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# A supplementary proof of $L^p$–logarithmic Sobolev inequality

Yasuhiro Fujita[1]

**Abstract.** — In this paper, we bridge a gap in the proof of the $L^p$–logarithmic Sobolev inequality obtained by Gentil [8, Theorem 1.1], and provide a supplementary proof. Our proof is based on a Hamilton–Jacobi equation and several approximations of functions in $W^{1,p}(\mathbb{R}^n)$.

**Résumé.** — Dans cet article, nous complétons la preuve de l'inégalité de Sobolev logarithmique $L^p$ obtenue par Gentil dans [8] et donnons aussi une preuve supplémentaire. Notre approche est basée sur une équation de Hamilton–Jacobi et sur plusieurs approximations de fonctions dans $W^{1,p}(\mathbb{R}^n)$.

## 1. Introduction

Let $n \in \mathbb{N}$. For a smooth enough function $f \geqslant 0$ on $\mathbb{R}^n$, we define the entropy of $f$ with respect to the Lebesgue measure by

$$\mathrm{Ent}(f) = \int f(x) \log f(x) dx - \int f(x) dx \ \log \int f(x) dx.$$

In this paper, the integral without its domain is always understood as the one over $\mathbb{R}^n$, and we interpret that $0 \log 0 = 0$.

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Let $p \geqslant 1$. We denote by $W^{1,p}(\mathbb{R}^n)$ the space of all weakly differentiable functions $f$ on $\mathbb{R}^n$ such that $f$ and $|Df|$ (the Euclidean length of the gradient $Df$ of $f$) are in $L^p(\mathbb{R}^n)$. For $f \in W^{1,p}(\mathbb{R}^n)$, the following $L^p$–logarithmic Sobolev inequality was shown for $p = 2$ by [10], $p = 1$ by [9], and $1 < p < n$ by [6]:

$$\operatorname{Ent}(|f|^p) \leqslant \frac{n}{p} \int |f(x)|^p dx \ \log \left( L_p \frac{\displaystyle\int |Df(x)|^p \, dx}{\displaystyle\int |f(x)|^p \, dx} \right). \tag{1.1}$$

Here,

$$L_p = \begin{cases} \dfrac{p}{n} \left( \dfrac{p-1}{e} \right)^{p-1} \pi^{-p/2} \left( \dfrac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(n \, \frac{p-1}{p} + 1\right)} \right)^{p/n}, & p > 1, \\ \dfrac{1}{n} \pi^{-1/2} \left[ \Gamma\left(\dfrac{n}{2}+1\right) \right]^{1/n}, & p = 1. \end{cases} \tag{1.2}$$

This is the best possible constant satisfying (1.1) for $1 \leqslant p < n$ (cf. [1, 6]).

For a general $p > 1$, with a deep insight, Gentil [8, Theorem 1.1] tried to give inequality (1.1) in the following way: First, he gave a hypercontractivity inequality for the unique viscosity solution to the Cauchy problem of the Hamilton-Jacobi equation

$$u_t(x,t) + \frac{1}{p}|Du(x,t)|^p = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{1.3}$$

$$u(\cdot,0) = \phi \qquad \text{in } \mathbb{R}^n. \tag{1.4}$$

Here, $\phi \in \operatorname{Lip}(\mathbb{R}^n)$. He showed that if there is a constant $\alpha > 0$ such that $e^{\phi} \in L^{\alpha}(\mathbb{R}^n)$, then $e^{u(\cdot,t)} \in L^{\beta}(\mathbb{R}^n)$ for any $\beta > \alpha$ and $t > 0$ and

$$\|e^{u(\cdot,t)}\|_{\beta} \leqslant \|e^{\phi}\|_{\alpha} \left( \frac{nL_p e^{p-1}(\beta-\alpha)}{p^p t} \right)^{\frac{n}{p}\frac{\beta-\alpha}{\alpha\beta}} \frac{\alpha^{\frac{n}{\alpha\beta}\left(\frac{\alpha}{p} + \frac{(p-1)\beta}{p}\right)}}{\beta^{\frac{n}{\alpha\beta}\left(\frac{\beta}{p} + \frac{(p-1)\alpha}{p}\right)}}, \tag{1.5}$$

where $L_p$ is the constant of (1.2) and

$$\|f\|_{\gamma} = \left( \int |f(x)|^{\gamma} dx \right)^{1/\gamma}, \quad \gamma > 0.$$

For completeness, we prove (1.5) in Section 2 for $\alpha = 1$ and $\beta > 1$; this case is sufficient to prove (1.1). Gentil [8, Theorem 1.1] tried to derive inequality (1.1) from inequality (1.5).

