the "Reidemeister-Schreier method".

First of all, it is worth noting that in [84, p. 14], the knot group is defined directly by indicating certain generators and relations, and for a proof of the fact that it is the fundamental group of the complement of the knot, Reidemeister refers to [5, p. 57-62]. The method used by Artin for constructing generators and relations is Wirtinger's; Artin says (p. 58) to have learned it from Schreier. Again, we find that the Vienna-Hamburg exchange played an important role in the development of an application of the fundamental group. [44] This historical context actually even played a role for Reidemeister's decision to adopt the strictly combinatorial viewpoint in his topological work (a feature which gives rise to very particular definitions of the basic notions); see [31, p. 322ff].

Reidemeister emphasizes that the knot group as such is not very useful as an invariant since among the many properties of such groups, only one is recognizable from the defining relations, namely the structure of the factor group of the commutator subgroup, or the "Poincaré numbers",[45] and these are identical for all knots. We would say that the abelianization of the fundamental group of $\mathbb{R}^3 \setminus K$ for every knot K is just $\mathbb{Z}$. But Reidemeister succeeds in determining defining relations for certain subgroups of the knot group whose Poincaré numbers are not trivial any longer.

Chandler and Magnus raised doubts about the claim that this strategy actually has been invented by Reidemeister.

> [Reidemeister in 1927] emphasizes the fact that the group H_n defined above is the fundamental group of an n-fold covering space of the space which has K as a fundamental group. Certainly, this is not a new insight. Probably, it is due to Poincaré; we have no reference and it is, of course very difficult to claim that something is *not* due to Poincaré. [20, p. 95]

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(42) See [31] for the history of knot theory in general and [20, p. 19] for a short account on Dehn's paper [27].
(43) This wording has been introduced by [31] in the present context.
(44) See [20, p. 90f] on the influence of Wirtinger on Reidemeister.
(45) This terminology is due to [101, p. 56ff] and also used by [104, p. 141].

But in the absence of any concrete evidence, I think we should continue to attribute the idea to Reidemeister. Poincaré in his study of the universal covering in uniformization theory does not even speak about the fundamental group he introduced elsewhere. Both Nielsen's and Sarkaria's reading of Poincaré discussed above only point to uses of the fundamental group in studies of the universal covering, not of coverings in general. It is clear that he studies such coverings in [76], but the only one important for the purpose is the universal covering since he needs a simply connected covering for the definition of the uniformization. The consideration of the group aspect is restricted to this case as well (see §9 of his paper). Reidemeister himself in the introduction to his monograph on combinatorial topology only very vaguely gives credit to Poincaré for the general idea, but clearly points to the accessibility of the subgroups of the fundamental group as something new in the field:

> The close relationship [between groups and complexes] has already been known since the fundamental work of Henri Poincaré. If it did not show up in all distinctiveness in the further development of combinatorial topology, this was due to the difficulty of those problems in which topology and group theory meet: it might have seemed pointless to investigate relations which at first glance only allowed to translate open topological questions into open group-theoretical questions. Such doubts would not be justified any longer today. Ever since, for a group given by generators and relations, generators and defining relations for subgroups can be determined, group theory forms a fruitful instrument of calculation for the topologist, allowing to submit many formerly intractable questions to a systematic investigation.[46]

(46) "Die enge Beziehung [zwischen Gruppen und Komplexen] ist schon seit den grundlegenden Arbeiten Henri Poincarés bekannt. Wenn sie in der Weiterentwicklung der kombinatorischen Topologie nicht immer mit nachdrücklicher Deutlichkeit hervortritt, so lag das an den Schwierigkeiten jener Probleme, in denen sich Topologie und Gruppentheorie berühren: Es mochte unfruchtbar erscheinen, Beziehungen nachzugehen, die zunächst nur ungelöste topologische Fragen in ungelöste gruppentheoretische Fragen zu übersetzen erlaubten. Solche Bedenken wären heute nicht mehr gerechtfertigt. Seitdem sich Erzeugende und definierende Relationen von Untergruppen einer durch Erzeugende und Relationen erklärten Gruppe bestimmen lassen, bildet die Gruppentheorie für den Topologen ein ertragreiches Recheninstrument, mit dem sich viele bisher unzugängliche Fragen einer systematischen Untersuchung unterwerfen lassen" [87, p. vii].

4. Composition of general paths and groupoids

> [...] the success of algebraic methods in topology [...] explains the preference for theories with "base point" and constrained deformation even though free deformation is a more natural concept [102, p. 182].

Whatever it means that free deformation[47] is "more natural" than constrained deformation, this quotation shows that talk about "algebraic methods" is easily taken to mean just the use of groups. From our present viewpoint, however, to restrict attention to group structures implies to impose a particular restriction on the kind of set of paths admitted for the use of structural methods. We can ask, historically, when this choice has been recognized as such, according to which criteria it has been made, and whether there have been exceptions. We find that there indeed have been exceptions in that the concept of fundamental group has given rise to the concept of fundamental groupoid. This concept was used at various places with variable effect, and there is an interesting but a bit intricate interaction with later generalizations of the concept of fundamental group. In the present section, we will retrace briefly both the history of the groupoid concept in general and more specifically of the fundamental groupoid.

4.1. Poincaré and partially defined group operations

We already have seen that Poincaré's use of the term "group" is quite informal. It is but another interesting fact about this use that he actually speaks about "groups" where the composition he has in mind actually only is partially defined. This is the case, for instance, for the "group" presented in [78, p. 198], whose elements are diffeomorphisms and which is actually a groupoid; see [92, p. 289] for discussion. (It is clear that Poincaré wants this "group" to play a role related to Klein's conception of a group as characterizing a geometry; see [31, p. 228].) But for the present paper, examples related to path composition would be more relevant.[48]

4.2. Reidemeister

Early usage of the notion of groupoid in topology can be found not only in Schreier's work discussed in section 3.4, but also in Reidemeister's monograph on combinatorial topology: "In some topological questions, a gener-

(47) A terminology that stems from [95, p. 174], actually.

(48) Volkert points to [78, p. 243] for a composition of not necessarily closed paths M_1AM_2, M_2BM_3 [106, p. 119f n.91], but Poincaré apparently does not discuss the algebraic structure of such a composition.

alization of the group concept, the groupoid, is a useful auxiliary notion"[49] [87, p. 27]. Reidemeister on p. 30 studies the generators of a groupoid, in analogy with the study of the generators of a group. On p. 107, Reidemeister considers the "Gruppoid aus den Klassen beliebiger äquivalenter Wege" (we would say now: the fundamental groupoid; actually, Reidemeister adopts the terminology "Wegegruppe" for the fundamental group), and indicates possible applications to surfaces of higher genus.

Like Schreier, Reidemeister gives explicit reference to the work of the algebraist Heinrich Brandt for the notion of groupoid. Thus, we should now take a look at this work, and at further evidence for its influence in topology.

4.3. Algebraic origin of the explicit groupoid concept

The concept of groupoid has first been explicitly defined by Heinrich Brandt.[50] In his dissertation (1913), Brandt generalized Gauss' notion of composition of equivalence classes of binary quadratic forms contained in the *Disquisitiones arithmeticae*[51] to the case of quaternary forms. While in the binary case this composition yields a finite group, it yields a finite groupoid in the quaternary case. Brandt extensively studied the structure of this groupoid in various publications and eventually in 1926 published a paper exclusively devoted to this generalization of the group concept [9]. The fact that the operation is only partially defined makes the basic axioms tedious, as in the case of associativity:

> If AB and BC exist, then $(AB)C$ and $A(BC)$ exist as well; if AB and $(AB)C$ exist, then BC and $A(BC)$ exist as well; if BC and $A(BC)$ exist, then AB and $(AB)C$ exist as well, and in each case $(AB)C = A(BC)$, such that one can write it as ABC as well.[52]

Brandt then postulates the existence, for any element A, of left and right unities ("Einheiten") E, E' with $AE = E'A = A$ and of (left) inverse elements. Brandt next shows that an element E is a left or right unity for some other element if and only if $EE = E$. Brandt's last axiom guarantees

(49) "Bei manchen topologischen Fragen ist eine Verallgemeinerung des Gruppenbegriffs, das Gruppoid, ein nützlicher Hilfsbegriff".

