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On the Configuration Spaces of Grassmannian Manifolds

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ABSTRACT. — Let $F^i_{h}(k, n)$ be the $i$-th ordered configuration space of all distinct points $H_1, \ldots, H_h$ in the Grassmannian $Gr(k, n)$ of $k$-dimensional subspaces of $C^n$, whose sum is a subspace of dimension $i$. We prove that $F^i_{h}(k, n)$ is (when non empty) a complex submanifold of $Gr(k, n)^h$ of dimension $i(n - i) + hk(i - k)$ and its fundamental group is trivial if $i = \min(n, hk)$, $hk \neq n$ and $n > 2$ and equal to the braid group of the sphere $CP^1$ if $n = 2$. Eventually we compute the fundamental group in the special case of hyperplane arrangements, i.e. $k = n - 1$.

RÉSUMÉ. — Soit $F^i_{h}(k, n)$ le $i$-ème espace de configuration ordonné de tous les points distincts $H_1, \ldots, H_h$ dans la Grassmannienne $Gr(k, n)$ de sous-espaces de dimension $k$ de $C^n$, dont la somme est un sous-espace de dimension $i$. Nous prouvons que $F^i_{h}(k, n)$ est (si non vide) une sous-variété complexe de $Gr(k, n)^h$ de dimension $i(n - i) + hk(i - k)$ et que son groupe fondamental est trivial si $i = \min(n, hk)$, $hk \neq n$ et $n > 2$ et égal au groupe de tresses de la sphère $CP^1$ si $n = 2$. Finalement, nous calculons le groupe fondamental dans le cas particulier des arrangements d’hyperplans, c’est-à-dire $k = n - 1$.

1. Introduction

Let $M$ be a manifold. The ordered configuration space

$$F_h(M) = \{(x_1, \ldots, x_h) \in M^h | x_i \neq x_j, \ i \neq j\}$$

of $h$ distinct points in $M$ has been widely studied after it has been introduced by Fadell and Neuwirth [5] and Fadell [3] in the sixties. It is well known that

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for a simply connected manifold $M$ of dimension greater or equal than 3, the pure braid group $\pi_1(F_h(M))$ on $h$ strings of $M$ is trivial. This is not the case when the dimension of $M$ is lower than 3 as, for example, the pure braid group of the sphere $S^2 \cong \mathbb{CP}^1$ with presentation:

$$\pi_1(F_h(\mathbb{CP}^1)) \cong \langle \alpha_{ij}, 1 \leq i < j \leq h - 1 \mid (YB3)_{h - 1}, (YB4)_{h - 1}, D^2_{h - 1} = 1 \rangle$$

where $D_k = \alpha_{12}(\alpha_{13}\alpha_{23})\alpha_{14}\alpha_{24}\alpha_{34} \cdots (\alpha_{1k}\alpha_{2k} \cdots \alpha_{k-1}k)$ and $(YB3)_n$ and $(YB4)_n$ are the Yang-Baxter relations (see [2] and [4]):

$$(YB3)_n: \quad \alpha_{ij}\alpha_{ik}\alpha_{jk} = \alpha_{ik}\alpha_{jk}\alpha_{ij} = \alpha_{jk}\alpha_{ij}\alpha_{ik}, \quad 1 \leq i < j < k \leq n,$$

$$(YB4)_n: \quad [\alpha_{kl}, \alpha_{ij}] = [\alpha_{il}, \alpha_{jk}] = [\alpha_{jl}, \alpha_{ik}^{-1}\alpha_{ik}\alpha_{jk}] = [\alpha_{jl}, \alpha_{kl}\alpha_{ik}\alpha_{kl}^{-1}] = 1,$$

where $1 \leq i < j < k < l \leq n$.

In a recent paper ([1]) Berceanu and Parveen introduced new configuration spaces. They stratify the classical configuration spaces $F_h(\mathbb{CP}^n)$ with complex submanifolds $F^i_h(\mathbb{CP}^n)$ defined as the ordered configuration spaces of all $h$ points in $\mathbb{CP}^n$ generating a projective subspace of dimension $i$. They prove that the fundamental groups $\pi_1(F^i_h(\mathbb{CP}^n))$ of these submanifolds are trivial except when $i = 1$ providing, in this last case, a presentation similar to those of the pure braid group of the sphere.

In a subsequent paper ([6]), authors apply similar techniques to the affine case, that is to the ordered configuration space $F^{i,n}_h = F^i_h(\mathbb{C}^n)$ of all $h$ points in $\mathbb{C}^n$ generating an affine subspace of dimension $i$. They prove that the spaces $F^{i,n}_h$ are simply connected except for $i = 1$ or $i = n = h - 1$ and, in the last cases, they provide a presentation of the fundamental groups $\pi_1(F^{i,n}_h)$.