However, his proof for inequality (1.1) seems to be valid only when $f \in W^{1,p}(\mathbb{R}^n)$ has the form $f = e^{\frac{1}{p}\phi}$ for $\phi \in \mathrm{Lip}(\mathbb{R}^n)$ of (1.4) with

$$\liminf_{s \to 0+} \frac{1}{s} \int [e^{(ks+1)u(x,s)} - e^{(ks+1)\phi(x)}]dx \geqslant -\frac{1}{p} \int e^{\phi(x)} |D\phi(x)|^p dx \quad (1.6)$$

for any $k > 0$, where $u$ is a viscosity solution to Cauchy problem (1.3) with (1.4). So, his paper proves (1.1) for a special class of functions $f \in W^{1,p}(\mathbb{R}^n)$.

Our aim in this paper is to bridge this gap in the proof of [8, Theorem 1.1] and provide a supplementary proof of inequality (1.1) for all $f \in W^{1,p}(\mathbb{R}^n)$ and $p > 1$. The strategy of our proof is the following: First, we show (1.1) for $f \in W^{1,p}(\mathbb{R}^n)$ such that

$$f \in C^1(\mathbb{R}^n),\ 0 < f \leqslant 1 \text{ in } \mathbb{R}^n, \text{ and } D(\log f) \text{ is bounded on } \mathbb{R}^n. \quad (1.7)$$

The point is that, under (1.7) for $f \in W^{1,p}(\mathbb{R}^n)$, inequality (1.6) is fulfilled for letting $\phi(\cdot) = p \log f(\cdot)$ (see the proof of Lemma 3.1 below). Such an argument was used in [3].

Second, we approximate $f \in W^{1,p}(\mathbb{R}^n)$ by a sequence of functions satisfying (1.7) by several steps. This is the key point to derive (1.1) from (1.5) (see Theorem 3.3 below). An important estimate is the following Fatou–type inequality: if a family $\{f_\epsilon\}_{0<\epsilon<1}$ of nonnegative and measurable functions on $\mathbb{R}^n$ approximates a function $f$ in some sense, then

$$\liminf_{\epsilon \to 0+} \int f_\epsilon(x)^p \log f_\epsilon(x) dx \geqslant \int f(x)^p \log f(x) dx. \quad (1.8)$$

We provide a sufficient condition on $\{f_\epsilon\}_{0<\epsilon<1}$ for (1.8) (see Lemmas 2.2 and 2.3 below). From this result, we provide a stability condition such that if $f_\epsilon$ satisfies (1.1), so does $f$.

Finally, by using these approximations, we show that $L^p$–logarithmic Sobolev inequality (1.1) holds true for all $f \in W^{1,p}(\mathbb{R}^n)$ and $p > 1$. This bridges a gap of the proof of [8, Theorem 1.1] for $L^p$–logarithmic Sobolev inequality (1.1) with $p > 1$.

The content of this paper is organized as follows: In Section 2, we provide preliminaries. In Section 3, we provide a supplementary proof of inequality (1.1) for all $f \in W^{1,p}(\mathbb{R}^n)$ and $p > 1$.

I express my hearty appreciation to Ivan Gentil for his encouragement.

## 2. Preliminaries

In this section, we provide preliminaries to the next section. In the following, we assume $p > 1$. Set $q = p/(p-1)$. We assume that

$$\phi \in \mathrm{Lip}(\mathbb{R}^n),\ \phi \leqslant 0 \text{ in } \mathbb{R}^n \text{ and } e^{\phi} \in L^1(\mathbb{R}^n). \tag{2.1}$$

We put

$$L := \|D\phi\|_\infty. \tag{2.2}$$

Here, $\|\cdot\|_\infty$ is the $L^\infty(\mathbb{R}^n;\mathbb{R}^n)$–norm. Under (2.1), Cauchy problem (1.3) with (1.4) admits the unique viscosity solution $u \in C(\mathbb{R}^n \times [0,\infty))$ with the following properties:

$$u(x,t) = \inf_{y\in\mathbb{R}^n}\left[\phi(y) + \frac{1}{qt^{q-1}}|x-y|^q\right], \quad x \in \mathbb{R}^n,\ t > 0. \tag{2.3}$$

$$|u(x,t) - u(y,t)| \leqslant L|x-y|, \quad x, y \in \mathbb{R}^n,\ t \geqslant 0. \tag{2.4}$$

$$|u(x,t) - \phi(x)| \leqslant Mt, \quad x \in \mathbb{R}^n,\ t \geqslant 0 \tag{2.5}$$

for some constant $M > 0$.

Hopf-Lax formula (2.3) is well-known for a viscosity solution to Cauchy problem (1.3) with (1.4). For inequalities (2.4) and (2.5), see [4, Theorem 1.3.2].

Next, under (2.1), we derive inequality (1.5) for completeness. Here, in (1.5), we take $\alpha = 1$ for simplicity, since this case is sufficient to prove (1.1). Following the idea due to Gentil [7, 8], we prove (1.5) by Prékopa–Leindler inequality. Note that we do not use (1.1) in this proof of (1.5).

