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JEAN GUTT

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Generalized Conley-Zehnder index

JEAN GUTT⁽¹⁾

ABSTRACT. — The Conley-Zehnder index associates an integer to any continuous path of symplectic matrices starting from the identity and ending at a matrix which does not admit 1 as an eigenvalue. Robbin and Salamon define a generalization of the Conley-Zehnder index for any continuous path of symplectic matrices; this generalization is half integer valued. It is based on a Maslov-type index that they define for a continuous path of Lagrangians in a symplectic vector space $(W, \overline{\Omega})$, having chosen a given reference Lagrangian V . Paths of symplectic endomorphisms of $(\mathbb{R}^{2n}, \Omega_0)$ are viewed as paths of Lagrangians defined by their graphs in $(W = \mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, \overline{\Omega} = \Omega_0 \oplus -\Omega_0)$ and the reference Lagrangian is the diagonal. Robbin and Salamon give properties of this generalized Conley-Zehnder index and an explicit formula when the path has only regular crossings. We give here an axiomatic characterization of this generalized Conley-Zehnder index. We also give an explicit way to compute it for any continuous path of symplectic matrices.

RÉSUMÉ. — L'indice de Conley-Zehnder associe un nombre entier à tout chemin de matrices symplectiques partant de l'identité et se terminant en une matrice n'admettant pas 1 comme valeur propre. Robbin et Salamon ont défini une généralisation de l'indice de Conley-Zehnder, définie pour tout chemin continu de matrices symplectiques; cette généralisation est à valeur demi entière. Elle est basée sur un indice de type Maslov qu'ils définissent pour un chemin continu de Lagrangiens dans un espace symplectique $(W, \overline{\Omega})$ ayant fixé un Lagrangien de référence V . Les chemins d'endomorphismes symplectiques de $(\mathbb{R}^{2n}, \Omega_0)$ sont vus comme les chemins de Lagrangiens définis par leur graphe dans $(W = \mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, \overline{\Omega} = \Omega_0 \oplus -\Omega_0)$. Le lagrangien de référence est la diagonale. Robbin et Salamon donnent des propriétés de cet indice de Conley-Zehnder

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⁽¹⁾ Département de Mathématique, Université Libre de Bruxelles, Campus Plaine, C. P. 218, Boulevard du Triomphe, B-1050 Bruxelles Belgium and Université de Strasbourg, IRMA, 7 rue René Descartes, 67000 Strasbourg, France.

généralisé et une formule explicite lorsque le chemin ne possède que des croisements réguliers. Nous donnons ici une caractérisation explicite de cet indice de Conley-Zehnder généralisé. Nous donnons également une manière explicite de calculer cet indice pour tout chemin de matrices symplectiques.

1. Introduction

The Conley-Zehnder index [2] associates an integer to any continuous path ψ defined on the interval $[0, 1]$ with values in the group $\mathrm{Sp}\left(\mathbb{R}^{2n}, \Omega_0 = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}\right)$ of $2n \times 2n$ symplectic matrices, starting from the identity and ending at a matrix which does not admit 1 as an eigenvalue. This index is used in the definition of the grading of Floer homology theories. If the path ψ were a loop with values in the unitary group, one could define an integer by looking at the degree of the loop in the circle defined by the (complex) determinant -or an integer power of it. The construction of the Conley-Zehnder index is based on this idea. One uses a continuous map ρ from the symplectic group $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ into S^1 and an “admissible” extension of ψ to a path $\tilde{\psi} : [0, 2] \rightarrow \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ in such a way that $\rho^2 \circ \tilde{\psi} : [0, 2] \rightarrow S^1$ is a loop. The Conley-Zehnder index of ψ is defined as the degree of this loop

$$\mu_{\mathrm{CZ}}(\psi) := \deg(\rho^2 \circ \tilde{\psi}).$$

We recall this construction in section 2 with the precise definition of the map ρ . The value of $\rho(A)$ involves the algebraic multiplicities of the real negative eigenvalues of A and the signature of natural symmetric 2-forms defined on the generalized eigenspaces of A for the non real eigenvalues lying on S^1 . We give alternative ways to compute this index.

In [4], Robbin and Salamon define a Maslov-type index for a continuous path Λ from the interval $[a, b]$ to the space $\mathcal{L}_{(W, \overline{\Omega})}$ of Lagrangian subspaces of a symplectic vector space $(W, \overline{\Omega})$, having chosen a reference Lagrangian L . They give a formula of this index for a path having only regular crossings. A crossing for Λ is a number $t \in [a, b]$ for which $\dim \Lambda_t \cap L \neq 0$, and a crossing t is regular if the crossing form $\Gamma(\Lambda, L, t)$ is nondegenerate. We recall the precise definitions in section 3.

In the same paper, Robbin and Salamon define the index of a continuous path of symplectic matrices $\psi : [0, 1] \rightarrow \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) : t \mapsto \psi_t$ as the index of the corresponding path of Lagrangians in $(W := \mathbb{R}^{2n} \times \mathbb{R}^{2n}, \overline{\Omega} = -\Omega_0 \times \Omega_0)$

defined by their graphs,

$$\Lambda = \text{Gr } \psi : [0, 1] \rightarrow \mathcal{L}_{(W, \overline{\Omega})} : t \mapsto \text{Gr } \psi_t = \{(x, \psi_t x) | x \in \mathbb{R}^{2n}\}.$$

The reference Lagrangian is the diagonal $\Delta = \{(x, x) | x \in \mathbb{R}^{2n}\}$. They prove that this index coincide with the Conley-Zehnder index on continuous paths of symplectic matrices which start from the identity and end at a matrix which does not admit 1 as an eigenvalue. To be complete, we include this in section 4. They also prove that this index vanishes on a path of symplectic matrices with constant dimensional 1-eigenspace.

We use the normal form of the restriction of a symplectic endomorphism to the generalized eigenspace of eigenvalue 1 obtained in [3] to construct special paths of symplectic endomorphisms with a constant dimension of the eigenspace of eigenvalue 1. This leads in section 5 to a characterization of the generalized half-integer valued Conley-Zehnder index defined by Robbin and Salamon :

THEOREM 1.1. — *The Robbin-Salamon index for a continuous path of symplectic matrices is characterized by the following properties:*

- (Homotopy) *it is invariant under homotopies with fixed endpoints;*
- (Catenation) *it is additive under catenation of paths;*
- (Zero) *it vanishes on any path $\psi : [a, b] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega)$ of matrices such that $\dim \text{Ker}(\psi(t) - \text{Id}) = k$ is constant on $[a, b]$;*
- (Normalization) *if $S = S^\tau \in \mathbb{R}^{2n \times 2n}$ is a symmetric matrix with all eigenvalues of absolute value $< 2\pi$ and if $\psi(t) = \exp(J_0 S t)$ for $t \in [0, 1]$ and $J_0 = \begin{pmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{pmatrix}$, then $\mu_{RS}(\psi) = \frac{1}{2} \text{Sign } S$ where $\text{Sign } S$ is the signature of S .*

The naturality property of this index (i.e. for any $\phi : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$, we have $\mu_{RS}(\phi\psi\phi^{-1}) = \mu_{RS}(\psi)$) follows from the other properties.

The same techniques lead in section 6 to a new formula for this index. As in Conley and Zehnder [2], we extend a given path of matrices to a path ending into $W^+ := -\text{Id}$ or $W^- := \text{diag}(2, -1, \dots, -1, \frac{1}{2}, -1, \dots, -1)$. We also use the function ρ , or, more generally, any continuous map $\tilde{\rho} : \text{Sp}(\mathbb{R}^{2n}, \Omega_0) \rightarrow S^1$ with the following properties:

1. $\tilde{\rho}$ coincides with the (complex) determinant $\det_{\mathbb{C}}$ on $U(n) = O(\mathbb{R}^{2n}) \cap \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$;
2. $\tilde{\rho}(W^-) \in \{\pm 1\}$;

3. $\deg(\tilde{\rho}^2 \circ \psi_{2-}) = n - 1$

for $\psi_{2-} : t \in [0, 1] \mapsto \exp t\pi J_0 \begin{pmatrix} 0 & 0 & -\frac{\log 2}{\pi} & 0 \\ 0 & \text{Id}_{n-1} & 0 & 0 \\ -\frac{\log 2}{\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{Id}_{n-1} \end{pmatrix}$.

THEOREM 1.2. — *Let $\psi : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ be a path of symplectic matrices. Decompose $\psi(0) = \psi^*(0) \tilde{\oplus} \psi^{(1)}(0)$ and $\psi(1) = \psi^*(1) \tilde{\oplus} \psi^{(1)}(1)$ where $\psi^*(\cdot)$ does not admit 1 as eigenvalue and $\psi^{(1)}(\cdot)$ is the restriction of $\psi(\cdot)$ to its generalized eigenspace of eigenvalue 1. Consider a continuous extension $\Psi : [-1, 2] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ of ψ , i.e. $\Psi(t) = \psi(t)$ for $t \in [0, 1]$, such that*

- $\Psi(-\frac{1}{2}) = \psi^*(0) \tilde{\oplus} \begin{pmatrix} e^{-1} \text{Id} & 0 \\ 0 & e \text{Id} \end{pmatrix}$ and $\Psi(t) = \psi^*(0) \tilde{\oplus} \phi_0(t)$ where $\phi_0(t)$ has only real positive eigenvalues for $t \in [-\frac{1}{2}, 0]$;
- $\Psi(\frac{3}{2}) = \psi^*(1) \tilde{\oplus} \begin{pmatrix} e^{-1} \text{Id} & 0 \\ 0 & e \text{Id} \end{pmatrix}$ and $\Psi(t) = \psi^*(1) \tilde{\oplus} \phi_1(t)$ where $\phi_1(t)$ has only real positive eigenvalues for $t \in [1, \frac{3}{2}]$;
- $\Psi(-1) = W^\pm$, $\Psi(2) = W^\pm$ and $\Psi(t)$ does not admit 1 as an eigenvalue for $t \in [-1, -\frac{1}{2}]$ and for $t \in [\frac{3}{2}, 2]$.