(50) Fuller accounts of the history of the groupoid concept can be found in [39] and [13]. The historical question whether Brandt's concept did play a role in the introduction of category theory or not will be treated elsewhere.

(51) For Gauss' work, see [112, p. 40ff] and [91].

(52) "Wenn AB und BC existiert, so existiert auch $(AB)C$ und $A(BC)$, wenn AB und $(AB)C$ existiert, so existiert auch BC und $A(BC)$, wenn BC und $A(BC)$ existiert, so existiert auch AB und $(AB)C$, und jedesmal ist $(AB)C = A(BC)$, so dass dafür auch ABC geschrieben werden kann" [9, p. 361].

transitivity in the following sense: for every two E, E' such that $EE = E$ and $E'E' = E'$ (hence, by the preceding proposition, for every two unities), there is an A with $AE = E'A = A$. This is a restriction compared with the now usual definition of a groupoid; as we have seen above, it was in this restricted sense that the term has been used by Schreier and Reidemeister.

Despite being published in the *Annalen*, Brandt's paper is only eight pages long and does not contain any "deep" theorem about the notion. After the definition, Brandt develops some basic groupoid theory by adapting in a more or less obvious way some notions from basic group theory (like homomorphisms, subgroups, factor groups and composition tables). While it would probably be easy to retrace these results to the original context of application (quaternary quadratic forms), it would be interesting to look for archival material explaining the fact that this paper has been accepted for publication in the *Annalen* at all.

The early reception of this paper by Brandt is visible through the *Jahrbuch*. Actually, there are only 22 hits for "Gruppoid"; these publications fall into the following categories:[53]

- 3 on groupoids in general: Brandt 1926, 1927, Richardson 1939;
- 1 concerning the fundamental groupoid: Reidemeister 1932 (in the Schreier review, the term is not present);
- 7 concerning applications of Brandt's notion in the theory of algebras and their ideals: Brandt 1928 (2 papers), Hasse 1931, Deuring 1935, Chevalley 1936, Eichler 1938, Asano 1939;
- 3 concerning other applications: Schmidt 1927, Baer 1929, Suschkewitsch 1930;
- 8 which do not use the term groupoid in Brandt's sense, but in a different sense. 3 papers by Birkhoff concern the following notion: "every ordered pair of elements corresponds uniquely to an element of the system as its product; multiplication is associative, and there is a left and right unity".[54] The papers Hausmann and Ore 1937, Boruvka 1941, Dubreil 1941 and 1942 concern a "set in which a multiplicatively written operation is possible"[55], and finally Vandiver 1940 concerns commutative semi-groups, in modern terms.

(53) Most of the publications listed here are not included in the list of references of the present paper since they are only mentioned here for statistical purposes. The reader can easily identify them by doing the corresponding query on the *Jahrbuch* web page.

(54) "Jedem geordneten Paar von Elementen entspricht eindeutig ein Element des Systems als Produkt; die Multiplikation ist assoziativ, und es gibt eine beiderseitige Einheit"; JFM60.1086.09.

(55) "Menge, in der eine als Multiplikation geschriebene Operation ausführbar ist".

Thus, we find that the use of Brandt's notion in topology suggested by Schreier and Reidemeister was an isolated one and had no followers, at least as far as the *Jahrbuch* data and the period up to 1942 is concerned. The only field were the notion acquainted some recognition (expressed in a wider use) is the theory of algebras and their ideals. Moreover, there was a variety of differing usages of the term; although they were even less influential than Brandt's usage, this shows that his usage was far from being well-established by then.

4.4. Veblen and Whitehead

Brandt was not the only author generalizing the group concept in the direction of partially defined composition. This conception played a central role also in the generalization of the *Erlanger Programm* for the purposes of relativity theory undertaken by Veblen and Whitehead [105]; see [22] chapter 13. In this context, they arrive at the concept of a pseudo-group of transformations:

> A set of transformations will be called a *pseudo-group* if it satisfies the conditions:
> (i) *If the resultant of two transformations in the set exists it is also in the set.*
> (ii) *The set contains the inverse of each transformation in the set.*
> [105, p. 38]

Towards the end of their book, this concept is applied in the following way:

> With each point P of an underlying regular manifold there is associated a space $S(P)$, and all these spaces are isomorphic. A family of displacements is a set of transformations $S(P) \to S(Q)$ [fulfilling four conditions]. Thus, a family of displacements is a pseudo-group of isomorphisms between associated spaces. The standard method of determining a family of displacements is by means of a geometric object, such as an affine connection, which, together with a curve joining two points P and Q, determines a transformation $S(P) \to S(Q)$ [105, p. 91].

Without going into the details of this definition, we can note that what is defined, implicitly, is a functor on a space of paths. Let us note further that on p. 92, they introduce the holonomy ("holonomic") group as a group contained in this pseudo-group.

4.5. Steenrod, local systems and axiomatic homotopy theory

Another appearance of the fundamental groupoid can be found a bit later in Steenrod's "local systems". Steenrod develops this concept in [97] and then devotes a paper on its own to it, entitled "Homology with local coefficients" [98]. This work belongs to the prehistory of sheaf theory.[56] Steenrod does not cite [105], but this book presumably is well known to him when introducing his "local systems". Actually, [97, p. 117] refers to [111] for the concept of differentiable manifold who in turn refers to [105].

The situation is the following:[57] for a point x of an arc-wise connected space R, F_x denotes the fundamental group of R with respect to x; there are isomorphisms $\alpha_{xy} : F_x \to F_y$ determined by the classes α_{xy} of homotopic paths from x to y. Steenrod generalizes this situation by speaking of a "system of local groups" when for each x there is a group G_x and for each class of paths a_{xy} there is a group isomorphism $G_x \to G_y$. The reader will at once recognize the great similarity of this situation with the one found in Veblen-Whitehead; this makes an influence of Veblen-Whitehead on Steenrod's work even more probable.

On the other hand, Steenrod's work is closely related to work by Reidemeister. Actually, Steenrod cites five publications by Reidemeister, one of which, namely [88], contains the notion of "Überdeckung"; according to Steenrod, this notion is equivalent to homology with local coefficients, but Reidemeister's paper only came to his attention after his own paper had been written [97, p. 124].

In modern terms, a local system as defined above is actually a functor from the fundamental groupoid to the category of groups, but Steenrod does not use any of these terms. Concerning the fundamental groupoid, it is not clear whether Steenrod knew the groupoid terminology by that time. He could have known from Reidemeister. But not only does Reidemeister in [88] not speak about the fundamental groupoid, but moreover in his setting of "Überdeckungen", the fundamental groupoid is not easily visible, first of all because the setting is strictly combinatorial. For Reidemeister, a "Überlagerung" of an n-dimensional simplicial complex $\mathfrak{K}$ is another such complex $\mathfrak{U}$ each of whose cells lays over a unique simplex of $\mathfrak{K}$. In a "Überdeckung", the cells over a simplex form a group whose com-

(56) See [41], [42], [54], [22].

