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On the arithmetic Chern character

H. Gillet[1], C. Soulé[2]

Abstract. — We consider a short sequence of hermitian vector bundles on some arithmetic variety. Assuming that this sequence is exact on the generic fiber we prove that the alternated sum of the arithmetic Chern characters of these bundles is the sum of two terms, namely the secondary Bott Chern class of the sequence and its Chern character with support on the finite fibers.
Next, we compute these classes in the situation encountered by the second author when proving a "Kodaira vanishing theorem" for arithmetic surfaces.

Résumé. — Nous considérons une suite courte de fibrés vectoriels hermitiens sur une variété arithmétique. Quand cette suite est exacte sur la fibre générique nous montrons que la somme alternée des caractères de Chern arithmétiques de ces fibrés est la somme de deux termes : la classe secondaire de Bott-Chern de la suite et son caractère de Chern à support dans les fibres finies. Nous calculons ces deux termes dans la situation rencontrée par le second auteur dans sa preuve d'un 'théorème d'annulation de Kodaira' pour les surfaces arithmétiques.

Let X be a proper and flat scheme over $\mathbb{Z}$, with smooth generic fiber $X_{\mathbb{Q}}$. In [4] we attached to every hermitian vector bundle $\overline{E} = (E, \| \,\|)$ on X a Chern character class lying in the arithmetic Chow groups of X:

$$\widehat{\mathrm{ch}}(\overline{E}) \in \bigoplus_{p \geqslant 0} \widehat{\mathrm{CH}}^p(X) \otimes \mathbb{Q}\,.$$

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Unlike the usual Chern character with values in the ordinary Chow groups, $\widehat{\mathrm{ch}}$ is not additive on exact sequences; indeed suppose that $\overline{E}_i$, $i=0,1,2$ is a triple of hermitian vector bundles on X, and that we are given an exact sequence

$$0 \to E_0 \to E_1 \to E_2 \to 0$$

of the underlying vector bundles on X, (*i.e.* in which we ignore the hermitian metrics). Then the difference $\widehat{\mathrm{ch}}(\overline{E}_0) + \widehat{\mathrm{ch}}(\overline{E}_2) - \widehat{\mathrm{ch}}(\overline{E}_1)$, is represented by a secondary characteristic class $\widetilde{\mathrm{ch}}$ first introduced by Bott and Chern [1] and defined in general in [2]. These Bott-Chern forms measure the defect in additivity of the Chern forms associated by Chern-Weil theory to the hermitian bundles in the exact sequence.

Assume now that the sequence

$$0 \to E_0 \to E_1 \to E_2 \to 0 \qquad (*)$$

is exact on the generic fiber $X_{\mathbb{Q}}$ but not on the whole of X. We shall prove here (Theorem 1) that $\widehat{\mathrm{ch}}(\overline{E}_0)+\widehat{\mathrm{ch}}(\overline{E}_2)-\widehat{\mathrm{ch}}(\overline{E}_1)$ is the sum of the class of $\widetilde{\mathrm{ch}}$ and the localized Chern character of $(*)$ (see [3], 18.1). This result fits well with the idea that characteristic classes with support on the finite fibers of X are the non-archimedean analogs of Bott-Chern classes (see [6]).

In Theorem 2 we compute more explicitly these secondary characteristic classes in a situation encountered when proving a "Kodaira vanishing theorem" on arithmetic surfaces ([7], 3.3).

Notation. — If A is an abelian group we let $A_{\mathbb{Q}} = A \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

1. A general formula

1.1. Let $S = \mathrm{Spec}(\mathbb{Z})$ and $f : X \to S$ a flat scheme of finite type over S. We assume that the generic fiber $X_{\mathbb{Q}}$ is smooth and equidimensional of dimension d. For every integer $p \geqslant 0$ we denote by $A^{pp}(X_{\mathbb{R}})$ the real vector space of smooth real differential forms α of type (p,p) on the complex manifold $X(\mathbb{C})$ such that $F_\infty^*(\alpha) = (-1)^p\,\alpha$, where F_∞ is the anti-holomorphic involution of $X(\mathbb{C})$ induced by complex conjugation. Let

$$A(X) = \bigoplus_{p\geqslant 0} A^{pp}(X_{\mathbb{R}})$$

and

$$\widetilde{A}(X) = \bigoplus_{p\geqslant 1} A^{p-1,p-1}(X_{\mathbb{R}})/(\mathrm{Im}(\partial)+\mathrm{Im}(\overline{\partial}))\,.$$

