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Bruno Kahn

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Algebraic tori as Nisnevich sheaves with transfers

Bruno Kahn[(1)]

Abstract. — We relate R-equivalence on tori with Voevodsky's theory of homotopy invariant Nisnevich sheaves with transfers and effective motivic complexes.

Résumé. — On relie la R-équivalence sur les tores aux faisceaux Nisnevich avec transferts invariants par homotopie et aux complexes motiviques effectifs, étudiés par Voevodsky.

Contents

1. Main results

Let k be a field and let T be a k-torus. The R-equivalence classes on T have been extensively studied by several authors, notably by Colliot-Thélène and Sansuc in a series of papers including [4] and [5]: they play a central rôle in many rationality issues. In this note, we show that Voevodsky's triangulated category of motives sheds a new light on this question: see Corollaries 1.3, 1.7 and 1.8 below.

(1) IMJ-PRG, UMR 7586, Case 247, 4 place Jussieu, 75252 Paris Cedex 05, France
bruno.kahn@imj-prg.fr

More generally, let G be a semi-abelian variety over k, which is an extension of an abelian variety A by a torus T. Denote by HI the category of homotopy invariant Nisnevich sheaves with transfers over k in the sense of Voevodsky [19]. Then G has a natural structure of an object of HI ([17, proof of Lemma 3.2], [1, Lemma 1.3.2]). Let L be the group of cocharacters of T.

Proposition 1.1. — *There is a natural isomorphism* $G_{-1} \xrightarrow{\sim} L$ *in* HI.

Here $_{-1}$ is the contraction operation of [18, p. 96], whose definition is recalled in the proof below.

Proof. — Recall that if $\mathcal{F}$ is a presheaf [with transfers] on smooth k-schemes, the presheaf [with transfers] $\mathcal{F}^p_{-1}$ is defined by

$$U \mapsto \operatorname{Coker}(\mathcal{F}(U \times \mathbf{A}^1) \to \mathcal{F}(U \times \mathbb{G}_m)).$$

If $\mathcal{F}$ is homotopy invariant, we may replace $U \times \mathbf{A}^1$ by U and the rational point $1 \in \mathbb{G}_m$ realises $\mathcal{F}^p_{-1}(U)$ as a functorial direct summand of $\mathcal{F}(U \times \mathbb{G}_m)$.

If $\mathcal{F}$ is a Nisnevich sheaf [with transfers], $\mathcal{F}_{-1}$ is defined as the sheaf associated to $\mathcal{F}^p_{-1}$.

Now $A(U \times \mathbf{A}^1) \xrightarrow{\sim} A(U \times \mathbb{G}_m)$ since A is an abelian variety, hence $A^p_{-1} = 0$. We therefore have an isomorphism of presheaves $T^p_{-1} \xrightarrow{\sim} G^p_{-1}$, and *a fortiori* an isomorphism of Nisnevich sheaves $T_{-1} \xrightarrow{\sim} G_{-1}$.

Let $p : \mathbb{G}_m \to \operatorname{Spec} k$ be the structural map. One easily checks that the *étale* sheaf $\operatorname{Coker}(T \xrightarrow{i} p_*p^*T)$ is canonically isomorphic to L. Since i is split, its cokernel is still L if we view it as a morphism of presheaves, hence of Nisnevich sheaves. □

From now on, we assume k perfect. Let $\mathrm{DM}^{\mathrm{eff}}_{-}$ be the triangulated category of effective motivic complexes introduced in [19]: it has a t-structure with heart HI. It also has a tensor structure and a (partially defined) internal Hom. We then have an isomorphism

$$L[0] = G_{-1}[0] \simeq \underline{\mathrm{Hom}}_{\mathrm{DM}^{\mathrm{eff}}_{-}}(\mathbb{G}_m[0], G[0])$$

[10, Rk. 4.4], hence by adjunction a morphism in $\mathrm{DM}^{\mathrm{eff}}_{-}$

$$L[0] \otimes \mathbb{G}_m[0] \to G. \tag{1.1}$$

Let $\nu_{\leqslant 0}G[0]$ denote the cone of (1.1): by [11, Lemma 6.3] or [8, §2], $\nu_{\leqslant 0}G[0]$ is the *birational motivic complex* associated to G. We want to compute its homology sheaves.

For this, consider a coflasque resolution[1]

$$0 \to Q \to L_0 \to L \to 0 \tag{1.2}$$

of L in the sense of [4, p. 179]. Taking a coflasque resolution of Q and iterating, we get a resolution of L by invertible lattices:

$$\dots \to L_n \to \dots \to L_0 \to L \to 0. \tag{1.3}$$

We set

$$Q_n = \begin{cases} Q & \text{for } n = 1 \\ \mathrm{Ker}(L_{n-1} \to L_{n-2}) & \text{for } n > 1. \end{cases}$$

THEOREM 1.2. — *a) Let T_n denote the torus with cocharacter group L_n. Then $\nu_{\leqslant 0}G[0]$ is isomorphic to the complex*

$$\dots \to T_n \to \dots \to T_0 \to G \to 0.$$

b) Let S_n be the torus with cocharacter group Q_n. For any connected smooth k-scheme X with function field K, we have

$$H_n(\nu_{\leqslant 0}G[0])(X) = \begin{cases} 0 & \text{if } n < 0 \\ G(K)/R & \text{if } n = 0 \\ S_n(K)/R & \text{if } n > 0. \end{cases}$$

The proof is given in Section 3.