In this paper we generalize the result in [1] to the Grassmannian manifold $Gr(k, n)$ parametrizing $k$-dimensional subspaces of $\mathbb{C}^n$. We define the $i$-th ordered configuration space $F^i_h(k, n)$ as the ordered configuration space of all distinct points $H_1, \ldots, H_h$ in the Grassmannian $Gr(k, n)$ such that the sum $(H_1 + \cdots + H_h)$ is an $i$-dimensional space.

We prove that the $i$-th ordered configuration space $F^i_h(k, n)$ is (when non empty) a complex submanifold of $Gr(k, n)^h$ and we compute its dimension.

As a corollary, we prove that if $n \neq hk$ and $i = \min(n, hk)$ then the $i$-th ordered configuration space $F^i_h(k, n)$ has trivial fundamental group except when $n = 2$, that is:

$$\begin{align*}
\pi_1(F^\min(n,hk)_h(k, n)) &= 0 \quad \text{if } (k, n) \neq (1, 2) \quad (1.1) \\
\pi_1(F^1_1(1, 2)) &= \pi_1(F_2(\mathbb{CP}^1)).
\end{align*}$$
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As a consequence, the fundamental group of the $i$-th ordered configuration space $\mathcal{F}_h^i(n-1, n)$ of hyperplane arrangements of cardinality $h$ vanishes except when $n = 2$.

Using a dual argument, we also get that the fundamental group of the ordered configuration space of all distinct $k$-dimensional subspaces $H_1, \ldots, H_h$ in $\mathbb{C}^n$ such that the intersection $(H_1 \cap \cdots \cap H_h)$ is an $i$-dimensional subspace is a simply connected manifold when $i = \max(0, n - hk)$, except when $n = 2$.

We conjecture that similar results to that obtained in [1] for projective spaces holds also for Grassmannian manifolds and the fundamental group of the $i$-th ordered configuration space $\mathcal{F}_h^i(k, n)$ vanishes except for low values of $i$. This will be the object of forthcoming publications.

2. Main Section

Let $Gr(k, n)$ be the Grassmannian manifold parametrizing $k$-dimensional subspaces of the $n$-dimensional complex space $\mathbb{C}^n$, $0 < k < n$, and $\mathcal{F}_h(Gr(k, n))$ be its ordered configuration spaces.

2.1. The spaces $\mathcal{F}_h^i(k, n)$

Let’s define the $i$-th ordered configuration space $\mathcal{F}_h^i(k, n)$ as the space of all distinct points $H_1, \ldots, H_h$ in the Grassmannian $Gr(k, n)$ whose sum is an $i$-dimensional subspace of $\mathbb{C}^n$, i.e.

$$\mathcal{F}_h^i(k, n) = \{(H_1, \ldots, H_h) \in \mathcal{F}_h(Gr(k, n)) \mid \dim(H_1 + \cdots + H_h) = i\}.$$ 

It is easy to see that the following results hold:

1. if $h = 1$ then $\mathcal{F}_h^1(k, n)$ is empty unless $i = k$, in which case $\mathcal{F}_1^k(k, n) = Gr(k, n)$;

2. if $i = 1$ then $\mathcal{F}_h^1(k, n)$ is empty unless $k = h = 1$ and we get $\mathcal{F}_1^1(1, n) = \mathbb{C}P^{n-1}$;

3. for $h \geq 2$, $\mathcal{F}_h^i(k, n) \neq \emptyset$ if and only if $i \geq k + 1$ and $i \leq \min(hk, n)$;

4. for $i = hk \leq n$, then the $h$ subspaces giving a point of $\mathcal{F}_h^{hk}(k, n)$ form a direct sum;

5. for $h \geq 2$, $\mathcal{F}_h(Gr(k, n)) = \prod_{i=2}^{n} \mathcal{F}_h^i(k, n)$;
6. for $h \geq 2$, the adjacency of the strata is given by

$$
\overline{F_i^h(k,n)} = F_i^h(k,n) \prod F_{i-1}^h(k,n) \prod \ldots \prod F_2^h(k,n).
$$

By above remarks, it follows that the case $h = 1$ is trivial, hence from now on, we will consider $h > 1$ (and hence $i > k$).

We want to show that $F_i^h(k,n)$ is (when non empty) a complex submanifold of $Gr(k,n)^h$ and compute its dimension. We need to briefly recall few easy facts and introduce some notations.