(57) A detailed account of the problem situation attacked by Steenrod (especially how Steenrod uses these systems to obtain local coefficients for homology) can be found in [23, p. 48f]. For our present purposes, it is sufficient to know that Steenrod is studying fibre bundles and that the coefficients come from the higher homotopy groups of the fibers. Accordingly, Steenrod in [99, p. 154] explains that homotopy groups form a special local system.

position corresponds in a certain way to the incidence relations in $\mathfrak{K}$, and it follows from the precise definition that all these groups are isomorphic. Reidemeister then explains how the incidence matrices of $\mathfrak{U}$ can be obtained from that group, and that they relate to the "Wegegruppe" in the same way exposed earlier by him for "Überlagerungen". All this presumably is re-translatable in the more structural setting of Steenrod. Groupoids are still absent from Steenrod's book on fibre bundles [99]. Local systems are now presented as "bundles of coefficients" (p. 155)—a setting actually closer to Reidemeister's—, and in this setting the role of paths is less obvious.

As concerns the terminology of functors and categories, as is well known, Steenrod was involved in Eilenberg's and Mac Lane's enterprise from an early stage on, but his paper on local systems was received January 1942, hence slightly before Eilenberg and Mac Lane's first writings on functors and categories could have induced him to use the term functor here.

The fundamental groupoid is explicitly mentioned in a presentation of local systems by Jean Frenkel (*Exposé* 10 in the *Séminaire Cartan* 1948/49).[58] Also Steenrod himself later used the fundamental groupoid explicitly in connection with local systems when sketching, with Eilenberg, a possible way to an axiomatization of homotopy theories following the model of their axiomatic homology theories. In the first 1952 edition of *Foundations of algebraic topology* one reads:

> In axiomatizing the homotopy groups one would need an additional basic concept, namely: the isomorphisms $\pi_q(X, A, x_0) \approx \pi_q(X, A, x_1)$ assigned to a homotopy class of paths in A from x_0 to x_1. One would deal not with single groups but with systems of groups connected by isomorphisms assigned to the fundamental groupoid of A [30, p. 49]

So Eilenberg and Steenrod thought local systems would be useful in axiomatizing homotopy theories (and eventually had adopted the terminology of fundamental groupoid by 1952, in agreement with Eilenberg's earlier use of the term in a Bourbaki draft to be described below).

However, when an axiomatization of homotopy theories actually was achieved somewhat later by John Milnor [60], local systems played no longer a role. Jean-Pierre Marquis says the Eilenberg-Steenrod proposal "turned out to be inadequate" [59] but gives no further evidence for that claim. The passage quoted above from [30] is still present in the second printing

(58) See http://www.numdam.org/numdam-bin/feuilleter?j=SHC.

of 1957, but a pointer to Milnor's paper is added. We will see in section 5.5 that Milnor's work eventually influenced later work generalizing the concept of fundamental group in an entirely different direction.

4.6. Groupoids in Bourbaki

The fundamental groupoid is not contained in any published part of the *Éléments de Mathématique*. But we can find traces of it in internal reports; and the groupoid concept actually is present in *Algèbre*. In the first 1942 edition of chapter I *(Structures algébriques)* of *Algèbre*, the very first definition reads as follows:

> We call an internal law of composition between elements of a set E a function f of a subset A of $E \times E$ into E.[59]

Thus, the Bourbaki members found it useful, by 1942, to provide for operations only partially defined (i.e., only defined on a subset of $E \times E$). The discussion is visible in the corresponding internal minutes, *La Tribu n°5, Congrès de Clermont (7-10 Décembre 1940)*. Concerning §1 of *livre II: Algèbre*, there is a handwritten commentary saying "provide for laws not everywhere defined, see groupoids, Fastringe and so on" ("prévoir lois non définies partout, cf. groupoïdes, Fastringe, etc.")[60] So far, I was not able to identify the hand. It is worth noting, in this context, that Claude Chevalley was among the authors contributing in the 1930s to the applications of Brandt's groupoid in the theory of algebras and their ideals ([21]; see above). It is an interesting fact that the passage "d'une partie A" disappears in later editions and thus is also absent from the 1974 english translation based on the 1971 edition. I know of no archival material documenting this change of mind.

But there has also been implanted an exercise in *Algèbre* (still existing in later editions) presenting the concept of groupoid.[61] In this case, archival material shows that the exercise has been added into the typescript by cutting and gluing. The relevant *rédaction* is n°33 which actually exists in Nancy (Archives Delsarte) in two versions, 33 (1) and 33 (2). Unfortunately, only 33 (1) is available on line,[62] but the readers can convince themselves

(59) "On appelle loi de composition interne entre éléments d'un ensemble E une application f d'une partie A de $E \times E$ dans E" [7].

(60) See http://mathdoc.emath.fr/archives-bourbaki/PDF/nbt_006.pdf. Readers not acquainted with the Bourbaki archival material might wish to read http://mathdoc.emath.fr/archives-bourbaki/a_propos.php, and section 1 of [53].

(61) See §6 *exercice* 22 in ^{1}1942 and ^{2}1951, or §4 *exercice* 23 in the *nouvelle édition* (^{3}1970).

(62) See http://mathdoc.emath.fr/archives-bourbaki/consulter.php?id=033_iecnr_040

that 33 (1) differs in important respects from the published 1942 version. The version used for printing actually is 33 (2), and there, the exercise is glued in at the appropriate place.

In another unpublished Bourbaki draft, the fundamental groupoid is used; see *rédaction* n^{o}103 entitled "Rapport SEAW sur la topologie préhomologique".[63] This is one of the first contributions of Samuel Eilenberg to the Bourbaki project, written together with André Weil for a *Congrès Bourbaki* in April 1949.[64] The major part of the 82 typewritten pages of the report is concerned with the theory of fibre spaces; the text however starts with a short section I on groupoids. There is a definition of the groupoid concept, together with a pointer to the corresponding exercise in *Algèbre*. The definition is Brandt's, with the exception that Brandt's transitivity axiom is not any longer postulated for groupoids in general; groupoids having this additional transitivity property are called "connexe". It is then noted that paths in a space identified if homotopic form a groupoid (actually using the terminology "fundamental groupoid") which is "connexe" for path-wise connected spaces.[65]

On p. 69 of that manuscript, there is a combinatorial definition of paths composed of abstractly given vertices ("sommets") and edges ("arêtes"); interestingly, they then say:

> Let [$\sim$] be the strongest equivalence relation on the set of paths which is compatible with the law of composition [...] and such that: 1) if b is an edge with vertices a, a', then $(a, b, a', b, a) \sim (a)$; 2) if c is a face with vertices a, a', a'' and edges b, b', b'', then $(a, b, a', b', a'', b'', a) \sim (a)$. [...] the equivalence classes of paths form [...] a groupoid, the *fundamental groupoid* of the complex [...].[66]

Note that by this definition the relation $\sim$ is chosen to be the finest relation guaranteeing that 1- and 2-simplexes are homotopically trivial. Thus, homotopy is overtly build in the definition of the groupoid structure. This groupoid is "connexe" for the space B supposed connected and locally con-

(63) See http://mathdoc.emath.fr/archives-bourbaki/PDF/103_nbr_014.pdf.

(64) Concerning this dating hypothesis, a detailed argumentation can be found in [55].

(65) This generalization of Brandt's original definition is essential in a later application of the fundamental groupoid, namely Ronald Brown's extension of the Seifert-van Kampen theorem to spaces whose intersection is not path-wise connected. This application was first presented in Brown's *Elements of modern topology* [12]; see [13] for a survey.

