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D. GAITSGORY

*Eisenstein series and quantum groups*

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# Eisenstein series and quantum groups

D. GAITSGORY<sup>(1)</sup>

*To V. Schechtman, with admiration*

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**RÉSUMÉ.** – Dans cette note on donne une esquisse de la démonstration d’une conjecture de [13] qui établit un lien entre le faisceau correspondant à la série d’Eisenstein géométrique et la cohomologie semi-infinie du petit groupe quantique à coefficients dans le module basculant pour le groupe quantique de Lusztig.

**ABSTRACT.** – We sketch a proof of a conjecture of [13] that relates the geometric Eisenstein series sheaf with semi-infinite cohomology of the small quantum group with coefficients in the tilting module for the big quantum group.

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<sup>1</sup>Department of Mathematics, Harvard University 1 Oxford street, Cambridge, MA USA

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## Introduction

### 0.1. The conjecture

A mysterious conjecture was suggested in the paper [13]. It tied two objects of very different origins associated with a reductive group  $G$ .

*0.1.1.* On the one hand, we consider the *geometric Eisenstein series* sheaf  $\mathrm{Eis}_{!,*}$ , which is an object of the derived category of constructible sheaves on  $\mathrm{Bun}_G$  for the curve  $X = \mathbb{P}^1$ . (Here and elsewhere  $\mathrm{Bun}_G$  denotes the moduli stack of  $G$ -bundles on  $X$ .) See Sect. 1.1, where the construction of geometric Eisenstein series is recalled. By the Decomposition Theorem,  $\mathrm{Eis}_{!,*}$  splits as a direct sum of (cohomologically shifted) irreducible perverse sheaves.

Now, for a curve  $X = \mathbb{P}^1$ , the stack  $\mathrm{Bun}_G$  has discretely many isomorphism classes of points, which are parameterized by dominant coweights of  $G$ . Therefore, irreducible perverse sheaves on  $\mathrm{Bun}_G$  are in bijection with dominant coweights of  $G$ : to each  $\lambda \in \Lambda^+$  we attach the intersection cohomology sheaf  $\mathrm{IC}^\lambda$  of the closure of the corresponding stratum.

In the left-hand side of the conjecture of [13] we consider the (cohomologically graded) vector space equal to the space of multiplicities of  $\mathrm{IC}^\lambda$  in  $\mathrm{Eis}_{!,*}$ .

*0.1.2.* On the other hand, we consider the big and small quantum groups,  $\mathfrak{U}_q(G)$  and  $\mathfrak{u}_q(G)$ , attached to  $G$ , where  $q$  is a root of unity of sufficiently high order. To the quantum parameter  $q$  one associates the action of the extended affine Weyl group  $W \ltimes \Lambda$  on the weight lattice  $\check{\Lambda}$ , and using this action, to a dominant coweight  $\lambda$  one attaches a particular dominant weight, denoted  $\min_\lambda(0)$ ; see Sect. 1.3 for the construction.

Consider the *indecomposable tilting module* over  $\mathfrak{U}_q(G)$  of highest weight  $\min_\lambda(0)$ ; denote it  $\mathfrak{T}_q^\lambda$ . The right-hand side of the conjecture of [13] is the *semi-infinite* cohomology of the small quantum group  $\mathfrak{u}_q(G)$  with coefficients in  $\mathfrak{T}_q^\lambda|_{\mathfrak{u}_q(G)}$ .

The conjecture of [13] says that the above two (cohomologically graded) vector spaces are canonically isomorphic. Because of the appearing of tilting modules, the above conjecture acquired a name of the “Tilting Conjecture”.

*0.1.3.* In this paper we will sketch a proof of the Tilting Conjecture. The word “sketch” should be understood in the following sense. We indicate<sup>1</sup> how to reduce it to two statements that we call “quasi-theorems”, Quasi-Theorem 7.9 and Quasi-Theorem 9.7. These are plausible statements of

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<sup>1</sup>Indicate=explain the main ideas, but far from supplying full details.

more general nature, which we hope will turn into actual theorems soon. We will explain the content of these quasi-theorems below, see Sect. 0.3.1 and Sect. 0.5.2, respectively.

*0.1.4.* This approach to the proof of the Tilting Conjecture is quite involved. It is very possible that if one does not aim for the more general Conjecture 6.1 (described in Sect. 0.2), a much shorter (and elementary) argument proving the Tilting Conjecture exists.

In particular, in a subsequent publication we will show that the Tilting Conjecture can be obtained as a formal consequence of the classical<sup>2</sup> geometric Langlands conjecture for curves of genus 0.

## 0.2. Our approach

We approach the Tilting Conjecture from the following perspective. Rather than trying to prove the required isomorphism directly, we first rewrite both sides so that they become amenable to generalization, and then proceed to proving the resulting general statement, Conjecture 6.1.

*0.2.1.* This generalized version of the Tilting Conjecture, i.e., Conjecture 6.1, takes the following form. First, our geometric input is a (smooth and complete) curve  $X$  of arbitrary genus, equipped with a finite collection of marked points  $x_1, \dots, x_n$ . Our representation-theoretic input is a collection  $\mathcal{M}_1, \dots, \mathcal{M}_n$  of representations of  $\mathfrak{U}_q(G)$ , so that we think of  $\mathcal{M}_i$  as sitting at  $x_i$ .

Starting with this data, we produce two (cohomologically graded) vector spaces.

*0.2.2.* The first vector space is obtained by combining the following steps.

(i) We apply the Kazhdan-Lusztig equivalence

$$\mathrm{KL}_G : \mathfrak{U}_q(G)\text{-mod} \simeq \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G^{(\mathcal{O})}}$$

to  $\mathcal{M}_1, \dots, \mathcal{M}_n$  and convert them to representations  $M_1, \dots, M_n$  of the Kac-Moody Lie algebra  $\widehat{\mathfrak{g}}_{\kappa'}$ , where  $\kappa'$  is a negative integral level corresponding to  $q$ .

(We recall that  $\widehat{\mathfrak{g}}_{\kappa'}$  is the central extension of  $\mathfrak{g}(\mathcal{K})$  equipped with a splitting over  $\mathfrak{g}(\mathcal{O})$ , with the bracket specified by  $\kappa'$ . Here and elsewhere  $\mathcal{O} = \mathbb{C}[[t]]$  and  $\mathcal{K} = \mathbb{C}((t))$ .)

---

<sup>2</sup>Classical=non-quantum.

(ii) Starting with  $M_1, \dots, M_n$ , we apply the *localization functor* and obtain a  $\kappa'$ -twisted D-module<sup>3</sup>  $\text{Loc}_{G, \kappa', x_1, \dots, x_n}(M_1, \dots, M_n)$  on  $\text{Bun}_G$ . Using the fact that  $\kappa'$  was integral, we convert  $\text{Loc}_{G, \kappa', x_1, \dots, x_n}(M_1, \dots, M_n)$  to a non-twisted D-module (by a slight abuse of notation we denote it by the same character).

(iii) We tensor  $\text{Loc}_{G, \kappa', x_1, \dots, x_n}(M_1, \dots, M_n)$  with  $\text{Eis}_{!*}$  and take its de Rham cohomology on  $\text{Bun}_G$ .

In Sect. 2 we explain that the space of multiplicities appearing in the Tilting Conjecture, is a particular case of this procedure, when we take  $X$  to be of genus 0,  $n = 1$  with the module  $\mathcal{M}$  being  $\mathfrak{T}_q^\Lambda$ .

This derivation is a rather straightforward application of the Kashiwara-Tanisaki equivalence between the (regular block of the) affine category  $\mathcal{O}$  and the category of D-modules on (the parabolic version) of  $\text{Bun}_G$ , combined with manipulation of various dualities.