For every $p \geqslant 0$ we let $\widehat{\mathrm{CH}}_p(X)$ be the p-th *arithmetic Chow homology* group of X ([5], §2.1, Definition 2). Elements of $\widehat{\mathrm{CH}}_p(X)$ are represented by pairs (Z, g) consisting of a p-dimensional cycle Z on X and a Green current g for $Z(\mathbb{C})$ on $X(\mathbb{C})$. Recall that here a Green current for $Z(\mathbb{C})$ is a current (*i.e.* a form with distribution coefficients) of type $(d-p-1, d-p-1)$ such that $dd^c(g) + \delta_Z$ is C^∞, δ_Z being the current of integration on $Z(\mathbb{C})$. There are canonical morphisms ([5], 2.2.1):

$$\begin{array}{rcl} z : \widehat{\mathrm{CH}}_p(X) & \to & \mathrm{CH}_p(X) \\ (Z, g) & \mapsto & Z \end{array}$$

and

$$\begin{array}{rcl} \omega : \widehat{\mathrm{CH}}_p(X) & \to & A^{d-p,d-p}(X_{\mathbb{R}}) \\ (Z, g) & \mapsto & dd^c(g) + \delta_Z . \end{array}$$

Let $\mathrm{CH}_p^{\mathrm{fin}}(X)$ be the Chow homology group of cycles on X the support of which does not meet $X_{\mathbb{Q}}$. There is a canonical morphism

$$b : \mathrm{CH}_p^{\mathrm{fin}}(X) \to \widehat{\mathrm{CH}}_p(X)$$

mapping the class of Z to the class of $(Z, 0)$. The composite morphism

$$z \circ b : \mathrm{CH}_p^{\mathrm{fin}}(X) \to \mathrm{CH}_p(X)$$

is the obvious map. Let

$$a : A^{d-p-1,d-p-1}(X_{\mathbb{R}}) \to \widehat{\mathrm{CH}}_p(X)$$

be the map sending η to the class of $(0, \eta)$. We have

$$\omega \circ a(\eta) = dd^c(\eta) .$$

1.2. We assume given a sequence

$$0 \to \overline{E}_0 \to \overline{E}_1 \to \overline{E}_2 \to 0$$

of hermitian vector bundles on X, the restriction of which to $X_{\mathbb{Q}}$ is exact. Let

$$\mathrm{ch}^{\mathrm{fin}}(E_\bullet) \cap [X] \in \mathrm{CH}_{\mathrm{fin}}(X)_{\mathbb{Q}} = \underset{p \geqslant 0}{\oplus} \ \mathrm{CH}_p^{\mathrm{fin}}(X)_{\mathbb{Q}}$$

be the *localized Chern character* of $E_\bullet$ ([3] 18.1), and

$$\widetilde{\mathrm{ch}}(\overline{E}_\bullet) \in \widetilde{A}(X)_{\mathbb{Q}}$$

the Bott-Chern secondary characteristic class [2], such that

$$dd^c\,\widetilde{\mathrm{ch}}(\overline{E}_\bullet) = \sum_{i=0}^{2}(-1)^i\,\mathrm{ch}(\overline{E}_{i,\mathbb{C}})\,,$$

where $\mathrm{ch}(\overline{E}_{i,\mathbb{C}}) \in A(X)$ is the differential form representing the Chern character of the restriction $E_{i,\mathbb{C}}$ of E_i to $X(\mathbb{C})$. Finally, if $i = 0,1,2$, we let

$$\widehat{\mathrm{ch}}(\overline{E}_i) \cap [X] \in \widehat{\mathrm{CH}}(X)_{\mathbb{Q}} = \underset{p \geqslant 0}{\oplus}\ \widehat{\mathrm{CH}}_p(X)_{\mathbb{Q}}$$

be the *arithmetic Chern character* of $\overline{E}_i$ ([4] 4.1, [5] Theorem 4).

