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Toric surfaces, vanishing Euler characteristic and Euler obstruction of a function

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Thaís Maria Dalbelo[1], Nivaldo de Góes Grulha Jr.[2] and Miriam Silva Pereira[3]

Abstract. — We define the vanishing Euler characteristic of a normal toric surface X_σ, we give a formula to compute it, and we relate this number with the second polar multiplicity of X_σ. We also present a formula for the Euler obstruction of a function $f : X_\sigma \to \mathbb{C}$ and for the difference between the Euler obstruction of the space X_σ and the Euler obstruction of a function f. As an application of this result we compute the Euler obstruction of a type of polynomial on a family of determinantal surfaces.

Résumé. — Nous définissons la caractéristique d'Euler évanescente d'une surface torique normale X_σ, nous donnons une formule pour la calculer, et nous associons ce nombre avec la seconde multiplicité polaire de X_σ. Nous présentons aussi une formule pour l'obstruction d'Euler d'une fonction $f : X_\sigma \to \mathbb{C}$ et pour la différence entre l'obstruction d'Euler de l'espace X_σ et l'obstruction d'Euler d'une fonction f. Comme application de ce résultat nous calculons l'obstruction d'Euler des polynômes d'un certain type sur une famille de surfaces déterminantales.

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1. Introduction

One of the most appealing aspects of toric varieties is the way that many questions that are difficult for general varieties, admit simple and concrete solutions in the toric case. The problem of finding resolutions of singularities is a perfect example [11].

We will work with the specific cases of toric surfaces and functions on toric surfaces. Let X_σ be a toric surface with isolated singularity associated to the cone $\sigma \subset \mathbb{R}^2$. This type of singular surface has many special properties, one of them, that we do not find in general, is that this singularity admits a smoothing associated to its minimal resolution [23]. Let Y be the generic fiber of this smoothing. Using continued fractions techniques, we give a very simple formula to compute the vanishing Euler characteristic of X_σ denoted by $\nu(X_\sigma)$ and we prove that $\nu(X_\sigma)$ is related to the polar multiplicities.

Since that X_σ is a normal singularity, it follows from a result of Greuel and Steenbrink [15] that $\beta_1(Y) = 0$, where β_1 is the first Betti number, and as Y has the homotopy type of a finite CW-complex of dimension $\leqslant 2$, one has $\dim H_2(Y) = \chi(Y) - 1$. Therefore, the vanishing Euler characteristic of X_σ equals the middle Betti number of Y, then in the case that X_σ has a unique smoothing, this number coincides with the Milnor number.

The Milnor number was defined by Milnor in [19]. Initially this invariant was associated to germs of analytic functions $f : (X, 0) \to (\mathbb{C}, 0)$ with isolated singularity, and to study isolated hypersurfaces singularities. However this invariant is well defined in many others contexts, for instance curves [8], isolated complete intersection singularities, or ICIS [16], [17] and determinantal varieties with codimension two [22]. When $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a germ of analytic function with isolated singularity at the origin the Milnor number of f, that we denote by $\mu(f)$, coincides with the number of Morse points of a Morsification of f.

Let us denote by $(X, 0)$ a germ of analytic singular space embedded in $\mathbb{C}^n$ and $f : (X, 0) \to (\mathbb{C}, 0)$ a germ of analytic function with isolated singularity at the origin. In this situation, Brasselet, Massey, Parameswaran and Seade introduced an invariant associated to f called the Euler obstruction of f [5]. In [27] the authors proved that the Euler obstruction of f is, up to sign, the number of Morse points of a Morsification of f on the regular part of X. Hence this invariant can be seen as a generalization of the Milnor number of f. In the last section, we give some formula to compute the Euler obstruction of a function $f : X_\sigma \to \mathbb{C}$ with isolated singularity at 0, and also for the difference between the Euler obstruction of the space

X_σ and the Euler obstruction of a function f, that as noticed in [5, 9] has interesting meanings, even if f has a non-isolated singularity.

2. Background Material

We present notions that we will use in the next sections. We recall first some objects developed to study the structure of toric surfaces. We describe their singularities and also present two examples.

2.1. Toric surfaces

For an overview about toric varieties see [11].

DEFINITION 2.1. — *Let $\sigma \subset \mathbb{R}^2$ be a rational, strongly convex, polyhedral cone and let*

$$\check{\sigma} = \left\{ v \in \mathbb{R}^2; \langle a, v \rangle \geqslant 0, \forall a \in \sigma \right\}$$

be the dual cone. The corresponding lattices are denoted by $N \subset \mathbb{R}^2$ and $M \subset \mathbb{R}^2$, respectively. Then the affine toric surface is defined as $X_\sigma = \operatorname{Spec} \mathbb{C}[\check{\sigma} \cap M]$.

Remark 2.2. — As generally known, the affine toric surfaces are exactly those affine, normal surfaces admitting a $(\mathbb{C}^*)^2$-action with an open, dense orbit.

A strongly convex cone in $\mathbb{R}^2$ has the following normal form that will simplify our study of the singularities of toric surfaces.

PROPOSITION 2.3. *(See [11]).* — *Let $\sigma \subset \mathbb{R}^2$ be a strongly convex cone, then σ is isomorphic to the cone generated by the vectors $v_1 = pe_1 - qe_2$ and $v_2 = e_2$, for some positive integers p, q which are coprime.*

Given a cone $\sigma \subset \mathbb{R}^2$, Riemenschneider proved in [23, 24] that the binomials which generate the ideal I_σ are given by *quasiminors* of a *quasimatrix*, where $X_\sigma = V(I_\sigma)$. In the following we recall the definition of *quasimatrix*.