(66) "Soit [$\sim$] la relation d'équivalence la plus stricte sur l'ensemble des chemins qui soit compatible avec la loi de composition [...] et telle que : 1) si b est une arête de sommets a, a', on ait $(a, b, a', b, a) \sim (a)$; 2) si c est une face de sommets a, a', a'' et d'arêtes b, b', b'', on ait $(a, b, a', b', a'', b'', a) \sim (a)$. [...] les classes d'équivalence de chemins forment [...] un groupoïde, le *groupoïde fondamental* du complexe [...]."

nected, and there are some results about representations of such a groupoid in a group, eventually applied to the group of a covering. From p. 71 on, the combinatorial setting is replaced by “la méthode des chemins”, including the “relèvement”, or more generally “chenilles” (which are defined on a simply connected space other than $[0, 1]$). The equivalence relation in this case again is chosen as “la plus stricte” compatible with the composition and fulfilling a supplementary condition, again guaranteeing homotopic triviality of certain simple parts of the space.[67]

From this data, we can conjecture that members of the *Séminaire Cartan* were aware of the Bourbaki report when reading Steenrod there, and therefore identified the fundamental groupoid implicit in Steenrod's construction of local systems. It is also probable that it was Eilenberg (who coauthored the report, after all) who introduced the explicit use of the fundamental groupoid in the discussion of local systems in Eilenberg-Steenrod.

5. Equivalence relations finer than Homotopy

5.1. The intuitive conception of continuous deformation

From today's point of view, the formally defined concept of homotopy of paths includes the homotopy of two different paths consisting of the same point set. In this section, I however intend to show that in the literature up to the first half of the twentieth century, continuous deformation of paths intuitively was conceived exclusively as a deformation of one point set into a different one. The methodological difficulty with this claim is the size of the corpus. The *Jahrbuch* has more than 3400 answers to the query “deform*”; this only shows how central the idea of deformation is to many fields of mathematics. But I see no simple way so far to pick out those relevant for my study. I thus cannot help to present just some more or less incidental findings which underpin my claim.

Let us see first how deformation of paths appears in Jordan's paper on the monodromy group [49, p. 101f]. In the following quote, x is the complex variable, and $y_1, \ldots, y_n$ are the solutions of the given equation.

> Let us now suppose that x, after having varied according to any determined law whatsoever, returns to its point of departure, and let us follow the variations of the integral y_1 and its derivatives. [...] The change produced on the system of inte-

(67) It would be interesting to compare in detail this report with the *exposés* on fibre space contained in the 1949/1950 *Séminaire Cartan*, in particular to check whether in the latter there were uses of the fundamental groupoid too.

grals $y_1, \ldots, y_n$ could be represented by a linear substitution [...]

Now, if we vary x in every possible way, in order to envelop successively the various singular points $\alpha, \beta \ldots$, we will obtain a certain number of substitutions the combination of which will form a group G.[68]

Jordan's wording does not necessarily imply the use of homotopy classes. The expression "varier suivant une loi déterminée quelconque" does not imply continuity of variation or variation along a path, and the expression "varier x de toutes les manières possibles, de manière à envelopper successivement les divers points singuliers" does not imply continuous deformation. It is just the intuitive idea to envelop the singularities. But my point here is not whether Jordan really considered noncontinuous paths and noncontinuous deformation or not; I just would like to stress that Jordan aims at exhausting the non-singular part of the plane, and in this situation it is certainly not useful to consider deformations of paths leaving the point set stable. This is what I would like to call the intended model of homotopy of paths; it will be shown to be assumed (sometimes tacitly, sometimes not) at key places in the literature under inspection.

We can find a similar but less detailed consideration of enveloping singularities in [71]; in [76], Poincaré very clearly presupposes that two different paths are different as a point set, since he says "they could be equivalent, that is they could bound a continuous area situated on D".[69] There is also evidence for the same phenomenon in Poincaré's writings on analysis situs. In [74], Poincaré says $M_0BM_0 \equiv 0$ if the closed contour M_0BM_0 constitutes the complete boundary of a 2-dimensional manifold contained in V [78, p. 241]. Also the following quote, taken from [75], shows that Poincaré thinks about deformations of one point set into another:[70]

When we write

$$C \equiv C'$$

(68) "Supposons maintenant que x, après avoir varié suivant une loi déterminée quelconque, revienne à son point de départ, et suivons les variations de l'intégrale y_1 et de ses dérivées. [...] L'altération produite sur le système des intégrales $y_1, \ldots, y_n$ pourra être représentée par une substitution linéaire [...]

Si maintenant nous faisons varier x de toutes les manières possibles, de manière à envelopper successivement les divers points singuliers $\alpha, \beta \ldots$, nous obtiendrons un certain nombre de substitutions dont la combinaison formera un groupe G."

(69) "ils pourront être *équivalents*, c'est-à-dire qu'ils pourront limiter une aire continue située sur D" [77, p. 91]

(70) The context of the passage is the one referred to in Nielsen's comment on the fifth complement discussed in section 3.3.

> we understand that the initial and endpoint of the closed cycle C is the same as that of C', and that there is a simply connected area between C and C' whose boundary consists of C and C'. In other words, we can pass from C to C' by making C vary in a continuous manner so that it always forms a single closed curve with fixed initial and endpoint. This is what we may call proper equivalence.[71]

According to Sarkaria, the subsequent notion of improper equivalence means free homotopy [90, p. 162]. Brouwer expressed much the same conception [11, p. 523].[72]

Now, while Poincaré still used many basic concepts of the theory of topological manifolds (including the concept of deformation) in an informal way, Dehn and Heegaard, as we have already stressed, replaced many of these informal conceptions by formally (and even strictly combinatorially) defined ones in [28]. We should ask whether there is still an intuitive residue in their treatment of deformation. Such a residue can be found in what they call the "Anschauungssubstrat". Recall that they postulate the equivalence of their purely combinatorial notion of homotopy and continuous deformability in the "Deformationsaxiom" (p. 169). Now, they do not define what continuous deformability means, but homotopy for them is an *external* transformation. This means the following (p. 164): let there be given on a manifold M_n a "Streckenkomplex" C_1^l constituted by "Strecken" $S_1^l = (P_0^i, P_0^k)$. Let there be given moreover a function mapping every point P_0^i to another point Q_0^i of M_n and every S_1^l to $T_1^l = (Q_0^i, Q_0^k)$; this gives another "Streckenkomplex" C_1''.[73] Now, suppose that one can join every two points P_0^i and Q_0^i by a "Strecke" U_1^i such that the *closed circle* $\{S_1^l, U_1^k, T_1^l, U_1^i\}$ bounds an elementary manifold. Then they say that C_1' is transformed into C_1'' by an external transformation. The fact that the condition involves a closed circle bounding a manifold together with the "Deformationsaxiom" means that deformation involves a change of point set.[74]

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(71) Quoted after [80, p. 200]. Original quote: "Quand nous écrivons

$$C \equiv C'$$

nous entendons que le point initial et final du cycle fermé C est le même que le point initial et final du cycle fermé C', et qu'il existe entre C et C' une aire simplement connexe dont la frontière complète est formé par C et C' en faisant varier C d'une manière continue de la façon que le cycle reste constamment formé d'une seule courbe fermée et que le point initial et final demeure invariable. C'est ce qu'on peut appeler l'*équivalence propre*" [78, p. 465f].

(72) We will come back on Brouwer's work in more detail in the next section. See also [102, p. 180].

(73) The indices represent dimensions.