Theorem 1.1. — *The following equality holds in* $\widehat{\mathrm{CH}}(X)_{\mathbb{Q}}$*:*

$$\sum_{i=0}^{2}(-1)^i\,\widehat{\mathrm{ch}}(\overline{E}_i) \cap [X] = b\,(\mathrm{ch}^{\mathrm{fin}}(E_\bullet) \cap [X]) + a\,(\widetilde{\mathrm{ch}}(\overline{E}_\bullet))\,.$$

1.3. This theorem is a special case of Lemma 21 in [5], though this may not be immediately apparent. Therefore, for the sake of completeness, we give a proof here.

1.4. To prove Theorem 1.1 we consider the Grassmannian graph construction applied to $E_\bullet$ ([3] 18.1, [5] 1.1). It consists of a proper surjective map

$$\pi : W \to X \times \mathbb{P}^1$$

such that, if $\phi \subset X$ is the support of the homology of $E_\bullet$ (hence $\phi_{\mathbb{Q}}$ is empty), the restriction of π onto $(X-\phi)\times\mathbb{P}^1$ and $X \times \mathbb{A}^1$ is an isomorphism. The effective Cartier divisor

$$W_\infty = \pi^{-1}(X \times \{\infty\})$$

is the union of the Zariski closure $\widetilde{X}$ of $(X-\phi)\times\{\infty\}$ with $Y = \pi^{-1}(\phi \times \{\infty\})$. The sequence $E_\bullet$ extends to a complex

$$0 \to \widetilde{E}_0 \to \widetilde{E}_1 \to \widetilde{E}_2 \to 0\,,$$

which is isomorphic to the pull-back of $E_\bullet$ over $X \times \mathbb{A}^1$. The restriction of $\widetilde{E}_\bullet$ to $\widetilde{X}$ is canonically split exact. On $W_{\mathbb{Q}} = X_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ the sequence $\widetilde{E}_\bullet$ is exact; it coincides with $E_\bullet$ (resp. $0 \to E_0 \to E_0 \oplus E_2 \to E_2 \to 0$) when restricted to $X_{\mathbb{Q}} \times \{0\}$ (resp. $X_{\mathbb{Q}} \times \{\infty\}$). We choose a metric on $\widetilde{E}_\bullet$ for which these isomorphisms are isometries.

1.5. Let

$$x = \sum_{i=0}^{2} (-1)^i \, \widehat{\mathrm{ch}}(\overline{\widetilde{E}}_i)\,,$$

and denote by t the standard parameter of $\mathbb{A}^1$. In the arithmetic Chow homology of W we have

$$0 = x \cap (W_0 - W_\infty, -\log|t|^2)\,.$$

If x is the class of (Z, g), with Z meeting properly W_0 and W_∞, we get

$$x \cap (W_0 - W_\infty, -\log|t|^2) = (Z \cap (W_0 - W_\infty), g * (-\log|t|^2))\,,$$

where the $*$-product is equal to

$$g * (-\log|t|^2) = g(\delta_{W_0} - \delta_{W_\infty}) - \mathrm{ch}(\overline{\widetilde{E}}_\bullet)\log|t|^2\,.$$

Since $W_\infty = \widetilde{X} \cup Y$, with $Y_{\mathbb{Q}} = \emptyset$, we get

$$\begin{aligned} 0 &= x \cap (W_0 - W_\infty, -\log|t|^2) \qquad (1.1)\\ &= (Z \cap W_0, g\,\delta_{W_0}) - (Z \cap \widetilde{X}, g\,\delta_{\widetilde{X}}) - (Z \cap Y, 0) - (0, \mathrm{ch}(\overline{\widetilde{E}}_\bullet)\log|t|^2)\,. \end{aligned}$$