Then the Robbin-Salamon index is given by

$$\mu_{RS}(\psi) = \deg(\tilde{\rho}^2 \circ \Psi) + \frac{1}{2} \sum_{k \geq 1} \text{Sign}(\hat{Q}_k^{(\psi(0))}) - \frac{1}{2} \sum_{k \geq 1} \text{Sign}(\hat{Q}_k^{(\psi(1))})$$

with $\tilde{\rho}$ as above, and with

$$\begin{aligned} \hat{Q}_k^A & : \text{Ker}((A - \text{Id})^{2k}) \times \text{Ker}((A - \text{Id})^{2k}) \rightarrow \mathbb{R} \\ & (v, w) \mapsto \Omega((A - \text{Id})^k v, (A - \text{Id})^{k-1} w). \end{aligned}$$

In the theorem above, we have used the notation $A \tilde{\oplus} B$ for the symplectic direct sum of two symplectic endomorphisms with the natural identification of $\text{Sp}(V', \Omega') \times \text{Sp}(V'', \Omega'')$ as a subgroup of $\text{Sp}(V' \oplus V'', \Omega' \oplus \Omega'')$. This writes in symplectic basis as

$$A \tilde{\oplus} B := \begin{pmatrix} A_1 & 0 & A_2 & 0 \\ 0 & B_1 & 0 & B_2 \\ A_3 & 0 & A_4 & 0 \\ 0 & B_3 & 0 & B_4 \end{pmatrix} \quad \text{for } A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}.$$

This paper is organized as follows. We recall the definition of the Conley-Zehnder index in section 2, with a new way of computing this index. In sections 3 and 4, we present known results about the Robbin-Salamon index of a path of Lagrangians and the Robbin-Salamon index of a path of symplectic matrices, including the fact that it is a generalization of the

Conley-Zehnder index; we stress the fact that another Maslov index introduced by Robbin and Salamon does not coincide with this generalization of the Conley-Zehnder index. In section 5, we give a characterization of the generalization of the Conley-Zehnder index (stated above as Theorem 1.1). Section 6 gives a new formula to compute this index (stated above as Theorem 1.2).

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2. The Conley-Zehnder index

The Conley-Zehnder index is an application which associates an integer to a continuous path of symplectic matrices starting from the identity and ending at a matrix in the set $\mathrm{Sp}^*(\mathbb{R}^{2n}, \Omega_0)$ of symplectic matrices which do not admit 1 as an eigenvalue.

DEFINITION 2.1 ([2]). — *We consider the set $\mathrm{SP}(n)$ of continuous paths of matrices in $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ linking the matrix Id to a matrix in $\mathrm{Sp}^*(\mathbb{R}^{2n}, \Omega_0)$:*

$$\mathrm{SP}(n) := \left\{ \psi : [0, 1] \rightarrow \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \mid \begin{array}{l} \psi(0) = \mathrm{Id} \text{ and} \\ 1 \text{ is not an eigenvalue of } \psi(1) \end{array} \right\}.$$

DEFINITION 2.2 ([6, 1]). — *Let $\rho : \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \rightarrow S^1$ be the continuous map defined as follows. Given $A \in \mathrm{Sp}(\mathbb{R}^{2n}, \Omega)$, we consider its eigenvalues $\{\lambda_i\}$. For an eigenvalue $\lambda = e^{i\varphi} \in S^1 \setminus \{\pm 1\}$, let $m^+(\lambda)$ be the number of positive eigenvalues of the symmetric non degenerate 2-form Q defined on the generalized eigenspace E_λ by*

$$Q : E_\lambda \times E_\lambda \rightarrow \mathbb{R} : (z, z') \mapsto Q(z, z') := \mathrm{Im} \Omega_0(z, \overline{z'}).$$

Then

$$\rho(A) := (-1)^{\frac{1}{2}m^-} \prod_{\lambda \in S^1 \setminus \{\pm 1\}} \lambda^{\frac{1}{2}m^+(\lambda)} \tag{2.1}$$

where m^- is the sum of the algebraic multiplicities $m_\lambda = \dim_{\mathbb{C}} E_\lambda$ of the real negative eigenvalues.

PROPOSITION 2.3 ([6, 1]). — *The map $\rho : \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \rightarrow S^1$ has the following properties:*

1. [determinant] ρ coincides with $\det_{\mathbb{C}}$ on the unitary subgroup

$$\rho(A) = \det_{\mathbb{C}} A \text{ if } A \in \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \cap O(2n) = U(n);$$

2. [invariance] ρ is invariant under conjugation :

$$\rho(kAk^{-1}) = \rho(A) \quad \forall k \in \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0);$$

3. [normalisation] $\rho(A) = \pm 1$ for matrices which have no eigenvalue on the unit circle;

4. [multiplicativity] ρ behaves multiplicatively with respect to direct sums: if $A = A' \oplus A''$ with $A' \in \mathrm{Sp}(\mathbb{R}^{2m}, \Omega_0)$, $A'' \in \mathrm{Sp}(\mathbb{R}^{2(n-m)}, \Omega_0)$ and \oplus expressing as before the obvious identification of $\mathrm{Sp}(\mathbb{R}^{2m}, \Omega_0) \times \mathrm{Sp}(\mathbb{R}^{2(n-m)}, \Omega_0)$ with a subgroup of $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ then

$$\rho(A) = \rho(A')\rho(A'').$$

The construction of the Conley-Zehnder index is based on the two following facts [2]:

- $\mathrm{Sp}^*(\mathbb{R}^{2n}, \Omega_0)$ has two connected components, one containing $W^+ = -\mathrm{Id}$, the other containing $W^- = \mathrm{diag}(2, -1, \dots, -1, \frac{1}{2}, -1, \dots, -1)$;
- any loop in $\mathrm{Sp}^*(\mathbb{R}^{2n}, \Omega_0)$ is contractible in $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$.

Thus any path $\psi : [0, 1] \rightarrow \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ in $\mathrm{SP}(n)$ can be extended to a path $\tilde{\psi} : [0, 2] \rightarrow \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ so that

- $\tilde{\psi}(t) = \psi(t)$ for $t \leq 1$;
- $\tilde{\psi}(t)$ is in $\mathrm{Sp}^*(\mathbb{R}^{2n}, \Omega_0)$ for any $t \geq 1$;
- $\tilde{\psi}(2) = W^\pm$.

Observe that $(\rho(\mathrm{Id}))^2 = 1$ and $(\rho(W^\pm))^2 = 1$ so that $\rho^2 \circ \tilde{\psi} : [0, 2] \rightarrow S^1$ is a loop in S^1 and the contractibility property shows that its degree does not depend on the extension chosen.

DEFINITION 2.4. — *The Conley-Zehnder index of ψ in $\mathrm{SP}(n)$ is defined by:*

$$\mu_{\mathrm{CZ}} : \mathrm{SP}(n) \rightarrow \mathbb{Z} : \psi \mapsto \mu_{\mathrm{CZ}}(\psi) := \deg(\rho^2 \circ \tilde{\psi}) \quad (2.2)$$

for an extension $\tilde{\psi}$ of ψ as above.

Remark 2.5. — In the original paper [2], the index is first defined on paths in $\mathrm{SP}(n)$ of the form $\psi_S(t) = \exp tJ_0S$ for a symmetric matrix S

so that the purely imaginary eigenvalues of J_0S are all distinct. It is then shown that in any component of $\mathrm{SP}(n)$ there exists such a path ψ_S and that the index of two such paths ψ_S and $\psi_{S'}$ are equal if and only if the paths are homotopic. The index is then defined to be constant on any connected component of $\mathrm{SP}(n)$. We have chosen in this presentation to start with the equivalent definition 2.4 given in [6].

PROPOSITION 2.6 ([5, 1]). — *The Conley-Zehnder index has the following properties:*

1. (Naturality) *For all path $\phi : [0, 1] \rightarrow \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ we have*

$$\mu_{\mathrm{CZ}}(\phi\psi\phi^{-1}) = \mu_{\mathrm{CZ}}(\psi);$$

2. (Homotopy) *The Conley-Zehnder index is constant on the components of $\mathrm{SP}(n)$;*
3. (Zero) *If $\psi(s)$ has no eigenvalue on the unit circle for $s > 0$ then*

$$\mu_{\mathrm{CZ}}(\psi) = 0;$$

4. (Product) *If $n' + n'' = n$, , if ψ' is in $\mathrm{SP}(n')$ and ψ'' in $\mathrm{SP}(n'')$, then*

$$\mu_{\mathrm{CZ}}(\psi' \tilde{\oplus} \psi'') = \mu_{\mathrm{CZ}}(\psi') + \mu_{\mathrm{CZ}}(\psi'');$$

with the identification of $\mathrm{Sp}(\mathbb{R}^{2n'}, \Omega_0) \times \mathrm{Sp}(\mathbb{R}^{2n''}, \Omega_0)$ with a subgroup of $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$;

5. (Loop) *If $\phi : [0, 1] \rightarrow \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ is a loop with $\phi(0) = \phi(1) = \mathrm{Id}$, then*

$$\mu_{\mathrm{CZ}}(\phi\psi) = \mu_{\mathrm{CZ}}(\psi) + 2\mu(\phi)$$

where $\mu(\phi)$ is the Maslov index of the loop ϕ , i.e. $\mu(\phi) = \deg(\rho \circ \phi)$;

6. (Signature) *If $S = S^\tau \in \mathbb{R}^{2n \times 2n}$ is a symmetric non degenerate matrix with all eigenvalues of absolute value $< 2\pi$ ($\|S\| < 2\pi$) and if $\psi(t) = \exp(J_0St)$ for $t \in [0, 1]$, then $\mu_{\mathrm{CZ}}(\psi) = \frac{1}{2} \mathrm{Sign}(S)$ where ($\mathrm{Sign}(S)$ is the signature of S).*

7. (Determinant) $(-1)^{n - \mu_{\mathrm{CZ}}(\psi)} = \mathrm{sign} \det(\mathrm{Id} - \psi(1))$

8. (Inverse) $\mu_{\mathrm{CZ}}(\psi^{-1}) = \mu_{\mathrm{CZ}}(\psi^\tau) = -\mu_{\mathrm{CZ}}(\psi)$

PROPOSITION 2.7 ([5, 1]). — *The properties 2., 5. and 6. of homotopy, loop and signature characterize the Conley-Zehnder index.*

Proof. — Assume $\mu' : \mathrm{SP}(n) \rightarrow \mathbb{Z}$ is a map satisfying those properties. Let $\psi : [0, 1] \rightarrow \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ be an element of $\mathrm{SP}(n)$; Since ψ is in the same

component of $\text{SP}(n)$ as its prolongation $\tilde{\psi} : [0, 2] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ we have $\mu'(\psi) = \mu'(\tilde{\psi})$.

Observe that $W^+ = \exp \pi(J_0 S^+)$ and $W^- = \exp \pi(J_0 S^-)$ with

$$S^+ = \text{Id} \quad S^- = \begin{pmatrix} 0 & 0 & -\frac{\log 2}{\pi} & 0 \\ 0 & \text{Id}_{n-1} & 0 & 0 \\ -\frac{\log 2}{\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{Id}_{n-1} \end{pmatrix}. \quad (2.3)$$

The catenation of $\tilde{\psi}$ and ψ_2^- (the path ψ_2 in the reverse order, i.e followed from end to beginning) when $\psi_2 : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ $t \mapsto \exp t\pi J_0 S^\pm$ is a loop ϕ . Hence $\tilde{\psi}$ is homotopic to the catenation of ϕ and ψ_2 , which is homotopic to the product $\phi\psi_2$.