DEFINITION 2.4. — *Given $A_i, B_i, C_{l,l+1} \in \mathbb{C}$ with $i = 1, \ldots, k$ and $l = 1, \ldots, k-1$, a quasimatrix with entries $A_i, B_i, C_{l,l+1}$ is written as*

$$A = \begin{pmatrix} A_1 & & A_2 & \cdots & A_{k-1} & & A_k \\ B_1 & & B_2 & \cdots & B_{k-1} & & B_k \\ & C_{1,2} & & \cdots & & C_{k-1,k} & \end{pmatrix}.$$

The quasiminors of the quasimatrix A are defined by

$$A_i B_j - B_i(C_{i,i+1} \ldots C_{j-1,j})A_j$$

for $1 \leqslant i < j \leqslant l$.

Given a cone $\sigma \subset \mathbb{R}^2$ generated by the vectors $v_1 = pe_1 - qe_2$ and $v_2 = e_2$, where $0 < q < p$ and p, q are coprime, let us consider the Hirzebruch-Jung continued fraction

$$\frac{p}{p-q} = a_2 - \cfrac{1}{a_3 - \cfrac{1}{\cdots - \cfrac{1}{a_{n-1}}}} = [[a_2, a_3, \ldots, a_{n-1}]]$$

where the integers $a_2, \ldots, a_{n-1}$ satisfy $a_i \geqslant 2$, for $i = 2, \ldots, n-1$. Riemenschneider proved the following:

PROPOSITION 2.5. *(See [24]).* — *The ideal I_σ is generated by the quasiminors of the quasimatrix*

$$\begin{pmatrix} z_1 & & z_2 & & z_3 & \cdots & z_{n-2} & & z_{n-1} \\ z_2 & & z_3 & & z_4 & \cdots & z_{n-1} & & z_n \\ & z_2^{a_2-2} & & z_3^{a_3-2} & & \cdots & & z_{n-1}^{a_{n-1}-2} & \end{pmatrix}.$$

Where the a_i are given by the Hirzebruch-Jung continued fraction of $\frac{p}{p-q}$. Moreover, this set of generators is minimal.

Then, if $a_i = 2$ for $i = 3, \ldots, n-2$, we have that X_σ is a determinantal surface [22], in particular if the minimal dimension of embedding of X_σ is 4, *i. e.*, if

$$\frac{p}{p-q} = a_2 - \frac{1}{a_3}$$

then X_σ is always determinantal and the ideal I_σ is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} z_1 & z_2 & z_3^{a_3-1} \\ z_2^{a_2-1} & z_3 & z_4 \end{pmatrix}.$$

Example 2.6. — Let $X_\sigma \subset \mathbb{C}^4$ be the toric surface associated to the cone $\sigma \subset \mathbb{R}^2$ generated by the vectors $v_1 = e_2$ and $v_2 = 14e_1 - 11e_2$. From the Hirzebruch-Jung continued fraction process we have

$$\frac{14}{3} = 5 - \frac{1}{3}$$

then $X_\sigma = V(I_\sigma)$ where I_σ is the ideal generated by the 2×2 minors of the matrix

$$\begin{pmatrix} z_1 & z_2 & z_3^2 \\ z_2^4 & z_3 & z_4 \end{pmatrix},$$

i.e., X_σ is a codimension 2 determinantal surface.

Example 2.7. — Let $X_\sigma \subset \mathbb{C}^5$ be the toric surface associated to the cone $\sigma \subset \mathbb{R}^2$ generated by the vectors $v_1 = e_2$ and $v_2 = 4e_1 - e_2$. From the Hirzebruch-Jung continued fraction process we have

$$\frac{4}{3} = 2 - \frac{1}{2 - \frac{1}{2}}$$

then $X_\sigma = V(I_\sigma)$ where I_σ is the ideal generated by the 2×2 minors of the matrix

$$\begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ z_2 & z_3 & z_4 & z_5 \end{pmatrix},$$

i.e., X_σ is a codimension 3 determinantal surface.

2.2. The Euler obstruction and Applications

An important invariant of singular varieties is the Euler obstruction, that was defined by MacPherson in [18] as a tool to prove the conjecture about existence and unicity of the Chern classes in the singular case. The Euler obstruction was deeply investigated by many authors, and for an overview about it see [2]. Let us now introduce some concepts in order to define the Euler obstruction.

Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be an equidimensional reduced complex analytic germ of dimension d in an open subset $U \subset \mathbb{C}^n$. We consider a complex analytic Whitney stratification $\mathcal{V} = \{V_i\}$ of U adapted to X and we assume that $\{0\}$ is a 0-dimensional stratum. We choose a small representative of $(X, 0)$ such that 0 belongs to the closure of all the strata. We still denote it by X and we will write $X = \cup_{i=0}^q V_i$ where $V_0 = \{0\}$ and $V_q = X_{\mathrm{reg}}$ is the set of regular points of X. We will assume that the strata $V_0, \ldots, V_{q-1}$ are connected and that the analytic sets $\overline{V_0}, \ldots, \overline{V_{q-1}}$ are reduced.

Let $G(d, n)$ denote the Grassmanian of complex d-planes in $\mathbb{C}^n$. On the regular part X_{reg} of X the Gauss map $\phi : X_{\mathrm{reg}} \to U \times G(d, n)$ is well defined by $\phi(x) = (x, T_x(X_{\mathrm{reg}}))$.

DEFINITION 2.8. — *The Nash transformation (or Nash blow up) $\widetilde{X}$ of X is the closure of the image $\mathrm{Im}(\phi)$ in $U \times G(d, n)$. It is a (usually singular)*

complex analytic space endowed with an analytic projection map $\nu : \widetilde{X} \to X$ which is a biholomorphism away from $\nu^{-1}(\mathrm{Sing}(X))$.