The restriction of $\overline{\widetilde{E}}_\bullet$ to $\widetilde{X}$ is split exact, therefore

$$(Z \cap \widetilde{X}, g\,\delta_{\widetilde{X}}) = 0\,.$$

Applying π_* to (1.1) we get

$$0 = \widehat{\mathrm{ch}}(\overline{E}_\bullet) - \pi_*((Z \cap Y), 0) - (0, \pi_*(\mathrm{ch}(\overline{\widetilde{E}}_\bullet)\log|t|^2)\,. \qquad (1.2)$$

By definition of the localized Chern character ([3], 18.1, (14), which remains valid over S by [3], 20.1, p. 395)

$$\pi_*(Z \cap Y) = \mathrm{ch}^{\mathrm{fin}}(E_\bullet) \cap [X] \qquad (1.3)$$

in $\mathrm{CH}^{\mathrm{fin}}(X)_{\mathbb{Q}}$. On the other hand we deduce from [4], (1.2.3.1), (1.2.3.2) that

$$-\,\pi_*(\mathrm{ch}(\overline{\widetilde{E}}_1)\log|t|^2) = \widetilde{\mathrm{ch}}(\overline{E}_\bullet)\,. \qquad (1.4)$$

and upon replacing t by $1/t$, as in the proof of (1.3.2) in [4], we see that

$$\pi_*(\mathrm{ch}(\overline{\widetilde{E}}_\bullet)\log|t|^2) = -\,\pi_*(\mathrm{ch}(\overline{\widetilde{E}}_1)\log|t|^2)\,. \qquad (1.5)$$

Theorem 1.1 follows from (1.2), (1.3), (1.4), (1.5).

2. A special case

2.1. We keep the hypotheses of the previous section, and we assume that X is normal, $d=1$, E_0 and E_2 have rank one and the metrics on E_0 and E_2 are induced by the metric on E_1. Finally, we assume that there exists a closed subscheme ϕ in X which is 0-dimensional and such that there is an exact sequence of sheaves on X

$$0 \to E_0 \to E_1 \to E_2 \otimes I_\phi \to 0\,, \tag{2.6}$$

where I_ϕ is the ideal of definition of ϕ.

Let

$$\widetilde{c}_2 \in A^{1,1}(X_{\mathbb{R}})/(\mathrm{Im}(\partial) + \mathrm{Im}(\overline{\partial}))$$

be the second Bott-Chern class of (2.6), $\Gamma(\phi, \mathcal{O}_\phi)$ the finite ring of functions on ϕ and $\#\,\Gamma(\phi, \mathcal{O}_\phi)$ its order. Let

$$f_* : \widehat{\mathrm{CH}}_0(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}_0(S) = \mathbb{R}$$

be the direct image morphism.

Theorem 2.1. — *We have an equality of real numbers*

$$f_*(\widehat{c}_2(\overline{E}_1) \cap [X]) = f_*(\widehat{c}_1(\overline{E}_0)\, \widehat{c}_1(\overline{E}_2) \cap [X]) - \int_{X(\mathbb{C})} \widetilde{c}_2 + \log \#\,\Gamma(\phi, \mathcal{O}_\phi)\,.$$

2.2. To prove Theorem 2.1 we remark first that

$$\widehat{c}_1(\overline{E}_1) = \widehat{c}_1(\overline{E}_0) + \widehat{c}_1(\overline{E}_2)\,,$$

because the metrics on E_0 and E_2 are induced from $\overline{E}_1$. Therefore, since $\mathrm{ch}_2 = -c_2 + \frac{c_1^2}{2}$, we get

$$\begin{aligned} \widehat{\mathrm{ch}}_2(\overline{E}_1) &= -\widehat{c}_2(\overline{E}_1) + \frac{(\widehat{c}_1(\overline{E}_0) + \widehat{c}_1(\overline{E}_2))^2}{2} \\ &= -\widehat{c}_2(\overline{E}_1) + c_1(\overline{E}_0)\, \widehat{c}_1(\overline{E}_2) + \widehat{\mathrm{ch}}_2(\overline{E}_0) + \widehat{\mathrm{ch}}_2(\overline{E}_2)\,. \end{aligned}$$