Thus we have $\mu'(\psi) = \mu'(\phi\psi_2)$. By the loop condition $\mu'(\phi\psi_2) = \mu'(\psi_2) + 2\mu(\phi)$ and by the signature condition $\mu'(\psi_2) = \frac{1}{2} \text{Sign}(S^\pm)$. Thus

$$\mu'(\psi) = 2\mu(\phi) + \frac{1}{2} \text{Sign}(S^\pm).$$

Since the same is true for $\mu_{\text{CZ}}(\psi)$, this proves uniqueness. \square

Remark 2.8. — Let $\psi \in \text{SP}(n)$ and let $\tilde{\psi} : [0, 2] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ be an extension as before. The Conley-Zehnder index of ψ is equal to the integer given by the degree of the map $\tilde{\rho}^2 \circ \tilde{\psi} : [0, 2] \rightarrow S^1$:

$$\mu_{\text{CZ}}(\psi) := \deg(\tilde{\rho}^2 \circ \tilde{\psi}) \quad (2.4)$$

for any continuous map $\tilde{\rho} : \text{Sp}(\mathbb{R}^{2n}, \Omega_0) \rightarrow S^1$ which coincide with the (complex) determinant $\det_{\mathbb{C}}$ on $\text{U}(n) = \text{O}(\mathbb{R}^{2n}) \cap \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$, such that $\tilde{\rho}(W^-) = \pm 1$, and such that

$$\deg(\tilde{\rho}^2 \circ \psi_{2-}) = n - 1 \quad \text{for } \psi_{2-} : t \in [0, 1] \mapsto \exp t\pi J_0 S^-.$$

This is a direct consequence of the fact that the map defined by $\deg(\tilde{\rho}^2 \circ \tilde{\psi})$ has the homotopy property, the loop property (since any loop is homotopic to a loop of unitary matrices where ρ and $\det_{\mathbb{C}}$ coincide) and we have added what we need of the signature property to characterize the Conley-Zehnder index, i.e.. the value of the Conley-Zehnder index on the paths $\psi_{2\pm} : t \in [0, 1] \mapsto \exp t\pi J_0 S^\pm$. Indeed $\frac{1}{2} \text{Sign} S^- = n - 1$, $S^+ = \text{Id}_{2n}$, $\frac{1}{2} \text{Sign} S^+ = n$ and $\exp t\pi J_0 S^+ = \exp t\pi \begin{pmatrix} 0 & -\text{Id}_n \\ \text{Id}_n & 0 \end{pmatrix} = \begin{pmatrix} \cos \pi t \text{Id}_n & -\sin \pi t \text{Id}_n \\ \sin \pi t \text{Id}_n & \cos \pi t \text{Id}_n \end{pmatrix}$ is in $\text{U}(n)$ so that $\tilde{\rho}^2(\exp t\pi \begin{pmatrix} 0 & -\text{Id}_n \\ \text{Id}_n & 0 \end{pmatrix}) = e^{2\pi i n t}$ and $\deg(\tilde{\rho}^2 \circ \psi_{2+}) = n$.

Example 2.9. — In particular, the Conley-Zehnder index of a path $\psi \in \text{SP}(n)$ is given by

$$\mu_{\text{CZ}}(\psi) := \deg(\det_{\mathbb{C}}^2 \circ U \circ \tilde{\psi}) \quad (2.5)$$

where $U : \text{Sp}(\mathbb{R}^{2n}, \Omega_0) \rightarrow U(n)$ is the projection defined by the polar decomposition $U(A) = AP^{-1}$ with P the unique symmetric positive definite matrix such that $P^2 = A^\tau A$. The map $\tilde{\rho} := \det_{\mathbb{C}} \circ U$ is indeed continuous, coincides obviously with $\det_{\mathbb{C}}$ on $U(n)$ and we have that $\exp t\pi J_0 \begin{pmatrix} 0 & -\frac{\log 2}{\pi} \\ -\frac{\log 2}{\pi} & 0 \end{pmatrix} = \begin{pmatrix} 2^t & 0 \\ 0 & 2^{-t} \end{pmatrix}$ is a positive symmetric matrix so that $U(\exp t\pi J_0 S^-) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \pi t \text{Id}_{n-1} & 0 & -\sin \pi t \text{Id}_{n-1} \\ 0 & 0 & 0 & 0 \\ 0 & \sin \pi t \text{Id}_{n-1} & 0 & \cos \pi t \text{Id}_{n-1} \end{pmatrix}$; hence $\det_{\mathbb{C}}^2 \circ U(\exp t\pi J_0 S^-) = e^{2\pi i(n-1)t}$ and $\deg(\det_{\mathbb{C}}^2 \circ U \circ \psi_{2-}) = n - 1$. An equivalent version of formula (2.5) already appears in [2].

Example 2.10 One obtains a new formula for the Conley-Zehnder index of a path $\psi \in \text{SP}(n)$ using the parametrization of the symplectic group introduced by Rawnsley:

$$\mu_{\text{CZ}}(\psi) := \deg(\hat{\rho}^2 \circ \tilde{\psi}) \tag{2.6}$$

where $\hat{\rho} : \text{Sp}(\mathbb{R}^{2n}, \Omega_0) \rightarrow S^1$ is the normalized complex determinant of the \mathbb{C} -linear part of the matrix:

$$\hat{\rho}(A) = \frac{\det_{\mathbb{C}}(\frac{1}{2}(A - J_0 A J_0))}{|\det_{\mathbb{C}}(\frac{1}{2}(A - J_0 A J_0))|}. \tag{2.7}$$

Let us observe that this is well defined : for any $A \in \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$, the element $C_A := \frac{1}{2}(A - J_0 A J_0)$, which clearly defines a complex linear endomorphism of \mathbb{C}^n since it commutes with J_0 , is invertible. Indeed, for any non-zero $v \in V$, one has

$$4\Omega_0(C_A v, J_0 C_A v) = 2\Omega_0(v, J_0 v) + \Omega_0(Av, J_0 Av) + \Omega_0(AJ_0 v, J_0 A J_0 v) > 0.$$

If $A \in U(n)$, then $C_A = A$ so that $\hat{\rho}(A) = \det_{\mathbb{C}}(A)$, hence $\hat{\rho}$ is a continuous map which coincide with $\det_{\mathbb{C}}$ on $U(n)$. Furthermore

$\frac{1}{2} \left(\begin{pmatrix} 2^t & 0 \\ 0 & 2^{-t} \end{pmatrix} - J_0 \begin{pmatrix} 2^t & 0 \\ 0 & 2^{-t} \end{pmatrix} J_0 \right) = \frac{1}{2} \begin{pmatrix} 2^t + 2^{-t} & 0 \\ 0 & 2^t + 2^{-t} \end{pmatrix}$ hence its complex determinant is equal to $\frac{1}{2}(2^t + 2^{-t})$ and its normalized complex determinant is equal to 1 so that $\hat{\rho}(\exp t\pi J_0 S^-) = e^{\pi i(n-1)t}$ and $\deg(\hat{\rho}^2 \circ \psi_{2-}) = n - 1$.

3. The Robbin-Salamon index for a path of Lagrangians

A *Lagrangian* in a symplectic vector space (V, Ω) of dimension $2n$ is a subspace L of V of dimension n such that $\Omega|_{L \times L} = 0$. Given any Lagrangian L in V , there exists a Lagrangian M (not unique!) such that $L \oplus M = V$. With the choice of such a supplementary M any Lagrangian L' in a neighborhood of L (any Lagrangian supplementary to M) can be identified

to a linear map $\alpha : L \rightarrow M$ through $L' = \{v + \alpha(v) \mid v \in L\}$, with α such that $\Omega(\alpha(v), w) + \Omega(v, \alpha(w)) = 0 \forall v, w \in L$. Hence it can be identified to a symmetric bilinear form $\underline{\alpha} : L \times L \rightarrow \mathbb{R} : (v, v') \mapsto \Omega(v, \alpha(v'))$. In particular the tangent space at a point L to the space $\mathcal{L}_{(V, \Omega)}$ of Lagrangians in (V, Ω) can be identified to the space of symmetric bilinear forms on L .

If $\Lambda : [a, b] \rightarrow \mathcal{L}_n := \mathcal{L}_{(\mathbb{R}^{2n}, \Omega_0)} : t \mapsto \Lambda_t$ is a smooth curve of Lagrangian subspaces in $(\mathbb{R}^{2n}, \Omega_0)$, we define $Q(\Lambda_{t_0}, \dot{\Lambda}_{t_0})$ to be the symmetric bilinear form on Λ_{t_0} defined by

$$Q(\Lambda_{t_0}, \dot{\Lambda}_{t_0})(v, v') = \left. \frac{d}{dt} \underline{\alpha}_t(v, v') \right|_{t_0} = \left. \frac{d}{dt} \Omega(v, \alpha_t(v')) \right|_{t_0} \quad (3.1)$$

where $\alpha_t : \Lambda_{t_0} \rightarrow M$ is the map corresponding to Λ_t for a decomposition $\mathbb{R}^{2n} = \Lambda_{t_0} \oplus M$ with M Lagrangian. Then [4] :

- the symmetric bilinear form $Q(\Lambda_{t_0}, \dot{\Lambda}_{t_0}) : \Lambda_{t_0} \times \Lambda_{t_0} \rightarrow \mathbb{R}$ is independent of the choice of the supplementary Lagrangian M to Λ_{t_0} ;
- if $\psi \in \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ then

$$Q(\psi \Lambda_{t_0}, \psi \dot{\Lambda}_{t_0})(\psi v, \psi v') = Q(\Lambda_{t_0}, \dot{\Lambda}_{t_0})(v, v') \quad \forall v, v' \in \Lambda_{t_0}. \quad (3.2)$$

Let us choose and fix a Lagrangian L in $(\mathbb{R}^{2n}, \Omega_0)$. Consider a smooth path of Lagrangians $\Lambda : [a, b] \rightarrow \mathcal{L}_n$. A *crossing* for Λ is a number $t \in [a, b]$ for which $\dim \Lambda_t \cap L \neq 0$. At each crossing time $t \in [a, b]$ one defines the *crossing form*

$$\Gamma(\Lambda, L, t) = Q(\Lambda_t, \dot{\Lambda}_t) \Big|_{\Lambda_t \cap L}. \quad (3.3)$$

A crossing t is called *regular* if the crossing form $\Gamma(\Lambda, L, t)$ is nondegenerate. In that case $\Lambda_s \cap L = \{0\}$ for $s \neq t$ in a neighborhood of t .

DEFINITION 3.1 ([4]). — *For a curve $\Lambda : [a, b] \rightarrow \mathcal{L}_n$ with only regular crossings the Robbin-Salamon index is defined as*

$$\mu_{RS}(\Lambda, L) = \frac{1}{2} \text{Sign } \Gamma(\Lambda, L, a) + \sum_{\substack{a < t < b \\ t \text{ crossing}}} \text{Sign } \Gamma(\Lambda, L, t) + \frac{1}{2} \text{Sign } \Gamma(\Lambda, L, b). \quad (3.4)$$

Robbin and Salamon show (Lemmas 2.1 and 2.2 in [4]) that two paths with only regular crossings which are homotopic with fixed endpoints have the same Robbin-Salamon index and that every continuous path of Lagrangians is homotopic with fixed endpoints to one having only regular crossings. The Robbin-Salamon index of a continuous path Λ of Lagrangians is defined as the Robbin-Salamon index of a path which has only regular crossings and which is homotopic to Λ with fixed endpoints.