COROLLARY 1.3. — *The assignment $Sm(k) \ni X \mapsto \bigoplus_{x \in X^{(0)}} G(k(x))/R$ provides G/R with the structure of a homotopy invariant Nisnevich sheaf with transfers. In particular, any morphism $\varphi : Y \to X$ of smooth connected k-schemes induces a morphism $\varphi^* : G(k(X))/R \to G(k(Y))/R$.*

This functoriality is essential to formulate Theorem 1.5 below. For φ a closed immersion of codimension 1, it recovers a specialisation map on R-equivalence classes with respect to a discrete valuation of rank 1 which was obtained (for tori) by completely different methods, *e.g.* [5, Th. 3.1 and Cor. 4.2] or [7]. (I am indebted to Colliot-Thélène for pointing out these references.)

COROLLARY 1.4. — *a) If k is finitely generated, the n-th homology sheaf of $\nu_{\leqslant 0}G[0]$ takes values in finitely generated abelian groups, and even in finite groups if $n > 0$ or G is a torus.*

b) If G is a torus, then $\nu_{\leqslant 0}G[0] = 0$ if G is split by a Galois extension E/k whose Galois group has cyclic Sylow subgroups. This condition is automatic if k is (quasi-)finite.

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(1) See Section 2 for this and further terminology.

The proof is also given in Section 3.

Given two semi-abelian varieties G, G', we would now like to understand the maps

$$\mathrm{Hom}_k(G,G') \to \mathrm{Hom}_{\mathrm{DM}_-^{\mathrm{eff}}}(\nu_{\leqslant 0}G[0], \nu_{\leqslant 0}G'[0]) \to \mathrm{Hom}_{\mathrm{HI}}(G/R, G'/R).$$

In Section 4, we succeed in elucidating the nature of their composition to a large extent, at least if G is a torus. Our main result, in the spirit of Yoneda's lemma, is

Theorem 1.5. — *Let G, G' be two semi-abelian varieties, with G a torus. Suppose given, for every function field K/k, a homomorphism $f_K : G(K)/R \to G'(K)/R$ such that f_K is natural with respect to the functoriality of Corollary 1.3. Then*

a) There exists an extension $\tilde G$ of G by a permutation torus, and a homomorphism $f : \tilde G \to G'$ inducing (f_K).

b) f_K is surjective for all K if and only if there exist extensions $\tilde G, \tilde G'$ of G and G' by permutation tori such that f_K is induced by a split surjective homomorphism $\tilde G \to \tilde G'$.

The proof is given in §4.3. See Proposition 4.7, Corollary 4.9, Remark 4.10 and Proposition 4.11 for complements.

This relates to questions of stable birationality studied by Colliot-Thélène and Sansuc in [4] and [5], providing alternate proofs and strengthening of some of their results (at least over a perfect field). More precisely, let us introduce the following terminology:

Definition 1.6. — a) A torus is quasi-invertible *if it is a quotient of a invertible torus by an permutation torus.*

b) An extension $0 \to T' \to T \to T'' \to 0$ of tori is Nisnevich-exact *if $T(K) \to T''(K)$ is surjective for any function field K/k.*

(a) was suggested by Xun Jiang; see also [2]. See §2 for "permutation torus" and "invertible torus".)

Thanks to [18, Cor. 4.18], Nisnevich-exact sequences of tori are exact in the Nisnevich topology and even in the Zariski topology. It is easy to see that an extension as in b) is Nisnevich-exact if T' is invertible, but not necessarily if T' is only quasi-invertible. Using [4, Th. 2], one sees that quasi-invertible tori are universally R-trivial. Conversely:

COROLLARY 1.7. — *a) Let G' be a semi-abelian k-variety such that $G'(K)/R = 0$ for any function field K/k. Then G' is a quasi-invertible torus.*

b) In Theorem 1.5 b), assume that f_K is bijective for all K/k. Then there exists an extension $\tilde{G}$ of G by a permutation torus and a Nisnevich-exact extension $\tilde{G}'$ of G' by a quasi-invertible torus such that f_K is induced by an isomorphism $\tilde{G} \xrightarrow{\sim} \tilde{G}'$.

Proof. — a) This is the special case $G = 0$ of Theorem 1.5 b).

b) By Theorem 1.5 b), we may replace G and G' by extensions by permutation tori such that f_K is induced by a split surjection $f : G \to G'$. Let $T = \mathrm{Ker} f$. Then $T/R = 0$ universally. By a), T is quasi-invertible. Replacing G' by $G' \times T$, we get the desired statement. □

Corollary 1.7 a) is a version of [5, Prop. 7.4] (taking [4, p. 199, Th. 2] into account). Theorem 1.5 was inspired by the desire to understand this result from a different viewpoint. Another characterisation of quasi-invertible tori in loc. cit. is that they are the retract-rational tori.