The fiber of the tautological bundle $\mathcal{T}$ over $G(d, n)$, at point $P \in G(d, n)$, is the set of vectors v in the d-plane P. We still denote by $\mathcal{T}$ the corresponding trivial extension bundle over $U \times G(d, n)$. Let $\widetilde{T}$ be the restriction of $\mathcal{T}$ to $\widetilde{X}$, with projection map π. The bundle $\widetilde{T}$ on $\widetilde{X}$ is called *the Nash bundle* of X.

An element of $\widetilde{T}$ is written (x, P, v) where $x \in U$, P is a d-plane in $\mathbb{C}^n$ based at x and v is a vector in P. We have the following diagram:

$$
\begin{array}{ccc}
\widetilde{T} & \hookrightarrow & \mathcal{T} \\
\pi \downarrow & & \downarrow \\
\widetilde{X} & \hookrightarrow & U \times G(d, n) \\
\nu \downarrow & & \downarrow \\
X & \hookrightarrow & U.
\end{array}
$$

Let us denote by $TU|_X$ the restriction to X of the tangent bundle of U. A stratified vector field v on X means a continuous section of $TU|_X$ such that if $x \in V_\alpha \cap X$ then $v(x) \in T_x(V_\alpha)$. By Whitney condition (a) one has the following:

LEMMA 2.9. *(See [6]). — Every stratified vector field v without zeros on a subset $A \subset X$ has a canonical lifting to a section $\tilde{v}$, of the Nash bundle $\tilde{T}$, without zeros on $\nu^{-1}(A) \subset \tilde{X}$.*

Now consider a stratified radial vector field $v(x)$ in a neighborhood of $\{0\}$ in X, *i.e.*, there is ε_0 such that for every $0 < \varepsilon \leqslant \varepsilon_0$, $v(x)$ is pointing outwards the ball B_ε over the boundary $S_\varepsilon := \partial B_\varepsilon$.

The following interpretation of the local Euler obstruction has been given by Brasselet and Schwartz in [6].

DEFINITION 2.10. — *Let v be a radial vector field on $X \cap S_\varepsilon$ and $\tilde{v}$ the lifting of v on $\nu^{-1}(X \cap S_\varepsilon)$ to a section of the Nash bundle. The local Euler obstruction (or simply the Euler obstruction) $\mathrm{Eu}_X(0)$ is defined to be the obstruction to extending $\tilde{v}$ as a nowhere zero section of $\tilde{T}$ over $\nu^{-1}(X \cap B_\varepsilon)$.*

More precisely, let $\mathcal{O}(\tilde{v}) \in H^{2d}(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon))$ be the obstruction cocycle to extending $\tilde{v}$ as a nowhere zero section of $\tilde{T}$ inside $\nu^{-1}(X \cap B_\varepsilon)$. The Euler obstruction $\mathrm{Eu}_X(0)$ is defined as the evaluation of the cocycle

$\mathcal{O}(\tilde{v})$ on the fundamental class of the pair $(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon))$. The Euler obstruction is an integer.

A Lefschetz type formula for the Euler obstruction was given by Brasselet, Lê and Seade. The formula shows that the Euler obstruction, as a constructible function, satisfies the Euler condition relatively to generic linear forms.

THEOREM 2.11. *(See [4]).* — *Let* $(X, 0)$ *and* $\{V_i\}$ *be given as before, then for each generic linear form* l*, there is* ε_0 *such that for any* ε *with* $0 < \varepsilon < \varepsilon_0$ *and* $t_0 \neq 0$ *sufficiently small, the Euler obstruction of* $(X, 0)$ *is equal to:*

$$\mathrm{Eu}_X(0) = \sum_{i=1}^{q} \chi\big(V_i \cap B_\varepsilon \cap l^{-1}(\delta)\big) \cdot \mathrm{Eu}_X(V_i),$$

where χ *denotes the Euler-Poincaré characteristic,* $\mathrm{Eu}_X(V_i)$ *is the value of the Euler obstruction of* X *at any point of* V_i*,* $i = 1, \ldots, q$*, and* $0 < |\delta| \ll \varepsilon \ll 1$*.*

In the last section we will study two generalizations of the Euler obstruction, the Euler obstruction of a function, defined in [5], and the Brasselet number, defined in [9]. Let us recall these two definitions.

Introduced by Brasselet, Massey, Parameswaran and Seade in [5], the Euler obstruction of a function measures how far the equality given in Theorem 2.11 is from being true if we replace the generic linear form l with some other function on X with at most an isolated stratified critical point at 0. Let $f : X \to \mathbb{C}$ be a holomorphic function which is the restriction of a holomorphic function $F : U \to \mathbb{C}$. A point x in X is a critical point of f if it is a critical point of $F_{|V(x)}$, where $V(x)$ is the stratum containing x. We assume that f has an isolated singularity (or an isolated critical point) at 0, *i.e.*, that f has no critical point in a punctured neighborhood of 0 in X. In order to define the new invariant the authors constructed a stratified vector field on X, denoted by $\overline{\nabla}_X f$. This vector field is homotopic to $\overline{\nabla} F|_X$ and one has $\overline{\nabla}_X f(x) \neq 0$ unless $x = 0$.

Let $\tilde{\zeta}$ be the lifting of $\overline{\nabla}_X f$ as a section of the Nash bundle $\widetilde{T}$ over $\widetilde{X}$ without singularity over $\nu^{-1}(X \cap S_\varepsilon)$. Let $\mathcal{O}(\tilde{\zeta}) \in H^{2d}\big(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon)\big)$ be the obstruction cocycle to the extension of $\tilde{\zeta}$ as a nowhere zero section of $\widetilde{T}$ inside $\nu^{-1}(X \cap B_\varepsilon)$.

DEFINITION 2.12. — *The Euler obstruction* $\mathrm{Eu}_{f,X}(0)$ *is the evaluation of* $\mathcal{O}(\tilde{\zeta})$ *on the fundamental class of the pair* $(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon))$.

The following result compares the Euler obstruction of the space X with the Euler obstruction of a function on X.