0.2.3. The second vector space is obtained by combining the following steps.

(i) We use the theory of *factorizable sheaves* of [12], thought of as a functor

$$\text{BFS}_{u_q}^{\text{top}} : \mathbf{u}_q(G)\text{-mod} \otimes \dots \otimes \mathbf{u}_q(G)\text{-mod} \rightarrow \text{Shv}_{\mathcal{G}_{q, \text{loc}}}(\text{Ran}(X, \check{\Lambda}))$$

(here  $\text{Ran}(X, \check{\Lambda})$  is the configuration space of  $\check{\Lambda}$ -colored divisors), and attach to attach to  $\mathcal{M}_1|_{u_q(G)}, \dots, \mathcal{M}_n|_{u_q(G)}$  a (twisted) constructible sheaf<sup>4</sup> on  $\text{Ran}(X, \check{\Lambda})$ , denoted,

$$\text{BFS}_{u_q}^{\text{top}}(\mathcal{M}_1|_{u_q(G)}, \dots, \mathcal{M}_n|_{u_q(G)}).$$

(ii) We apply the direct image functor with respect to the Abel-Jacobi map

$$\text{AJ} : \text{Ran}(X, \check{\Lambda}) \rightarrow \text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}$$

and obtain a (twisted) sheaf

$$\text{AJ}_!(\text{BFS}_{u_q}^{\text{top}}(\mathcal{M}_1|_{u_q(G)}, \dots, \mathcal{M}_n|_{u_q(G)})) \tag{0.1}$$

on  $\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}$ .

---

<sup>3</sup>For the duration of the introduction we will ignore the difference between the two versions of the derived category of (twisted) D-modules on  $\text{Bun}_G$  that occurs because the latter stack is non quasi-compact.

<sup>4</sup>The twisting is given by a canonically defined gerbe over  $\text{Ran}(X, \check{\Lambda})$ , denoted  $\mathcal{G}_{q, \text{loc}}$ .

(iii) We tensor (0.1) with a canonically defined (twisted<sup>5</sup>) local system  $\mathcal{E}_{q^{-1}}$  on  $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}$ , and take cohomology along  $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}$ .

In Sect. 4 we explain why the above procedure, applied in the case when  $X$  has genus 0,  $n = 1$  and  $\mathcal{M} = \mathfrak{T}_q^\Lambda$ , recovers the right-hand side of the Tilting Conjecture.

In fact, this derivation is immediate from one of the main results of the book [12] that gives the expression for the semi-infinite cohomology of  $\mathfrak{u}_q(G)$  in terms of the procedure indicated above when  $X$  has genus 0.

0.2.4. Thus, Conjecture 6.1 states that the two procedures, indicated in Sects. 0.2.2 and 0.2.3 above, are canonically isomorphic as functors

$$\mathfrak{U}_q(G)\text{-mod} \times \dots \times \mathfrak{U}_q(G)\text{-mod} \rightarrow \mathrm{Vect}.$$

The second half of this paper is devoted to the outline of the proof of Conjecture 6.1. As was already mentioned, we do not try to give a complete proof, but rather show how to deduce Conjecture 6.1 from Quasi-Theorems 7.9 and 9.7.

### 0.3. KL vs. BFS via BRST

The two most essential ingredients in the functors in Sects. 0.2.2 and 0.2.3 are the Kazhdan-Lusztig equivalence

$$\mathrm{KL}_G : \mathfrak{U}_q(G)\text{-mod} \simeq \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G^{(0)}} \tag{0.2}$$

(in the case of the former<sup>6</sup>) and the [12] construction

$$\mathrm{BFS}_{\mathfrak{u}_q}^{\mathrm{top}} : \mathfrak{u}_q(G)\text{-mod} \times \dots \times \mathfrak{u}_q(G)\text{-mod} \rightarrow \mathrm{Shv}_{\mathfrak{g}_{q,\mathrm{loc}}}(\mathrm{Ran}(X, \check{\Lambda})), \tag{0.3}$$

(in the case of the latter).

In order to approach Conjecture 6.1 we need to understand how these two constructions are related. The precise relationship is given by Quasi-Theorem 7.9, and it goes through a *particular version* of the functor of BRST reduction of  $\widehat{\mathfrak{g}}_{\kappa'}$ -modules with respect to the Lie subalgebra  $\mathfrak{n}(\mathcal{K}) \subset \widehat{\mathfrak{g}}_{\kappa'}$ :

$$\mathrm{BRST}_{\mathfrak{n},!} : \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G^{(0)}} \rightarrow \widehat{\mathfrak{t}}_{\kappa'}\text{-mod}^{T^{(0)}},$$

---

<sup>5</sup>By means of the inverse gerbe, so that the tensor product is a usual sheaf, for which it make sense to take cohomology.

<sup>6</sup>Here and elsewhere  $\widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G^{(0)}}$  denotes the category of Harish-Chandra modules for the pair  $(\widehat{\mathfrak{g}}_{\kappa'}, G^{(0)})$ . This is the category studied by Kazhdan and Lusztig in the series of papers [22].

introduced<sup>7</sup> in Sect. 7.4, using the theory of D-modules on *the semi-infinite flag space*.

0.3.1. Quasi-Theorem 7.9 is a local assertion, which may be thought of as a characterization of the Kazhdan-Lusztig equivalence. It says that the following diagram of functors commutes

$$\begin{array}{ccc} \mathfrak{U}_q(G)\text{-mod} & \xrightarrow{\text{KL}_G} & \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G^{(\circ)}} \\ \text{Inv}_{\mathfrak{u}_q(N^+)} \circ \text{Res}^{\text{big} \rightarrow \text{small}} \downarrow & & \downarrow \text{BRST}_{\mathfrak{n},!,*} \\ \mathfrak{U}_q(T)\text{-mod} & \xrightarrow{\text{KL}_T} & \widehat{\mathfrak{t}}_{\kappa'}\text{-mod}^{T^{(\circ)}}. \end{array}$$

In this diagram,  $\mathfrak{U}_q(T)\text{-mod}$  is the category of representations of the quantum torus, denoted in the main body of the paper  $\text{Rep}_q(T)$ . The functor  $\text{KL}_T$  is Kazhdan-Lusztig equivalence for  $T$ , which is more or less tautological. The functor

$$\text{Inv}_{\mathfrak{u}_q(N^+)} \circ \text{Res}^{\text{big} \rightarrow \text{small}} : \mathfrak{U}_q(G)\text{-mod} \rightarrow \mathfrak{U}_q(T)\text{-mod}$$

is the following: we restrict a  $\mathfrak{U}_q(G)$ -module to  $\mathfrak{u}_q(G)$ , and then take (derived) invariants with respect to the subalgebra  $\mathfrak{u}_q(N^+)$ .

Thus, the upshot of Quasi-Theorem 7.9 is that the Kazhdan-Lusztig equivalences for  $G$  and  $T$ , respectively, intertwine the functor  $\text{BRST}_{\mathfrak{n},!,*}$  and the functor of taking invariants with respect to  $\mathfrak{u}_q(N^+)$ .

0.3.2. Let us now explain how Quasi-Theorem 7.9 allows to relate the functors  $\text{KL}_G$  and  $\text{BFS}_{\mathfrak{u}_q}^{\text{top}}$ . This crucially relies in the notions of *factorization category*, and of the category over the Ran space, attached to a given factorization category. We refer the reader to [26] for background on these notions.

First, the equivalence  $\text{KL}_T$  (combined with Riemann-Hilbert correspondence) can be viewed as a functor

$$\text{Shv}_{\mathcal{G}_q, \text{loc}}(\text{Ran}(X, \check{\Lambda})) \xrightarrow{(\text{KL}_T)_{\text{Ran}(X)}} (\widehat{\mathfrak{t}}_{\kappa'}\text{-mod}^{T^{(\circ)}})_{\text{Ran}(X)},$$

where  $(\widehat{\mathfrak{t}}_{\kappa'}\text{-mod}^{T^{(\circ)}})_{\text{Ran}(X)}$  is the category over the Ran space attached to  $\widehat{\mathfrak{t}}_{\kappa'}\text{-mod}^{T^{(\circ)}}$ , when the latter is viewed as a factorization category.