COROLLARY 1.8. — *Let $f : G \dashrightarrow G'$ be a rational map of semi-abelian varieties, with G a torus. Then the following conditions are equivalent:*

(i) $f_* : \nu_{\leqslant 0} G[0] \to \nu_{\leqslant 0} G'[0]$ *is an isomorphism (see Proposition 4.7).*

(ii) $f_* : G(K)/R \to G'(K)/R$ *is bijective for any function field K/k.*

(iii) *f is an isomorphism, up to Nisnevich-exact extensions of G and G' by quasi-invertible tori and up to a translation. (See Lemma 4.4.)*

Acknowledgements. — Part of Theorem 1.2 was obtained in the course of discussions with Takao Yamazaki during his stay at the IMJ in October 2010: I would like to thank him for inspiring exchanges. I also thank Daniel Bertrand for a helpful discussion, Xun Jiang for pointing out some errors and the referee for suggesting expository improvements. Finally, I wish to acknowledge inspiration from the work of Colliot-Thélène and Sansuc, which will be obvious throughout this paper.

2. Review of terminology for tori

We take this terminology from [4] and [5].

Definition 2.1 Let G be a profinite group.

a) A lattice *is a G-module which is finitely generated and free over* $\mathbf{Z}$.

b) A lattice L is

- permutation *if it affords a G-invariant* $\mathbf{Z}$*-basis.*
- invertible *if it is isomorphic to a direct summand of a permutation lattice.*
- coflasque *if $H^1(H, L) = 0$ for any open (hence closed) subgroup $H \subseteq G$.*
- flasque *if the dual lattice L^* is coflasque.*

c) A coflasque resolution *of a lattice L is a short exact sequence of lattices*

$$0 \to Q \to P \to L \to 0$$

where P is permutation and Q is coflasque. Dually, we have flasque [co]resolutions

$$0 \to L \to P \to F \to 0$$

with P permutation and F flasque.

Proposition 2.2 ([4, p. 181, lemme 3]). — *Any lattice has a flasque and a coflasque resolution.*

In [5, Lemma 0.6], the first statement of c) is extended to G-modules which are finitely generated over $\mathbf{Z}$ but not necessatily free.

Let k_s be a separable closure of the field k and take $G = Gal(k_s/k)$. Let T be a k-torus: we shall say that it is *permutation, invertible, flasque, coflasque*, if its character group is (Colliot-Thélène and Sansuc use *quasi-trivial* for "permutation"). Any permutation torus is of the form $R_{E/k}\mathbb{G}_m$ (Weil restriction of scalars) for some étale k-algebra E.

3. Proofs of Theorem 1.2 and Corollary 1.4

Lemma 3.1. — *The exact sequence*

$$0 \to T(k) \to G(k) \to A(k)$$

induces an exact sequence

$$0 \to T(k)/R \xrightarrow{i} G(k)/R \to A(k).$$

Proof. — Let $f : \mathbf{P}^1 \dashrightarrow G$ be a k-rational map defined at 0 and 1. Its composition with the projection $G \to A$ is constant: thus the image of f lies in a T-coset of G defined by a rational point. This implies the injectivity of i, and the rest is clear. □

Let NST denote the category of Nisnevich sheaves with transfers. Recall that $\mathrm{DM}_-^{\mathrm{eff}}$ may be viewed as a localisation of $D^-(\mathrm{NST})$, and that its tensor structure is a descent of the tensor structure on the latter category [19, Prop. 3.2.3].

Lemma 3.2. — *If G is an invertible torus, there is a canonical isomorphism in $D^-(\mathrm{NST})$*

$$L[0] \otimes \mathbb{G}_m \xrightarrow{\sim} G[0].$$

In particular, $\nu_{\leqslant 0} G[0] = 0$.

Proof. — We reduce to the case $T = R_{E/k}\mathbb{G}_m$, where E is a finite extension of k. Let us write more precisely $\mathrm{NST}(k)$ and $\mathrm{NST}(E)$. There is a pair of adjoint functors

$$\mathrm{NST}(k) \xrightarrow{f^*} \mathrm{NST}(E), \quad \mathrm{NST}(E) \xrightarrow{f_*} \mathrm{HI}(k)$$

where $f : \mathrm{Spec}E \to \mathrm{Spec}k$ is the projection. Clearly,

$$f_*\mathbf{Z} = \mathbf{Z}_{\mathrm{tr}}(\mathrm{Spec}E), \quad f_*\mathbb{G}_m = T$$

where $\mathbf{Z}_{\mathrm{tr}}(\mathrm{Spec}E)$ is the Nisnevich sheaf with transfers represented by $\mathrm{Spec}E$. Since $\mathbf{Z}_{\mathrm{tr}}(\mathrm{Spec}E) = L$, this proves the claim. □

Proof of Theorem 1.2. — a) Recall that L_0 is an invertible lattice chosen so that $L_0(E) \to L(E)$ is surjective for any extension E/k. In particular, (1.2) and (1.3) are exact as sequences of Nisnevich sheaves; hence $L[0]$ is isomorphic in $D^-(\mathrm{NST})$ to the complex

$$L. = \ldots \to L_n \to \ldots \to L_0 \to 0.$$

(We may view (1.3) as a version of Voevodsky's "canonical resolutions" as in [19, §3.2 p. 206].)

