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Systole growth for finite area hyperbolic surfaces

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Abstract. — In this note, we observe that the maximum value achieved by the systole function over all complete finite area hyperbolic surfaces of a given signature (g, n) is greater than a function that grows logarithmically in terms of the ratio g/n.

Résumé. — Dans cette note, nous observons que le maximum de la fonction systole sur l'espace des surfaces hyperboliques complètes et d'aire finie de signature donnée (g, n) est plus grand qu'une fonction qui croît de façon logarithmique en g/n.

1. Introduction

Consider two natural integers g and n such that $2g - 2 + n$ is positive. For an orientable complete finite area hyperbolic surface M of signature (g, n), where g is the genus and n the number of cusps, the *systole* is defined as length of the (or a) shortest closed geodesic and denoted by $\mathrm{sys}(M)$.

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Because there is an upper bound on the systole which only depends on the topology of the surface, the quantity

$$\mathrm{sys}(g,n) :=$$

$$\sup\{\mathrm{sys}(M) \mid M \text{ is a finite area hyperbolic surface of signature } (g,n)\}$$

is finite and we are interested in its asymptotic behavior in terms of g and n. Observe that this supremum is in fact a maximum following Mahler's compactness theorem, see [5].

Schmutz Schaller [6, Theorem 14] proved the following upper bound

$$\mathrm{sys}(g,n) \leqslant 4 \operatorname{arccosh}\left(\frac{6g-6+3n}{n}\right) \tag{1.1}$$

for $n \geqslant 2$ and $(g,n) \neq (0,3)$ (see [1] for an alternate proof). For closed surfaces, Buser and Sarnak proved in [3] that there exists a constant $U > 0$ such that

$$\mathrm{sys}(g,0) \geqslant U \log g$$

for all genus $g \geqslant 2$.

In this note, using Buser and Sarnak's lower bound in the compact case and an easy packing argument, we prove that Schmutz Schaller's upper bound has indeed the correct asymptotic growth.

Theorem. — *For any $g \geqslant 2$ and $n \geqslant 1$,*

$$\mathrm{sys}(g,n) > U \log\left(\frac{g}{n+1}\right)$$

where U denotes Buser and Sarnak's constant.

This together with inequality (1.1) implies that for any sequence n_g of integers such that $\lim_{g\to\infty} g/n_g = +\infty$, the function $\mathrm{sys}(g,n_g)$ grows roughly like $\log(g/n_g)$.

Along the way we also prove the following inequalities.

Proposition. — *For all (g,n) such that $2g-2+n > 0$ and $n \leqslant 2$,*

$$\mathrm{sys}(g,n) < \mathrm{sys}(g,n+1).$$

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2. Systole vs. adding cusps

Fix a signature (g, n) with $g, n \geqslant 2$ and let M be a hyperbolic closed surface of genus g with n points $\{p_1, \dots, p_n\}$ and M' be a hyperbolic punctured surface with n cusps. Suppose that M' and $M \setminus \{p_1, \dots, p_n\}$ are conformally equivalent. We denote by $D_i := D(p_i, r)$ the disks in M of radius r centered at the p_is.

LEMMA 2.1. — *For any r such that the disks $D_1, \dots, D_n$ of radius r are embedded and pairwise disjoint,*

$$\mathrm{sys}(M') > \min\{\mathrm{sys}(M), 4r\}.$$

Proof. — A consequence of the Pick-Schwarz lemma – see for instance [2, Theorem 3.2] – is that the natural map $M' \to M$ is strictly contracting. So any closed geodesic of M' whose image under the map $M' \to M$ is not homotopically trivial has length strictly bounded from below by $\mathrm{sys}(M)$. We will prove that the remaining homotopy classes have geodesic representatives of length greater than $4r$. Among such classes, a class whose geodesic representative is of minimal length is necessarily simple as $g \geqslant 2$. Consider such a class in M'. This class defines a unique homotopy class in $M \setminus \{p_1, \dots, p_n\}$ which in turn is uniquely represented by a minimal piecewise geodesic in M where the singular points of the curve belong to $\{p_1, \dots, p_n\}$. The number of these singular points is at least two and so, as two points of the p_is are at distance at least $2r$, its length is at least $4r$. To conclude the proof, we again use that the natural map $M' \to M$ is strictly contracting. □

Using the same argument, we see that if M is a hyperbolic surface of signature (g, n) with $2g - 2 + n > 0$ and admitting an embedded open disk of radius $\frac{\mathrm{sys}(M)}{2}$ centered at a point p, then the unique hyperbolic surface M' of signature $(g, n+1)$ and conformally equivalent to $M \setminus \{p\}$ satisfies

$$\mathrm{sys}(M') > \mathrm{sys}(M).$$

Observe that the above discussion and the following lemma prove the proposition of the introduction.

LEMMA 2.2. — *On a hyperbolic surface M of genus g with at most two cusps, there always exists an embedded open disk of radius $\frac{\mathrm{sys}(M)}{2}$.*

Proof. — If there is no cusp, this is a straightforward consequence of the definition of systole.