This index depends on the choice of the reference Lagrangian L . Robbin and Salamon show ([4], Theorem 2.3):

THEOREM 3.2 ([4]). — *The index μ_{RS} has the following properties:*

1. (*Naturality*) For $\psi \in \text{Sp}(\mathbb{R}^{2n}, \Omega)$ $\mu_{RS}(\psi\Lambda, \psi L) = \mu_{RS}(\Lambda, L)$.
2. (*Catenation*) For $a < c < b$, $\mu_{RS}(\Lambda, L) = \mu_{RS}(\Lambda|_{[a,c]}, L) + \mu_{RS}(\Lambda|_{[c,b]}, L)$.
3. (*Product*) If $n' + n'' = n$, identify $\mathcal{L}_{n'} \times \mathcal{L}_{n''}$ as a submanifold of \mathcal{L}_n in the obvious way. Then $\mu_{RS}(\Lambda' \oplus \Lambda'', L' \oplus L'') = \mu_{RS}(\Lambda', L') + \mu_{RS}(\Lambda'', L'')$.
4. (*Localization*) If $L = \mathbb{R}^n \times \{0\}$ and $\Lambda(t) = \text{Gr}(A(t))$ where $A(t)$ is a path of symmetric matrices, then the index of Λ is given by $\mu_{RS}(\Lambda, L) = \frac{1}{2} \text{Sign } A(b) - \frac{1}{2} \text{Sign } A(a)$.
5. (*Homotopy*) Two paths $\Lambda_0, \Lambda_1 : [a, b] \rightarrow \mathcal{L}_n$ with $\Lambda_0(a) = \Lambda_1(a)$ and $\Lambda_0(b) = \Lambda_1(b)$ are homotopic with fixed endpoints if and only if they have the same index.
6. (*Zero*) Every path $\Lambda : [a, b] \rightarrow \Sigma_k(V)$, with $\Sigma_k(V) = \{M \in \mathcal{L}_n \mid \dim M \cap L = k\}$, has index $\mu_{RS}(\Lambda, L) = 0$.

4. The Robbin-Salamon index for a path of symplectic matrices

4.1. Generalized Conley-Zehnder index

Consider the symplectic vector space $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \overline{\Omega} = -\Omega_0 \times \Omega_0)$. Given any linear map $\psi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$, its graph

$$\text{Gr } \psi = \{(x, \psi x) \mid x \in \mathbb{R}^{2n}\}$$

is a $2n$ -dimensional subspace of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ which is Lagrangian if and only if ψ is symplectic ($\psi \in \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$).

A particular Lagrangian is given by the diagonal

$$\Delta = \text{Gr Id} = \{(x, x) \mid x \in \mathbb{R}^{2n}\}. \tag{4.1}$$

Remark that $\text{Gr}(-\psi)$ is a Lagrangian subspace which is always supplementary to $\text{Gr } \psi$ for $\psi \in \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$. In fact $\text{Gr } \phi$ and $\text{Gr } \psi$ are supplementary if and only if $\phi - \psi$ is invertible.

DEFINITION 4.1. — *The Robbin-Salamon index of a continuous path of symplectic matrices $\psi : [0, 1] \rightarrow \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) : t \mapsto \psi_t$ is defined as the Robbin-Salamon index of the path of Lagrangians in $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \overline{\Omega})$,*

$$\Lambda = \mathrm{Gr} \psi : [0, 1] \rightarrow \mathcal{L}_{2n} : t \mapsto \mathrm{Gr} \psi_t$$

when the fixed Lagrangian is the diagonal Δ :

$$\mu_{RS}(\psi) := \mu_{RS}(\mathrm{Gr} \psi, \Delta). \tag{4.2}$$

Note that this index is defined for any continuous path of symplectic matrices but can have half integer values. It is used in the definition of the grading of some homologies, for instance Morse-Bott symplectic homology.

A crossing for a smooth path $\mathrm{Gr} \psi$ is a number $t \in [0, 1]$ for which 1 is an eigenvalue of ψ_t and

$$\mathrm{Gr} \psi_t \cap \Delta = \{ (x, x) \mid \psi_t x = x \}$$

is in bijection with $\mathrm{Ker}(\psi_t - \mathrm{Id})$.

The properties of homotopy, catenation and product of theorem 3.2 imply that [4]

- μ_{RS} is invariant under homotopies with fixed endpoints,
- μ_{RS} is additive under catenation of paths and
- μ_{RS} has the product property $\mu_{RS}(\psi' \widetilde{\oplus} \psi'') = \mu_{RS}(\psi') + \mu_{RS}(\psi'')$ as in proposition 2.6.

The zero property of the Robbin-Salamon index of a path of Lagrangians becomes:

PROPOSITION 4.2. — *If $\psi : [a, b] \rightarrow \mathrm{Sp}(\mathbb{R}^{2n}, \Omega)$ is a path of matrices such that $\dim \mathrm{Ker}(\psi(t) - \mathrm{Id}) = k$ for all $t \in [a, b]$ then $\mu_{RS}(\psi) = 0$.*

Indeed, $\mathrm{Gr} \psi_t \cap \Delta = \{v \in \mathbb{R}^{2n} \mid \psi_t v = v\}$ so $\dim(\mathrm{Gr} \psi_t \cap \Delta) = k$ if and only if $\dim \mathrm{Ker}(\psi(t) - \mathrm{Id}) = k$.

PROPOSITION 4.3 (Naturality). — *Consider two continuous paths of symplectic matrices $\psi, \phi : [0, 1] \rightarrow \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ and define $\psi' = \phi\psi\phi^{-1}$. Then*

$$\mu_{RS}(\psi') = \mu_{RS}(\psi)$$

Proof. — One has

$$\begin{aligned}
 \Lambda'_t := \text{Gr } \psi'_t &= \{(x, \phi_t \psi_t \phi_t^{-1} x) \mid x \in \mathbb{R}^{2n}\} \\
 &= \{(\phi_t y, \phi_t \psi_t y) \mid y \in \mathbb{R}^{2n}\} \\
 &= (\phi_t \times \phi_t) \text{Gr } \psi_t \\
 &= (\phi_t \times \phi_t) \Lambda_t
 \end{aligned}$$

and $(\phi_t \times \phi_t) \Delta = \Delta$. Furthermore $(\phi_t \times \phi_t) \in \text{Sp}(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \overline{\Omega})$. Hence $t \in [0, 1]$ is a crossing for the path of Lagrangians $\Lambda' = \text{Gr } \psi'$ if and only if $\dim \text{Gr } \psi'_t \cap \Delta \neq 0$ if and only if $\dim(\phi_t \times \phi_t)(\text{Gr } \psi_t \cap \Delta) \neq 0$ if and only if t is a crossing for the path of Lagrangian $\Lambda = \text{Gr } \psi$.

By homotopy with fixed endpoints, we can assume that Λ has only regular crossings and ϕ is locally constant around each crossing t so that

$$\frac{d}{dt}(\phi \psi \phi^{-1})(t) = \phi_t \dot{\psi}_t \phi_t^{-1}.$$

Then at each crossing

$$\begin{aligned}
 \Gamma(\text{Gr } \psi', \Delta, t) &= Q(\Lambda'_t, \dot{\Lambda}'_t)|_{\text{Gr } \psi'_t \cap \Delta} \\
 &= Q((\phi_t \times \phi_t) \Lambda_t, (\phi_t \times \phi_t) \dot{\Lambda}_t)|_{(\phi_t \times \phi_t) \text{Gr } \psi_t \cap \Delta} \\
 &= Q(\Lambda_t, \dot{\Lambda}_t)|_{\text{Gr } \psi_t \cap \Delta} \circ (\phi_t^{-1} \times \phi_t^{-1}) \otimes (\phi_t^{-1} \times \phi_t^{-1})
 \end{aligned}$$

in view of (3.2), so that

$$\text{Sign } \Gamma(\text{Gr } \psi', \Delta, t) = \text{Sign } \Gamma(\text{Gr } \psi, \Delta, t).$$

□

DEFINITION 4.4. — For any smooth path ψ of symplectic matrices, define a path of symmetric matrices S through

$$\dot{\psi}_t = J_0 S_t \psi_t.$$

This is indeed possible since $\psi_t \in \text{Sp}(\mathbb{R}^{2n}, \Omega_0) \forall t$, thus $\psi_t^{-1} \dot{\psi}_t$ is in the Lie algebra $\mathfrak{sp}(\mathbb{R}^{2n}, \Omega_0)$ and every element of this Lie algebra may be written in the form $J_0 S$ with S symmetric.

The symmetric bilinear form $Q(\text{Gr } \psi, \frac{d}{dt} \text{Gr } \psi)$ is given as follows. For any $t_0 \in [0, 1]$, write $\mathbb{R}^{2n} \times \mathbb{R}^{2n} = \text{Gr } \psi_{t_0} \oplus \text{Gr}(-\psi_{t_0})$. The linear map $\alpha_t : \text{Gr } \psi_{t_0} \rightarrow \text{Gr}(-\psi_{t_0})$ corresponding to $\text{Gr } \psi_t$ is obtained from:

$$(x, \psi_t x) = (y, \psi_{t_0} y) + \alpha_t(y, \psi_{t_0} y) = (y, \psi_{t_0} y) + (\tilde{\alpha}_t y, -\psi_{t_0} \tilde{\alpha}_t y)$$

if and only if $(\text{Id} + \tilde{\alpha}_t)y = x$ and $\psi_{t_0}(\text{Id} - \tilde{\alpha}_t)y = \psi_t x$, hence $\psi_{t_0}^{-1}\psi_t(\text{Id} + \tilde{\alpha}_t) = \text{Id} - \tilde{\alpha}_t$ and

$$\tilde{\alpha}_t = (\text{Id} + \psi_{t_0}^{-1}\psi_t)^{-1}(\text{Id} - \psi_{t_0}^{-1}\psi_t) \quad ; \quad \left. \frac{d}{dt}\tilde{\alpha}_t \right|_{t_0} = -\frac{1}{2}\psi_{t_0}^{-1}\dot{\psi}_{t_0}.$$

Thus

$$\begin{aligned} & Q\left(\text{Gr } \psi_{t_0}, \frac{d}{dt} \text{Gr } \psi_{t_0}\right)((v, \psi_{t_0}v), (v', \psi_{t_0}v')) \\ &= \left. \frac{d}{dt}\overline{\Omega}((v, \psi_{t_0}v), \alpha_t(v', \psi_{t_0}v')) \right|_{t_0} \\ &= \left. \frac{d}{dt}\overline{\Omega}((v, \psi_{t_0}v), (\tilde{\alpha}_t v', -\psi_{t_0}\tilde{\alpha}_t v')) \right|_{t_0} \\ &= -2\Omega_0\left(v, \left. \frac{d}{dt}\tilde{\alpha}_t \right|_{t_0} v'\right) \\ &= \Omega_0(v, \psi_{t_0}^{-1}\dot{\psi}_{t_0}v') \\ &= \Omega_0(\psi_{t_0}v, J_0 S_{t_0} \psi_{t_0}v'). \end{aligned}$$

Hence the restriction of Q to $\text{Ker}(\psi_{t_0} - \text{Id})$ is given by

$$Q\left(\text{Gr } \psi_{t_0}, \frac{d}{dt} \text{Gr } \psi_{t_0}\right)((v, \psi_{t_0}v), (v', \psi_{t_0}v')) = v^T S_{t_0} v' \quad \forall v, v' \in \text{Ker}(\psi_{t_0} - \text{Id}).$$

A crossing $t_0 \in [0, 1]$ is thus regular for the smooth path $\text{Gr } \psi$ if and only if the restriction of S_{t_0} to $\text{Ker}(\psi_{t_0} - \text{Id})$ is nondegenerate.