Recall Prékopa–Leindler inequality (cf. [5, Theorem 2]): Let $h_0, h_1 : \mathbb{R}^n \to \mathbb{R}$ be Borel measurable and nonnegative functions, and $\theta \in (0,1)$ a constant. Assume that $h : \mathbb{R}^n \to \mathbb{R}$ is a Borel measurable and nonnegative function such that

$$h_0(x_0)^{1-\theta} h_1(x_1)^{\theta} \leqslant h((1-\theta)x_0 + \theta x_1), \quad x_0, x_1 \in \mathbb{R}^n. \tag{2.6}$$

If $h_0, h_1, h \in L^1(\mathbb{R}^n)$, then

$$\left(\int h_0(x)dx\right)^{1-\theta}\left(\int h_1(x)dx\right)^{\theta} \leqslant \int h(x)dx. \tag{2.7}$$

Now, let $\beta > 1$ and $t > 0$. Under (2.1), we consider the functions $h_0, h_1, h$ defined by

$$\begin{aligned}
h_0(x) &= \exp\{\beta u(x,t)\},\\
h_1(x) &= \exp\left\{-\beta\,(\beta-1)^{q-1}\,\frac{|x|^q}{qt^{q-1}}\right\},\\
h(x) &= \exp\{\phi(\beta x)\}.
\end{aligned}$$

Since $u(x,t) \leqslant \phi(x) \leqslant 0$ for $(x,t) \in \mathbb{R}^n \times [0,\infty)$ by (2.3), we have

$$\beta u(x,t) \leqslant \beta\phi(x) \leqslant \phi(x), \quad (x,t) \in \mathbb{R}^n \times [0,\infty), \tag{2.8}$$

so that $h_0 \in L^1(\mathbb{R}^n)$ by (2.1). It is clear that $h_1 \in L^1(\mathbb{R}^n)$. Since $e^{\phi} \in L^1(\mathbb{R}^n)$, we have $h \in L^1(\mathbb{R}^n)$. Furthermore, let $\theta = (\beta-1)/\beta \in (0,1)$. By (2.3), we have, for $x_0, x_1 \in \mathbb{R}^n$,

$$\begin{aligned} h_0(x_0)^{1-\theta} h_1(x_1)^{\theta} &= \exp\left\{ u(x_0,t) - \frac{1}{qt^{q-1}}\,|(\beta-1)x_1|^q \right\} \\ \leqslant\ & \exp\left\{\phi\left(\beta[(1-\theta)x_0 + \theta x_1]\right)\right\} = h((1-\theta)x_0+\theta x_1). \end{aligned}$$

Thus, (2.6) holds for these $h_0, h_1, h$. Note that

$$\left(\int h_0(x)dx\right)^{1-\theta} = \|e^{u(\cdot,t)}\|_{\beta}, \quad \int h(x)dx = \frac{\|e^{\phi}\|_1}{\beta^n}.$$

By (1.2) and a slightly long calculation, we have

$$\begin{aligned} & \int h_1(x)dx = \int e^{-C|x|^q}dx \quad \left(C = \frac{\beta(\beta-1)^{q-1}}{qt^{q-1}}\right) \\ = \ & \frac{\sigma_{n-1}}{q\,C^{\frac{n}{q}}}\,\Gamma(n/q) \quad (\sigma_{n-1} = \text{the surface area of the unit ball of } \mathbb{R}^n) \\ = \ & \left[\beta^{p-1}\,\frac{nL_p e^{p-1}(\beta-1)}{p^p t}\right]^{-\frac{n}{p}}. \end{aligned}$$

Thus, by (2.7), we conclude (1.5) for $\alpha = 1$, $\beta > 1$ and $t > 0$.

We prepare three lemmas for the next section.

Lemma 2.1. — *Assume that $\phi \in C^1(\mathbb{R}^n)$ and $D\phi$ is bounded on $\mathbb{R}^n$. Let $u \in C(\mathbb{R}^n \times [0,\infty))$ be the unique viscosity solution to the Cauchy problem (1.3) with (1.4). Then, we have*

$$u(x,s) - \phi(x) \geqslant -\frac{s}{p}\left[\max_{|z-x|\leqslant Cs} |D\phi(z)|\right]^p, \ (x,s) \in \mathbb{R}^n \times (0,\infty),$$

*where $C = (qL)^{\frac{1}{q-1}}$ and $L$ is the constant of (2.2).*

*Proof.* — Fix $(x,s) \in \mathbb{R}^n \times (0,\infty)$ arbitrarily. Let $\hat{y} \in \mathbb{R}^n$ be a minimizer of the Hopf-Lax formula

$$u(x,s) = \inf_{y\in\mathbb{R}^n}\left[\phi(x-y) + \frac{|y|^q}{qs^{q-1}}\right] = \phi(x-\hat{y}) + \frac{|\hat{y}|^q}{qs^{q-1}}.$$