Second, the functor  $\text{BRST}_{\mathfrak{n},!,*}$ , viewed as a factorization functor gives rise to a functor

$$(\text{BRST}_{\mathfrak{n},!,*})_{\text{Ran}(X)} : (\widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G^{(\circ)}})_{\text{Ran}(X)} \rightarrow (\widehat{\mathfrak{t}}_{\kappa'}\text{-mod}^{T^{(\circ)}})_{\text{Ran}(X)}.$$

---

<sup>7</sup>One actually needs to replace  $\widehat{\mathfrak{t}}_{\kappa'}$  by its version that takes into account the critical twist and the  $\rho$ -shift, but we will ignore this for the duration of the introduction.

Now, it is a formal consequence of Quasi-Theorem 7.9 (and the interpretation of the functor  $\text{BFS}_{\mathfrak{u}_q}^{\text{top}}$  via  $\text{Inv}_{\mathfrak{u}_q(N^+)}$  that we explain in Sect. 5.1 ) that we have the following commutative diagram of categories:

$$\begin{array}{ccc}
 \mathfrak{U}_q(G)\text{-mod} \times \dots \times \mathfrak{U}_q(G)\text{-mod} & \xrightarrow{\text{KLG}} & \widehat{\mathfrak{g}}_{\kappa', x_1}\text{-mod}^{G(\circ x_1)} \times \dots \times \widehat{\mathfrak{g}}_{\kappa', x_n}\text{-mod}^{G(\circ x_n)} \\
 \downarrow \text{Res}^{\text{big} \rightarrow \text{small}} \times \dots \times \text{Res}^{\text{big} \rightarrow \text{small}} & & \downarrow \\
 \mathfrak{u}_q(G)\text{-mod} \times \dots \times \mathfrak{u}_q(G)\text{-mod} & & (\widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\circ)})_{\text{Ran}(X)} \\
 \downarrow \text{BFS}_{\mathfrak{u}_q}^{\text{top}} & & \downarrow (\text{BRST}_{\mathfrak{n}, !*})_{\text{Ran}(X)} \\
 \text{Shv}_{\mathcal{G}_{q, \text{loc}}}(\text{Ran}(X, \check{\Lambda})) & \xrightarrow{(\text{KLT})_{\text{Ran}(X)}} & (\widehat{\mathfrak{t}}_{\kappa'}\text{-mod}^{T(\circ)})_{\text{Ran}(X)}.
 \end{array} \tag{0.4}$$

### 0.4. Disposing of quantum groups

We shall now show how to use the commutative diagram (0.4) to rewrite Conjecture 6.1 as a statement that is purely algebraic, i.e., one that only deals with D-modules as opposed to constructible sheaves, and in particular one that does not involve quantum groups, but only Kac-Moody representations.

*0.4.1.* The commutative diagram (0.4) gets us one step closer to the proof of Conjecture 6.1. Namely, it gives an interpretation of Step (i) in the procedure of Sect. 0.2.3 in terms of Kac-Moody algebras. In order to make it possible to compare the entire procedure of Sect. 0.2.3 with that of Sect. 0.2.2 we need to give a similar interpretation of Steps (ii) and (iii).

This is done by means of combining Riemann-Hilbert correspondence with Fourier-Mukai transform. Namely, we claim that we have the following two commutative diagrams.

One diagram is:

$$\begin{array}{ccc}
 \text{Shv}_{\mathcal{G}_{q, \text{loc}}}(\text{Ran}(X, \check{\Lambda})) & \xrightarrow{(\text{KLT})_{\text{Ran}(X)}} & (\widehat{\mathfrak{t}}_{\kappa'}\text{-mod}^{T(\circ)})_{\text{Ran}(X)} \\
 \text{AJ!} \downarrow & & \downarrow \text{Loc}_{T, \kappa', \text{Ran}(X)} \\
 \text{Shv}_{\mathcal{G}_{q, \text{glob}}}(\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) & \xrightarrow{\text{FM} \circ \text{RH}} & \text{D-mod}_{\kappa'}(\text{Bun}_T).
 \end{array}$$

Here RH stands for the Riemann-Hilbert functor, and the subscript  $\mathcal{G}_{q, \text{glob}}$  stands for an appropriate gerbe on  $\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}$ . The commutativity of this diagram follows from the standard properties of the Fourier-Mukai transform.

The other diagram is:

$$\begin{array}{ccc}
 \mathrm{Shv}_{\mathcal{G}_{q,\mathrm{glob}}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) & \xrightarrow{\mathrm{FM}\circ\mathrm{RH}} & \mathrm{D} - \mathrm{mod}_{\kappa'}(\mathrm{Bun}_T) \\
 \downarrow -\otimes \mathcal{E}_{q-1} & & \downarrow \sim \\
 \mathrm{Shv}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) & & \mathrm{D} - \mathrm{mod}(\mathrm{Bun}_T) \\
 \Gamma(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}, -) \downarrow & & \downarrow \Gamma_{\mathrm{dr}}(\mathrm{Bun}_T, -) \\
 \mathrm{Vect} & \xrightarrow{\mathrm{Id}} & \mathrm{Vect}.
 \end{array}$$

In this diagram<sup>8</sup>  $\mathcal{E}_{q-1}$  is the (twisted) local system from Step (iii) in Sect. 0.2.3.

The equivalence  $\mathrm{D} - \mathrm{mod}_{\kappa'}(\mathrm{Bun}_T) \rightarrow \mathrm{D} - \mathrm{mod}(\mathrm{Bun}_T)$ , appearing in the above diagram, comes from the fact that the twisting  $\kappa'$  is *integral*. The commutativity of the diagram follows from the definition of the (twisted) local system  $\mathcal{E}_{q-1}$ .

0.4.2. Putting the above two diagrams together with (0.4), we obtain a commutative diagram

$$\begin{array}{ccc}
 \mathcal{U}_q(G) - \mathrm{mod} \times \dots \times \mathcal{U}_q(G) - \mathrm{mod} & \xrightarrow{\mathrm{KL}_G} & \widehat{\mathfrak{g}}_{\kappa', x_1} - \mathrm{mod}^{G(\mathcal{O}_{x_1})} \times \dots \times \widehat{\mathfrak{g}}_{\kappa', x_n} - \mathrm{mod}^{G(\mathcal{O}_{x_n})} \\
 \downarrow \mathrm{Res}^{\mathrm{big} \rightarrow \mathrm{small}} \times \dots \times \mathrm{Res}^{\mathrm{big} \rightarrow \mathrm{small}} & & \downarrow \\
 \mathfrak{u}_q(G) - \mathrm{mod} \times \dots \times \mathfrak{u}_q(G) - \mathrm{mod} & & (\widehat{\mathfrak{g}}_{\kappa'} - \mathrm{mod}^{G(\mathcal{O})})_{\mathrm{Ran}(X)} \\
 \downarrow \mathrm{BFS}_{\mathfrak{u}_q}^{\mathrm{top}} & & \downarrow (\mathrm{BRST}_{\mathfrak{n}, !*})_{\mathrm{Ran}(X)} \\
 \mathrm{Shv}_{\mathcal{G}_{q,\mathrm{loc}}}(\mathrm{Ran}(X, \check{\Lambda})) & & (\widehat{\mathfrak{t}}_{\kappa'} - \mathrm{mod}^{T(\mathcal{O})})_{\mathrm{Ran}(X)} \\
 \mathrm{AJ}_1 \downarrow & & \downarrow \mathrm{Loc}_{T, \kappa', \mathrm{Ran}(X)} \\
 \mathrm{Shv}_{\mathcal{G}_{q,\mathrm{glob}}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) & & \mathrm{D} - \mathrm{mod}_{\kappa'}(\mathrm{Bun}_T) \\
 \downarrow -\otimes \mathcal{E}_{q-1} & & \downarrow \sim \\
 \mathrm{Shv}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) & & \mathrm{D} - \mathrm{mod}(\mathrm{Bun}_T) \\
 \Gamma(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}, -) \downarrow & & \downarrow \Gamma_{\mathrm{dr}}(\mathrm{Bun}_T, -) \\
 \mathrm{Vect} & \xrightarrow{\mathrm{Id}} & \mathrm{Vect}, \tag{0.5}
 \end{array}$$

in which the left composed vertical arrow is the procedure of Sect. 0.2.3.