DEFINITION 4.5 ([4]). — *Let $\psi : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0) : t \mapsto \psi_t$ be a smooth path of symplectic matrices. Write $\dot{\psi}_t = J_0 S_t \psi_t$ with $t \mapsto S_t$ a path of symmetric matrices. A number $t \in [0, 1]$ is called a crossing if $\det(\psi_t - \text{Id}) = 0$. For $t \in [0, 1]$, the crossing form $\Gamma(\psi, t)$ is defined as the quadratic form which is the restriction of S_t to $\text{Ker}(\psi_t - \text{Id})$. A crossing t_0 is called regular if the crossing form $\Gamma(\psi, t_0)$ is nondegenerate.*

PROPOSITION 4.6 ([4]). — *For a smooth path $\psi : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0) : t \mapsto \psi_t$ having only regular crossings, the Robbin-Salamon index introduced in definition 4.1 is given by*

$$\mu_{RS}(\psi) = \frac{1}{2} \text{Sign } \Gamma(\psi, 0) + \sum_{\substack{t \text{ crossing} \\ t \in]0, 1[}} \text{Sign } \Gamma(\psi, t) + \frac{1}{2} \text{Sign } \Gamma(\psi, 1). \quad (4.3)$$

PROPOSITION 4.7 ([4]). — *Let $\psi : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ be a continuous path of symplectic matrices such that $\psi(0) = \text{Id}$ and such that 1 is not an*

eigenvalue of $\psi(1)$ (i.e. $\psi \in \text{SP}(n)$). The Robbin-Salamon index of ψ defined by (4.2) coincides with the Conley-Zehnder index of ψ . In particular, for a smooth path $\psi \in \text{SP}(n)$ having only regular crossings, the Conley-Zehnder index is given by

$$\begin{aligned} \mu_{CZ}(\psi) &= \frac{1}{2} \text{Sign} \Gamma(\psi, 0) + \sum_{\substack{t \text{ crossing,} \\ t \in]0, 1[}} \text{Sign} \Gamma(\psi, t) \\ &= \frac{1}{2} \text{Sign}(S_0) + \sum_{\substack{t \text{ crossing,} \\ t \in]0, 1[}} \text{Sign} \Gamma(\psi, t) \end{aligned} \quad (4.4)$$

with $S_0 = -J_0 \dot{\psi}_0$.

Proof. — Since the Robbin-Salamon index for paths of Lagrangians is invariant under homotopies with fixed endpoints, the Robbin-Salamon index for paths of symplectic matrices is also invariant under homotopies with fixed endpoints.

Its restriction to $\text{SP}(n)$ is actually invariant under homotopies of paths in $\text{SP}(n)$ since for any path in $\text{SP}(n)$, the starting point $\psi_0 = \text{Id}$ is fixed and the endpoint ψ_1 can only move in a connected component of $\text{Sp}^*(\mathbb{R}^{2n}, \Omega_0)$ where no matrix has 1 as an eigenvalue.

To show that this index coincides with the Conley-Zehnder index, it is enough, in view of proposition 2.7, to show that it satisfies the loop and signature properties.

Let us prove the signature property. Let $\psi_t = \exp(tJ_0S)$ with S a symmetric nondegenerate matrix with all eigenvalues of absolute value $< 2\pi$, so that $\text{Ker}(\exp(tJ_0S) - \text{Id}) = \{0\}$ for all $t \in]0, 1]$. Hence the only crossing is at $t = 0$, where $\psi_0 = \text{Id}$ and $\psi_t = J_0S\psi_t$ so that $S_t = S$ for all t and

$$\mu_{CZ}(\psi) = \frac{1}{2} \text{Sign} S_0 = \frac{1}{2} \text{Sign} S.$$

To prove the loop property, note that μ_{RS} is additive for catenation and invariant under homotopies with fixed endpoints. The path $(\phi\psi)$ is homotopic to the catenation of ϕ and ψ ; it is thus enough to show that the Robbin-Salamon index of a loop is equal to $2 \deg(\rho \circ \phi)$. Since two loops ϕ and ϕ' are homotopic if and only if $\deg(\rho \circ \phi) = \deg(\rho \circ \phi')$, it is enough to consider the loops ϕ_n defined by

$$\phi_n(t) := \begin{pmatrix} \cos 2\pi nt & -\sin 2\pi nt \\ \sin 2\pi nt & \cos 2\pi nt \end{pmatrix} \oplus \begin{pmatrix} a(t) \text{Id} & 0 \\ 0 & a(t)^{-1} \text{Id} \end{pmatrix}$$

with $a : [0, 1] \rightarrow \mathbb{R}^+$ a smooth curve with $a(0) = a(1) = 1$ and $a(t) \neq 1$ for $t \in]0, 1[$. Since $\rho(\phi_n(t)) = e^{2\pi i n t}$, we have $\deg(\phi_n) = n$.

The crossings of ϕ_n arise at $t = \frac{m}{n}$ with m an integer between 0 and n . At such a crossing, $\text{Ker}(\phi_n(t))$ is \mathbb{R}^2 for $0 < t < 1$ and is \mathbb{R}^{2n} for $t = 0$ and $t = 1$. We have

$$\dot{\phi}_n(t) = \left(\begin{pmatrix} 0 & -2\pi n \\ 2\pi n & 0 \end{pmatrix} \tilde{\oplus} \begin{pmatrix} \frac{\dot{a}(t)}{a(t)} \text{Id} & 0 \\ 0 & -\frac{\dot{a}(t)}{a(t)} \text{Id} \end{pmatrix} \right) \phi_n(t)$$

so that, extending $\tilde{\oplus}$ to symmetric matrices in the obvious way,

$$S(t) = \begin{pmatrix} 2\pi n & 0 \\ 0 & 2\pi n \end{pmatrix} \tilde{\oplus} \begin{pmatrix} 0 & -\frac{\dot{a}(t)}{a(t)} \text{Id} \\ -\frac{\dot{a}(t)}{a(t)} \text{Id} & 0 \end{pmatrix}.$$

Thus $\text{Sign } \Gamma(\phi_n, t) = 2$ for all crossings $t = \frac{m}{n}$, $0 \leq m \leq n$. From equation (4.3) we get

$$\begin{aligned} \mu_{RS}(\phi_n) &= \frac{1}{2} \text{Sign } \Gamma(\phi_n, 0) + \sum_{0 < m < n} \text{Sign } \Gamma(\phi_n, \frac{m}{n}) + \frac{1}{2} \text{Sign } \Gamma(\phi_n, 1) \\ &= 1 + 2(n - 1) + 1 = 2n = 2 \deg(\rho \circ \phi_n) \end{aligned}$$

and the loop property is proved. Thus the Robbin-Salamon index for paths in $\text{Sp}(n)$ coincides with the Conley-Zehnder index.

The formula for the Conley-Zehnder index of a path $\psi \in \text{Sp}(n)$ having only regular crossings, follows then from (4.3). Indeed, we have $\text{Ker}(\psi_1 - \text{Id}) = \{0\}$, while $\text{Ker}(\psi_0 - \text{Id}) = \mathbb{R}^{2n}$ and $\Gamma(\psi, 0) = S_0$. \square

DEFINITION 4.8. — *A symplectic shear is a path of symplectic matrices of the form $\psi_t = \begin{pmatrix} \text{Id} & B(t) \\ 0 & \text{Id} \end{pmatrix}$ with $B(t)$ symmetric.*

PROPOSITION 4.9. — *The Robbin-Salamon index of a symplectic shear $\psi_t = \begin{pmatrix} \text{Id} & B(t) \\ 0 & \text{Id} \end{pmatrix}$, with $B(t)$ symmetric, is equal to*

$$\mu_{RS}(\psi) = \frac{1}{2} \text{Sign } B(0) - \frac{1}{2} \text{Sign } B(1).$$

Proof. — We write $B(t) = A(t)^\tau D(t) A(t)$ with $A(t) \in O(\mathbb{R}^n)$ and $D(t)$ a diagonal matrix. The matrix $\phi_t = \begin{pmatrix} A(t)^\tau & 0 \\ 0 & A(t) \end{pmatrix}$ is in $\text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ and

$$\psi'_t := \phi_t \psi_t \phi_t^{-1} = \begin{pmatrix} \text{Id} & D(t) \\ 0 & \text{Id} \end{pmatrix}.$$

By proposition 4.3 $\mu_{RS}(\psi) = \mu_{RS}(\psi')$; by the product property it is enough to show that $\mu_{RS}(\psi) = \frac{1}{2} \text{Sign } d(0) - \frac{1}{2} \text{Sign } d(1)$ for the path

$$\psi : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^2, \Omega_0) : t \mapsto \psi_t = \begin{pmatrix} 1 & d(t) \\ 0 & 1 \end{pmatrix}.$$

Since μ_{RS} is invariant under homotopies with fixed endpoints, we may assume $\psi_t = \begin{pmatrix} a(t) & d(t) \\ c(t) & a(t)^{-1}(1+d(t)c(t)) \end{pmatrix}$ with a and c smooth functions such that $a(0) = 1$, $a(1) = 1$, $\dot{a}(0) \neq 0$, $\dot{a}(1) \neq 0$ and $a(t) > 1$ for $0 < t < 1$; $c(0) = c(1) = 0$, $c(t)d(t) \geq 0 \forall t$ and $\dot{c}(t) \neq 0$ (resp. = 0) when $d(t) \neq 0$ (resp. = 0) for $t = 0$ or 1 .

The only crossings are $t = 0$ and $t = 1$ since the trace of $\psi(t)$ is > 2 for $0 < t < 1$. Now, at those points ($t = 0$ and $t = 1$) $\dot{\psi}_t = \begin{pmatrix} \dot{a}(t) & \dot{d}(t) \\ \dot{c}(t) & -\dot{a}(t)+d(t)\dot{c}(t) \end{pmatrix}$ so that $S_t = -J_0 \dot{\psi}_t \psi_t^{-1} = \begin{pmatrix} \dot{c}(t) & -\dot{a}(t) \\ -\dot{a}(t) & \dot{a}(t)d(t)-\dot{d}(t) \end{pmatrix}$.