Such a $\hat{y}$ surely exists, since $q > 1$ and $D\phi$ is bounded on $\mathbb{R}^n$. Since $u(x,s) \leqslant \phi(x)$ by (2.3), we have

$$\frac{|\hat{y}|^q}{qs^{q-1}} \leqslant \phi(x) - \phi(x-\hat{y}) \leqslant L|\hat{y}|,$$

so that $|\hat{y}| \leqslant Cs$. Note that, when $|y| \leqslant Cs$, we have

$$\begin{aligned} \phi(x-y) - \phi(x) &= \int_0^1 \frac{d}{d\theta}\phi(x-\theta y)d\theta = \int_0^1 D\phi(x-\theta y)\cdot(-y)d\theta \\ \geqslant & \quad -|y| \max_{|z-x|\leqslant Cs} |D\phi(z)|. \end{aligned}$$

Thus,

$$\begin{aligned} u(x,s) - \phi(x) &= \inf_{|y|\leqslant Cs} \left[\phi(x-y) - \phi(x) + \frac{|y|^q}{qs^{q-1}}\right] \\ \geqslant & \quad \inf_{|y|\leqslant Cs} \left[-|y| \max_{|z-x|\leqslant Cs} |D\phi(z)| \; + \frac{|y|^q}{qs^{q-1}}\right] \\ \geqslant & \quad \inf_{y\in\mathbb{R}^n} \left[-|y| \max_{|z-x|\leqslant Cs} |D\phi(z)| \; + \frac{|y|^q}{qs^{q-1}}\right] \\ = & \quad -\frac{s}{p}\left[\max_{|z-x|\leqslant Cs} |D\phi(z)|\right]^p. \end{aligned}$$

□

LEMMA 2.2. — *Let $\{f_\epsilon\}_{0<\epsilon<1}$ be a family of nonnegative and measurable functions on $\mathbb{R}^n$ such that $f := \lim_{\epsilon\to 0+} f_\epsilon$ exists a.e. on $\mathbb{R}^n$. Assume that there exists a constant $\delta \in (0,p)$ such that $f_\epsilon, f \in L^{p-\delta}(\mathbb{R}^n)$ and*

$$\lim_{\epsilon\to 0+} \int f_\epsilon(x)^{p-\delta}dx = \int f(x)^{p-\delta}dx. \tag{2.9}$$

*Then, we have (1.8).*

*Proof.* — Note that the inequality

$$t^\delta \log t + \frac{1}{\delta e} \geqslant 0, \quad t \geqslant 0,\ \delta > 0$$

holds. Thus, applying the Fatou's lemma to

$$\int \left(f_\epsilon^p \log f_\epsilon + \frac{1}{\delta e} f_\epsilon^{p-\delta}\right) dx = \int f_\epsilon^{p-\delta}\left(f_\epsilon^\delta \log f_\epsilon + \frac{1}{\delta e}\right) dx,$$

we have

$$\liminf_{\epsilon\to 0+}\int\left(f_\epsilon^p\log f_\epsilon+\frac{1}{\delta e}f_\epsilon^{p-\delta}\right)dx\geqslant\int f^{p-\delta}\left(f^\delta\log f+\frac{1}{\delta e}\right)dx.$$

By our assumption, the left-hand side of this inequality is equal to

$$\liminf_{\epsilon\to 0+}\int f_\epsilon^p\log f_\epsilon dx+\frac{1}{\delta e}\lim_{\epsilon\to 0+}\int f_\epsilon^{p-\delta}dx.$$

Therefore, we conclude (1.8) by (2.9). □

Lemma 2.3. — *For $0\leqslant f\in L^p(\mathbb{R}^n)$, let*

$$f_\epsilon(x)=\lambda(\epsilon x)f(x),\quad x\in\mathbb{R}^n,\ 0<\epsilon<1,$$

*where $\lambda$ is a $C(\mathbb{R}^n)$–function such that $\lambda(0)=1$ and $0\leqslant\lambda\leqslant 1$ on $\mathbb{R}^n$. Then, we have (1.8).*

*Proof.* — We have

$$\begin{aligned}
&\liminf_{\epsilon\to 0+}\int f_\epsilon^p\log f_\epsilon dx\\
=\ &\liminf_{\epsilon\to 0+}\left[\int\lambda(\epsilon\,\cdot)^pf^p\log\lambda(\epsilon\,\cdot)dx+\int_{\{f\geqslant 1\}}\lambda(\epsilon\,\cdot)^pf^p\log f dx\right.\\
&\qquad\qquad\left.+\int_{\{f<1\}}\lambda(\epsilon\,\cdot)^pf^p\log f dx\right]\\
\geqslant\ &\liminf_{\epsilon\to 0+}\int\lambda(\epsilon\,\cdot)^pf^p\log\lambda(\epsilon\,\cdot)dx+\liminf_{\epsilon\to 0+}\int_{\{f\geqslant 1\}}\lambda(\epsilon\,\cdot)^pf^p\log f dx\\
&\qquad\qquad+\liminf_{\epsilon\to 0+}\int_{\{f<1\}}\lambda(\epsilon\,\cdot)^pf^p\log f dx\\
\equiv\ &I+J+K.
\end{aligned}$$