<sup>8</sup>In the lower right vertical arrow, as well as elsewhere in the paper, the notation  $\Gamma_{\mathrm{dr}}(-, -)$  stands for the functor of de Rham cohomology.

Thus, in order to prove Conjecture 6.1, it remains to show the right composed vertical arrow in (0.5) is canonically isomorphic to the composition of Steps (ii) and (iii) in the procedure of Sect. 0.2.2. Recall, however, that the latter functor involves  $\text{Eis}_{!*}$  and thus contains the information about the intersection cohomology (a.k.a. IC) sheaf on Drinfeld's compactification  $\overline{\text{Bun}}_B$ .

Note that, as promised, the latter assertion only involves algebraic objects.

### 0.5. Bringing the semi-infinite flag space into the game

We now outline the remaining steps in the derivation of Conjecture 6.1.

*0.5.1.* In order to compare the right vertical composition in (0.5) with the functor

$$\widehat{\mathfrak{g}}_{\kappa', x_1} \text{-mod}^{G(\mathcal{O}_{x_1})} \times \dots \times \widehat{\mathfrak{g}}_{\kappa', x_n} \text{-mod}^{G(\mathcal{O}_{x_n})} \rightarrow \text{Vect},$$

given by composing Steps (ii) and (iii) in Sect. 0.2.2, it is convenient to rewrite both sides using the notion of dual functor, see Sect. 0.7.3.

Let

$$\text{CT}_{!*} : \text{D} - \text{mod}(\text{Bun}_G) \rightarrow \text{D} - \text{mod}(\text{Bun}_T)$$

denote the functor *dual* to  $\text{Eis}_{!*}$ . Let

$$\text{CT}_{\kappa', !*} : \text{D} - \text{mod}_{\kappa'}(\text{Bun}_G) \rightarrow \text{D} - \text{mod}_{\kappa'}(\text{Bun}_T)$$

denote its  $\kappa'$ -twisted counterpart (we remind that because the level  $\kappa'$  was assumed integral, the twisted categories are canonically equivalent to the non-twisted ones).

It then follows formally that the required isomorphism of functors is equivalent to the commutativity of the next diagram:

$$\begin{array}{ccc} (\widehat{\mathfrak{g}}_{\kappa'} \text{-mod}^{G(\mathcal{O})})_{\text{Ran}(X)} & \xrightarrow{(\text{BRST}_{\mathfrak{n}, !*})_{\text{Ran}(X)}} & (\widehat{\mathfrak{t}}_{\kappa'} \text{-mod}^{T(\mathcal{O})})_{\text{Ran}(X)} \\ \text{Loc}_{G, \kappa', \text{Ran}(X)} \downarrow & & \downarrow \text{Loc}_{T, \kappa', \text{Ran}(X)} \\ \text{D} - \text{mod}_{\kappa'}(\text{Bun}_G) & \xrightarrow{\text{CT}_{\kappa', !*}} & \text{D} - \text{mod}_{\kappa'}(\text{Bun}_T) \end{array} \quad (0.6)$$

*0.5.2.* Now, it turns out that the commutation of the diagram (0.6) is a particular case of a more general statement.

In Sect. 7 we introduce the category, denoted  $\mathcal{C}_{\kappa'}^{T(\mathcal{O})}$ , to be thought of as the category of twisted D-modules on the double quotient

$$N(\mathcal{K}) \backslash G(\mathcal{K}) / G(\mathcal{O}).$$

This is also a factorization category, and we denote by  $(\mathcal{C}_{\kappa'}^{T(\mathcal{O})})_{\text{Ran}(X)}$  the corresponding category over the Ran space.

In Sect. 9 we show that to any object

$$\mathbf{c} \in (\mathcal{C}_{\kappa'}^{T(\mathcal{O})})_{\text{Ran}(X)}$$

we can attach a functor

$$\text{CT}_{\kappa', \mathbf{c}} : \text{D} - \text{mod}_{\kappa'}(\text{Bun}_G) \rightarrow \text{D} - \text{mod}_{\kappa'}(\text{Bun}_T),$$

and also a functor

$$\text{BRST}_{\mathbf{c}} : (\widehat{\mathfrak{g}}_{\kappa'} - \text{mod}^{G(\mathcal{O})})_{\text{Ran}(X)} \rightarrow (\widehat{\mathfrak{t}}_{\kappa'} - \text{mod}^{T(\mathcal{O})})_{\text{Ran}(X)}.$$

We now have the following statement, Quasi-Theorem 9.7, that says that the following diagram is commutative for any  $\mathbf{c}$  as above:

$$\begin{array}{ccc} (\widehat{\mathfrak{g}}_{\kappa'} - \text{mod}^{G(\mathcal{O})})_{\text{Ran}(X)} & \xrightarrow{(\text{BRST}_{\mathbf{n}, \mathbf{c}})_{\text{Ran}(X)}} & (\widehat{\mathfrak{t}}_{\kappa'} - \text{mod}^{T(\mathcal{O})})_{\text{Ran}(X)} \\ \text{Loc}_{G, \kappa', \text{Ran}(X)} \downarrow & & \downarrow \text{Loc}_{T, \kappa', \text{Ran}(X)} \\ \text{D} - \text{mod}_{\kappa'}(\text{Bun}_G) & \xrightarrow{\text{CT}_{\kappa', \mathbf{c}}} & \text{D} - \text{mod}_{\kappa'}(\text{Bun}_T). \end{array} \quad (0.7)$$

*0.5.3. The IC object on the semi-infinite flag space.* Let us explain how the commutation of the diagram (0.7) implies the desired commutation of the diagram (0.6).

It turns out that the category  $(\mathcal{C}_{\kappa'}^{T(\mathcal{O})})_{\text{Ran}(X)}$  contains a particular object<sup>9</sup>, denoted  $J_{\kappa', 0, !*} \in (\mathcal{C}_{\kappa'}^{T(\mathcal{O})})_{\text{Ran}(X)}$ . It should be thought of as the IC sheaf on the semi-infinite flag space.

Now, on the one hand, the  $(\text{BRST}_{\mathbf{n}, !*})_{\text{Ran}(X)}$  is the functor  $(\text{BRST}_{\mathbf{n}, \mathbf{c}})_{\text{Ran}(X)}$  for  $\mathbf{c} = J_{\kappa', 0, !*}$  (this is in fact the definition of the functor  $(\text{BRST}_{\mathbf{n}, !*})_{\text{Ran}(X)}$ ).

On the other hand, the object  $J_{\kappa', 0, !*}$  is closely related to the IC sheaf on  $\overline{\text{Bun}}_B$ ; this relationship is expressed via the isomorphism

$$\text{CT}_{\kappa', !*} = \text{CT}_{\kappa', \mathbf{c}} \text{ for } \mathbf{c} = J_{\kappa', 0, !*}.$$

---

<sup>9</sup>This object actually belongs to a certain completion of  $\mathcal{C}_{\kappa'}^{T(\mathcal{O})}$ ; we ignore this issue in the introduction.

Thus, taking  $\mathbf{c} = j_{\kappa', 0, !*}$  in the diagram (0.7), we obtain the diagram (0.6).