Clearly, at the crossings, we have $\text{Ker } \psi_t = \mathbb{R}^2$ iff $d(t) = 0$ and $\text{Ker } \psi_t$ is spanned by the first basis element iff $d(t) \neq 0$, so that from definition 4.5 $\Gamma(\psi, t) = (\dot{c}(t))$ when $d(t) \neq 0$ and $\Gamma(\psi, t) = \begin{pmatrix} 0 & -\dot{a}(t) \\ -\dot{a}(t) & 0 \end{pmatrix}$ when $d(t) = 0$. Hence both crossings are regular and $\text{Sign } \Gamma(\psi, t) = \text{Sign } \dot{c}(t)$ when $d(t) \neq 0$ and $\text{Sign } \Gamma(\psi, t) = 0$ when $d(t) = 0$. Since $d(t)c(t) \geq 0$ for all t , we clearly have $\text{Sign } \dot{c}(0) = \text{Sign } d(0)$ and $\text{Sign } \dot{c}(1) = -\text{Sign } d(1)$. Proposition 4.6 then gives $\mu_{RS}(\psi) = \frac{1}{2}\Gamma(\psi, 0) + \frac{1}{2}\text{Sign } \Gamma(\psi, 1) = \frac{1}{2}\text{Sign } d(0) - \frac{1}{2}\text{Sign } d(1)$. \square

Remark 4.10. — Robbin and Salamon introduce another index μ'_{RS} , called Maslov index, for paths of symplectic matrices built from their index for paths of Lagrangians. It is associated to the choice of a Lagrangian. Considering the fixed Lagrangian $L = \{0\} \times \mathbb{R}^n$ in $(\mathbb{R}^{2n}, \Omega_0)$, and observing that AL is Lagrangian for any $A \in \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$, they define, for $\psi : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$

$$\mu'_{RS}(\psi) := \mu_{RS}(\psi L, L). \tag{4.5}$$

They prove the following properties for this Maslov index.

- It is invariant under homotopies with fixed endpoints and two paths with the same endpoints are homotopic with fixed endpoints if and only if they have the same μ'_{RS} index;
- it is additive under catenation of paths;
- it has the product property $\mu_{RS}(\psi' \tilde{\oplus} \psi'') = \mu_{RS}(\psi') + \mu_{RS}(\psi'')$;
- it vanishes on a path whose image lies in

$$\{A \in \text{Sp}(\mathbb{R}^{2n}, \Omega_0) \mid \dim AL \cap L = k\}$$

for a given $k \in \{0, \dots, n\}$;

- $\mu'_{RS}(\psi) = \frac{1}{2}\text{Sign } B(0) - \frac{1}{2}\text{Sign } B(1)$ when $\psi_t = \begin{pmatrix} \text{Id } B(t) \\ 0 & \text{Id} \end{pmatrix}$.

They show that those properties characterize this index.

Remark that this Maslov index, being associated to a path of symplectic

matrices and the choice of a Lagrangian, is not invariant by conjugaison. Indeed, consider the path $\psi : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0) : t \mapsto \psi_t = \begin{pmatrix} \text{Id} & 0 \\ C(t) & \text{Id} \end{pmatrix}$. Since $\psi_t L \cap L = L \quad \forall t$, $\mu_{\text{RS}2}(\psi) = 0$. On the other hand, if $\phi = \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix}$ and $\psi' = \phi \psi \phi^{-1}$, then $\psi'_t = \begin{pmatrix} \text{Id} & -C(t) \\ 0 & \text{Id} \end{pmatrix}$ and $\mu'_{\text{RS}}(\psi') = \frac{1}{2} \text{Sign } C(1) - \frac{1}{2} \text{Sign } C(0)$ which is in general different from $\mu'_{\text{RS}}(\psi)$.

Hence the two indices μ_{RS} and μ'_{RS} defined on paths of symplectic matrices DO NOT coincide in general. The index μ'_{RS} vanishes on a path whose image lies into one of the $(n+1)$ strata defined by $\{A \in \text{Sp}(\mathbb{R}^{2n}, \Omega_0) \mid \dim AL \cap L = k\}$ for $0 \leq k \leq n$, whereas the index μ_{RS} vanishes on a path whose image lies into one of the $(2n + 1)$ strata defined by the set of symplectic matrices whose eigenspace of eigenvalue 1 has dimension k (for $0 \leq k \leq 2n$).

Remark however, that the two indices μ_{RS} and μ'_{RS} coincide on symplectic shears.

5. Characterization of the Robbin-Salamon index

In this section, we prove theorem 1.1 stated in the introduction. Before proving this theorem, we show that the Robbin-Salamon index is characterized by the fact that it extends Conley-Zehnder index and has all the properties stated in the previous section. This is made explicit in Lemma 5.2. We then use the characterization of the Conley-Zehnder index given in Proposition 2.7 to give in Lemma 5.3 a characterization of the Robbin-Salamon index in terms of six properties. We use explicitly the normal form of the restriction of a symplectic endomorphism to its generalized eigenspace of eigenvalue 1 that we have proven in [3] and that we summarize in the following proposition

PROPOSITION 5.1 (Normal form for $A|_{V_{[\lambda]}}$ for $\lambda = \pm 1$). — *Let $\lambda = \pm 1$ be an eigenvalue of $A \in \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ and let $V_{[\lambda]}$ be the generalized eigenspace of eigenvalue λ . There exists a symplectic basis of $V_{[\lambda]}$ in which the matrix associated to the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of matrices of the form*

$$\begin{pmatrix} J(\lambda, r_j)^{-1} & C(r_j, d_j, \lambda) \\ 0 & J(\lambda, r_j)^\tau \end{pmatrix}$$

where $J(\lambda, r)$ is the $r \times r$ elementary Jordan matrix with eigenvalue λ and where $C(r_j, d_j, \lambda) := J(\lambda, r_j)^{-1} \text{diag}(0, \dots, 0, d_j)$ with $d_j \in \{0, 1, -1\}$. If $d_j = 0$, then r_j is odd. The dimension of the eigenspace of eigenvalue 1 is given by $2 \text{Card}\{j \mid d_j = 0\} + \text{Card}\{j \mid d_j \neq 0\}$.

For any integer $k \geq 1$, the bilinear form on $\text{Ker}((A - \lambda \text{Id})^{2k})$ defined by

$$\begin{aligned} \hat{Q}_k &: \text{Ker}((A - \lambda \text{Id})^{2k}) \times \text{Ker}((A - \lambda \text{Id})^{2k}) \rightarrow \mathbb{R} \\ (v, w) &\mapsto \Omega((A - \lambda \text{Id})^k v, (A - \lambda \text{Id})^{k-1} w) \end{aligned} \quad (5.1)$$

is symmetric and we have

$$\sum_j d_j = \lambda \sum_{k \geq 1} \text{Signature}(\hat{Q}_k) \quad (5.2)$$

LEMMA 5.2. — *The Robbin-Salamon index is characterized by the following properties:*

1. (Generalization) *it is a correspondence μ_{RS} which associates a half integer to any continuous path $\psi : [a, b] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ of symplectic matrices and it coincides with μ_{CZ} on paths starting from the identity matrix and ending at a matrix for which 1 is not an eigenvalue;*
2. (Naturality) *if $\phi, \psi : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$, we have $\mu_{RS}(\phi\psi\phi^{-1}) = \mu_{RS}(\psi)$;*
3. (Homotopy) *it is invariant under homotopies with fixed endpoints;*
4. (Catenation) *it is additive under catenation of paths;*
5. (Product) *it has the product property $\mu_{RS}(\psi' \tilde{\oplus} \psi'') = \mu_{RS}(\psi') + \mu_{RS}(\psi'')$;*
6. (Zero) *it vanishes on any path $\psi : [a, b] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega)$ of matrices such that $\dim \text{Ker}(\psi(t) - \text{Id}) = k$ is constant on $[a, b]$;*
7. (Shear) *on a symplectic shear, $\psi : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ of the form*

$$\psi_t = \begin{pmatrix} \text{Id} & -tB \\ 0 & \text{Id} \end{pmatrix} = \exp t \begin{pmatrix} 0 & -B \\ 0 & 0 \end{pmatrix} = \exp t J_0 \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$$

with B symmetric, it is equal to $\mu_{RS}(\psi) = \frac{1}{2} \text{Sign } B$.

Proof. — We have seen in the previous section that the index μ_{RS} defined by Robbin and Salamon satisfies all the above properties. To see that those properties characterize this index, it is enough to show (since the group $\text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ is connected and since we have the catenation property) that those properties determine the index of any path starting from the identity. Since it must be a generalization of the Conley-Zehnder index and must be additive for catenations of paths, it is enough to show that any symplectic matrix A which admits 1 as an eigenvalue can be linked to a

matrix B which does not admit 1 as an eigenvalue by a continuous path whose index is determined by the properties stated. From proposition 5.1, there is a basis of \mathbb{R}^{2n} such that A is the symplectic direct sum of a matrix which does not admit 1 as eigenvalue and matrices of the form

$$A_{r_j, d_j}^{(1)} := \begin{pmatrix} J(1, r_j)^{-1} & J(1, r_j)^{-1} \text{diag}(0, \dots, 0, d_j) \\ 0 & J(1, r_j)^\tau \end{pmatrix};$$

with d_j equal to 0, 1 or -1 . The dimension of the eigenspace of eigenvalue 1 for $A_{r_j, d_j}^{(1)}$ is equal to 1 if $d_j \neq 0$ and is equal to 2 if $d_j = 0$. In view of the naturality and the product property of the index, we can consider a symplectic direct sum of paths with the constant path on the symplectic subspace where 1 is not an eigenvalue and we just have to build a path in $\text{Sp}(\mathbb{R}^{2r_j}, \Omega_0)$ from $A_{r_j, d_j}^{(1)}$ to a matrix which does not admit 1 as eigenvalue and whose index is determined by the properties given in the statement. This we do by the catenation of three paths : we first build the path $\psi_1 : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2r_j}, \Omega_0)$ defined by

$$\psi_1(t) := \begin{pmatrix} D(t, r_j)^{-1} & D(t, r_j)^{-1} \text{diag}(c(t), 0, \dots, 0, d(t)) \\ 0 & D(t, r_j)^\tau \end{pmatrix}$$

with $D(t, r_j) = \begin{pmatrix} 1 & 1-t & 0 & \dots & \dots & 0 \\ 0 & e^t & 1-t & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & e^t & 1-t & 0 \\ 0 & \dots & \dots & 0 & e^t & 1-t \\ 0 & \dots & \dots & \dots & 0 & e^t \end{pmatrix},$

and with $c(t) = td_j$, $d(t) = (1-t)d_j$. Observe that $\psi_1(0) = A_{r_j, d_j}^{(1)}$ and $\psi_1(1)$ is the symplectic direct sum of $\begin{pmatrix} 1 & c(1)=d_j \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} e^{-1} \text{Id}_{r_j-1} & 0 \\ 0 & e \text{Id}_{r_j-1} \end{pmatrix}$ and this last matrix does not admit 1 as eigenvalue.