By Theorem 1.1, this implies that

$$\widehat{c}_2(\overline{E}_1) \cap [X] = \widehat{c}_1(\overline{E}_0)\, \widehat{c}_1(\overline{E}_2) \cap [X] + b\,(\mathrm{ch}^{\mathrm{fin}}(E_\bullet) \cap [X]) + a\,(\widetilde{\mathrm{ch}}(\overline{E}_\bullet))\,. \tag{2.7}$$

Since $\widetilde{\mathrm{ch}}_0(\overline{E}_\bullet)$ and $\widetilde{\mathrm{ch}}_1(\overline{E}_\bullet)$ vanish we have

$$\widetilde{\mathrm{ch}}(\overline{E}_\bullet) = -\,\widetilde{c}_2\,.$$

Therefore, if we apply f_* to (2.7), we get

$$f_*(\widehat{c}_2(\overline{E}_1)\cap[X]) = f_*(\widehat{c}_1(\overline{E}_0)\,\widehat{c}_1(\overline{E}_2)\cap[X]) - \int_{X(\mathbb{C})} \widetilde{c}_2 + f_*(b(\mathrm{ch}^{\mathrm{fin}}(E_\bullet)\cap[X]))\,,$$

and we are left with showing that

$$f_* \circ b\,(\mathrm{ch}^{\mathrm{fin}}(E_\bullet)\cap[X]) = \log\#\,\Gamma(\phi,\mathcal{O}_\phi)\,. \tag{2.8}$$

Let $|\phi| = \{P_1,\cdots,P_n\} \subset X$ be the support of ϕ and $\psi = f(|\phi|) \subset S$. The following diagram is commutative:

$$\begin{tikzcd} \mathrm{CH}_0(\phi) \arrow[r, "b"] \arrow[d, "f_*"] & \widehat{\mathrm{CH}}_0(X) \arrow[d, "f_*"] \\ \mathrm{CH}_0(\psi) \arrow[r, "b"] & \widehat{\mathrm{CH}}_0(S) = \mathbb{R}\,, \end{tikzcd}$$

where

$$b : \mathrm{CH}_0(\psi) = \mathbb{Z}^\psi \to \mathbb{R}$$

maps $(n_p)_{p\in\psi}$ to $\sum\limits_p n_p \log(p)$.

For any prime $p \in \psi$ we let $\mathbb{Z}_{(p)}$ be the local ring of S at p and we let $\ell_p = \ell_p(\phi) \geqslant 0$ be the length of the finite $\mathbb{Z}_{(p)}$-module $\Gamma(\phi,\mathcal{O}_\phi)\otimes\mathbb{Z}_{(p)}$. Clearly

$$\log\#\,\Gamma(\phi,\mathcal{O}_\phi) = \sum_{p\in\psi} \ell_p \log(p)\,,$$

hence it is enough to prove that

$$f_*(\mathrm{ch}^{\mathrm{fin}}(E_\bullet)\cap[X]) = (\ell_p) \in \mathrm{CH}_0(\psi)_{\mathbb{Q}} = \mathbb{Q}^\psi\,. \tag{2.9}$$

The complex $E_\bullet$ defines an element

$$[E_\bullet] = \sum_{i=1}^n [\mathcal{O}_{\phi,P_i}] \in K_0^\phi(X) = \bigoplus_{i=1}^n K_0^{P_i}(X)\,.$$

To prove (2.9), by replacing X by an affine neighbourhood of P, one can assume that $|\phi| = \{P\}$, and it is enough to show that, if $p = f(P)$,

$$f_*(\mathrm{ch}^{\mathrm{fin}}(\mathcal{O}_{\phi,P})\cap[X]) = \ell_p(\mathcal{O}_{\phi,P})[p]\,.$$

Now recall that, if $\mathcal{F}$ is a coherent sheaf on a scheme X of finite type over S, supported on a finite set of closed points, the associated 0-cycle

$$[\mathcal{F}] = \sum_{P \in |\mathcal{F}|} \ell_p(\mathcal{F}_P)[P] \in Z_0(X)$$

is such that, if $f : X \to Y$ is a proper morphism of schemes of finite type over S,

$$f_*[\mathcal{F}] = [f_*(\mathcal{F})]$$

([3], 15.1.5). Hence it is enough to show that

$$\mathrm{ch}^P(\mathcal{O}_{\phi,P}) = \ell_p(\mathcal{O}_{\phi,P})[P] \in \mathrm{CH}_0(P)_{\mathbb{Q}} \simeq \mathbb{Q}\,. \tag{2.10}$$