By Lemma 3.2, $L_n[0] \otimes \mathbb{G}_m[0] \simeq T_n[0]$ is homologically concentrated in degree 0 for all n. It follows that the complex

$$T. = \ldots \to T_n \to \ldots \to T_0 \to 0$$

is isomorphic to $L[0] \otimes \mathbb{G}_m[0]$ in $D^-(\mathrm{NST})$, hence *a fortiori* in $\mathrm{DM}_-^{\mathrm{eff}}$.

b) For any nonempty open subscheme $U \subseteq X$ we have isomorphisms

$$H_n(\nu_{\leqslant 0}G[0])(X) \xrightarrow{\sim} H_n(\nu_{\leqslant 0}G[0])(U) \xrightarrow{\sim} H_n(\nu_{\leqslant 0}G[0])(K) \tag{3.1}$$

(e.g. [8, p. 912]). By a), the right hand term is the n-th homology group of the complex

$$\ldots \to T_n(K) \to \ldots \to T_0(K) \to G(K) \to 0$$

with $G(K)$ in degree 0. By [4, p. 199, Th. 2], the sequences

$$0 \to S_1(K) \to T_0(K) \to T(K) \to T(K)/R \to 0$$
$$0 \to S_{n+1}(K) \to T_n(K) \to S_n(K) \to S_n(K)/R \to 0$$

are all exact. Using Lemma 3.1 for H_0, the conclusion follows from an easy diagram chase. □

Remark 3.3. — As a corollary to Theorem 1.2, $S_n(K)/R$ only depends on G. This can be seen without mentioning $\mathrm{DM}_-^{\mathrm{eff}}$: in view of the reasoning just above, it suffices to construct a homotopy equivalence between two resolutions of the form (1.3), which easily follows from the definition of coflasque modules.

Proof of Corollary 1.4. — a) This follows via Theorem 1.2 and Lemma 3.1 from [4, p. 200, Cor. 2] and the Mordell-Weil-Néron theorem. b) We may choose the L_n, hence the S_n split by E/k. The conclusion now follows from Theorem 1.2 and [4, p. 200, Cor. 3]. The last claim is clear. □

Remark 3.4. — In characteristic $p > 0$, all finitely generated perfect fields are finite. To give some contents to Corollary 1.4 a) in this characteristic, one may pass to the perfect [one should say radicial] closure k of a finitely generated field k_0. If G is a semi-abelian k-variety, it is defined over some finite extension k_1 of k_0. If k_2/k_1 is a finite (purely inseparable) subextension of k/k_1, then the composition

$$G(k_2) \xrightarrow{N_{k_2/k_1}} G(k_1) \to G(k_2)$$

equals multiplication by $[k_2 : k_1]$. Hence Corollary 1.4 a) remains true at least after inverting p.

4. Stable birationality

If X is a smooth variety over a field k, we write $\mathrm{Alb}(X)$ for its generalised Albanese variety in the sense of Serre [16]: it is a semi-abelian variety, and a rational point $x_0 \in X$ determines a morphism $X \to \mathrm{Alb}(X)$ which is universal for morphisms from X to semi-abelian varieties sending x_0 to 0.

We also write $\mathrm{NS}(X)$ for the group of cycles of codimension 1 on X modulo algebraic equivalence. This group is finitely generated if k is algebraically closed [9, Th. 3].

4.1. Well-known lemmas

I include proofs for lack of reference.

Lemma 4.1. — *a) Let G, G' be two semi-abelian k-varieties. Then any k-morphism $f : G \to G'$ can be written uniquely $f = f(0) + f'$, where f' is a homomorphism.*

b) For any semi-abelian k-variety G, the canonical map $G \to \mathrm{Alb}(G)$ sending 0 to 0 is an isomorphism.

Proof. — a) amounts to showing that if $f(0) = 0$, then f is a homomorphism. By an adjunction game, this is equivalent to b). Let us give two proofs: one of a) and one of b).

Proof of a). — We may assume k to be a universal domain. The statement is classical for abelian varieties [15, p. 41, Cor. 1] and an easy computation for tori. In the general case, let T, T' be the toric parts of G and G' and A, A' be their abelian parts. Let $g \in G(k)$. As any morphism from T to A' is constant, the k-morphism

$$\varphi_g : T \ni t \mapsto f(g+t) - f(g) \in G'$$

(which sends 0 to 0) lands in T', hence is a homomorphism. Therefore it only depends on the image of g in $A(k)$. This defines a morphism $\varphi : A \to \underline{\mathrm{Hom}}(T, T')$, which must be constant with value $\varphi_0 = f$. It follows that

$$(g, h) \mapsto f(g+h) - f(g) - f(h)$$

induces a morphism $A \times A \to T'$. Such a morphism is constant, of value 0.

Proof of b). — This is true if G is abelian, by rigidity and the equivalence between a) and b). In general, any morphism from G to an abelian variety is trivial on T. This shows that the abelian part of $\mathrm{Alb}(G)$ is A. Let $T' = \mathrm{Ker}(\mathrm{Alb}(G) \to A)$. We also have the counit morphism $\mathrm{Alb}(G) \to G$, and the composition $G \to \mathrm{Alb}(G) \to G$ is the identity. Thus T is a direct summand of T'. It suffices to show that $\dim T' = \dim T$. Going to the algebraic closure, we may reduce to $T = \mathbb{G}_m$.

Then consider the line bundle completion $\bar{G} \to A$ of the $\mathbb{G}_m$-bundle $G \to A$. It is sufficient to show that the kernel of

$$\mathrm{Alb}(G) \to \mathrm{Alb}(\bar{G}) = A$$

is 1-dimensional. This follows for example from [1, Cor. 10.5.1]. □

Lemma 4.2. — *Suppose k algebraically closed, and let G be a semi-abelian k-variety. Let A be the abelian quotient of G. Then the map*

$$\mathrm{NS}(A) \to \mathrm{NS}(G) \tag{4.2}$$

is an isomorphism.