Clearly $\dim \ker(\psi_1(t) - \text{Id}) = 2$ for all $t \in [0, 1]$ when $d_j = 0$; we now prove that $\dim \ker(\psi_1(t) - \text{Id}) = 1$ for all $t \in [0, 1]$ when $d_j \neq 0$. Hence the index of ψ_1 must always be zero by the zero property.

To prove that $\dim \ker(\psi_1(t) - \text{Id}) = 1$ we have to show the non vanishing of the determinant of the $2r_j - 1 \times 2r_j - 1$ matrix

$$\begin{pmatrix} E_{12}^t & \dots & \dots & E_{1r_j}^t & c(t) & 0 & \dots & 0 & E_{1r_j}^t d(t) \\ e^{-t}-1 & E_{23}^t & \ddots & E_{2r_j}^t & 0 & 0 & \dots & 0 & E_{2r_j}^t d(t) \\ 0 & \ddots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \ddots & e^{-t}-1 & E_{r_j-1r_j}^t & 0 & 0 & \dots & 0 & E_{r_j-1r_j}^t d(t) \\ 0 & \dots & 0 & e^{-t}-1 & 0 & 0 & \dots & 0 & e^{-t}d(t) \\ 0 & \dots & 0 & 0 & 1-t & e^t-1 & 0 & \ddots & 0 \\ \vdots & \dots & \vdots & 0 & 0 & 1-t & e^t-1 & \ddots & 0 \\ \vdots & \dots & \vdots & 0 & \dots & 0 & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1-t & e^t-1 \end{pmatrix}$$

where $E^t := D(t, r_j)^{-1}$ is upper triangular. This determinant is equal to

$$(-1)^{r_j+1}c(t)(e^{-t}-1)^{r_j-1}(e^t-1)^{r_j-1} + (-1)^{r_j-1}d(t)(1-t)^{r_j-1} \det E'(t)$$

where $E'(t)$ is obtained by deleting the first column and the last line in $E(t) - \text{Id}$ so given by the $(r_j - 1) \times (r_j - 1)$ matrix

$$\begin{pmatrix} (t-1)e^{-t} & (t-1)^2e^{-2t} & \dots & \dots & (t-1)^{r_j-1}e^{-(r_j-1)t} \\ e^{-t}-1 & (t-1)e^{-2t} & (t-1)^2e^{-3t} & \dots & (t-1)^{r_j-2}e^{-(r_j-1)t} \\ 0 & e^{-t}-1 & (t-1)e^{-2t} & \ddots & (t-1)^{r_j-3}e^{-(r_j-2)t} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & e^{-t}-1 & (t-1)e^{-2t} & (t-1)^2e^{-3t} \\ 0 & \dots & 0 & e^{-t}-1 & (t-1)e^{-2t} \end{pmatrix}$$

Thus $\det E'(t) = (t-1)(e^{-t} - (e^{-t}-1)) \det F_{r_j-2}(t)$ where

$$F_m(t) := \begin{pmatrix} (t-1)e^{-2t} & (t-1)^2e^{-3t} & \dots & \dots & (t-1)^{r_j-2}e^{-(r_j-1)t} \\ e^{-t}-1 & (t-1)e^{-2t} & \ddots & \dots & (t-1)^{r_j-3}e^{-(r_j-2)t} \\ 0 & e^{-t}-1 & (t-1)e^{-2t} & \ddots & (t-1)^{r_j-3}e^{-(r_j-2)t} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & e^{-t}-1 & (t-1)e^{-2t} & (t-1)^2e^{-3t} \\ 0 & \dots & 0 & e^{-t}-1 & (t-1)e^{-2t} \end{pmatrix}$$

and we have $\det F_m(t) = ((t-1)e^{-2t} - (e^{-t}-1)(t-1)e^{-t}) \det F_{m-1}(t) = (t-1)e^{-t} \det F_{m-1}(t)$ so that, by induction on m , $\det F_m(t) = (t-1)^m e^{-(m+1)t}$ hence the determinant we have to study is

$(-1)^{r_j-1}c(t)(2 - e^t - e^{-t})^{r_j-1} + d(t)(t-1)^{r_j} \det F_{r_j-2}(t)$ which is equal to $(-1)^{r_j-1}c(t)(2 - e^t - e^{-t})^{r_j-1} + d(t)(t-1)^{r_j}(t-1)^{r_j-2}e^{-(r_j-1)t}$ hence to

$$c(t)(e^t + e^{-t} - 2)^{r_j-1} + d(t)(1-t)^{2r_j-2}e^{-(r_j-1)t}$$

which never vanishes if $c(t) = td_j$ and $d(t) = (1-t)d_j$ since $e^t + e^{-t} - 2$ and $(1-t)$ are ≥ 0 .

We then construct a path $\psi_2 : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2r_j}, \Omega_0)$ which is constant on the symplectic subspace where 1 is not an eigenvalue and which is a symplectic shear on the first two dimensional symplectic vector space, i.e.

$$\psi_2(t) := \begin{pmatrix} 1 & (1-t)d_j \\ 0 & 1 \end{pmatrix} \tilde{\oplus} \begin{pmatrix} e^{-1} \text{Id}_{r_j-1} & 0 \\ 0 & e \text{Id}_{r_j-1} \end{pmatrix};$$

then the index of ψ_2 is equal to $\frac{1}{2} \text{Sign } d_j$. Observe that ψ_2 is constant if $d_j = 0$; then the index of ψ_2 is zero. In all cases $\psi_2(1) = \text{Id}_2 \tilde{\oplus} \begin{pmatrix} e^{-1} \text{Id}_{r_j-1} & 0 \\ 0 & e \text{Id}_{r_j-1} \end{pmatrix}$.

We then build $\psi_3 : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2r_j}, \Omega_0)$ given by

$$\psi_3(t) := \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \tilde{\oplus} \begin{pmatrix} e^{-1} \text{Id}_{r_j-1} & 0 \\ 0 & e \text{Id}_{r_j-1} \end{pmatrix}$$

which is the direct sum of a path whose Conley-Zehnder index is known and a constant path whose index is zero. Clearly 1 is not an eigenvalue of $\psi_3(1)$. \square

Combining the above with the characterization of the Conley-Zehnder index, we now prove:

LEMMA 5.3. — *The Robbin-Salamon index for a path of symplectic matrices is characterized by the following properties:*

- (Homotopy) *it is invariant under homotopies with fixed endpoints;*
- (Catenation) *it is additive under catenation of paths;*
- (Zero) *it vanishes on any path $\psi : [a, b] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega)$ of matrices such that $\dim \text{Ker}(\psi(t) - \text{Id}) = k$ is constant on $[a, b]$;*
- (Product) *it has the product property $\mu_{RS}(\psi' \tilde{\oplus} \psi'') = \mu_{RS}(\psi') + \mu_{RS}(\psi'')$;*
- (Signature) *if $S = S^\tau \in \mathbb{R}^{2n \times 2n}$ is a symmetric non degenerate matrix with all eigenvalues of absolute value $< 2\pi$ and if $\psi(t) = \exp(J_0 S t)$ for $t \in [0, 1]$, then $\mu_{RS}(\psi) = \frac{1}{2} \text{Sign } S$ where $\text{Sign } S$ is the signature of S ;*
- (Shear) *if $\psi_t = \exp t J_0 \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$ for $t \in [0, 1]$, with B symmetric, then $\mu_{RS}(\psi) = \frac{1}{2} \text{Sign } B$.*

Proof. — Remark first that the invariance by homotopies with fixed endpoints, the additivity under catenation and the zero property imply the naturality; they also imply the constancy on the components of $\text{SP}(n)$. The signature property stated above is the signature property which arose in the characterization of the Conley-Zehnder index given in proposition 2.7. To be sure that our index is a generalization of the Conley-Zehnder index, there remains just to prove the loop property. Since the product of a loop ϕ and a path ψ starting at the identity is homotopic to the catenation of ϕ and ψ , it is enough to prove that the index of a loop ϕ with $\phi(0) = \phi(1) = \text{Id}$ is given by $2 \deg(\rho \circ \phi)$. Since two loops ϕ and ϕ' are homotopic if and only if $\deg(\rho \circ \phi) = \deg(\rho \circ \phi')$, it is enough to consider the loops ϕ_n defined by $\phi_n(t) := \begin{pmatrix} \cos 2\pi nt & -\sin 2\pi nt \\ \sin 2\pi nt & \cos 2\pi nt \end{pmatrix} \tilde{\oplus} \text{Id}$; since $\phi_n(t) = (\phi_1(t))^n$, it is enough to show, using the homotopy, catenation, product and zero properties that the index of the loop given by $\phi(t) = \begin{pmatrix} \cos 2\pi t & -\sin 2\pi t \\ \sin 2\pi t & \cos 2\pi t \end{pmatrix}$ for $t \in [0, 1]$ is equal to 2. This is true, using the signature property, writing ϕ as the catenation of the path $\psi_1(t) := \phi(\frac{t}{2}) = \exp tJ_0 \begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}$ for $t \in [0, 1]$ whose index is 1 and the path $\psi_2(t) := \phi(\frac{t}{2}) = \exp tJ_0 \begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}$ for $t \in [1, 2]$. We introduce the path in the reverse direction $\psi_2^-(t) := \exp -tJ_0 \begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}$ for $t \in [0, 1]$ whose index is -1 ; since the catenation of ψ_2^- and ψ_2 is homotopic to the constant path whose index is zero, the index of ϕ_1 is given by the index of ψ_1 minus the index of ψ_2^- hence is equal to 2. \square

We are now ready to prove the characterization of the Robbin-Salamon index stated in the introduction.

Proof of theorem 1.1. — Observe that any symmetric matrix can be written as the symplectic direct sum of a non degenerate symmetric matrix S and a matrix S' of the form $\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$ where B is symmetric and may be degenerate. The index of the path $\psi_t = \exp tJ_0 S'$ is equal to the index of the path $\psi'_t = \exp t\lambda J_0 S'$ for any $\lambda > 0$. Hence the signature and shear conditions, in view of the product condition, can be simultaneously written as: if $S = S^\tau \in \mathbb{R}^{2n \times 2n}$ is a symmetric matrix with all eigenvalues of absolute value $< 2\pi$ and if $\psi(t) = \exp(J_0 S t)$ for $t \in [0, 1]$, then $\mu_{\text{RS}}(\psi) = \frac{1}{2} \text{Sign } S$. This is the normalization condition stated in the theorem.

From Lemma 5.3, we just have to prove that the product property is a consequence of the other properties. We prove it for paths with values in $\text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ by induction on n , the case $n = 1$ being obvious. Since $\psi' \tilde{\oplus} \psi''$ is homotopic with fixed endpoints to the catenation of $\psi' \tilde{\oplus} (\psi''(0))$ and $(\psi'(1)) \tilde{\oplus} \psi''$, it is enough to show that the index of $A \tilde{\oplus} \psi$ is equal to the index of ψ for any fixed $A \in \text{Sp}(\mathbb{R}^{2n'}, \Omega_0)$ with $n' < n$ and any continuous path $\psi : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2n''), \Omega_0)$ with $n'' < n$.