(74) Van den Eynde gave a thorough conceptual comparison between the Dehn-

The "Anschauungssubstrat" by Dehn-Heegaard is also referred to by Tietze [101, p. 2]. Tietze underlines that "one will consider the idea of the cell system as taken from intuition".[75] He then notes that this "Anschauung" is drawn on in deductions to a large extent. Later on, he notes that "Anschauung" occasionally has been drawn on for the sake of clarity at places where a purely logical deduction would have been possible without difficulty either (p. 4). But which intuitive features are lost in the strictly combinatorial approach? As we have seen, Tietze stressed that the multivalued-functions context is among the things one loses. Things that flow directly from this context as long as it is present have to be proved as soon as it is absent. But in this context, deformations of a path inside its point set are of no use, since this changes nothing with respect to the singularities.

5.2. From paths to general mappings

The observations made in the preceding section incidentally motivate to focus a bit on an interesting aspect of the modern formal conceptual apparatus, concerning the very notion of path. Formally, a path in a space X is not just a point set but a continuous function $[0, 1] \to X$. The curve is determined and tied to the continuum of a real interval by a parametrization. This guarantees for the "continuity" and the "1-dimensionality" of the path, but also for the existence of a sense in which to run through it.[76] Historically, these features certainly were needed because paths are typically paths of integration. We have seen in what precedes that paths have been defined in this way by various protagonists of our story, *e.g.* Weyl and Schreier.

But once paths are defined as such mappings, one needs to know what a continuous deformation of such a mapping into another is. (Note that Dehn-Heegaard and still Veblen defined homotopy of complexes rather than of mappings, and saw paths as a special type of 1-cells.) And then one might start to think about continuous deformations of arbitrary continuous mappings into other arbitrary continuous mappings. This step has first been made by Brouwer [11, p. 527]. Vanden Eynde, after giving the corresponding quotation, comments:

Heegaard conception of homotopy and Jordan's [102, p. 165f]. But the results of this comparison do not alter the point I am making here.

(75) "die Vorstellung des Zellensystems [...] wird man [...] als der Anschauung entnommen [...] ansehen".

(76) Philosophically speaking, we could say that the mathematically precise definition of "path" seems to involve necessarily a sidestep to the real numbers. We do not understand what a one-dimensional continuum *in general* is; we only can recognize continua by relating them to a "sample continuum", the continuum of real numbers.

> The explicit statement which says that the position of a point is a continuous function of its initial position and the parameter is the mathematically rigorous definition of the deformation process. It marks the transition of the intuitive understanding of this process to a rigorously defined concept and allows for the extension of the homotopy concept from paths to maps in general. [102, p. 179]

I agree that it is the explicit statement mentioned by Vanden Eynde which "allows" for that extension, but this does not tell us anything about the motives for this extension. At least, the intuitive understanding of the process of deformation of a map in general is much less clear than in the case of a path. Anyway, Vanden Eynde describes in some detail how Brouwer uses the device for the study of mappings. He defines homotopy classes of maps, and shows in particular that his central tool for the study of mappings, the degree, characterizes the homotopy classes of maps of the 2-sphere in itself [102, p. 178f].

As soon as the general notion is available, the homotopy of paths becomes a simple (but still crucial) example of it. This approach is chosen, for instance, in the influential textbook by Seifert and Threlfall [95]: they first introduce the concept of a homotopy of two mappings in general (p. 113) and consider path homotopy as a special case of homotopy of mappings (p. 148).

This new conceptual situation had a variety of consequences. It made possible the introduction of higher homotopy groups by Hurewicz (and Čech independently of him). While Čech gave a recursive definition, Hurewicz defined them as fundamental groups of iterated loop spaces: $\pi_i(M) = \pi_1(\Omega^{i-1}M)$ [90, p. 144].[77] Another consequence is a stress put on mappings instead of spaces, yielding new concepts based on mappings and of fundamental importance for topology, like homotopy equivalence of spaces and homotopy invariance of functors. The general concept of homotopy equivalence of spaces was introduced by Hurewicz [43], [44] (see [103, p. 98]): Two spaces X and Y have the same homotopy type if there exist mappings $f : X \to Y$ and $g : Y \to X$ such that the composed mappings are homotopic to the identity mappings.

(77) To write the history of this generalization of the fundamental group would require a study of its own (a task in which other authors actually are more advanced than myself) and will thus not be extensively discussed in the present paper – even if (at least in the Hurewicz definition) it is also about structures on classes of paths. Recall, however, the role played by higher homotopy groups in one of the contexts where the fundamental groupoid is useful (section 4.5).

As to homotopy invariance of functors, I was not able to find out who first used this notion, and as a historical topic, this certainly needs further elucidation. But recall that one of the aims of this paper spelled out in the introduction is to find the true motivation to focus on homotopy of paths as the crucial equivalence relation on appropriate sets of paths of a space. Technically, the choice of homotopy as the crucial equivalence relation can be justified by the observation that typical topological tools used for the classification of spaces and similar tasks are homotopy-invariant in the functorial or some looser sense. Can this line of thought provide more than just a technical, namely a historical answer to our question?

Just to set the stage: Poincaré introduced the fundamental group precisely because it yields a finer invariant than the Betti numbers. Thus, in a rather loose, informal sense, homology is homotopy-invariant from the outset. Tietze, when describing the outcome of his paper [101], notes in particular that he has shown that from the fundamental group of a two-sided closed three-dimensional manifold, all other topological invariants known (by 1908) can be deduced (p. 2). But how about homotopy invariance in the technical sense? Recall Poincaré's definition of the fundamental group as a group of transformations acting on a set of functions from 1892 and 1895 discussed above. In [78, p. 190], he argues that the group actually does not depend on the functions chosen, but only on the hypersurface. From this he concludes: "The group G can then serve to define the form of the surface and it is called the group of the surface"[78] [80, p. 2]. And then he makes the following remark:

> It is clear that if two surfaces can each be transformed to the other by a continuous transformation, then their groups are isomorphic. The converse, though less evident, is again true for closed surfaces [78, p. 190].[79]

This passage in a typical way exhibits the difficulties in interpreting Poincaré's writings. The translation "continuous transformation" for "déformation continue" is debatable. If Poincaré means homeomorphism, his statement affirms the invariance up to homeomorphism of the fundamental group (and in this case the converse, contrary to Poincaré's second statement, would be wrong for 3-manifolds, see [2]). If he means isotopy, it affirms a preformal type of homotopy invariance (and the converse would be wrong *a fortiori*). But arguably Poincaré never distinguished clearly between homeo-

(78) "Le groupe G peut donc servir à définir la forme de la surface et s'appeler le groupe de la surface".

(79) Quoted after [80, p. 2]. Original quote: "Il est clair que si deux surfaces peuvent se transformer l'une dans l'autre par voie de déformation continue, leurs groupes sont isomorphes. La réciproque, quoique moins évidente, est encore vraie [...]."

morphism and isotopy. And then there is the problem that his notion of isomorphism is weaker than the now standard one, of course (see section 2.3). The fact that homotopy is not used otherwise in the 1892 paper but everything is expressed in the monodromy language rather runs against the stronger interpretation. But even if the stronger reading turns out to be an over-interpretation, it has become clear that the matter of homotopy invariance of functors deserves further historiographical attention.

To sum up, we find that the transition from paths as 1-cells to paths as functions is more than just a necessary step in the "transition of the intuitive understanding of [the deformation] process to a rigorously defined concept", as Vanden Eynde puts it. It is the decision to abandon the intuitive conceivability of such a process in favour of making it more tractable and easier to generalize by embedding it in a rich theoretical framework (given by the composition of arrows, in category-theoretic terms).

5.3. Paths and changes of parameter

The observations of section 5.1 notwithstanding, it is possible to define on one and the same point set different paths formally homotopic to each other. And these are not just "pathological" instances of the concept of homotopy of paths, but play a crucial role in the proof of the group property of the fundamental group, as we will see below. In order to get there, we shall first study the notion of paths related by a change of parameter.