Since $f\in L^p(\mathbb{R}^n)$, we have $I=0$ by Lebesgue's dominated convergence theorem. By Fatou's lemma, we have

$$J\geqslant\int_{\{f\geqslant 1\}}f^p\log f dx.$$

Since $0\leqslant\lambda\leqslant 1$ on $\mathbb{R}^n$, we have

$$K\geqslant\int_{\{f<1\}}f^p\log f dx,$$

so that

$$I+J+K \geqslant \int f^p \log f dx.$$

Therefore, we conclude (1.8). $\square$

## 3. Proof of inequality (1.1)

In this section, we provide a complete proof of inequality (1.1) for all $f \in W^{1,p}(\mathbb{R}^n)$ and $p > 1$. First, we show (1.1) for $f \in W^{1,p}(\mathbb{R}^n)$ satisfying (1.7). We put $\phi := p \log f$. Then, $\phi$ fulfills

$$\begin{aligned} &\phi \in C^1(\mathbb{R}^n),\ \phi \leqslant 0 \text{ in } \mathbb{R}^n,\ e^{\phi} \in L^1(\mathbb{R}^n), \text{ and} \\ &D\phi \text{ is bounded on } \mathbb{R}^n. \end{aligned} \tag{3.1}$$

Further, note that (3.1) implies (2.1). Thus, if $f \in W^{1,p}(\mathbb{R}^n)$ fulfills (1.7), Cauchy problem (1.3) with (1.4) for $\phi := p \log f$ admits the unique viscosity solution $u \in C(\mathbb{R}^n \times [0,\infty))$.

Lemma 3.1. — *Let $p > 1$ and $k > 0$. Assume that $f \in W^{1,p}(\mathbb{R}^n)$ fulfills (1.7). Let $u \in C(\mathbb{R}^n \times [0,\infty))$ be the unique viscosity solution of Cauchy problem (1.3) with (1.4) for $\phi = p \log f$. We define the function $F$ on $[0,\infty)$ by*

$$F(s) = \int e^{(ks+1)u(x,s)} dx, \quad s \geqslant 0.$$

*If* $\mathrm{Ent}(e^{\phi}) > -\infty$, *then we have*

$$\liminf_{s \to 0+} \frac{F(s)-F(0)}{s} \geqslant -\frac{1}{p}\int e^{\phi(x)} |D\phi(x)|^p dx + k\int \phi(x) e^{\phi(x)} dx. \tag{3.2}$$

*Proof.* — 1. Since $\phi \leqslant 0$ in $\mathbb{R}^n$, we have, by (2.8),

$$e^{(ks+1)u(x,s)} \leqslant e^{(ks+1)\phi(x)} \leqslant e^{\phi(x)} \in L^1(\mathbb{R}^n), \quad s \geqslant 0. \tag{3.3}$$

Thus, $F$ is well–defined. Furthermore, note that

$$0 \leqslant -\int \phi(x) e^{\phi(x)} dx < \infty, \tag{3.4}$$

since $\mathrm{Ent}(e^{\phi}) > -\infty$. Thus, by (2.5), (3.3) and (3.4), we have, for $(x,s) \in \mathbb{R}^n \times (0,\infty)$,

$$0 \leqslant (ks+1)|u(x,s)| e^{(ks+1)u(x,s)} \leqslant (ks+1)(|\phi(x)| + Ms) e^{\phi(x)} \in L^1(\mathbb{R}^n).$$

2. We show that

$$F(s) - F(0) \geqslant -\frac{s}{p}(ks+1)\int e^{(ks+1)\phi(x)}\left[\max_{|z-x|\leqslant Cs}|D\phi(z)|^p\right]dx \quad (3.5)$$
$$+\int\int_0^s k\phi(x)e^{(k\theta+1)\phi(x)}d\theta dx$$

(note that all terms in (3.5) are well–defined by the arguments above). In order to show (3.5), we see that

$$\begin{aligned} & F(s) - F(0) \\ = & \int[e^{(ks+1)u(x,s)} - e^{(ks+1)\phi(x)}]dx + \int[e^{(ks+1)\phi(x)} - e^{\phi(x)}]dx =: I + J. \end{aligned}$$

Using the inequalities $u(x,s) \leqslant \phi(x)$ and

$$|e^b - e^a| = \left|\int_a^b e^t dt\right| \leqslant \max\{e^a, e^b\}|b-a|, \quad a, b \in \mathbb{R},$$

we have

$$\begin{aligned} & 0 \leqslant -[e^{(ks+1)u(x,s)} - e^{(ks+1)\phi(x)}] = |e^{(ks+1)u(x,s)} - e^{(ks+1)\phi(x)}| \\ \leqslant & (ks+1)\max\{e^{(ks+1)u(x,s)}, e^{(ks+1)\phi(x)}\}|u(x,s) - \phi(x)| \\ \leqslant & (ks+1)e^{(ks+1)\phi(x)}[\phi(x) - u(x,s)], \end{aligned}$$

so that, by Lemma 2.1,

$$\begin{aligned} & e^{(ks+1)u(x,s)} - e^{(ks+1)\phi(x)} \geqslant (ks+1)e^{(ks+1)\phi(x)}[u(x,s) - \phi(x)] \\ \geqslant & -\frac{s}{p}(ks+1)e^{(ks+1)\phi(x)}\left[\max_{|z-x|\leqslant Cs}|D\phi(z)|\right]^p. \end{aligned}$$