THEOREM 2.13. *(See [5]).* — *Let $(X,0)$ and $\{V_i\}$ given as before and let $f : (X,0) \to (\mathbb{C},0)$ be a function with an isolated singularity at 0. For $0 < |\delta| \ll \varepsilon \ll 1$ we have:*

$$\operatorname{Eu}_{f,X}(0) = \operatorname{Eu}_X(0) - \left(\sum_{i=1}^{q} \chi\big(V_i \cap B_\varepsilon \cap f^{-1}(\delta)\big) \cdot \operatorname{Eu}_X(V_i) \right).$$

For an overview about Euler obstruction of a function see [3, 5].

In [9] Dutertre and Grulha defined the Brasselet number $B_{f,X}(0)$. In the general case this definition involves some technical elements, however when f has an isolated singularity, this number is equal to the difference $\operatorname{Eu}_X(0) - \operatorname{Eu}_{f,X}(0)$.

2.3. The Generic Fiber

In this section we remember the definition of smoothing of an analytic variety and some results related to it.

Let $X_0 \subset \mathbb{C}^n$ be a germ of an analytic d-dimensional variety, in some open subset of $\mathbb{C}^n$ with isolated singularity at the origin.

DEFINITION 2.14. — *We say that a germ of analytic variety $(X_0,0)$ with isolated singularity of complex dimension $d \geqslant 1$ has a smoothing, if there exist an open ball $B_\varepsilon(0) \subset \mathbb{C}^n$ centered at the origin, a closed subspace $X \subset B_\varepsilon(0) \times D$, where $D \subset \mathbb{C}$ is an open disc with center at zero and a proper analytic map $F : X \longrightarrow D$, with the restriction to X of the projection $p : B_\varepsilon(0) \times D \longrightarrow D$ such that*

a) *F is flat;*

b) *$(F^{-1}(0),0)$ is isomorphic to $(X_0,0)$;*

c) *$F^{-1}(t)$ is non singular for $t \neq 0$.*

It follows from the above definition that X has isolated singularity at the origin and it is a normal variety if X_0 is normal at zero. Moreover,

$$F \mid_{F^{-1}(D - \{0\})} : F^{-1}(D - \{0\}) \longrightarrow D - \{0\}$$

is a fiber bundle whose fibers $X_t = F^{-1}(t)$ are non singular.

The topology of the generic fiber has been intensively studied. For determinantal varieties, for instance, the following result was proved by Wahl.

THEOREM 2.15. *(See [29]). — Let $(X,0)$ be a determinantal variety with isolated singularity at the origin defined by $t \times t$ minors of an $s \times p$ matrix M, whose entries are on the ring of convergent power series on $\mathbb{C}^n$, and $2 \leqslant t \leqslant s \leqslant p$. If $dim(X) < s + p - 2t + 3$, then X has a smoothing.*

Remark 2.16. — In particular, it follows from Theorem 2.15 that if $(X,0)$ is Cohen-Macaulay with $\dim(X,0) = 2, 3$ and $\operatorname{codim}(X,0) \leqslant 2$, then $(X,0)$ admits a smoothing. This remark can be also found in ([15], p. 537).

The following result was proved by Greuel and Steenbrink.

THEOREM 2.17. *(See [15]). — Let X_t be the Milnor fiber of a smoothing of a normal singularity, then $\beta_1(X_t) = 0$.*

In [10] Ebeling and Gusein-Zade introduced the notion of a determinantal variety with an *essentially isolated determinantal singularity* (EIDS) ([10], Section 1), and they observed that, if an EIDS $X \subset \mathbb{C}^n$ is defined by $t \times t$ minors of an $s \times p$ matrix M, then X has an isolated singularity at the origin if, and only if, $n \leqslant (s - t + 2)(p - t + 2)$. Moreover, according to the Thom transversality theorem an EIDS always admits an *essential smoothing* ([10], Section 1), and in the specific case that $n < (s - t + 2)(p - t + 2)$ the *essential smoothing* is a genuine smoothing. The topology of the general fiber of a smoothing of an isolated determinantal variety was also studied by Ballesteros, Okamoto and Tomazella in [21].

Remark 2.18. — A very important fact is that varieties such as X do not admit all a smoothing. For examples about nonsmoothable singularities see [15].

3. The vanishing Euler characteristic of a toric surface

Throughout this section we denote by $X_\sigma \subset \mathbb{C}^n$ a toric surface. Considering σ into standard form (generated by e_2, $pe_1 - qe_2$, $0 \leqslant q < p$ with $gcd(p,q) = 1$) we obtain a refinement Δ of σ adding r new vertices $v_1, \ldots, v_r$ between the given vertices $v_0 = e_2$, $v_{r+1} = pe_1 - qe_2$, where the number r came from the Hirzebruch-Jung continued fraction

$$\frac{p}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\cdots - \frac{1}{b_r}}} = [[b_1, b_2, \ldots, b_r]].$$

The fan Δ provides a new smooth toric surface $X(\Delta)$. Using this construction Fulton provides the following result.

PROPOSITION 3.1. *(See [11]).* — *If Δ is the subdivision of σ given as above then $X(\Delta)$ is the minimal equivariant resolution of singularities of X_σ.*

Remark 3.2. — A toric surface X_σ, which is a cyclic quotient singularity, always possesses a smoothing which is locally diffeomorphic to its resolution $X(\Delta)$ (see [23], Satz 10). The problem is that this smoothing needs not be unique. For instance, in [29] Wahl gives an example of a smoothing for the toric surface $X_\sigma \subset \mathbb{C}^5$ associated to the cone $\sigma \subset \mathbb{R}^2$ generated by the vectors $v_1 = e_2$ and $v_2 = 4e_1 - e_2$, whose fiber X_t has Euler characteristic equal to 1. But in [21] the authors gave an example of a smoothing of the same X_σ such that $\chi(X_t) = 2 = \chi(X(\Delta))$.