We start this study with a discussion of Weyl's usage of the term "Weg" (path) in [107]. This usage deviates from the one now standard, in the sense that Weyl rather considers equivalence classes of paths in the latter sense as "Wege". Actually, we have seen before that what we call path now has been called "Kurve" by Weyl, and this term is used throughout Weyl's development of the theory of coverings at places where we would speak about paths. The term "Weg", on the other hand, has a different meaning which is not used throughout the book, but only appears at a very early stage, namely among the examples for definition by abstraction ("Definition durch Abstraktion"). This is a technique of forming new objects central not only to Weyl's book[80], but to modern mathematics as a whole as well; it consists in considering a particular equivalence relation on given objects, and the corresponding equivalence classes. For a modern reader, this might seem to be a quite common idea, but the way Weyl presents the idea incidentally shows that he did take greatest care to make his students understand it; after all, the lecture course was delivered in a time when axiomatics and

(80) Weyl's main use of the idea is to define the notion of "analytic form" ("analytisches Gebilde") by abstraction.

structural mathematics were quite young, and the average student far from being used to such an approach. Here is Weyl's definition of "Weg":

> A "motion" (of a point) is specified if the position of the moving point p is given at each instant λ of a certain time interval $\lambda_0 \leqslant \lambda \leqslant \lambda_1$: $p = p(\lambda)$. If one has two such motions, $p = p(\lambda)$, $q = q(\mu)$, then one says these motions travel the same "*path*" if and only if λ, the time parameter of the first motion, can be expressed as a continuous monotone increasing function of the time parameter μ of the second motion, $\lambda = \lambda(\mu)$, such that thereby the first motion becomes the second: $p(\lambda(\mu)) \equiv q(\mu)$. Here it is the concept of "path" which is to be defined [110, p. 6].[81]

At this place, Weyl in a note stresses that such a "Weg" is not to be confounded with the underlying point set:

> This concept is something more than that of the *point set* which consists of all points passed in the motion. We are concerned with the same distinction as, in the case of a pedestrian, that between the path traced (which, as long as he walks, is in statu nascendi) and the path (long since existing) *on which* he walks [110, p. 6].[82]

The equivalence relation Weyl uses for defining his concept of "Weg" actually is also studied in a textbook written by Newman [63][83]. In contrast with Weyl, however, Newman does not define a path to be an equivalence class with respect to this equivalence relation. Rather, he defines a path in a metrisable space S as a mapping $s(\tau)$ of the segment $[0, 1]$ into S (p. 175), thus in the way which is now standard. Newman also speaks about composition of paths (defined as usual; p. 176) and some of its properties, but without considering the totality of homotopy classes explicitly as a

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(81) Original passage: "Eine "Bewegung" (eines Punktes) ist gegeben, wenn die Lage des beweglichen Punktes p in jedem Moment λ eines gewissen Zeitintervalls $\lambda_0 \leqslant \lambda \leqslant \lambda_1$ gegeben ist: $p = p(\lambda)$. Hat man zwei solche Bewegungen $p = p(\lambda)$, $q = q(\mu)$, so sagt man dann und nur dann, diese Bewegungen durchlaufen denselben "*Weg*", falls λ, der Zeitparameter der ersten Bewegung, sich derart als stetige monoton wachsende Funktion des Zeitparameters μ der zweiten Bewegung ansetzen läßt: $\lambda = \lambda(\mu)$, daß dadurch die erste Bewegung in die zweite übergeht: $p(\lambda(\mu)) \equiv q(\mu)$. Hier ist es der Begriff des "Weges", der auf diese Weise definiert werden soll" [107, p. 7].

(82) Original passage: "Dieser Begriff meint etwas Anderes als die *Punktmenge*, welche aus allen während der Bewegung passierten Punkten besteht. Es handelt sich hier um den gleichen Unterschied wie zwischen dem von einem Fußgänger zurückgelegten Wege (der, solange der Fußgänger marschiert, in statu nascendi ist) und dem (seit langem existierenden) Wege, *auf dem* er marschiert" [107, p. 7].

(83) We quote from the second 1951 edition, but the quoted passages are already in the first 1939 edition.

group (probably in accordance with the very basic level of his text). It is with respect to the problem of non-associativity that the equivalence relation enters the stage. Newman first says what it means that two paths are related by change of parameter (defined pretty much like Weyl's equivalence of two "Bewegungen") and then proves that if s_1 and s_2 are related by a change of parameter, they are deformable into each other with fixed endpoints and inside the "track" (the point-set; p. 179). This theorem is commented on like this:

> The inconveniences that might be expected to arise from the non-associative addition of paths do not occur in view of this theorem, and the fact that it is *homotopies* [...] and not identities between paths that are interesting."

So in a way we have a careful analysis with a result not carefully spelled out: Newman finds presumably the finest equivalence relation on paths yielding associativity of composition but then does not make any use of this result since it is homotopy classes that are "interesting" anyway. Let us now see how this way of presenting the matter repeatedly can be found in the textbook literature.

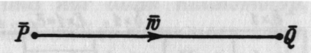
5.4. Homotopic paths and the group(oid) property

Our next step is to look at the role of the passage to homotopy classes of loops in the proof (or rather: the rhetoric surrounding it) of the group property of the fundamental group in various textbook presentations. I did not make a systematic inspection of the available textbook literature. A first observation to be made is that the full details of this proof (including the definition of the homotopies between the products of paths on the two sides of the group axioms) are very often left to the reader; we have seen in section 3.1 that the presentations in [104] and [57] belong to this class, and two more such cases will be discussed below (but also two examples for the contrary case). Put differently, there seem to be relatively few textbooks where an explicit discussion of the role of this passage can be found.

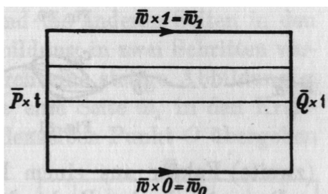
The example of [63] has been considered above; Newman was quite explicit about the details of the proof for associativity but did not discuss the other group axioms. A fuller discussion is contained in [95]. Seifert and Threlfall explicitly give the unit element and the inverse elements of the group; as to associativity, however, they succinctly say "the associative law is obviously valid"[84] (p. 153). But there is another aspect of their treatment worth to be discussed here. As we have noted above, Seifert and Threlfall

(84) "das assoziative Gesetz ist offenbar erfüllt".

consider homotopy of paths as a special case of homotopy of mappings. Actually, already in the more general context, they included an alternative description of a homotopy of mappings as a single mapping defined on the "Deformationskomplex". In the case of a homotopy of paths, this means the following (p. 150). A path for them is a continuous mapping w of an oriented straight line ("Strecke") $\overline{w}$ in a complex $\mathfrak{K}^n$. The homotopic deformation of such a path w_0 into a path w_1 is given by a set of continuous mappings $g_t : \overline{w} \to \mathfrak{K}^n$ for every value $0 \leqslant t \leqslant 1$ with the initial point and end point fixed. Now, this set of mappings g_t can be replaced by a single mapping f defined on the "Deformationskomplex" $\overline{w} \times \mathfrak{t}$ where $\mathfrak{t}$ is the unit line ("Einheitsstrecke") $0 \leqslant t \leqslant 1$. In particular, one can choose for $\overline{w} \times \mathfrak{t}$ a rectangle from the euclidean plane, the "deformation rectangle". This yields a new type of picture, called a "deformation rectangle":



Urbildstrecke
Fig. 77.



Deformationsrechteck
Fig. 78.

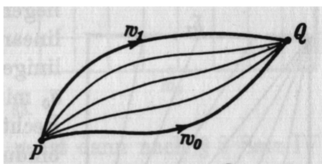


Abbildung in $\mathfrak{K}^n$
Fig. 79.