Proof. — Let $T = \mathrm{Ker}(G \to A)$ and $X(T)$ be its character group. Choosing a basis (e_i) of $X(T)$, we may complete the $\mathbb{G}_m^n$-torsor G into a product of line bundles $\bar{G} \to A$. The surjection

$$\mathrm{Pic}(A) \xrightarrow{\sim} \mathrm{Pic}(\bar{G}) \twoheadrightarrow \mathrm{Pic}(G)$$

show the surjectivity of (4.2). Its kernel is generated by the classes of the irreducible components D_i of the divisor with normal crossings $\bar{G} - G$. These components correspond to the basis elements e_i. Since the corresponding $\mathbb{G}_m$-bundle is a group extension of A by $\mathbb{G}_m$, the class of the 0 section of its line bundle completion lies in $\mathrm{Pic}^0(A)$, hence goes to 0 in $\mathrm{NS}(A)$. □

Lemma 4.3. — *Let X be a smooth k-variety, and let $U \subseteq X$ be a dense open subset. Then there is an exact sequence of semi-abelian varieties*

$$0 \to T \to \mathrm{Alb}(U) \to \mathrm{Alb}(X) \to 0$$

with T a torus. If $\mathrm{NS}(U \otimes_k \bar{k}) = 0$ (this happens if U is small enough), there is an exact sequence of character groups

$$0 \to X(T) \to \bigoplus_{x \in X^{(1)} - U^{(1)}} \mathbf{Z} \to \mathrm{NS}(\bar{X}) \to 0.$$

Proof. — This follows for example from [1, Cor. 10.5.1]. □

Lemma 4.4. — *Let $f : G \dashrightarrow G'$ be a rational map between semi-abelian k-varieties, with G a torus. Then there exists an extension $\tilde{G}$ of G by a permutation torus and a homomorphism $\tilde{f} : \tilde{G} \to G'$ which extends f up to translation in the following sense: there exists a rational section $s : G \dashrightarrow \tilde{G}$ of the projection $\pi : \tilde{G} \to G$ and a rational point $g' \in G'(k)$ such that $f = \tilde{f}s + g'$. If f is defined at 0_G and sends it to $0_{G'}$, then $g' = 0$.*

Proof. — Let U be an open subset of G where f is defined. We define $\tilde{G} = \mathrm{Alb}(U)$. Applying Lemmas 4.3 and 4.1 b) and using $\mathrm{NS}(G \otimes_k \bar{k}) = 0$, we get an extension

$$0 \to P \to \tilde{G} \to G \to 0$$

where P is a permutation torus, as well as a morphism $\tilde{f} = \mathrm{Alb}(f) : \tilde{G} \to G'$.

Let us first assume k infinite. Then $U(k) \neq \emptyset$ because G is unirational. A rational point $g \in U$ defines an Albanese map $s : U \to \tilde{G}$ sending g to $0_{\tilde{G}}$. Since P is a permutation torus, $g \in G(k)$ lifts to $\tilde{g} \in \tilde{G}(k)$ (Hilbert 90) and we may replace s by a morphism sending g to $\tilde{g}$. Then s is a rational section of π. Moreover, $f = \tilde{f}s + g'$ with $g' = f(g) - \tilde{f}(\tilde{g})$. The last assertion follows.

If k is finite, then U has at least a zero-cycle g of degree 1, which is enough to define the Albanese map s. We then proceed as above (lift every closed point involved in g to a closed point of $\tilde{G}$ with the same residue field). □

Lemma 4.5. — *Let G be a finite group, and let A be a finitely generated G-module. Then*

a) There exists a short exact sequence of G-modules $0 \to P \to F \to A \to 0$, with F torsion-free and flasque, and P permutation.

b) Let B be another finitely generated G-module, and let $0 \to P' \to E \to B \to 0$ be an exact sequence with P' an invertible module. Then any G-morphism $f : A \to B$ lifts to $\tilde{f} : F \to E$.

Proof. — a) is the contents of [5, Lemma 0.6, (0.6.2)]. b) The obstruction to lifting f lies in $\operatorname{Ext}^1_G(F, P') = 0$ [4, p. 182, Lemme 9]. □

4.2. Functoriality of $\nu_{\leqslant 0}G$

We now assume k perfect.

Lemma 4.6. — *Let*

$$0 \to P \to G \to H \to 0 \tag{4.3}$$

be an exact sequence of semi-abelian varieties, with P an invertible torus. Then $\nu_{\leqslant 0}G[0] \xrightarrow{\sim} \nu_{\leqslant 0}H[0]$.