2.2. The determinantal variety

Let’s recall that the determinantal variety $D_r(m,m')$ is the variety of $m \times m'$ matrices with complex entries of rank less than or equal to $r \leq \min(m,m')$. It is an analytic (algebraic, in fact) variety of dimension $r(m + m' - r)$ whose set of singular points is given by those matrices of rank less than $r$. From now on, $D_r(m,m')^*$ will denote the set of non-singular points of the determinantal variety $D_r(m,m')$, that is the set of $m \times m'$ matrices of rank equal to $r$.

2.3. A system of local coordinates for $Gr(k,n)^h$

Let $V_0 \subset \mathbb{C}^n$ be a subspace of dimension $\dim V_0 = n - k$, then the set

$$
U_{V_0} = \{H \in Gr(k,n) \mid H \oplus V_0 = \mathbb{C}^n\}
$$

is an open dense subset of $Gr(k,n)$.

Let $B = \{w_1, \ldots, w_k, v_1, \ldots, v_{n-k}\}$ be a basis of $\mathbb{C}^n$ such that $\{v_1, \ldots, v_{n-k}\}$ is a basis of $V_0$. We get a (complex) coordinate system on $U_{V_0}$ as follows.

Let $H$ be an element in $U_{V_0}$, then the affine subspace $V_0 + w_j$ intersects $H$ in one point $u_j$ for any $j = 1, \ldots, k$ and $\{u_1, \ldots, u_k\}$ form a basis of $H$. Hence $H$ is uniquely determined by a $n \times k$ matrix of the form $\left( \begin{array}{c} I \\ A \end{array} \right)$, where $I$ is the $k \times k$ identity matrix and $A$ is the $(n-k) \times k$ matrix of the coordinates of $u_1 - w_1, \ldots, u_k - w_k$ with respect to vectors $\{v_1, \ldots, v_{n-k}\}$. The coefficients of $A$ give complex coordinates in $U_{V_0} \cong \mathbb{C}^{k(n-k)}$.

Let $(H_1, \ldots, H_h)$ be a point in $Gr(k,n)^h$, the open sets $U_{H_1}, \ldots, U_{H_h}$ in the Grassmannian manifold $Gr(n-k,n)$ have non empty intersection, that is there exists an element $V_0 \in Gr(n-k,n)$ such that $V_0 \oplus H_j = \mathbb{C}^n$ for all
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$j = 1, \ldots, h$. Thus, $Gr(k, n)^{h}$ is covered by the open sets $U_{V_0}^h$ as $V_0$ varies in $Gr(n - k, n)$. Taking a basis as defined above, each element in $U_{V_0}^h$ is uniquely determined by a $n \times h k$ matrix of the form

$$
\begin{pmatrix}
I & I & \cdots & I \\
A_1 & A_2 & \cdots & A_h
\end{pmatrix}
$$

and the coefficients of $(A_1, A_2, \ldots, A_h)$ give complex coordinates in $U_{V_0}^h \cong \mathbb{C}^{hk(n-k)}$.

2.4. A system of local coordinates for $F_i^h(k, n)$

In terms of the above coordinates, $(H_1, \ldots, H_h) \in U_{V_0}^h$ belongs to $F_i^h(k, n)$ if and only if $A_j \neq A_l$ when $j \neq l$ and rank

$$
\begin{pmatrix}
I & I & \cdots & I \\
A_1 & A_2 & \cdots & A_h
\end{pmatrix}
$$

$= i$. Let us remark that

\[
\text{rank}
\begin{pmatrix}
I & I & \cdots & I \\
A_1 & A_2 & \cdots & A_h
\end{pmatrix}
= \text{rank}
\begin{pmatrix}
I & I & \cdots & I \\
0 & A_2 - A_1 & \cdots & A_h - A_1
\end{pmatrix}
= k + \text{rank} (A_2 - A_1 \cdots A_h - A_1).
\]

Then the coefficients of $B_j = A_j - A_1$ are new coordinates, in which the intersection $U_{V_0} \cap F_i^h(k, n)$ corresponds, in $\mathbb{C}^{hk(n-k)}$, to the product $\mathbb{C}^{k(n-k)} \times D_{1-k}(n - k, hk - k)^*$ minus the closed sets given by $B_j = 0$ for $2 \leq j \leq h$ and by $B_j = B_l$ for $2 \leq j, l \leq h, j \neq l$. We get the following theorem.

**Theorem 2.1.** — The $i$-th ordered configuration space $F_i^h(k, n)$ is a complex submanifold of the Grassmannian manifold $Gr(k, n)$ of dimension

$$
d_i^h(k, n) = i(n - i) + hk(i - k). \quad (2.2)
$$

Equation (2.2) is an easy consequence of the equality:

$$
k(n - k) + (i - k)(n - k + hk - k - (i - k)) = i(n - i) + hk(i - k).
$$

Let us remark that the dimension $d_i^h(k, n)$ attains its maximum $hk(n - k)$ if and only if $i = n$ or $i = hk$. Hence $d_i^h(k, n)$ is a strictly increasing function of $i$ when $i \leq \min(n, hk)$.