If $X_\sigma \subset \mathbb{C}^n$ is an ICIS (when $p = q + 1$), $X(\Delta)$ can be seen as the Milnor fiber of X_σ. In this case, $X(\Delta)$ has the homotopy type of a bouquet of spheres and the Milnor number of X_σ is the number of such spheres (see [16]). In this case, the Milnor number coincides with the so-called vanishing Euler characteristic, that is

$$\mu(X_\sigma) = \beta_2(X(\Delta)) = \chi(X(\Delta)) - 1$$

where $\beta_2(X(\Delta))$ denotes the second Betti number of $X(\Delta)$. Based on [21] and supported by the previous remark, we give the following definition now in the general case of a toric surface.

DEFINITION 3.3. — *The vanishing Euler characteristic of a toric surface X_σ is defined by*
$$\nu(X_\sigma) := \chi(X(\Delta)) - 1.$$

THEOREM 3.4. — *Let $\sigma \subset \mathbb{R}^2$ be the cone generated by $v_1 = e_2$ and $v_2 = pe_1 - qe_2$, where $0 < q < p$ and p, q are coprimes, then*

$$\nu(X_\sigma) = (a_2 - 2) + \cdots + (a_{n-2} - 2) + (a_{n-1} - 1),$$

where $a_2, \ldots, a_{n-1}$ are the integers coming from the Hirzebruch-Jung continued fraction of $\frac{p}{p-q}$.

Proof. — We know that

$$\chi(X(\Delta)) = \beta_0(X(\Delta)) - \beta_1(X(\Delta)) + \beta_2(X(\Delta)) = 1 - \beta_1(X(\Delta)) + \beta_2(X(\Delta)).$$

Since X_σ is normal, by Theorem 2.17, $\beta_1(X(\Delta)) = 0$, then

$$\beta_2(X(\Delta)) = \chi(X(\Delta)) - 1 = \nu(X_\sigma).$$

By [1], we have
$$\dim H_2^{cld}(X(\Delta)) = d_1 - 2,$$
where d_1 denotes the number of 1-dimensional cones in Δ. But, $X(\Delta)$ is a smooth surface, then
$$H_2^{cld}(X(\Delta)) = H_2(X(\Delta)),$$
i.e., $\beta_2(X(\Delta)) = d_1 - 2 = r$. As a consequence of [23, 24], we have
$$\sum_{i=1}^{r} (b_i - 1) = \sum_{j=2}^{n-1} (a_j - 1),$$
so
$$r = (a_2 - 2) + \cdots + (a_{n-2} - 2) + (a_{n-1} - 1).$$
$\square$

In particular, if a variety X of dimension d has a unique smoothing, then the Milnor number of X is defined as the dth Betti number $\beta_d(X_t)$ of the generic fiber X_t of a smoothing of X, whenever X_t has homology only in the middle dimension, that is
$$\mu(X) := \beta_d(X_t).$$

Thus, we have the following consequence.

COROLLARY 3.5. — *If X_σ is a toric surface that admits a unique smoothing, then*
$$\nu(X_\sigma) = \beta_2(X(\Delta)) = \mu(X_\sigma).$$

Example 3.6. — Let $\sigma \subset \mathbb{R}^2$ be the cone generated by $v_1 = e_2$ and $v_2 = pe_1 - qe_2$, where $0 < q < p$ and p, q are coprime, such that
$$\frac{p}{p - q} = a - \frac{1}{b}.$$
Then by Theorem 3.4 one has
$$\nu(X_\sigma) = (a - 2) + (b - 1).$$
But, from [23, 24] we know that X_σ is a determinantal surface in $\mathbb{C}^4$ given by the 2×2 minors of the matrix
$$\begin{pmatrix} z_1 & z_2 & z_3^{b-1} \\ z_2^{a-1} & z_3 & z_4 \end{pmatrix}$$

and in [22] Pereira and Ruas proved that when $X \subset \mathbb{C}^4$ is a determinantal surface X has a unique smoothing. The authors also present a formula to compute the Milnor number in this case, that coincides with our formula to compute ν.

Given X a smoothable isolated singularity we know that $\chi(X_t)$ does not depend on t. When we consider a radial continuous vector field v on X with isolated singularity at 0, we can relate the number $\chi(X_t)$ with the GSV index of v in X. The GSV index was introduced by Gómez-Mont, Seade and Verjovsky in [13, 25] for hypersurface germs, and extended in [26] to complete intersections. In Section 3 of [7] we can find the definition of this index for the case that X admits a smoothing, which depends on the smoothing given by F. They also proved that $\mathrm{Ind}_{GSV}(v, X, F) = \chi(X_t)$, then we have the following.

COROLLARY 3.7. — *Let $\sigma \subset \mathbb{R}^2$ be the cone generated by $v_1 = e_2$ and $v_2 = pe_1 - qe_2$, where $0 < q < p$ and p, q are coprimes, and consider v a radial continuous vector field on X_σ with isolated singularity at 0. Let us consider the smoothing defined by F_{Res}, whose the fiber is the resolution of X_σ. Then,*

$$\mathrm{Ind}_{GSV}(v, X_\sigma, F_{Res}) = (a_2 - 2) + \cdots + (a_{n-2} - 2) + (a_{n-1}),$$

where $a_2, \ldots, a_{n-1}$ are the integers coming from the Hirzebruch-Jung continued fraction of $\frac{p}{p-q}$ and $\mathrm{Ind}_{GSV}(v, X_\sigma, F_{Res})$ is the GSV index of v at 0 relative to the smoothing whose fiber is $X(\Delta)$.