Proof. — As P is invertible, (4.3) is exact in NST hence defines an exact triangle

$$P[0] \to G[0] \to H[0] \xrightarrow{+1}$$

in $\mathrm{DM}^{\mathrm{eff}}_-$. The conclusion then follows from Lemma 3.2. □

Proposition 4.7. — *Let G, G' be two semi-abelian k-varieties, with G a torus. Then a rational map $f : G \dashrightarrow G'$ induces a morphism $f_* : \nu_{\leqslant 0}G[0] \to \nu_{\leqslant 0}G'[0]$, hence a homomorphism $f_* : G(K)/R \to G'(K)/R$ for any extension K/k. If K is infinite, f_* agrees up to translation with the morphism induced by f via the isomorphism $U(K)/R \xrightarrow{\sim} G(K)/R$ from [4, p. 196 Prop. 11], where U is an open subset of definition of f.*

Proof. — By Lemma 4.4, f induces a homomorphism $\tilde{G} \to G'$ where $\tilde{G}$ is an extension of G by a permutation torus. By Lemma 4.6, the induced morphism

$$\nu_{\leqslant 0}\tilde{G}[0] \to \nu_{\leqslant 0}G'[0]$$

factors through a morphism $f_* : \nu_{\leqslant 0}G[0] \to \nu_{\leqslant 0}G'[0]$.

The claims about R-equivalence classes follow from Theorem 1.2 b) and Lemma 4.4. □

Remark 4.8. — The proof shows that $f'_* = f_*$ if f' differs from f by a translation by an element of $G(k)$ or $G'(k)$.

Corollary 4.9. — *If T and T' are birationally equivalent k-tori, then $\nu_{\leqslant 0}T[0] \simeq \nu_{\leqslant 0}T'[0]$. In particular, the groups $T(k)/R$ and $T'(k)/R$ are isomorphic.*

Proof. — The proof of Proposition 4.7 shows that $f \mapsto f_*$ is functorial for composable rational maps between tori. Let $f : T \dashrightarrow T'$ be a birational isomorphism, and let $g : T' \dashrightarrow T$ be the inverse birational isomorphism. Then we have $g_*f_* = 1_{\nu_{\leqslant 0}T[0]}$ and $f_*g_* = 1_{\nu_{\leqslant 0}T'[0]}$. The last claim follows from Theorem 1.2. □

Remark 4.10. — It is proven in [4] that a birational isomorphism of tori $f : T \dashrightarrow T'$ induces a set-theoretic bijection $f_* : T(k)/R \xrightarrow{\sim} T'(k)/R$ (p. 197, Cor. to Prop. 11) and that the *group* $T(k)/R$ is abstractly a birational invariant of T (p. 200, Cor. 4). The proof above shows that f_* is an isomorphism of groups if f respects the origins of T and T'. This solves the question raised in [4, mid. p. 397]. The proofs of Lemma 4.4 and Proposition 4.7 may be seen as dual to the proof of [4, p. 189, Prop. 5], and are directly inspired from it.

4.3. Faithfulness and fullness

Proposition 4.11. — *Let $f : G \dashrightarrow G'$ be a rational map between semi-abelian varieties, with G a torus. Assume that the map $f_* : G(K)/R \to G'(K)/R$ from Proposition 4.7 is identically 0 when K runs through the finitely generated extensions of k. Then there exists a permutation torus P and a factorisation of f as*

$$G \overset{\tilde{f}}{\dashrightarrow} P \xrightarrow{g} G'$$

where $\tilde{f}$ is a rational map and g is a homomorphism. If f is a morphism, we may choose $\tilde{f}$ to be a homomorphism.

Conversely, if there is such a factorisation, then $f_* : \nu_{\leqslant 0}G[0] \to \nu_{\leqslant 0}G'[0]$ *is the* 0 *morphism.*

Proof. — By Lemma 4.4, we may reduce to the case where f is a homomorphism. Let $K = k(G)$. By hypothesis, the image of the generic point $\eta_G \in G(K)$ is R-equivalent to 0 on $G'(K)$. By a lemma of Gille [6, Lemme II.1.1 b)], it is directly R-equivalent to 0: in other words, there exists a rational map $h : G \times \mathbf{A}^1 \dashrightarrow G'$, defined in the neighbourhood of 0 and 1, such that $h_{|G\times\{0\}} = 0$ and $h_{|G\times\{1\}} = f$.

Let $U \subseteq G \times \mathbf{A}^1$ be an open set of definition of h. The 0 and 1-sections of $G \times \mathbf{A}^1 \to G$ induce sections

$$s_0, s_1 : G \to \mathrm{Alb}(U)$$

of the projection $\pi : \mathrm{Alb}(U) \to \mathrm{Alb}(G \times \mathbf{A}^1) = G$ such that $\mathrm{Alb}(h) \circ s_0 = 0$ and $\mathrm{Alb}(h) \circ s_1 = f$. If $P = \mathrm{Ker}\pi$, then $s_1 - s_0$ induces a homomorphism $\tilde{f} : G \to P$ such that the composition

$$G \xrightarrow{\tilde{f}} P \to \mathrm{Alb}(U) \xrightarrow{\mathrm{Alb}(h)} G'$$

equals f. Finally, P is a permutation torus by Lemma 4.3.

The last claim follows from Lemma 3.2. □

Proof of Theorem 1.5. — a) Take $K = k(G)$. The image of the generic point η_G by f_K lifts to a (non unique) rational map $f : G \dashrightarrow G'$. Using Lemma 4.4, we may extend f to a homomorphism

$$\tilde{f} : \tilde{G} \to G'$$

where $\tilde{G}$ is an extension of G by a permutation torus P. Since $\tilde{G}(K)/R \xrightarrow{\sim} G(K)/R$, we reduce to $\tilde{G} = G$ and $\tilde{f} = f$.