Replacing X by an affine neighbourhood of P, we may assume that we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\alpha} \mathcal{O}_X^2 \xrightarrow{\beta} \mathcal{O}_X \longrightarrow \mathcal{O}_\phi \longrightarrow 0\,. \tag{2.11}$$

Hence the ideal $I_\phi \subset \mathcal{O}_X(X)$ is generated by two elements β_1 and β_2. Since X is normal, its local rings satisfy Serre property S_2 and, as $\dim(X) = 2$, X is Cohen-Macaulay. Since β_1 and β_2 span an ideal of height two, (β_1, β_2) is a regular sequence and the sequence (2.11) is isomorphic to the Koszul resolution of $\mathcal{O}_\phi = \mathcal{O}_X/(\beta_1, \beta_2)$. Now (2.10) can be deduced from the following general fact:

Lemma 2.2. — *Let $X = \mathrm{Spec}\,(A)$ be an affine scheme and $Z \subset X$ a closed subset such that the ideal $I_Z = (x_1, \dots, x_n)$ is generated by a regular sequence $(x_1, \dots, x_n)$. Let $K_\bullet(x_1, \dots, x_n)$ be the Koszul complex associated to $(x_1, \dots, x_n)$. Then*

$$\mathrm{ch}_n^Z(K_\bullet(x_1, \dots, x_n)) = [\mathcal{O}_Z] \in \mathrm{CH}_0(Z)_{\mathbb{Q}}\,.$$

Proof. — The Grassmannian-graph construction on $K_\bullet(x_1, \dots, x_n)$ coincides with the deformation to the normal bundle of Z in X. If W is defined as in 1.4,

$$W_\infty = \widetilde{X} \cup \widehat{\mathbb{P}}(N_{Z/X})\,,$$

where $\widetilde{X}$ is the blow up of X along Z, and $\widehat{P}(N_{Z/X})$ is the projective completion of the normal bundle of Z in X. The pull back of the Koszul complex $K_\bullet(x_1, \dots, x_n)$ to $W \backslash W_\infty$ extends to a complex $\widetilde{K}_\bullet(x_1, \dots, x_n)$ on W. The restriction of $\widetilde{K}_\bullet(x_1, \dots, x_n)$ to $\widetilde{X}$ is acyclic while the restriction of $\widetilde{K}_\bullet(x_1, \dots, x_n)$ to $\widehat{\mathbb{P}}(N_{Z/X})$ is a resolution of the structure sheaf of the zero section $Z \subset N_{Z/X} \subset \widehat{\mathbb{P}}(N_{Z/X})$.

Now observe that $Z \subset \widehat{\mathbb{P}}(N_{Z/X})$ is an intersection of Cartier divisors $D_1, \ldots, D_n$, hence

$$\begin{aligned} & \operatorname{ch}(\widetilde{K}_\bullet(x_1, \ldots, x_n) \mid_{\widehat{\mathbb{P}}(N_{Z/X})}) \\ = & \prod_{i=1}^{n} \operatorname{ch}(\mathcal{O}(-D_i) \to \mathcal{O}_{\widehat{\mathbb{P}}(N_{Z/X})}) \\ = & \prod_{i=1}^{n} \operatorname{ch}(\mathcal{O}(D_i))\,. \end{aligned}$$

Since

$$\operatorname{ch}(\mathcal{O}(D_i)) = \operatorname{ch}_1(\mathcal{O}(D_i)) + y_i = [D_i] + y_i$$

where y_i has degree $\geqslant 2$, we get

$$\operatorname{ch}(\widetilde{K}_\bullet(x_1, \ldots, x_n) \mid_{\widehat{\mathbb{P}}(N_{Z/X})}) = [D_1] \ldots [D_n] = [Z]\,.$$

This ends the proof of Lemma 1 and Theorem 2.

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