2.5. The fundamental group of $F_{h}^{\min(n,hk)}(k, n)$

The space $F_{h}^{\min(n,hk)}(k, n)$ is an open subset of the ordered configuration space $F_{h}(Gr(k, n))$ and all other (non void) $F_{h}^{j}(k, n)$ have strictly lower dimension. Moreover, if $i = n$ the difference of dimensions $d_i^h(k, n) - d_{i-1}^h(k, n)$
equals $1 + hk - n$ and if $i = hk$ it equals $1 + n - hk$. Then if $n \neq hk$, all (non void) $F_h^j(k,n)$ with $j < \min(n,hk)$ have real codimension at least 4 in $F_h(Gr(k,n))$. Then, if $n \neq hk$ and $i = \min(n,hk)$, the fundamental group of $F_h^j(k,n) = F_h(Gr(k,n)) \setminus F_h^{j-1}(k,n)$ is the same as the fundamental group of $F_h(Gr(k,n))$ (since, by the adjacency of the strata, the closure $F_h^{j-1}(k,n)$ is the finite union of complex subvarieties of $F_h(Gr(k,n))$ of real codimension at least 4).

Let us recall that the complex Grassmannian manifolds $Gr(k,n)$ are simply connected and have real dimension at least 4 except $Gr(1,2) = \mathbb{CP}^1$ and that for a simply connected manifold of real dimension at least 3 the pure braid groups vanish, i.e. $\pi_1(F_h(Gr(k,n))) = 0$ if $(k,n)$ $\neq (1,2)$. We get the following corollary.

**Corollary 2.2.** — The fundamental group of the $i$-th ordered configuration space $F_i^h(k,n)$ vanishes if $n \neq hk$ and $i = \min(n,hk)$ except when $n = 2$ in which it is the pure braid group of the sphere.

### 2.6. The dual case

Let $Gr(k,n)^*$ be the Grassmannian manifold parametrizing $k$-dimensional subspaces in the dual space $\mathbb{C}^n^*$. Then we can define the $i$-th dual ordered configuration space $F_i^h(k,n)^*$ as

$$F_i^h(k,n)^* = \{(H_1, \ldots, H_h) \in F_h(Gr(k,n)^*) \mid \dim(H_1 \cap \cdots \cap H_h) = i\}.$$ 

The spaces $F_i^h(k,n)^*$ stratify the ordered configuration space $F_h(Gr(k,n)^*)$ of the Grassmannian manifold $Gr(k,n)^*$.

The annihilators define homeomorphisms $Ann: Gr(n-k,n) \to Gr(k,n)^*$ which induce homeomorphisms between the $(n-i)$th ordered configuration space $F_{h-i}^{n-i}(n-k,n)$ and the $i$-th dual ordered configuration space $F_i^h(k,n)^*$. As a consequence the spaces $F_{h}^{\max(0,n-hk)}(n-k,n)^*$ are simply connected manifolds except when $n = 2$. In this case the fundamental group is the pure braid group of the sphere.

### 2.7. $i$-th ordered configuration spaces of hyperplane arrangements

If $k = n - 1$ points in the ordered configuration space $F_h(Gr(n-1,n))$ are $h$-uple of hyperplanes in $\mathbb{C}^n$, i.e. ordered arrangements of hyperplanes. In this case, $h = 1$ implies $i = n - 1$ and the $i$-th ordered configuration space is the Grassmannian manifold, i.e. $F_1^{n-1}(n-1,n) = Gr(n-1,n)$. While $h > 1$ implies $i = n$, since the sum of two (different) hyperplanes is
the whole space $\mathbb{C}^n$, and the following equalities hold

$$F_{n}^{h}(n - 1, n) = F_{h}(Gr(n − 1, n)) = F_{h}(\mathbb{C}P^{n-1}).$$

Hence, the fundamental group of the $i$-th ordered configuration space of hyperplane arrangements $F_{h}^i(n - 1, n)$ vanishes except when $n = 2$. In this case it is the fundamental group of the sphere $\mathbb{C}P^1$.

In the dual case there are homeomorphisms $F_{h}^i(n - 1, n)^* \cong F_{h}^{n-i}(1, n)$ and fundamental groups of $F_{h}^{n-i}(1, n)$ are zero except if $i = n - 1$ (see [1]). Hence the space of $h$-uples of distinct hyperplanes in $\mathbb{C}^n$ whose intersection has dimension equal to $i$ is simply connected except if $i = n - 1$.

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