This implies that

$$I \geqslant -\frac{s}{p}(ks+1)\int e^{(ks+1)\phi(x)}\left[\max_{|z-x|\leqslant Cs}|D\phi(z)|^p\right]dx.$$

On the other hand, we have

$$\begin{aligned} J = \int[e^{(ks+1)\phi(x)} - e^{\phi(x)}]dx = \int\int_0^s \frac{d}{d\theta}e^{(k\theta+1)\phi(x)}d\theta dx \\ = \int\int_0^s k\phi(x)e^{(k\theta+1)\phi(x)}d\theta dx. \end{aligned}$$

Thus, we have obtained (3.5). Then, by Lebesgue's dominated convergence theorem, we conclude (3.2). □

Proposition 3.2. — *Let $p > 1$. Then, inequality (1.1) holds true for all $f \in W^{1,p}(\mathbb{R}^n)$ satisfying (1.7).*

*Proof.* — By (1.7), we put $\phi(x) = p \log f(x)$. When $\mathrm{Ent}(f^p) = -\infty$, (1.1) is trivial. So, we may assume that $\mathrm{Ent}(e^\phi) = \mathrm{Ent}(f^p) > -\infty$.

For any $k > 0$, we consider the functions $F$ of Lemma 3.1 and

$$B(s) = \left(\frac{nL_p e^{p-1} k}{p^p}\right)^{\frac{nks}{p}} (ks+1)^{-\frac{n(ks+p)}{p}}, \quad s \geqslant 0.$$

Note that (1.5) with $\alpha = 1$ and $\beta = ks + 1$ can be rewritten as

$$F(s) \leqslant F(0)^{ks+1} B(s).$$

Since $B(0) = 1$, we have

$$\liminf_{s \to 0+} \frac{F(s) - F(0)}{s} \leqslant F(0) \liminf_{s \to 0+} \frac{F(0)^{ks} B(s) - B(0)}{s}.$$

Note that

$$\begin{aligned} &\liminf_{s \to 0+} \frac{F(0)^{ks} B(s) - B(0)}{s} = \frac{d}{ds} \left[F(0)^{ks} B(s)\right]\Big|_{s=0} \\ = \;& k \log\left(\int e^{\phi(x)} dx\right) + \frac{nk}{p} \log\left(\frac{nL_p k}{p^p e}\right). \end{aligned}$$

Therefore, by Lemma 3.1, we obtain

$$\begin{aligned} &-\frac{1}{p} \int e^{\phi(x)} \left|D\phi(x)\right|^p dx + k \int \phi(x) e^{\phi(x)} dx \\ \leqslant\; & \int e^{\phi(x)} dx \left[k \log\left(\int e^{\phi(x)} dx\right) + \frac{nk}{p} \log\left(\frac{nL_p k}{p^p e}\right)\right], \end{aligned}$$

so that

$$k\mathrm{Ent}(e^\phi) \leqslant \frac{1}{p} \int e^{\phi(x)} \left|D\phi(x)\right|^p dx + \int e^{\phi(x)} dx \; \frac{nk}{p} \log\left(\frac{nL_p k}{p^p e}\right).$$

Since $e^{\phi(x)} = f(x)^p$ and $e^{\phi(x)}|D\phi(x)|^p = p^p|Df(x)|^p$ in $\mathbb{R}^n$, we have obtained

$$\mathrm{Ent}(f^p) \leqslant \frac{p^{p-1}}{k} \int |Df(x)|^p dx + \frac{n}{p} \int f(x)^p dx \; \log\left(\frac{nL_p k}{p^p e}\right).$$

Minimizing the right-hand side with respect to $k > 0$ over $(0, \infty)$, we obtain (1.1) for $f \in W^{1,p}(\mathbb{R}^n)$ satisfying (1.7). $\square$

Now, we state the theorem of this paper.

THEOREM 3.3. — *Let $p > 1$. Inequality (1.1) holds true for all $f \in W^{1,p}(\mathbb{R}^n)$.*

*Proof.* — We divide the proof of Theorem 3.3 into six steps as follows:

(i) We show (1.1) for $f \in W^{1,p}(\mathbb{R}^n)$ satisfying

$$f \in C^1(\mathbb{R}^n),\ 0 < f \text{ in } \mathbb{R}^n, \text{ and } D(\log f) \text{ is bounded on } \mathbb{R}^n. \tag{3.6}$$

(ii) We show (1.1) for $0 \leqslant f \in C_0^1(\mathbb{R}^n)$, where $C_0^1(\mathbb{R}^n)$ is the set of all $C^1(\mathbb{R}^n)$–functions with compact supports in $\mathbb{R}^n$.