## 0.6. Structure of the paper

The proof of the Tilting Conjecture, sketched in the main body of the paper, follows the same steps as those described above, but not necessarily in the same order. We shall now review the contents of this paper, section-by-section.

*0.6.1.* In Sect. 1 we recall the definition of the geometric Eisenstein series functor, the set-up for quantum groups, and state the Tilting Conjecture.

At the end of that section we rewrite the space of multiplicities, appearing in the left-hand side of the Tilting Conjecture as a Hom space from a certain canonically defined object  $\tilde{\mathcal{P}}^\lambda \in \mathbf{D} - \text{mod}(\text{Bun}_G)$  to our Eisenstein series object  $\text{Eis}_{!,*}$ .

In Sect. 2 we show that the object  $\tilde{\mathcal{P}}^\lambda$ , or rather its  $\kappa'$ -twisted counterpart, can be obtained as the localization of a projective object  $\mathcal{P}_\kappa^\lambda$  in the category  $\widehat{\mathfrak{g}}_\kappa - \text{mod}^{G(0)}$  (here  $\kappa$  is the *positive* level, related to  $\kappa'$  by the formula  $\kappa = -\kappa_{\text{Kil}} - \kappa'$ ).

We then perform a duality manipulation and replace  $\mathcal{H}om(\tilde{\mathcal{P}}^\lambda, \text{Eis}_{!,*})$  by the cohomology over  $\text{Bun}_G$  of the D-module obtained by tensoring the Eisenstein sheaf  $\text{Eis}_{!,*}$  with the localization (at the negative level  $\kappa'$ ) of the tilting object  $T_{\kappa'}^\lambda \in \widehat{\mathfrak{g}}_{\kappa'} - \text{mod}^{G(0)}$ .

In Sect. 3 we further rewrite  $\mathcal{H}om(\tilde{\mathcal{P}}^\lambda, \text{Eis}_{!,*})$  as the cohomology over  $\text{Bun}_T$  of the D-module obtained by applying the *constant term functor* to the localization of  $T_{\kappa'}^\lambda$ . The reason for making this (completely formal) step is that we will eventually generalize Conjecture 1.4 to a statement that certain two functors from the category of Kac-Moody representations to  $\mathbf{D} - \text{mod}(\text{Bun}_T)$  are isomorphic.

*0.6.2.* In Sect. 4 we review the Bezrukavnikov-Finkelberg-Schechtman realization of representations of the small quantum group as factorizable sheaves.

The theory developed in [12] enables us to replace the semi-infinite cohomology appearing in the statement of the Tilting Conjecture by a certain geometric expression: sheaf cohomology on the space of colored divisors.

In Sect. 5 we give a reinterpretation of the main construction of [12], i.e., the functor (0.3), as an instance of Koszul duality. This is needed in order to eventually compare it with the Kazhdan-Lusztig equivalence, i.e., the functor (0.2).

*0.6.3.* In Sect. 6 we reformulate and generalize the Tilting Conjecture as Conjecture 6.1, which is a statement that two particular functors from the category of Kac-Moody representations to Vect are canonically isomorphic. This reformulation uses the Kazhdan-Lusztig equivalence between quantum groups and Kac-Moody representations.

We then further reformulate Conjecture 6.1 and Conjecture 6.5 in a way that gets rid of quantum groups altogether, and compares two functors from the category of Kac-Moody representations to that of twisted D-modules on  $\text{Bun}_T$ .

The goal of the remaining sections is to sketch the proof of Conjecture 6.5.

*0.6.4.* In Sect. 7 we introduce the category of D-modules on the semi-infinite flag space and explain how objects of this category give rise to (the various versions of) the functor of BRST reduction from modules over  $\widehat{\mathfrak{g}}_{\kappa'}$  to modules over  $\widehat{\mathfrak{t}}_{\kappa'}$ .

We then formulate a crucial result, Quasi-Theorem 7.9 that relates one specific such functor, denoted  $\text{BRST}_{n,!*}$ , to the functor of  $\mathfrak{u}_q(N^+)$ -invariants for quantum groups.

In Sect. 8 we describe the particular object in the category of D-modules on the semi-infinite flag space that gives rise to the functor  $\text{BRST}_{n,!*}$ . This is the “IC sheaf” on the semi-infinite flag space.

*0.6.5.* Finally, in Sect. 9, we show how the functor  $\text{BRST}_{n,!*}$  interacts with the localization functors for  $G$  and  $T$ , respectively. It turns out that this interaction is described by the functor of *constant term*  $\text{CT}_{\kappa',!*}$ . We show how this leads to the proof of Conjecture 6.5.

## 0.7. Conventions

*0.7.1.* Throughout the paper we will be working over the ground field  $\mathbb{C}$ .

We let  $X$  be an arbitrary smooth projective curve; at some (specified) places in the paper we will take  $X$  to be  $\mathbb{P}^1$ .

Given an algebraic group  $H$ , we denote by  $\text{Bun}_H$  the moduli *stack* of principal  $H$ -bundles on  $X$ .

0.7.2. We let  $G$  be a reductive group (over  $\mathbb{C}$ ). We shall assume that the derived group of  $G$  is simply-connected (so that the half-sum of positive roots  $\check{\rho}$  is a weight of  $G$ ).

We let  $\Lambda$  denote the *coweight* lattice of  $G$ , and  $\check{\Lambda}$  the dual lattice, i.e., the weight lattice. Let  $\Lambda^+ \subset \Lambda$  denote the monoid of dominant coweights. This should not be confused with  $\Lambda^{\text{pos}}$ , the latter being the monoid generated by simple coroots.

We denote by  $B$  a (fixed) Borel subgroup of  $G$  and by  $T$  the Cartan quotient of  $B$ . We let  $N$  denote the unipotent radical of  $B$ .

We let  $W$  denote the Weyl group of  $G$ .

0.7.3. This paper does not use derived algebraic geometry, but it does use higher category theory in an essential way: whenever we say “category” we mean a DG category. We refer the reader to [10, Sect. 1], where the theory of DG categories is reviewed.

In particular, we need the reader to be familiar with the notions of: (i) compactly generated DG category (see [10, Sect. 1.2]); (ii) ind-completion of a given (small) DG category (see [10, Sect. 1.3]); (iii) *dual* category and *dual* functor (see [10, Sect. 1.5]); (iv) the limit of a diagram of DG categories (see [10, Sect. 1.6]).

We let  $\text{Vect}$  denote the DG category of chain complexes of  $\mathbb{C}$ -vector spaces.

Given a DG category  $\mathbf{C}$  and a pair of objects  $\mathbf{c}_1, \mathbf{c}_2 \in \mathbf{C}$  we let  $\mathcal{H}om(\mathbf{c}_1, \mathbf{c}_2) \in \text{Vect}$  denote their Hom complex (this structure embodies the enrichment of every DG category over  $\text{Vect}$ ).

If a DG category  $\mathbf{C}$  is endowed with a t-structure, we denote by  $\mathbf{C}^\heartsuit$  its heart, and by  $\mathbf{C}^{\leq 0}$  (resp.,  $\mathbf{C}^{\geq 0}$ ) the connective (resp., coconnective) parts.

0.7.4. Some of the geometric objects that we consider transcend the traditional realm of algebraic geometry: in addition to schemes and algebraic stacks, we will consider arbitrary prestacks.

By definition, a prestack is an arbitrary functor

$$(\text{Sch}_{\text{ft}}^{\text{aff}})^{\text{op}} \rightarrow \infty\text{-Groupoids}.$$

A prime example of a prestack that appears in this paper is the Ran space of  $X$ , denoted  $\text{Ran}(X)$ .

0.7.5. For a given prestack  $\mathcal{Y}$ , we will consider the DG category of D-modules on  $\mathcal{Y}$ , denoted  $D - \text{mod}(\mathcal{Y})$ ; whenever we say “D-module on  $\mathcal{Y}$ ” we mean an object of  $D - \text{mod}(\mathcal{Y})$ .