So suppose that there is at least one cusp on the surface and proceed by contradiction. Consider a maximal embedded disk D and denote by p its center. The existence of such a disk is guaranteed by compactness (the thick part of the surface is compact) and the fact that there is an upper bound on the radius of an embedded disk. By assumption, its radius r is strictly less than $\frac{\mathrm{sys}(M)}{2}$. Due to the maximality of D, its boundary admits at least one point of self-intersection which we call a *self-bumping point* of the disk. Consider the geodesic loop formed by the two radii of the disk joining p to some self-bumping point. As its length is $2r < \mathrm{sys}(M)$, this *bumping loop* must be parallel to a cusp. If there is only one such bumping loop, it is easy to see that we get a contradiction: by moving our point p in the appropriate direction, one can increase the radius of our maximal embedded open disk. So there are at least two bumping loops. Now observe that because the disk is embedded, two such loops only intersect in p and thus both loops are non homotopic (see for instance the bigon criterion [4]). So both loops surround distinct cusps. In particular for the case of a surface with one cusp we have reached a contradiction. We now suppose for the sequel that M has two cusps.

Observe that the existence of a third self-bumping point is impossible, as there are no more cusps to surround. We will get a contradiction in the event that there are only two self-bumping loops on the maximal embedded disk by proving that we can move its center in an appropriate direction in order to increase its radius. Recall that two horocyclic neighborhoods of area 1 around the cusps are disjoint. Now consider the two geodesic loops γ_1, γ_2 based at p and surrounding the two cusps. Consider, for each of the neighborhoods, the unique shortest geodesic between the neighborhood and the point p entirely contained in the cylinder bounded by the cusp and the corresponding geodesic loop (see figure 1).

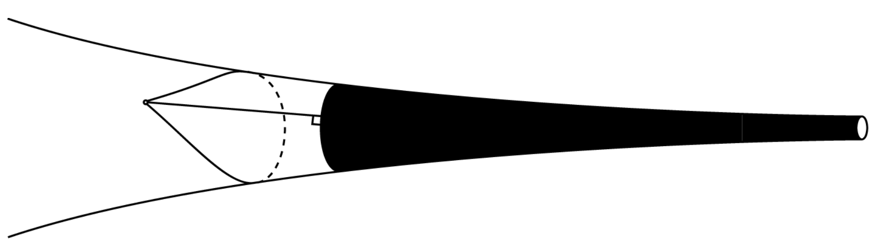

Figure 1. — A geodesic loop surrounding a cusp

For each loop γ_k, denote this distance d_k and its associated path c_k. Via standard hyperbolic trigonometry considerations, the geometry of this hyperbolic cylinder between one of the geodesic loops and the associated cusp is completely determined by either $\ell(\gamma_k)$ or d_k. More precisely, there exists a strictly increasing function that associates to the length $\ell(\gamma_k)$ the distance d_k. In particular, as $\ell(\gamma_1) = \ell(\gamma_2)$, we deduce that $d_1 = d_2$. Now consider the two distance realizing paths c_1 and c_2 emanating from p. They locally divide the surface around p into two parts, and as such one can move the point p in the appropriate part in order to increase its distance to each of the horocyclic neighborhoods. This leads to a contradiction. $\square$

We now prove our main theorem. It follows easily from the fact that $\mathrm{sys}(g,1) > \mathrm{sys}(g,0)$ and the following lower bound.

LEMMA 2.3. — *For all signature (g,n) with $g,n \geqslant 2$,*

$$\mathrm{sys}(g,n) > \min\left\{U \log g, 2\operatorname{arccosh}\left(\frac{2(g-1)}{n}+1\right)\right\}$$

where U denotes Buser and Sarnak's constant.

Proof. — Fix $n \geqslant 2$ and a maximal surface M of genus $g \geqslant 2$. By [3]

$$\mathrm{sys}(M) \geqslant U \log g.$$

Consider a maximal system $\{D_i\}_{i=1}^N$ of pairwise disjoint disks of radius

$$r := \min\left\{\frac{U}{4}\log g, \frac{1}{2}\operatorname{arccosh}\left(\frac{2(g-1)}{n}+1\right)\right\}.$$

Observe that $r \leqslant \frac{\mathrm{sys}(M)}{4}$ and denote by p_i the center of the disk D_i. By the maximality of such a system, M must be covered by the disks $\{2D_i\}_{i=1}^N$, where $2D_i$ denotes the disk of center p_i and radius $2r$. We derive that

$$N \cdot 2\pi(\cosh 2r - 1) \geqslant N \cdot A(2D_i) \geqslant A(M) = 4\pi(g-1),$$

so

$$N \geqslant n.$$

Now let M' be the unique surface with n cusps such that $M \setminus \{p_1, \dots, p_n\}$ and M' are conformally equivalent. We conclude by applying Lemma 2.1. $\square$

Remark that in particular we have

$$\mathrm{sys}(g, [g^\alpha]) \geqslant c_\alpha \log g,$$

for any $\alpha \in [0,1[$ with $c_\alpha := \min\{U, 2(1-\alpha)\}$.

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