In [21] the authors defined the Milnor number of a function f with isolated singularity defined on Isolated Determinantal Singularity X, as

$$\mu(f|_X) = \#\Sigma(\tilde{f}|_{X_t}),$$

where X_t is a fiber of an smoothing of X and $\tilde{f}|_{X_t}$ is a morsefication of f and $\#\Sigma(\tilde{f}|_{X_t})$ denote the number of Morse points of $\tilde{f}$ on X_t.

PROPOSITION 3.8. — *Let $f : X \to \mathbb{C}$ be a function with isolated singularity defined in a smoothable Isolated Determinantal Singularity X, and consider v the vector field given by the gradient of the function f, then*

$$\mu(f|_X) = \mathrm{Ind}_{GSV}(v, X, F),$$

where F is the flat map associated to the smoothing of X.

Proof. — Follows directly from the definition of the GSV index in the general case of smoothable varieties (see [7]). □

Based on this last result we can extend the Milnor number of Ballesteros, Okamoto and Tomazella to the case of toric surfaces.

4. Relation with polar multiplicities and Euler obstruction

Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a d-dimensional variety with isolated singularity at the origin. Suppose that X has a smoothing, that is, there exists a family $\Pi : \mathfrak{X} \to D \subset \mathbb{C}$, restriction of the projection $\Phi : B_\varepsilon(0) \times D \to D$, such that $X_t = \Pi^{-1}(t)$ is smooth for all $t \neq 0$ and $X_0 = X$.

The variety $\mathfrak{X}$ also has an isolated singularity at the origin. Let p be a complex analytic function defined in X with isolated singularity at the origin. Let us define

$$
\begin{array}{rcccc}
\tilde{p} : \mathfrak{X} & \subset & \mathbb{C}^n \times \mathbb{C} & \longrightarrow & \mathbb{C} \\
& & (x, t) & \longmapsto & \tilde{p}(x, t)
\end{array}
$$

such that $\tilde{p}(x, 0) = p(x)$ and for all $t \neq 0$, $\tilde{p}(., t) = p_t$ is a Morse function in X_t. Thus we have the following diagram:

$$
\begin{array}{ccccc}
X_t \subset & \mathfrak{X} & \subset \mathbb{C}^n \times \mathbb{C} \\
\downarrow p_t & & \downarrow (\Pi, p) & & (4.1) \\
\mathbb{C} \times \{t\} & & \mathbb{C} \times \mathbb{C}
\end{array}
$$

Notice that the number of critical points of p_t is finite. In fact, x is a critical point of p_t if and only if x is a critical point of the real part of p_t. Since $Re(p_t) : X_t \to \mathbb{R}$ is an analytic function on X_t, the number of critical points of $Re(p_t)$ and, hence of p_t, is finite. Pereira and Ruas proved the following:

PROPOSITION 4.1. *(See [22]). — Let X be a d-dimensional variety with isolated singularity at the origin admitting a smoothing and $p_t : X_t \to \mathbb{C}$, $p_t = \tilde{p}(., t)$ as above. Then,*

$$
\chi(X_t) = \chi(p_t^{-1}(0)) + (-1)^d n_\sigma
$$

where n_σ is the number of critical points of p_t and $\chi(X_t)$ denotes the Euler characteristic of X_t.

The above formula can also be expressed replacing n_σ by $m_d(X)$ the d-th polar multiplicity of X. We refer to [28] for the definition and properties of polar varieties and to [12] for the definition of the d-th polar multiplicity.

Let $\mathfrak{X} \subset \mathbb{C}^n \times \mathbb{C}^s$ be a complex analytic variety of complex dimension $d + s$ and $\Pi : \mathfrak{X} \to \mathbb{C}^s$ an analytic function such that $\Pi^{-1}(0) = X$. Let

$\tilde{p} : \mathfrak{X} \subset \mathbb{C}^n \times \mathbb{C}^s, 0 \to \mathbb{C}^s, 0$ be such that $\tilde{p}|_X$ has isolated singularity at the origin. Then, we can define $m_d(X, \tilde{p}, \Pi) = m_0(P_d(\Pi, \tilde{p}))$, where $P_d(\Pi, \tilde{p})$ is the polar variety of $\mathfrak{X}$ with respect to $(\Pi, \tilde{p})$.

In general, $m_d(X, \tilde{p}, \Pi)$ depends on the choices of $\mathfrak{X}$ and $\tilde{p}$, but when $\mathfrak{X}$ is a versal deformation of X or in the case that X has a unique smoothing, m_d depends only on X and $\tilde{p}$. Furthermore, if $\tilde{p}$ is a generic linear projection, m_d is an invariant of the analytic variety X, which we denote by $m_d(X)$.

When $s = 1$ and $\tilde{p}$ is a generic linear projection, we have a diagram similar to (4.1) and we can relate n_σ and $m_d(X)$. In fact, the following result is a direct consequence of the definitions of these two invariants.

PROPOSITION 4.2. (See [22]). — *Under the conditions of Proposition 4.1, $n_\sigma = m_d(X)$.*

THEOREM 4.3. — *Let $\sigma \subset \mathbb{R}^2$ be the cone generated by $v_1 = e_2$ and $v_2 = pe_1 - qe_2$, where $0 < q < p$ and p, q are coprime, then*

$$m_2(X_\sigma) = (a_2 - 1) + \cdots + (a_{n-2} - 1) + (a_{n-1}),$$

where $a_2, \ldots, a_{n-1}$ are the integers coming from the Hirzebruch-Jung continued fraction of $\frac{p}{p-q}$.