This category is defined as the *limit* of the categories  $D - \text{mod}(S)$  over the category of schemes (of finite type)  $S$  over  $\mathcal{Y}$ . We refer the reader to [21] where a comprehensive review of the theory is given.

The category of D-modules is contravariantly functorial with respect to the  $!$ -pullback: for a morphism of prestacks  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  we have the functor

$$f^! : D - \text{mod}(\mathcal{Y}_2) \rightarrow D - \text{mod}(\mathcal{Y}_1).$$

For a given  $\mathcal{Y}$  we let  $\omega_{\mathcal{Y}}$  denote the canonical object of  $D - \text{mod}(\mathcal{Y})$  equal to the  $!$ -pullback of  $\mathbb{C} \in \text{Vect} = D - \text{mod}(\text{pt})$ .

0.7.6. In addition, we will need the notion of *twisting* on a prestack and, given a twisting, of the category of twisted D-modules. We refer the reader to [21, Sects. 6 and 7], where these notions are developed.

0.7.7. Given a prestack  $\mathcal{Y}$ , we will also consider the DG category of *constructible sheaves* on it, denoted  $\text{Shv}(\mathcal{Y})$ ; whenever we say “sheaf on  $\mathcal{Y}$ ” we mean an object of  $\text{Shv}(\mathcal{Y})$ .

When  $\mathcal{Y} = S$  is a scheme of finite type, we let  $\text{Shv}(S)$  be the ind-completion of the (standard DG model of the) constructible derived category of sheaves in the analytic topology on  $S(\mathbb{C})$  with  $\mathbb{C}$ -coefficients.

For an arbitrary prestack the definition is obtained by passing to the limit over the category of schemes mapping to it, as in the case of D-modules. We refer the reader to [17, Sect. 1] for further details.

The usual Riemann-Hilbert correspondence (for schemes) gives rise to the fully faithful embedding

$$\text{Shv}(\mathcal{Y}) \xrightarrow{\text{RH}} D - \text{mod}(\mathcal{Y}).$$

The notions of  $\mathbb{C}^*$ -gerbe over a prestack and of the DG category of sheaves twisted by a given gerbe are obtained by mimicking the D-module context of [21].

0.7.8. In several places in this paper we mention algebro-geometric objects of infinite type, such as the loop group  $G(\mathcal{K})$ , where  $\mathcal{K} = \mathbb{C}((t))$ .

We do *not* consider  $D - \text{mod}(-)$  or  $\text{Shv}(-)$  on such objects directly. Rather we approximate them by objects of finite type in a specified way.

## 0.8. Acknowledgements

It is an honor to dedicate this paper to Vadim Schechtman. Our central theme – geometric incarnations of quantum groups – originated in his works [28, 29] and [12]. His other ideas, such as factorization of sheaves and anomalies of actions of infinite-dimensional Lie algebras, are also all-pervasive here.

The author learned about the main characters in this paper (such factorizable sheaves and their relation to quantum groups, the semi-infinite flag space and its relation to Drinfeld’s compactifications, and the Tilting Conjecture) from M. Finkelberg. I would like to thank him for his patient explanations throughout many years.

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## 1. Statement of the conjecture

### 1.1. Eisenstein series functors

*1.1.1.* Let  $X$  be a smooth projective curve, and  $G$  a reductive group. We will be concerned with the moduli stack  $\mathrm{Bun}_G$  classifying principal  $G$ -bundles on  $X$ , and specifically with the DG category  $\mathrm{D} - \mathrm{mod}(\mathrm{Bun}_G)$  of  $\mathrm{D}$ -modules on  $\mathrm{Bun}_G$ .

There are (at least) three functors  $\mathrm{D} - \mathrm{mod}(\mathrm{Bun}_T) \rightarrow \mathrm{D} - \mathrm{mod}(\mathrm{Bun}_G)$ , denoted  $\mathrm{Eis}_!$ ,  $\mathrm{Eis}_*$  and  $\mathrm{Eis}_{!,*}$ , respectively. Let us recall their respective definitions.

*1.1.2.* Consider the diagram

$$\begin{array}{ccc}
 & \mathrm{Bun}_B & \\
 \swarrow \mathrm{q} & & \searrow \mathrm{p} \\
 \mathrm{Bun}_T & & \mathrm{Bun}_G
 \end{array} \tag{1.1}$$

The functors  $\text{Eis}_*$  and  $\text{Eis}_!$  are defined to be

$$\text{Eis}_*(-) := \mathbf{p}_*(\mathbf{q}^!(\mathcal{F}) \otimes^! \text{IC}_{\text{Bun}_B})[\dim(\text{Bun}_T)] \simeq \mathbf{p}_* \circ \mathbf{q}^!(\mathcal{F})[-\dim.\text{rel.}(\text{Bun}_B/\text{Bun}_T)]$$

and

$$\text{Eis}_!(\mathcal{F}) := \mathbf{p}_!(\mathbf{q}^!(\mathcal{F}) \otimes^! \text{IC}_{\text{Bun}_B})[\dim(\text{Bun}_T)] \simeq \mathbf{p}_! \circ \mathbf{q}^*(\mathcal{F})[\dim.\text{rel.}(\text{Bun}_B/\text{Bun}_T)],$$

respectively.

*Remark 1.1.* — The isomorphisms inserted into the above formulas are due to the fact that the stack  $\text{Bun}_B$  is smooth, so  $\text{IC}_{\text{Bun}_B}$  is the constant D-module  $\omega_{\text{Bun}_B}[-\dim(\text{Bun}_B)]$ , and the morphism  $\mathbf{q}$  is smooth as well. The above definition of  $\text{Eis}_*$  (resp.,  $\text{Eis}_!$ ) differs from that in [11] by a cohomological shift (that depends on the connected component of  $\text{Bun}_B$ ).

1.1.3. To define the compactified Eisenstein series functor  $\text{Eis}_{!*}$ , we consider the diagram

$$\begin{array}{ccc} \text{Bun}_B & \xrightarrow{j} & \overline{\text{Bun}}_B \\ & \searrow \bar{q} & \swarrow \bar{p} \\ & \text{Bun}_T & \text{Bun}_G, \end{array} \quad (1.2)$$

where  $\overline{\text{Bun}}_B$  is stack classifying  $G$ -bundles, equipped with a *generalized reduction* to  $B$ ; see [7, Sect. 1.2] for the definition.

In the above diagram  $\bar{p} \circ j = \mathbf{p}$  and  $\bar{q} \circ j = \mathbf{q}$ , while the morphism  $\bar{p}$  is proper. We set

$$\text{Eis}_{!*}(\mathcal{F}) = \bar{p}_* \left( \bar{q}^!(\mathcal{F}) \otimes^! \text{IC}_{\overline{\text{Bun}}_B} \right) [\dim(\text{Bun}_T)].$$

Note that we can rewrite

$$\text{Eis}_*(\mathcal{F}) = \bar{p}_* \left( \bar{q}^!(\mathcal{F}) \otimes^! J_*(\text{IC}_{\text{Bun}_B}) \right) [\dim(\text{Bun}_T)].$$

*Remark 1.2.* — According to [7, Theorem 5.1.5], the object  $j_!(\text{IC}_{\text{Bun}_B}) \in \text{D-mod}(\overline{\text{Bun}}_B)$  is *universally locally acyclic* with respect to the morphism  $\bar{q}$ . This implies that we also have

$$\text{Eis}_!(\mathcal{F}) = \bar{p}_* \left( \bar{q}^!(\mathcal{F}) \otimes^! J_!(\text{IC}_{\text{Bun}_B}) \right) [\dim(\text{Bun}_T)].$$

The maps

$$J_!(\text{IC}_{\text{Bun}_B}) \rightarrow \text{IC}_{\overline{\text{Bun}}_B} \rightarrow J_*(\text{IC}_{\text{Bun}_B})$$

induce the natural transformations

$$\mathrm{Eis}_! \rightarrow \mathrm{Eis}_{!*} \rightarrow \mathrm{Eis}_*.$$

## 1.2. What do we want to study?

In this subsection we specialize to the case when  $X$  is of genus 0.