Let L/k be a fonction field, and let $g \in G(L)$. Then g arises from a morphism $g : X \to G$ for a suitable smooth model X of L. By assumption on $K \mapsto f_K$, the diagram

$$\begin{array}{ccc} G(K)/R & \xrightarrow[f_K]{} & G'(K)/R \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ G(L)/R & \xrightarrow[f_L]{} & G'(L)/R \end{array}$$

commutes. Applying this to $\eta_K \in G(K)$, we find that $f_L([g]) = [g \circ f]$, which means that f_L is the map induced by f.

b) The hypothesis implies that $G'(E)/R = 0$ for any algebraically closed extension E/k, which in turn implies that G' is also a torus. Applying a), we may, and do, convert f into a true homomorphism by replacing G by a suitable extension by a permutation torus. Applying Lemma 4.5 a) to the cocharacter group of G, we then get a resolution $0 \to P_1 \to Q \to G \to 0$ with Q coflasque and P_1 permutation. Hence we may (and do) further assume G coflasque.

Let $K = k(G')$ and choose some $g \in G(K)$ mapping modulo R-equivalence to the generic point of G'. Then g defines a rational map $g : G' \dashrightarrow G$ such that fg is R-equivalent to $1_{G'}$. It follows that the induced map

$$1 - fg : G'/R \to G'/R \tag{4.4}$$

is identically 0.

Reapplying Lemma 4.4, we may find an extension $\tilde{G}'$ of G' by a suitable permutation torus which converts g into a true homomorphism. Since G is coflasque, Lemma 4.5 b) shows that $f : G \to G'$ lifts to $\tilde{f} : G \to \tilde{G}'$. Then (4.4) is still identically 0 when replacing (G', f) by $(\tilde{G}', \tilde{f})$.

Summarising: we have replaced the initial G and G' by suitable extensions by permutation tori, such that f lifts to these extensions and there is a homomorphism $g : G' \to G$ such that (4.4) vanishes identically. Hence $1 - fg$ factors through a permutation torus P thanks to Proposition 4.11. Write $u : G' \to P$ and $v : P \to G'$ for homomorphisms such that $1 - fg = vu$. Let $G_1 = G \times P$ and consider the maps

$$f_1 = (f, v) : G_1 \to G', \qquad g_1 = (\, g \,)\, u : G' \to G_1.$$

Then $f_1 g_1 = 1$ and G' is a direct summand of G_1 as requested. □

5. Some open questions

Question 5.1. — Are lemma 4.4 and Proposition 4.7 still true when G is not a torus?

This is far from clear in general, starting with the case where G is an abelian variety and G' a torus. Let me give a positive answer in the case of an elliptic curve.

PROPOSITION 5.2. — *The answer to Question 5.1 is yes if the abelian part A of G is an elliptic curve.*

Proof. — Arguing as in the proof of Proposition 4.7, we get for an open subset $U \subseteq G$ of definition for f an exact sequence

$$0 \to \mathbb{G}_m \to P \to \mathrm{Alb}(U) \to G \to 0$$

where P is a permutation torus. Here we used that $\mathrm{NS}(\bar{G}) \simeq \mathbf{Z}$, which follows from Lemma 4.2.

The character group $X(P)$ has as a basis the geometric irreducible components of codimension 1 of $G - U$. Up to shrinking U, we may assume that $G - U$ contains the inverse image D of $0 \in A$. As the divisor class of 0 generates $\mathrm{NS}(\bar{A})$, D provides a Galois-equivariant splitting of the map $\mathbb{G}_m \to P$. Thus its cokernel is still a permutation torus, and we conclude as before. □

Question 5.3. — Can one formulate a version of Theorem 1.5 and Corollary 1.7 providing a description of the groups $\mathrm{Hom}_{\mathrm{DM}_-^{\mathrm{eff}}}(\nu_{\leqslant 0}G[0], \nu_{\leqslant 0}G'[0])$ and $\mathrm{Hom}_{\mathrm{HI}}(G/R, G'/R)$ (at least when G and G' are tori)?

The proof of Theorem 1.5 suggests the presence of a closed model structure on the category of tori (or lattices), which might provide an answer to this question.

For the last question, let G be a semi-abelian variety. Forgetting its group structure, it has a motive $M(G) \in \mathrm{DM}_-^{\mathrm{eff}}$. Recall the canonical morphism

$$M(G) \to G[0]$$

induced by the "sum" maps

$$c(X, G) \xrightarrow{\sigma} G(X) \tag{5.1}$$

for smooth varieties X ([17, (6), (7)], [1, §1.3]).

The morphism (5.1) has a canonical section

$$G(X) \xrightarrow{\gamma} c(X, G) \tag{5.2}$$

given by the graph of a morphism: this section is functorial in X but is not additive.

Consider now a smooth equivariant compactification $\bar{G}$ of G. It exists in all characteristics. For tori, this is written up in [3]. The general case reduces to this one by the following elegant argument I learned from M. Brion: if G is an extension of an abelian variety A by a torus T, take a smooth projective equivariant compactification Y of T. Then the bundle $G \times^T Y$ associated to the T-torsor $G \to A$ also exists: this is the desired compactification.