(iii) We show (1.1) for $0 \leqslant f \in W^{1,p}(\mathbb{R}^n) \bigcap C^1(\mathbb{R}^n)$.

(iv) We show (1.1) for $0 \leqslant f \in W^{1,p}(\mathbb{R}^n) \bigcap L^{p-\delta}(\mathbb{R}^n)$ with some $\delta \in (0, p-1)$.

(v) We show (1.1) for $0 \leqslant f \in W^{1,p}(\mathbb{R}^n)$.

(vi) We show (1.1) for $f \in W^{1,p}(\mathbb{R}^n)$.

Here, in (iv) and (v), $f \geqslant 0$ means that $f \geqslant 0$ a.e. in $\mathbb{R}^n$. In (iv), we consider a constant $\delta \in (0, p-1)$, although we considered the case $\delta \in (0, p)$ in Lemma 2.2.

(i) Let $f \in W^{1,p}(\mathbb{R}^n)$ be a function satisfying (3.6). We denote by $L_0$ the Lipschitz constant of $\log f$. Note that there exists a constant $M > 0$ such that $\log f(x) \leqslant M$ on $\mathbb{R}^n$. If not, we find a sequence $\{x_j\}$ of $\mathbb{R}^n$ such that $\log f(x_j) \geqslant j+1$ for each $j \in \mathbb{N}$. Fix $j \in \mathbb{N}$ arbitrarily. Since $\log f$ is Lipschitz continuous on $\mathbb{R}^n$, we have

$$\log f(x_j) - \log f(x) \leqslant L_0 |x - x_j| \leqslant 1, \quad |x - x_j| \leqslant \frac{1}{L_0},\ j \in \mathbb{N},$$

so that $j \leqslant \log f(x)$ on $\{|x - x_j| \leqslant 1/L_0\}$. Thus,

$$\infty > \int f(x)^p dx = \int e^{p \log f(x)} dx \geqslant \int_{\{|x-x_j| \leqslant 1/L_0\}} e^{p \log f(x)} dx \geqslant e^{pj} \omega_n \left( \frac{1}{L_0} \right)^n,$$

where $\omega_n$ is the volume of the unit ball of $\mathbb{R}^n$. Since $j \in \mathbb{N}$ is arbitrary, this is a contradiction. Hence, there exists a constant $M > 0$ such that $\log f(x) \leqslant M$ on $\mathbb{R}^n$. Set

$$f_M(x) = f(x) e^{-M} = e^{\log f(x) - M}, \quad x \in \mathbb{R}^n.$$

It is easy to see that $f_M \in W^{1,p}(\mathbb{R}^n)$ fulfills (1.7). Thus, we have, by Proposition 3.2,

$$\mathrm{Ent}(f_M^p) \leqslant \frac{n}{p} \int f_M(x)^p dx \ \log \left( L_p \frac{\displaystyle\int |Df_M(x)|^p \, dx}{\displaystyle\int f_M(x)^p \, dx} \right).$$

Since $\mathrm{Ent}(f_M^p) = e^{-pM}\mathrm{Ent}(f^p)$, we have shown (1.1) for $f \in W^{1,p}(\mathbb{R}^n)$ satisfying (3.6).

(ii) Let $0 \leqslant f \in C_0^1(\mathbb{R}^n)$. We set

$$f_\epsilon(x) = \left[ f(x)^p + \epsilon e^{-\langle x \rangle} \right]^{1/p}, \quad x \in \mathbb{R}^n, \ 0 < \epsilon < 1,$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$. Then, $0 < f_\epsilon \in W^{1,p}(\mathbb{R}^n) \bigcap C^1(\mathbb{R}^n)$. Since $f$ has a compact support in $\mathbb{R}^n$, $D(\log f_\epsilon)$ is bounded on $\mathbb{R}^n$. Thus, $f_\epsilon$ belongs to $W^{1,p}(\mathbb{R}^n)$ and fulfills (3.6). By (i), we see that $f_\epsilon$ satisfies

$$\begin{aligned} &\int f_\epsilon^p dx \ \log \int f_\epsilon^p dx + \frac{n}{p} \int f_\epsilon^p dx \ \log \left( L_p \frac{\displaystyle\int |Df_\epsilon|^p \, dx}{\displaystyle\int f_\epsilon^p \, dx} \right) \qquad (3.7) \\ \geqslant \ &\int f_\epsilon^p \log f_\epsilon^p dx. \end{aligned}$$

Let $\delta \in (0, p-1)$. Using the inequality

$$(a+b)^\kappa \leqslant a^\kappa + b^\kappa \quad a, b \geqslant 0, \ 0 < \kappa < 1,$$

we have

$$|f_\epsilon(x)|^{p-\delta} \leqslant f(x)^{p-\delta} + e^{-\frac{p-\delta}{p}\langle x \rangle}.$$