Using the proof of lemma 5.2, any symplectic matrix A can be linked by a path $\phi(s)$ with constant dimension of the 1-eigenspace to a matrix of the form $\exp(J_0 S')$ with S' a symmetric $n' \times n'$ matrix with all eigenvalues of absolute value $< 2\pi$. The index of $A \tilde{\oplus} \psi$ is equal to the index of $\exp(J_0 S') \tilde{\oplus} \psi$; indeed $A \tilde{\oplus} \psi$ is homotopic with fixed endpoints to the catenation of the three paths $\phi_s \tilde{\oplus} \psi(0)$, $\exp(J_0 S') \tilde{\oplus} \psi$ and the path $\phi_s \tilde{\oplus} \psi(1)$ in the reverse order, and the index of the first and third paths are zero since the dimension of the 1-eigenspace does not vary along those paths.

Hence it is enough to show that the index of $\exp(J_0 S') \tilde{\oplus} \psi$ is the same as the index of ψ . This is true because the map μ sending a path ψ in $\text{Sp}(\mathbb{R}^{2n'}, \Omega_0)$ (with $n' < n$) to the index of $\exp(J_0 S') \tilde{\oplus} \psi$ has the four properties stated in the theorem, and these characterize the Robbin-Salamon index for those paths by induction hypothesis. It is clear that μ is invariant under homotopies, additive for catenation and equal to zero on paths ψ for which the dimension of the 1-eigenspace is constant. Furthermore $\mu(\exp t(J_0 S))$ which is the index of $\exp(J_0 S') \tilde{\oplus} \exp t(J_0 S)$ is equal to $\frac{1}{2} \text{Sign } S$, because the path $\exp t J_0(S' \tilde{\oplus} S)$ whose index is $\frac{1}{2} \text{Sign}(S' \tilde{\oplus} S) = \frac{1}{2} \text{Sign } S' + \frac{1}{2} \text{Sign } S$ is homotopic with fixed endpoints with the catenation of $\exp t(J_0 S') \tilde{\oplus} \text{Id} = \exp t J_0(S' \tilde{\oplus} 0)$, whose index is $\frac{1}{2} \text{Sign } S'$, and the path $\exp(J_0 S') \tilde{\oplus} \exp t(J_0 S)$. \square

6. A formula for the Robbin-Salamon index

Let $\psi : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ be a path of symplectic matrices. The symplectic transformation $\psi(1)$ of $V = \mathbb{R}^{2n}$ decomposes as

$$\psi(1) = \psi^*(1) \tilde{\oplus} \psi^{(1)}(1)$$

where $\psi^*(1)$ does not admit 1 as eigenvalue and $\psi^{(1)}(1)$ is the restriction of $\psi(1)$ to the generalized eigenspace of eigenvalue 1

$$\psi(1)|_{V_{[1]}}.$$

By proposition 5.1, there exists a symplectic matrix A such that $A\psi^{(1)}(1)A^{-1}$ is equal to

$$\begin{aligned} \psi^*(1) \tilde{\oplus} \left(\begin{array}{cc} J(1, r_1)^{-1} C(r_1, d_1^{(1)}, 1) & \\ 0 & J(1, r_1)^\tau \end{array} \right) \tilde{\oplus} \dots \tilde{\oplus} \left(\begin{array}{cc} J(1, r_k)^{-1} C(r_k, d_k^{(1)}, 1) & \\ 0 & J(1, r_k)^\tau \end{array} \right) & \quad (6.1) \\ \tilde{\oplus} \left(\begin{array}{cc} J(1, s_1)^{-1} & 0 \\ 0 & J(1, s_1)^\tau \end{array} \right) \tilde{\oplus} \dots \tilde{\oplus} \left(\begin{array}{cc} J(1, s_l)^{-1} & 0 \\ 0 & J(1, s_l)^\tau \end{array} \right) \end{aligned}$$

with each $d_j^{(1)} = \pm 1$. Since $\text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ is connected, there is a path $\varphi : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ such that $\varphi(0) = \text{Id}$ and $\varphi(1) = A$. We define

$$\psi_I : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0) : t \mapsto \varphi(t)\psi(t)(\varphi(t))^{-1}.$$

It is a path from $\psi(1)$ to the matrix defined in 6.1. Clearly, $\mu_{RS}(\psi_I) = 0$ and ρ is constant on ψ_I .

Let $\psi_{II} : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ be the path from $\psi_I(1)$ to

$$\psi^*(1) \tilde{\oplus} \begin{pmatrix} 1 & d_1^{(1)} \\ 0 & 1 \end{pmatrix} \tilde{\oplus} \dots \tilde{\oplus} \begin{pmatrix} 1 & d_k^{(1)} \\ 0 & 1 \end{pmatrix} \tilde{\oplus} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{\oplus} \dots \tilde{\oplus} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{\oplus} \begin{pmatrix} e^{-1} \text{Id} & 0 \\ 0 & e \text{Id} \end{pmatrix}$$

defined as in the proof of lemma 5.2 in each block by

$$\begin{pmatrix} D(t, r_j)^{-1} & D(t, r_j)^{-1} \text{diag}(td_j^{(1)}, 0, \dots, 0, (1-t)d_j^{(1)}) \\ 0 & D(t, r_j)^\tau \end{pmatrix}$$

with $D(t, r_j) = \begin{pmatrix} 1 & 1-t & 0 & \dots & \dots & 0 \\ 0 & e^t & 1-t & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & e^t & 1-t & 0 \\ 0 & \dots & \dots & 0 & e^t & 1-t \\ 0 & \dots & \dots & \dots & 0 & e^t \end{pmatrix}$. Note that $\mu_{RS}(\psi_{II}) = 0$ since the

eigenspace of eigenvalue 1 has constant dimension and ρ is constant on ψ_{II} . We define $\psi_{III} : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ from $\psi_{II}(1)$ to

$$\psi^*(1) \tilde{\oplus} \begin{pmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix} \tilde{\oplus} \begin{pmatrix} e^{-1} \text{Id} & 0 \\ 0 & e \text{Id} \end{pmatrix}$$

which is given on each block $\begin{pmatrix} 1 & d_j^{(1)} \\ 0 & 1 \end{pmatrix}$ by $\begin{pmatrix} 1 & (1-t)d_j^{(1)} \\ 0 & 1 \end{pmatrix}$. Note that $\mu_{RS}(\psi_{III}) = \frac{1}{2} \sum_j d_j^{(1)}$ by proposition 4.9 and ρ is constant on ψ_{III} .

Finally, consider $\psi_{IV} : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ from $\psi_{III}(1)$ to

$$\psi^*(1) \tilde{\oplus} \begin{pmatrix} e^{-1} \text{Id} & 0 \\ 0 & e \text{Id} \end{pmatrix}$$

which is given by $\psi^*(1) \tilde{\oplus} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \tilde{\oplus} \begin{pmatrix} e^{-1} \text{Id} & 0 \\ 0 & e \text{Id} \end{pmatrix}$. Note that $\mu_{RS}(\psi_{IV}) = 0$, ρ is constant on ψ_{IV} and $\psi_{IV}(1)$ is in $\text{Sp}^*(\mathbb{R}^{2n}, \Omega_0)$. Since two paths of matrices with fixed ends are homotopic if and only if their image under ρ are homotopic, the catenation of the paths ψ_{III} and ψ_{IV} is homotopic to any path from $\psi_I(1)$ to $\psi^*(1) \tilde{\oplus} \begin{pmatrix} e \text{Id} & 0 \\ 0 & e^{-1} \text{Id} \end{pmatrix}$ of the form $\psi^*(1) \tilde{\oplus} \phi_1(t)$ where $\phi_1(t)$ has only real positive eigenvalues. We proceed similarly for $\psi(0)$ and we get

THEOREM 6.1. — *Let $\psi : [0, 1] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ be a path of symplectic matrices. Decompose $\psi(0) = \psi^*(0) \tilde{\oplus} \psi^{(1)}(0)$ and $\psi(1) = \psi^*(1) \tilde{\oplus} \psi^{(1)}(1)$ where $\psi^*(0)$ (resp. $\psi^*(1)$) does not admit 1 as eigenvalue and $\psi^{(1)}(0)$ (resp. $\psi^{(1)}(1)$) is the restriction of $\psi(0)$ (resp. $\psi(1)$) to the generalized eigenspace of eigenvalue 1 of $\psi(0)$ (resp. $\psi(1)$). Consider a prolongation $\Psi : [-1, 2] \rightarrow \text{Sp}(\mathbb{R}^{2n}, \Omega_0)$ of ψ such that*

- $\Psi(t) = \psi(t) \forall t \in [0, 1]$;

- $\Psi(-\frac{1}{2}) = \psi^*(0) \tilde{\oplus} \begin{pmatrix} e^{-1} \text{Id} & 0 \\ 0 & e \text{Id} \end{pmatrix}$ and $\Psi(t) = \psi^*(0) \tilde{\oplus} \phi_0(t)$ where $\phi_0(t)$ has only real positive eigenvalues for $t \in [-\frac{1}{2}, 0]$;
- $\Psi(\frac{3}{2}) = \psi^*(1) \tilde{\oplus} \begin{pmatrix} e^{-1} \text{Id} & 0 \\ 0 & e \text{Id} \end{pmatrix}$ and $\Psi(t) = \psi^*(1) \tilde{\oplus} \phi_1(t)$ where $\phi_1(t)$ has only real positive eigenvalues for $t \in [1, \frac{3}{2}]$;
- $\Psi(-1) = W^\pm$, $\Psi(2) = W^\pm$ and $\Psi(t) \in \text{Sp}^*(\mathbb{R}^{2n}, \Omega_0)$ for $t \in [-1, -\frac{1}{2}] \cup [\frac{3}{2}, 2]$.

Then

$$\mu_{RS}(\psi) = \text{deg}(\rho^2 \circ \Psi) + \frac{1}{2} \sum d_i^{(0)} - \frac{1}{2} \sum d_j^{(1)}.$$

Remark that we can replace in the formula above ρ by $\tilde{\rho}$ as in proposition 2.8.

By proposition 5.1, we have theorem 1.2 :

$$\mu_{RS}(\psi) = \text{deg}(\rho^2 \circ \Psi) + \frac{1}{2} \sum_{k=1}^{\dim V} \text{Sign}(\hat{Q}_k^{(\psi(0))}) - \frac{1}{2} \sum_{k=1}^{\dim V} \text{Sign}(\hat{Q}_k^{(\psi(1))}).$$

Remark 6.2. — The advantage of this new formula is that to compute the index of a path whose crossing with the Maslov cycle is non transverse we do not need to perturb the path. The drawback is that we have to extend the initial path.

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