*1.2.1.* Recall that in the case of a curve of genus 0, Grothendieck’s classification of  $G$ -bundles implies that the stack  $\mathrm{Bun}_G$  is stratified by locally closed substacks  $\mathrm{Bun}_G^\lambda$  where  $\lambda$  ranges over  $\Lambda^+$ , the semi-group of dominant weights.

For  $\lambda \in \Lambda^+$ , let  $\mathrm{IC}^\lambda \in \mathrm{D} - \mathrm{mod}(\mathrm{Bun}_G)^\heartsuit$  denote the corresponding irreducible object.

*1.2.2.* Since the morphism  $\bar{\rho}$  is proper, the Decomposition Theorem implies that the object

$$\mathrm{Eis}_{!*}(\mathrm{IC}_{\mathrm{Bun}_T}) = \mathrm{Eis}_{!*}(\omega_{\mathrm{Bun}_T})[-\dim(\mathrm{Bun}_T)] \simeq \bar{\rho}_*(\mathrm{IC}_{\mathrm{Bun}_B}) \in \mathrm{D} - \mathrm{mod}(\mathrm{Bun}_G)$$

can be written as

$$\bigoplus_{\lambda} V^\lambda \otimes \mathrm{IC}^\lambda, \quad V^\lambda \in \mathrm{Vect}. \tag{1.3}$$

The goal is to understand the vector spaces  $V^\lambda$ , i.e., the multiplicity of each  $\mathrm{IC}^\lambda$  in  $\mathrm{Eis}_{!*}(\omega_{\mathrm{Bun}_T})$ .

*1.2.3.* Below we state a conjecture from [13, Sect. 7.8] that describes this (cohomologically graded) vector space in terms of the semi-infinite cohomology of the small quantum group.

As was mentioned in the introduction, the goal of this paper is to sketch a proof of this conjecture.

## 1.3. The “q”-parameter

*1.3.1.* Let  $\mathrm{Quad}(\check{\Lambda}, \mathbb{Z})^W$  be the lattice of integer-valued  $W$ -invariant quadratic forms of the weight lattice  $\check{\Lambda}$ .

We fix an element

$$q \in \mathrm{Quad}(\check{\Lambda}, \mathbb{Z})^W \otimes_{\mathbb{Z}} \mathbb{C}^*.$$

Let  $b_q : \check{\Lambda} \otimes \check{\Lambda} \rightarrow \mathbb{C}^*$  be the corresponding symmetric bilinear form. One should think of  $b_q$  as the square of the braiding on the category of

representations of the quantum torus, whose lattice of characters is  $\check{\Lambda}$ ; in what follows we denote this category by  $\text{Rep}_q(T)$ .

1.3.2. We will assume that  $q$  is torsion. Let  $G^\sharp$  be the recipient of Lusztig's quantum Frobenius. I.e., this is a reductive group, whose weight lattice is the kernel of  $b_q$ .

Let  $\Lambda^\sharp$  denote the coweight lattice of  $G^\sharp$ , so that at the level of lattices, the quantum Frobenius defines a map

$$\text{Frob}_{\Lambda,q} : \Lambda \rightarrow \Lambda^\sharp.$$

1.3.3. In what follows we will assume that  $q$  is such that  $G^\sharp$  equals the Langlands dual  $\check{G}$  of  $G$ . In particular,  $\Lambda^\sharp \simeq \check{\Lambda}$ , and we can think of the quantum Frobenius as a map

$$\text{Frob}_{\Lambda,q} : \Lambda \rightarrow \check{\Lambda}.$$

Thus, we obtain that the extended affine Weyl group

$$W_{q,\text{aff}} := W \ltimes \Lambda$$

acts on  $\check{\Lambda}$ , with  $\Lambda$  acting via  $\text{Frob}_{\Lambda,q}$ . (We are considering the “dotted” action, so that the fixed point of the action of the finite Weyl group  $W$  is  $-\check{\rho}$ .)

1.3.4. For  $\lambda \in \Lambda$  let  $\min_\lambda \in W_{q,\text{aff}}$  be the *shortest* representative in the double coset of

$$\lambda \in \Lambda \subset W_{q,\text{aff}}$$

with respect to  $W \subset W_{q,\text{aff}}$ .

Consider the corresponding weight  $\min_\lambda(0) \in \check{\Lambda}$ .

## 1.4. Quantum groups

1.4.1. Let

$$\mathfrak{U}_q(G)\text{-mod and } \mathfrak{u}_q(G)\text{-mod}$$

be the categories of representations of the big (Lusztig's) and small<sup>10</sup> quantum groups, respectively, attached to  $q$ .

Consider the *indecomposable tilting module*

$$\mathfrak{T}_q^\lambda \in \mathfrak{U}_q(G)\text{-mod}^\heartsuit$$

with highest weight  $\min_\lambda(0)$ .

<sup>10</sup>We are considering the graded version of the small quantum group, i.e., we have a forgetful functor  $\mathfrak{u}_q(G)\text{-mod} \rightarrow \text{Rep}_q(T)$ .

1.4.2. We have the tautological forgetful functor

$$\mathrm{Res}^{\mathrm{big} \rightarrow \mathrm{small}} : \mathfrak{U}_q(G)\text{-mod} \rightarrow \mathfrak{u}_q(G)\text{-mod}.$$

Recall now that there is a canonically defined functor

$$C^{\frac{\infty}{2}} : \mathfrak{u}_q(G)\text{-mod} \rightarrow \mathrm{Vect},$$

see [1]. We have

$$H^\bullet(C^{\frac{\infty}{2}}(\mathcal{M})) = H^{\frac{\infty}{2} + \bullet}(\mathcal{M}), \quad \mathcal{M} \in \mathfrak{u}_q(G)\text{-mod}.$$

*Remark 1.3.* The functor  $C^{\frac{\infty}{2}}$  is the functor of semi-infinite cochains with respect to the non-graded version of  $\mathfrak{u}_q(G)$ . In particular, its natural target is the category  $\mathrm{Rep}(T^\sharp)$  if representations of the Cartan group  $T^\sharp$  of  $G^\sharp$ .

1.4.3. The following is the statement of the tilting conjecture from [13]:

CONJECTURE 1.4. — *For  $\lambda \in \Lambda^+$  we have a canonical isomorphism*

$$V^\lambda \simeq C^{\frac{\infty}{2}} \left( \mathfrak{u}_q(G), \mathrm{Res}^{\mathrm{big} \rightarrow \mathrm{small}}(\mathfrak{T}_q^\lambda) \right), \quad (1.4)$$

where  $V^\lambda$  is as in (1.3).

*Remark 1.5.* — According to Remark 1.3, the right-hand side in (1.4) is naturally an object of  $\mathrm{Rep}(T^\sharp)$ , and since due to our choice of  $q$  we have  $T^\sharp = \check{T}$ , we can view it as a  $\Lambda$ -graded vector space. This grading corresponds to the grading on the left hand side, given by the decomposition of  $\mathrm{Eis}_{!*}(\mathrm{IC}_{\mathrm{Bun}_T})$  according to connected components of  $\mathrm{Bun}_T$ .

*Remark 1.6.* — One can strengthen the previous remark as follows: both sides in (1.4) carry an action of the Langlands dual Lie algebra  $\check{\mathfrak{g}}$ : on the right-hand side this action comes from the quantum Frobenius, and on the left-hand side from the action of  $\check{\mathfrak{g}}$  on  $\mathrm{Eis}_{!*}(\mathrm{IC}_{\mathrm{Bun}_T})$  from [13, Sect. 7.4]. One can strengthen the statement of Conjecture 1.4 by requiring that these two actions be compatible. Although our methods allow to deduce this stronger statement, we will not pursue it in this paper.