Thus, $f_\epsilon, f \in L^{p-\delta}(\mathbb{R}^n)$. By Lemma 2.2, we see that (1.8) holds for this $\{f_\epsilon\}$ and $f$. Since $f_\epsilon, f \in W^{1,p}(\mathbb{R}^n)$ fulfill

$$\lim_{\epsilon \to 0+} \int f_\epsilon(x)^p dx = \int f(x)^p dx, \quad \lim_{\epsilon \to 0+} \int |Df_\epsilon(x)|^p dx = \int |Df(x)|^p dx, \qquad (3.8)$$

we have shown (1.1) for $0 \leqslant f \in C_0^1(\mathbb{R}^n)$ by letting $\epsilon$ to $0+$ in (3.7).

(iii) Let $0 \leqslant f \in W^{1,p}(\mathbb{R}^n) \bigcap C^1(\mathbb{R}^n)$. Let $\rho$ be a $C_0^1(\mathbb{R}^n)$–function with $\rho(0) = 1$ and $0 \leqslant \rho \leqslant 1$ on $\mathbb{R}^n$. We set

$$f_\epsilon(x) = \rho(\epsilon x) f(x), \quad x \in \mathbb{R}^n, \ 0 < \epsilon < 1.$$

Then, $0 \leqslant f_\epsilon \in C_0^1(\mathbb{R}^n)$. Thus, by (ii), we see that (3.7) holds for this function $f_\epsilon$. Since $f_\epsilon$ and $f$ satisfy (3.8), we conclude (1.1) for $0 \leqslant f \in W^{1,p}(\mathbb{R}^n) \bigcap C^1(\mathbb{R}^n)$ by using Lemma 2.3 and letting $\epsilon$ to $0+$ in (3.7).

(iv) Let $0 \leqslant f \in W^{1,p}(\mathbb{R}^n) \bigcap L^{p-\delta}(\mathbb{R}^n)$ with some $\delta \in (0, p-1)$. Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be a nonnegative function such that $\displaystyle\int \eta(x)dx = 1$. For a sufficiently small $\epsilon > 0$, we define $f_\epsilon$ by

$$f_\epsilon(x) = \frac{1}{\epsilon^n} \int f(y)\, \eta\left(\frac{x-y}{\epsilon}\right) dy, \quad x \in \mathbb{R}^n.$$

Then, $0 \leqslant f_\epsilon \in W^{1,p}(\mathbb{R}^n) \bigcap C^1(\mathbb{R}^n) \bigcap L^{p-\delta}(\mathbb{R}^n)$ with some $\delta \in (0, p-1)$, since $p - \delta > 1$. Thus, by (iii), we see that (3.7) holds for this function $f_\epsilon$.

Next, since $f_\epsilon \to f$ in $L^{p-\delta}(\mathbb{R}^n)$, we find a sequence $\{\epsilon_j\} \subset (0,1)$ such that $\epsilon_j \to 0$ as $j \to \infty$ and $f_{\epsilon_j} \to f$ a.e. on $\mathbb{R}^n$. Thus, by Lemma 2.2, we have

$$\liminf_{j\to\infty} \int f_{\epsilon_j}(x)^p \log f_{\epsilon_j}(x)dx \geqslant \int f(x)^p \log f(x)dx.$$

Since (3.8) is fulfilled, we have shown (1.1) for $0 \leqslant f \in W^{1,p}(\mathbb{R}^n) \bigcap L^{p-\delta}(\mathbb{R}^n)$ with some $\delta \in (0, p-1)$ by using Lemma 2.2 and letting $\epsilon$ to $0+$ in (3.7).

(v) Let $0 \leqslant f \in W^{1,p}(\mathbb{R}^n)$. Set

$$f_\epsilon(x) = \rho(\epsilon x) f(x), \quad x \in \mathbb{R}^n,\ 0 < \epsilon < 1.$$

Here, $\rho$ is a $C_0^1(\mathbb{R}^n)$–function with $\rho(0) = 1$ and $0 \leqslant \rho \leqslant 1$ on $\mathbb{R}^n$. Then, it is easy to see that $0 \leqslant f_\epsilon \in W^{1,p}(\mathbb{R}^n) \bigcap L^{p-\delta}(\mathbb{R}^n)$ for all $\delta \in (0, p-1)$. Thus, by (iv), (3.7) holds for this function $f_\epsilon$. By the same arguments as those of (iii), we conclude (1.1) for $0 \leqslant f \in W^{1,p}(\mathbb{R}^n)$.

(vi) We show (1.1) for $f \in W^{1,p}(\mathbb{R}^n)$. Note that if $f \in W^{1,p}(\mathbb{R}^n)$ then $|f| \in W^{1,p}(\mathbb{R}^n)$. Hence, by (v) and the fact that $|D|f|| \leqslant |Df|$ a.e. in $\mathbb{R}^n$, we conclude (1.1) for $f \in W^{1,p}(\mathbb{R}^n)$. $\square$

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