Then we have a diagram of birational motives

$$\begin{array}{ccc} \nu_{\leqslant 0} M(G) & \xrightarrow{\sim} & \nu_{\leqslant 0} M(\bar{G}) \\ {\scriptstyle \nu_{\leqslant 0}\sigma}\downarrow & & \\ \nu_{\leqslant 0} G[0]. & & \end{array} \tag{5.3}$$

By [11], we have $H_0(\nu_{\leqslant 0} M(\bar{G}))(X) = CH_0(\bar{G}_{k(X)})$ for any smooth connected X. Hence the above diagram induces a homomorphism

$$CH_0(\bar{G}_{k(X)}) \to G(k(X))/R \tag{5.4}$$

which is natural in X for the action of finite correspondences (compare Corollary 1.3). One can probably check that this is the homomorphism of [12, (17) p. 78], reformulating [4, Proposition 12 p. 198]. Similarly, the set-theoretic map

$$G(k(X))/R \to CH_0(\bar{G}_{k(X)}) \tag{5.5}$$

of [4, p. 197] can presumably be recovered as a birational version of (5.2), using perhaps the homotopy category of schemes of Morel and Voevodsky [14].

In [12], Merkurjev shows that (5.4) is an isomorphism for G a torus of dimension at most 3. This suggests:

Question 5.4. — Is the map $\nu_{\leqslant 0}\sigma$ of Diagram (5.3) an isomorphism when G is a torus of dimension $\leqslant 3$?

In [13], Merkurjev gives examples of tori G for which (5.5) is not a homomorphism; hence its (additive) left inverse (5.4) cannot be an isomorphism. Merkurjev's examples are of the form $G = R^1_{K/k}\mathbb{G}_m \times R^1_{L/k}\mathbb{G}_m$, where K and L are distinct biquadratic extensions of k. This suggests:

Question 5.5. — Can one study Merkurjev's examples from the above viewpoint? More generally, what is the nature of the map $\nu_{\leqslant 0}\sigma$ of Diagram (5.3)?

We leave all these questions to the interested reader.

Bibliography

[1] Barbieri-Viale (L.), Kahn (B.). — On the derived category of 1-motives, arXiv:1009.1900.

[2] Borovoi (M.), Kunyavskiĭ (B. E'), Lemire (N.), Reichstein (Z.). — Stably Cayley groups in characteristic zero, IMRN (2013).

[3] Colliot-Thélène (J.-L.), Harari (D.), Skorobogatov (A.). — Compactification équivariante d'un tore (d'après Brylinski et Künnemann), Expo. Math. 23, p. 161-170 (2005).

[4] Colliot-Thélène (J.-L.), Sansuc (J.-J.). — La R-équivalence sur les tores, Ann. Sci. Éc. Norm. Sup. 10, p. 175-230 (1977).

[5] Colliot-Thélène (J.-L.), Sansuc (J.-J.). — Principal homogeneous spaces under flasque tori; applications, J. Alg. 106, p. 148-205 (1987).

[6] Gille (P.). — La R-équivalence pour les groupes algébriques réductifs définis sur un corps global, Publ. Math. IHÉS 86, p. 199-235 (1997).

[7] Gille (P.). — Spécialisation de la R-équivalence pour les groupes réductifs, Trans. Amer. Math. Soc. 356, p. 4465-4474 (2004).

[8] Huber (A.) and Kahn (B.). — The slice filtration and mixed Tate motives, Compositio Math. 142, p. 907-936 (2006).

[9] Kahn (B.). — Sur le groupe des classes d'un schéma arithmétique (avec un appendice de Marc Hindry), Bull. Soc. Math. France 134 (2006), p. 395-415.

[10] Kahn (B.), Yamazaki (T.). — Voevodsky's motives and Weil reciprocity, Duke Math. J. 162, p. 2751-2796 (2013).

[11] Kahn (B.), Sujatha (R.). — Birational motives, I (preliminary version), preprint, 2002, http://www.math.uiuc.edu/K-theory/0596/.

[12] Merkurjev (A. S.). — R-equivalence on three-dimensional tori and zero-cycles, Algebra Number Theory 2, p. 69-89 (2008).

[13] Merkurjev (A. S.). — Zero-cycles on algebraic tori, in The geometry of algebraic cycles, 119-122, Clay Math. Proc., 9, Amer. Math. Soc., Providence, RI (2010).

[14] Morel (F.), Voevodsky (V.). — $\mathbf{A}^1$-homotopy theory of schemes, Publ. Math. IHÉS 90, p. 45-143 (1999).

[15] Mumford (D.). — Abelian varieties (corrected reprint), TIFR - Hindustan Book Agency (2008).

[16] Serre J.-P.. — Morphismes universels et variétés d'Albanese, in Exposés de séminaires, 1950-1989, Doc. mathématiques 1, SMF, p. 141-160 (2001).

[17] Spiess (M.), T. Szamuely (T.). — On the Albanese map for smooth quasi-projective varieties, Math. Ann. 325, p. 1-17 (2003).

[18] Voevodsky (V.). — Cohomological theory of presheaves with transfers, in E. Friedlander, A. Suslin, Voevodsky (V.) Cycles, transfers and motivic cohomology theories, Ann. Math. Studies 143, Princeton University Press, 88-137 (2000).

[19] Voevodsky (V.). — Triangulated categories of motives over a field, in E. Friedlander, A. Suslin, Voevodsky (V.) Cycles, transfers and motivic cohomology theories, Ann. Math. Studies 143, Princeton University Press, p. 188-238 (2000).