Proof. — Let $p : X_\sigma \to \mathbb{C}$ be a generic linear function, then by Propositions 4.1 and 4.2

$$\chi(X(\Delta)) = \chi(p_t^{-1}(0)) + (-1)^d n_\sigma = \chi(X(\Delta) \cap p_t^{-1}(0)) + m_2(X_\sigma),$$

but $X(\Delta) \cap p_t^{-1}(0)$ is homeomorphic to $X \cap p^{-1}(c)$, since

$$f := (\Pi, \tilde{p}) : \mathfrak{X} \subset \mathbb{C}^n \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$$

is a bundle in the punctured disk and

$$X(\Delta) \cap p_t^{-1}(0) = f^{-1}(t, 0) \quad \text{and} \quad X \cap p^{-1}(c) = f^{-1}(0, c)$$

are fibers of this fibration. Then

$$\chi(X(\Delta)) = \chi(X \cap p^{-1}(c)) + m_2(X_\sigma)$$

but by [4] we have $\chi(X \cap p^{-1}(c)) = \mathrm{Eu}_{X_\sigma}(0)$. In [14] Gonzalez-Sprinberg proved that $\mathrm{Eu}_{X_\sigma}(0) = 3 - n$, then by Theorem 3.4 we have

$$m_2(X_\sigma) = (a_2 - 1) + \cdots + (a_{n-2} - 1) + (a_{n-1}).$$

□

Next, we give an illustration, with an example, of the computation of m_2 using Theorem 4.3. The next example was also computed in [10, 21, 22], but we provide this computation in an easier way. For instance, in [21] a software was used to compute this invariant and here we use only the continued fractions.

Example 4.4. — Let $X_\sigma \subset \mathbb{C}^4$ be the toric surface associated to the cone $\sigma \subset \mathbb{R}^2$ generated by the vectors $v_1 = e_2$ and $v_2 = 3e_1 - e_2$. From the Hirzebruch-Jung continued fraction process we have

$$\frac{3}{2} = 2 - \frac{1}{2},$$

then $X_\sigma = V(I_\sigma)$ where I_σ is the ideal generated by the 2×2 minors of the matrix

$$\begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_3 & z_4 \end{pmatrix}$$

i.e., X_σ is a codimension 2 determinantal surface. Then,

$$m_2(X_\sigma) = (a_2 - 1) + a_3 = 3.$$

In a more general way, we have the following.

COROLLARY 4.5. — *Consider $Y \subset \mathbb{C}^{n+1}$ the determinantal surface given by the 2×2 minors of the matrix*

$$A = \begin{pmatrix} z_1 & z_2 & \cdots & z_{n-1} & z_n^b \\ z_2^a & z_3 & \cdots & z_n & z_{n+1} \end{pmatrix},$$

where $n \geqslant 2$ and a, b are positive integers. Then, $m_2(Y) = a + b + n - 2$.

Proof. — In [23, 24] Riemenschneider proved that $Y = X_\sigma$, where $\sigma \subset \mathbb{R}^2$ is the cone generated by the vectors $v_1 = e_2$ and $v_2 = w_{n+1}e_1 - u_{n+1}e_2$ with

$$\begin{aligned}
u_1 e_1 + w_1 e_2 &= e_1, \\
u_2 e_1 + w_2 e_2 &= e_1 + e_2, \\
u_3 e_1 + w_3 e_2 &= ((a+1)u_2 - u_1)e_1 + ((a+1)w_2 - w_1)e_2, \\
u_4 e_1 + w_4 e_2 &= (2u_3 - u_2)e_1 + (2w_3 - w_2)e_2, \\
u_5 e_1 + w_5 e_2 &= (2u_4 - u_3)e_1 + (2w_4 - w_3)e_2, \\
&\ \ \vdots \\
u_n e_1 + w_n e_2 &= (2u_{n-1} - u_{n-2})e_1 + (2w_{n-1} - w_{n-2})e_2, \\
u_{n+1} e_1 + w_{n+1} e_2 &= ((b+1)u_n - u_{n-1})e_1 + ((b+1)w_n - w_{n-1})e_2.
\end{aligned}$$

i.e., the integers coming from the Hirzebruch-Jung continued fraction of $\frac{w_{n+1}}{w_{n+1}-u_{n+1}}$ are

$$a_2 = a + 1, \quad a_3 = 2, \ldots, \quad a_{n-1} = 2, \quad a_n = b + 1.$$

Therefore, from Theorem 4.3 we have that $m_2(Y) = a + b + n - 2$. $\qquad\square$

5. The Euler obstruction of a function on a toric surface

Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be an algebraic variety and let $f : (X, 0) \to (\mathbb{C}, 0)$ be a function with isolated singularity at the origin. The Euler obstruction of f can be viewed as a generalization of the Milnor number of the function f, that usually is denoted by $\mu(f)$. In this section, we compute the Euler obstruction of f on a toric surface X_σ.

From now on we will consider the following setup.

Let $\sigma \subset \mathbb{R}^2$ be the cone generated by $v_1 = e_2$ and $v_2 = pe_1 - qe_2$, where $0 < q < p$ and p, q are coprimes, and consider $a_2, \ldots, a_{n-1}$ the integers coming from the Hirzebruch-Jung continued fraction of $\frac{p}{p-q}$. We will denote by $\mathcal{O}_{(0,\ldots,0)}$, $\mathcal{O}_{(1,0,\ldots,0)}$, $\mathcal{O}_{(0,\ldots,0,1)}$ and $\mathcal{O}_{(1,\ldots,1)}$ the four orbits of the action $\tilde{\varphi} : (\mathbb{C}^*)^2 \times X_\sigma \longrightarrow X_\sigma$ given by

$$\tilde{\varphi}((t_1, t_2), (x_1, \ldots, x_n)) = (t_1 x_1, t_1 t_2 x_2, t_1^{u_3} t_2^{v_3} x_3, \ldots, t_1^{u_n} t_2^{v_n} x_n)$$

where $\{(1, 0), (1, 1), (u_3, v_3), \ldots, (u_n, v_n)\}$ is the minimal basis of the monoid $\check{\sigma} \cap \mathbb{Z}^2$. Using Theorem 2.13 we will prove the following.