Note that their figure 79 displays a deformation actually affecting the point set of the path, hence a deformation like the ones in the "intended model" of continuous deformations of paths identified in section 5.1 above. Figure 78, however, would also apply to a different situation, namely a deformation leaving the point set invariant (like one by a simple change of parameter). We could say that by making available this more widely applicable type of illustration, Seifert and Threlfall extend the scope of an intuitive grasp of the concept of deformation to a type of instances where the original way of displaying the effect of the deformation by a picture (like in figure 79) would not apply. We will see that deformation rectangles have been used later precisely for this purpose.[85]

(85) Newman in [63] instead of deformation rectangles uses a different kind of figure

Let us now see two examples of textbooks whose authors clearly spelled out informal motivations in the context of this proof. Pontrjagin in his 1939 textbook on topological groups says the following:

> One should not think that the totality of all paths given in the space G forms a group. First, multiplication is not always possible. Moreover, the product does not satisfy the associative law, and the product of the path l by its inverse l^{-1} is not a null path; nor is the product of the path l by a null path the path l, but rather something new. Because of this the paths themselves will not interest us a great deal. What will be important for our purposes are the classes of equivalent or homotopic paths. Certain totalities of these classes also form a group, namely the fundamental group. [82, p. 218][86]

Thus, Pontrjagin stresses that composition on the paths themselves has not the desired properties, but that composition on homotopy classes of paths has. (A sketch of the proof, including the definition of the homotopies needed, is present in both editions of Pontrjagin's book.) At this stage, the reader of Pontrjagin's text might easily get the impression that this fact is precisely the reason for the passage to homotopy classes. One can get a similar impression when reading the textbook by Spanier [96]:

> We should like to form a category whose objects are the points of X, whose morphisms from x_0 to x_1 are the paths from x_0 to x_1, and with the composite defined to be the product path. With these definitions, neither axiom of a category is satisfied. That is, there need not be an identity morphism for each point, and it is generally not true that the associative law for product paths holds [...]. A category can be obtained, however, if the morphisms are defined not to be the paths themselves, but instead, homotopy classes of paths. [96, p. 46]

We notice at once the greater generality of Spanier's overall approach, couched in the language of category theory. Much like Pontrjagin, Spanier does not even mention the existence of relations like being related by a change of parameter, but immediately presents a proof of the fact that homotopy classes of paths form a groupoid. In contrast with many other presentations, this proof is given in great detail, providing both for exact

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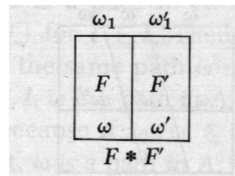
which however is "equivalent" to deformation rectangles as far as the scope of the displayable situations is concerned.

(86) It is worth noting that the second english edition of 1966 contains a similar passage but (due to the new translation) with a quite different wording [83, p. 346].

definitions of the homotopies needed,[87] and carefully made deformation rectangles (like in Fig.78 from [95] reproduced above). In fact, the "homotopies" used in Spanier's proof just could not be displayed by a picture like Fig.79 ibid. Here are the definitions and pictures for the particular assertions to be checked:

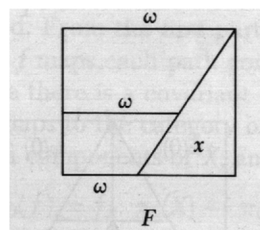
(1) Compatibility of path composition with homotopy:

$$(F * F')(t,t') = \begin{cases} F(2t,t') & 0 \leqslant t \leqslant 1/2 \\ F(2t-1,t') & 1/2 \leqslant t \leqslant 1 \end{cases}$$



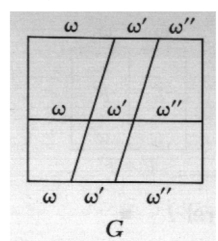
(2) Construction of identities: use the homotopy class of the constant map $\epsilon_x : [0,1] \mapsto x$. The proof that $\omega * \epsilon_x \cong \omega$ goes like this:

$$F(t,t') = \begin{cases} \omega\left(\dfrac{2t}{t'+1}\right) & 0 \leqslant t \leqslant \dfrac{t'+1}{2} \\ x & \dfrac{t'+1}{2} \leqslant t \leqslant 1 \end{cases}$$



(3) Proof of the associativity $(\omega * \omega') * \omega'' \cong \omega * (\omega' * \omega'')$:

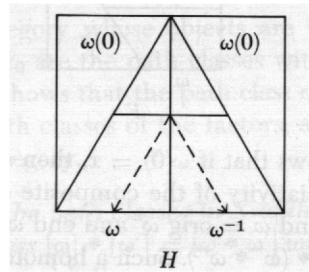
$$G(t,t') = \begin{cases} \omega\left(\dfrac{4t}{t'+1}\right) & 0 \leqslant t \leqslant \dfrac{t'+1}{4} \\ \omega'\left(4t - t' - 1\right) & \dfrac{t'+1}{4} \leqslant t \leqslant \dfrac{t'+2}{4} \\ \omega''\left(\dfrac{4t-2-t'}{2-t'}\right) & \dfrac{t'+2}{4} \leqslant t \leqslant 1 \end{cases}$$



(4) Existence of inverses: define $\omega^{-1}(t) := \omega(1-t)$; to prove that $[\omega^{-1}] = [\omega]^{-1}$, one shows that $\omega * \omega^{-1} \cong \epsilon_{\omega(0)}$:

[87] Recall that formally, a homotopy of two paths ω, ω' is a continuous function $F : [0,1] \times [0,1] \to X$ such that $F(t,0) = \omega(t)$, $F(t,1) = \omega'(t)$ for all $t \in [0,1]$.

$$H(t,t') = \begin{cases} \omega(0) & 0 \leqslant t \leqslant \dfrac{t'}{2} \\ \omega(2t-t') & \dfrac{t'}{2} \leqslant t \leqslant \dfrac{1}{2} \\ \omega(2-2t-t') & \dfrac{1}{2} \leqslant t \leqslant 1-\dfrac{t'}{2} \\ \omega(0) & 1-\dfrac{t'}{2} \leqslant t \leqslant 1 \end{cases}$$



What does this proof actually prove? If a path ω is related to another ω' by change of parameter in the sense of Weyl-Newman explained above,[88] we shall write $\omega \cong_{cp} \omega'$. This is clearly an equivalence relation (that's why Weyl spoke about it at all), actually much finer than homotopy. But the exclusive use of deformation rectangles reveals what an inspection of the formulæ confirms: Spanier's proof shows that the $\cong_{cp}$-classes of paths form a category, for

- composition of paths respects $\cong_{cp}$;
- $\omega * \epsilon_x \cong_{cp} \omega$;
- $(\omega * \omega') * \omega'' \cong_{cp} \omega * (\omega' * \omega'')$.

Does anyone care about the structure of the set of $\cong_{cp}$-classes of paths? We could say that this relation is not interesting since there is a problem with inversion:

$$\omega * \omega^{-1} \ncong_{cp} \epsilon_{\omega_0}.$$

But we still can have

$$\omega * \omega^{-1} \sim \epsilon_{\omega_0}$$

for a relation $\sim$ coarser than $\cong_{cp}$, but much finer than homotopy, e.g. "point set-stable deformation" (deform only inside the point set; two homotopic paths in general are not point set-stably homotopic, but two paths in the same $\cong_{cp}$-class are).

It is interesting that the textbooks so unanimously skip such finer relations and rather step immediately to "true" homotopies (namely homotopies in agreement with what has been called the "intended model" above). This

(88) *I.e.*, if there is a homeomorphism

$$\tau : [0,1] \to [0,1] \text{ with } \tau(0)=0, \tau(1)=1$$

such that

$$\forall t \in [0,1] : \omega(t) = \omega'(\tau(t))$$

notwithstanding, such finer relations $\sim$ later on indeed *have* been used, as we will see in the next section.

5.5. "Thin" homotopies

We have seen in section 4 that the fundamental groupoid was at first supposed to play a role in the axiomatization of homotopy theories, but that this axiomatization as finally presented by Milnor took a somewhat different road. It is interesting, however, that Milnor's work much later influenced papers from theoretical physics by Barrett [6], who developed an approach in turn taken up by Mackaay and Picken.[89] Before entering a (necessarily very short) discussion of these recent papers, we should very briefly speak about holonomy groups.

Recall that Sarkaria among Poincaré's approaches to the fundamental group mentioned another one which hardly appeared so far in our study, namely the treatment of the fundamental group as a certain homology group. More than in the other cases, this seems to be a retrospective ascription, since the very concept of holonomy group was introduced only in the 1920s by Elie Cartan. The origin of the notion is Cartan's effort to put Riemannian geometries into the scope of a (suitably generalized) Erlangen program. We have seen a quote from his seminal paper [18] for Cartan's usage of the terminology "groupe fondamental" in this respect; in the sequel of this passage, Cartan very clearly explains the strategy of such a generalization. The problem of including Riemannian geometries into the Erlangen program might be very succinctly described thus: how to make compatible the local Riemannian data (tangent spaces) with Klein's global approach. Cartan's idea is to introduce a "connection" between the tangent spaces. Now, passing from one point of the space to another by different paths might yield different results on the level of the connection of the tangent spaces; Cartan's speaks about the "non-holonomy" of the space, and it is this non-holonomy which is measured by the holonomy group. In [17], Cartan had shown that such a group is given by infinitesimal closed paths; the construction was taken up by Veblen-Whitehead, as we have seen.

Now, let us try to see what happens in Barrett's and Mackaay-Picken's work. The central objects are bundles on a smooth base manifold M with structure group G (the "fundamental group" in Cartan's sense) and the holonomy map $H : \Omega M \to G$ defined on the loop space of M. Barrett's mathematical result (with applications in theoretical physics) is a reconstruction theorem allowing to reconstruct a fiber bundle and a connection from an H fulfilling some conditions [6, p. 1178]. Barrett relates his result

(89) I am indebted to Tim Porter for a pointer to Picken's work.

to earlier work, notably Milnor's, in which a similar result is contained in a topological (not differentiable) framework. We will not at present try to understand how such a consideration related to Milnor's goal of an axiomatization of homotopy theories. Rather let us see the consideration of a certain kind of homotopies in the work of Mackaay and Picken motivated by Barrett's construction:

> [...] the holonomy of a connection in a principal G-bundle [...], defined over a connected smooth manifold M assigns an element of G to each smooth (based) loop in M. [...] two homotopic loops have different holonomies in general. However, when there is a homotopy between the loops whose differential has rank at most 1 everywhere, the holonomies around the two loops are the same. We call these homotopies *thin* homotopies. One glance at any introductory book on algebraic topology shows that the homotopies used in the proof that the fundamental group obeys the group axioms are all thin (after smoothing at a finite number of non-differentiable points). Therefore the holonomy map descends to a group homomorphism from the *thin* fundamental group of M [...] to G. [58, p. 288]

In general, two different loops have different holonomies, but two thinly homotopic loops have the same holonomy. The relation "thinly homotopic" is a finer equivalence relation than just "homotopic". And Mackaay-Picken's pointer to "any introductory book" reminds us of the observation of the preceding section that the standard proof of the group property of the fundamental group applies to such finer equivalence relations. So there is by now a serious mathematical application of them.

6. Conclusions

The aim of the present paper spelled out in the introduction was to provide a test case for the significance of the historical category "structuralism", especially concerning the process (guided by "intuition", according to Bourbaki), of invention of new structures and identification of unexpected structures in more classical settings. This test case was obtained by focussing on the late introduction and comparatively marginal use of the notion of fundamental groupoid and the even later consideration of equivalence relations finer than homotopy of paths, in contrast with the "importance" or "success" of the notion of fundamental group.

It is relatively easy to enumerate a variety of reasons for that "importance" or "success". First of all, the group concept was well established before the notion of fundamental group was introduced, and it was (or quickly became) known to provide a rich theory (subgroups, generators and relations). Next, the notion of fundamental group and the corresponding theory was relevant for central problems in fields like knot theory, classification of 3-manifolds, structure of Lie groups etc. And then there are various different ways of presenting (and making use of) the fundamental group. By consequence, there is more to the concept of fundamental group than just the formal definition; when using the concept, one has also these background ideas in mind — and this explains the stability of the concept, as we have seen.

In contrast, the notion of groupoid neither was equally well established nor endowed with an equally rich theory when topologists started to work with it. The fundamental groupoid, in particular, was neither presented in various different ways nor used in many fields. Also, Brandt's notion of groupoid was too restricted to cover cases where one really needs the fundamental groupoid (Brown).

As to the case of the finer equivalence relations, we have seen that the conceptual situation changed considerably from the beginnings of our history in the 19th century to the present state of affairs. In the setting of the 19th century, the objects of study were properties of multivalued complex functions and similar analytic problems, and the tools were algebraic descriptions of the topological properties of the corresponding Riemann surfaces. The algebraic structure on the set of homotopy classes of paths proved useful for a study of the properties of these (hyper-)surfaces in a neighbourhood of singularities, but equivalence relations leaving the point set untouched were not interesting since the interesting things (singularities) are located outside the point set of the path. Accordingly, the crucial type of homotopy was homotopy between paths different as point sets, and the relevant pictures (visual representations) resembled Fig.79 from [95] reproduced above. Still in early 20th century algebraic topology, homotopy was considered as the crucial equivalence relation, probably because many topological tools are homotopy invariant; again, finer equivalence relations are not useful.

In the modern setting, an intuitive vision of a path as a special type of point set is no longer sufficient, but paths are now precisely described as continuous functions $\omega : [0,1] \to X$. In the proof that sets of such paths have some kind of algebraic structure, a special kind of homotopy is needed which yields the same point set without yielding the same path (the same ω). Consequently, the relevant pictures in this case are deformation rectangles like in Fig.78 of [95] or in Spanier's figures. By transition from paths as

1-cells to paths as functions, the intuitive conceivability of the deformation process has been abandoned in favour of embedding it in a rich theoretical framework (given by the composition of arrows, in category-theoretic terms).

Thus, the proof that there actually is an algebraic structure on the set of homotopy classes of paths only uses instances of homotopy of paths which are pathological with respect to the operations related to the originally intended analytic and topological applications (like exhaustion of the non-singular part of the space by continuous deformation). In a first phase of historical development, this state of affairs apparently has been taken as a necessary evil, the key interest being to prove the group property of the set of homotopy classes. In a second phase, one gradually became aware of the usefulness of group(oids) defined for other equivalence relations. Thus, a group structure was eventually discovered which was not used before (unlike the fundamental group in the guise of the monodromy group) — but became used later.

Hopefully these considerations make clear why I think it is helpful in historical investigations to focus on the ways mathematical concepts are used and to distinguish between formal extension and intended uses. In this way, the historian is no longer committed to resort to finesses like Bourbaki's "intuition" when encountering phenomena in the development of structural mathematics asking for historical explanation.

Acknowledgements. — This paper greatly benefited from the critical remarks made by the anonymous referees, the editors of the present volume, the participants of the 2011 Paris and Toulouse workshops on homotopy, and Philippe Nabonnand.

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