THEOREM 5.1. — *Let* $f : (X_\sigma, 0) \to (\mathbb{C}, 0)$ *be a function with isolated singularity at the origin, then*

$$\mathrm{Eu}_{f, X_\sigma}(0) = 3 - n - \chi(\gamma) - \#B,$$

where γ *is the curve whose trace is the solution of equation*

$$f(t_1, t_1 t_2, t_1^{u_3} t_2^{v_3}, \ldots, t_1^{u_n} t_2^{v_n}) - t_0$$

and $B = X_\sigma^{reg} \cap B_\varepsilon \cap f^{-1}(t_0) \cap \mathcal{O}_{(0,\ldots,0,1)}$, *with* $X_\sigma^{reg} = X_\sigma \setminus \{0\}$ *and* $t_0 \neq 0$.

Proof. — Since X_σ has isolated singularity at the origin, by Theorem 2.13

$$\mathrm{Eu}_{f, X_\sigma}(0) = \mathrm{Eu}_{X_\sigma}(0) - \chi\big(X_\sigma^{reg} \cap B_\varepsilon \cap f^{-1}(t_0)\big).$$

We know that the orbit $\mathcal{O}_{(1,\ldots,1)}$ of the action $\tilde{\varphi}$ is homeomorphic to $(\mathbb{C}^*)^2$. Now, consider the application

$$\varphi : \mathbb{C}^2 \to X_\sigma$$

given by $\varphi(t_1, t_2) = (t_1, t_1 t_2, t_1^{u_3} t_2^{v_3}, \ldots, t_1^{u_n} t_2^{v_n})$, then $\varphi(\mathbb{C}^2) = X_\sigma \backslash \mathcal{O}_{(0,\ldots,0,1)}$. Furthermore,

$$\varphi|_{\mathbb{C}^* \times \mathbb{C}} : \mathbb{C}^* \times \mathbb{C} \to X_\sigma \setminus \left\{ \mathcal{O}_{(0,\ldots,0,1)} \cup \mathcal{O}_{(0,\ldots,0)} \right\}$$

is a bijection. Then, consider the curve $\gamma : \mathbb{C}^2 \to \mathbb{C}$ given by

$$\gamma(t_1, t_2) = f(t_1, t_1 t_2, t_1^{u_3} t_2^{v_3}, \ldots, t_1^{u_n} t_2^{v_n}) - t_0$$

we have that

$$\chi\big(X_\sigma^{reg} \cap B_\varepsilon \cap f^{-1}(t_0)\big) = \chi(\gamma) + \#B,$$

where $B = X_\sigma^{reg} \cap B_\varepsilon \cap f^{-1}(t_0) \cap \mathcal{O}_{(0,\ldots,0,1)}$. By [14] we know that $\mathrm{Eu}_{X_\sigma}(0) = 3 - n$, therefore $\mathrm{Eu}_{f, X_\sigma}(0) = 3 - n - \chi(\gamma) - \#B$. $\qquad \square$

Example 5.2. — Let $\sigma \subset \mathbb{R}^2$ be the cone generated by $v_1 = e_2$ and $v_2 = ne_1 - e_2$, where $n > 1$, *i.e.*, X_σ is the determinantal surface given by the 2×2 minors of the matrix

$$\begin{pmatrix} z_1 & z_2 & z_3 & \cdots & z_{n-1} & z_n \\ z_2 & z_3 & z_4 & \cdots & z_n & z_{n+1} \end{pmatrix}$$

and consider $f : (X_\sigma, 0) \to (\mathbb{C}, 0)$ the polynomial with isolated singularity at the origin given by

$$f(x_1, \ldots, x_{n+1}) = x_1^{n+1} + x_{n+1} + \sum_{l=1}^{k} a_l x_2^{d_2^l} \ldots x_n^{d_n^l}$$

where $d_2^i, \ldots, d_n^i$ satisfy the condition

$$n + 1 = \sum_{j=2}^{n} d_j^1 (u_j + v_j) = \cdots = \sum_{j=2}^{n} d_j^k (u_j + v_j)$$

and $a_l \in \mathbb{C}^*$ for $l = 1, \ldots, k$. In this case, we know that

$$(u_3, v_3) = (1, 2), \quad (u_4, v_4) = (1, 3), \ldots, (u_{n+1}, v_{n+1}) = (1, n)$$

then the curve in $\mathbb{C}^2$

$$\tilde{\gamma}(t_1, t_2) = f(t_1, t_1 t_2, t_1^{u_3} t_2^{v_3}, \ldots, t_1^{u_n} t_2^{v_n})$$

is given by

$$\tilde{\gamma}(t_1, t_2) = t_1^{n+1} + t_1 t_2^n + \sum_{l=1}^{k} a_l (t_1 t_2)^{d_2^l} \ldots (t_1 t_2^{n-1})^{d_n^l}$$

that is a homogeneous polynomial of degree $n+1$ with isolated singularity at the origin. Now, recall that the curve γ is the fiber of a smoothing of curve $\tilde{\gamma}$, then by [20] $\mu(\tilde{\gamma}) = n^2$. Since $\chi(\gamma) = 1 - \mu(\tilde{\gamma})$, we have that $\chi(\gamma) = 1 - n^2$. Note that, $\#B = 1$, then by Theorem 5.1

$$\mathrm{Eu}_{f,X_\sigma}(0) = n^2 - n.$$

COROLLARY 5.3. — *Let $f : X_\sigma \to \mathbb{C}$ be a function with isolated singularity at the origin, then*

$$\mathrm{Eu}_{X_\sigma}(0) - \mathrm{Eu}_{f,X_\sigma}(0) = \chi(\gamma) + \#B$$

where γ and B are as above.

Corollary 5.3 concern the difference between the Euler obstruction of a space and the Euler obstruction of a function, that as noticed in [5, 9] has interesting meanings, even if f has a non-isolated singularity.

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