*Remark 1.7.* — Note that the LHS in (1.4) is, by construction, independent of the choice of  $q$ , whereas the definition of the RHS explicitly depends on  $q$ .

However, one can show (by identifying the regular blocks of the categories  $\mathfrak{U}_q(G)\text{-mod}$  for different  $q$ 's), that the vector space  $C^{\frac{\infty}{2}} \left( \mathfrak{u}_q(G), \mathrm{Res}^{\mathrm{big} \rightarrow \mathrm{small}}(\mathfrak{T}_q^\lambda) \right)$  is also independent of  $q$ .

*Remark 1.8.* — For our derivation of the isomorphism (1.4) we have to take our ground field to be  $\mathbb{C}$ , since it relies on Riemann-Hilbert correspondence. It is an interesting question to understand whether the resulting isomorphism can be defined over  $\mathbb{Q}$  (or some small extension of  $\mathbb{Q}$ ).

### 1.5. Multiplicity space as a Hom

The definition of the left-hand side in Conjecture 1.4 as a space of multiplicities is not very convenient to work with. In this subsection we will rewrite it a certain Hom space, the latter being more amenable to categorical manipulations.

*1.5.1.* Fix a point  $x_0 \in X$ . Let  $\text{Bun}_G^{N,x_0}$  (resp.,  $\text{Bun}_G^{B,x_0}$ ) be the moduli of  $G$ -bundles on  $X$ , equipped with a reduction of the fiber at  $x_0 \in X$  to  $N$  (resp.,  $B$ ). Note that  $\text{Bun}_G^{N,x_0}$  is equipped with an action of  $T$ . We let

$$\text{D-mod}(\text{Bun}_G^{N,x_0})^{T\text{-mon}} \subset \text{D-mod}(\text{Bun}_G^{N,x_0})$$

denote the full subcategory consisting of  $T$ -monodromic objects, i.e., the full subcategory generated by the image of the pullback functor

$$\text{D-mod}(\text{Bun}_G^{B,x_0}) = \text{D-mod}(T \backslash (\text{Bun}_G^{N,x_0})) \rightarrow \text{D-mod}(\text{Bun}_G^{N,x_0}).$$

Let

$$\pi : \text{Bun}_G^{N,x_0} \rightarrow \text{Bun}_G$$

denote the tautological projection. We consider the resulting pair of adjoint functors

$$\pi_![\dim(G/N)] : \text{D-mod}(\text{Bun}_G^{N,x_0})^{T\text{-mon}} \rightleftarrows \text{D-mod}(\text{Bun}_G) : \pi^![-\dim(G/N)].$$

*1.5.2.* It is known that for  $(X, x_0) = (\mathbb{P}^1, 0)$ , the category  $\text{D-mod}(\text{Bun}_G^{N,x_0})^{T\text{-mon}}$  identifies with the derived DG category of the heart of the natural t-structure<sup>11</sup> (see [5, Corollary 3.3.2]).

Let

$$\mathcal{P}^\lambda \in \text{D-mod}(\text{Bun}_G^{N,x_0})^\heartsuit$$

denote the projective cover of the irreducible  $\pi^!(\text{IC}^\lambda)[- \dim(G/N)]$ . Set

$$\tilde{\mathcal{P}}^\lambda := \pi_!(\mathcal{P}^\lambda)[\dim(G/N)] \in \text{D-mod}(\text{Bun}_G).$$

It is clear that if  $\mathcal{F} \in \text{D-mod}(\text{Bun}_G)$  is a semi-simple object equal to

$$\bigoplus_{\lambda} V_{\mathcal{F}}^{\lambda} \otimes \text{IC}^{\lambda},$$

---

<sup>11</sup>This is because the inclusions of the strata are affine morphisms.

then

$$\mathcal{H}om(\tilde{\mathcal{P}}^\lambda, \mathcal{F}) \simeq \mathcal{H}om(\mathcal{P}^\lambda, \pi^!(\mathcal{F})[-\dim(G/N)]) \simeq V_{\mathcal{F}}^\lambda.$$

1.5.3. Thus, we can restate Conjecture 1.4 as one about the existence of a canonical isomorphism

$$\mathcal{H}om(\tilde{\mathcal{P}}^\lambda, \mathrm{Eis}_{!*}(\mathrm{IC}_{\mathrm{Bun}_T})) \simeq \mathbb{C}^{\frac{\infty}{2}} \left( \mathfrak{u}_q(G), \mathrm{Res}^{\mathrm{big} \rightarrow \mathrm{small}}(\mathfrak{I}_q^\lambda) \right). \quad (1.5)$$

## 2. Kac-Moody representations, localization functors and duality

Conjecture 1.4 compares an algebraic object (semi-infinite cohomology of the quantum group) with a geometric one (multiplicity spaces in geometric Eisenstein series). The link between the two will be provided by the category of representations of the Kac-Moody algebra.

On the one hand, Kac-Moody representations will be related to modules over the quantum group via the Kazhdan-Lusztig equivalence. On the other hand, they will be related to D-modules on  $\mathrm{Bun}_G$  via *localization functors*.

In this section we will introduce the latter part of the story: Kac-Moody representations and the localization functors to  $D - \mathrm{mod}(\mathrm{Bun}_G)$ .

### 2.1. Passing to twisted D-modules

In this subsection we will introduce a *twisting* on D-modules into our game. Ultimately, this twisting will account for the  $q$  parameter in the quantum group via the Kazhdan-Lusztig equivalence.

2.1.1. Let  $\kappa$  be a *level* for  $G$ , i.e., a  $G$ -invariant symmetric bilinear form

$$\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}.$$

To the datum of  $\kappa$  one canonically attaches a *twisting* on  $\mathrm{Bun}_G$  (resp.  $\mathrm{Bun}_G^{N,x_0}$ ), see [20, Proposition-Construction 1.3.6]. Let

$$D - \mathrm{mod}_\kappa(\mathrm{Bun}_G^{N,x_0})^{T-\mathrm{mon}} \text{ and } D - \mathrm{mod}_\kappa(\mathrm{Bun}_G)$$

denote the corresponding DG categories of twisted D-modules.

2.1.2. Suppose now that  $\kappa$  is an integral multiple of the Killing form,

$$\kappa = c \cdot \kappa_{\mathrm{Kil}}, \quad c \in \mathbb{Z}.$$

In this case we have canonical equivalences

$$\begin{aligned} D - \mathrm{mod}_\kappa(\mathrm{Bun}_G) &\simeq D - \mathrm{mod}(\mathrm{Bun}_G) \\ \text{and } D - \mathrm{mod}_\kappa(\mathrm{Bun}_G^{N,x_0})^{T-\mathrm{mon}} &\simeq D - \mathrm{mod}(\mathrm{Bun}_G^{N,x_0})^{T-\mathrm{mon}}, \end{aligned}$$

given by tensoring by the  $c$ -th power of the determinant line bundle on  $\text{Bun}_G$ , denoted  $\mathcal{L}_{G,\kappa}$ , and its pullback to  $\text{Bun}_G^{N,x_0}$ , respectively.

Let  $\mathcal{P}_\kappa^\lambda \in \text{D-mod}_\kappa(\text{Bun}_G^{N,x_0})^{T\text{-mon}}$ ,

$\widetilde{\mathcal{P}}_\kappa^\lambda, \text{Eis}_{\kappa,!}(\text{IC}_{\text{Bun}_T}), \text{Eis}_{\kappa,*}(\text{IC}_{\text{Bun}_T})$  and  $\text{Eis}_{\kappa,!*}(\text{IC}_{\text{Bun}_T}) \in \text{D-mod}_\kappa(\text{Bun}_G)$ ,

denote the objects that correspond to  $\mathcal{P}^\lambda \in \text{D-mod}(\text{Bun}_G^{N,x_0})^{T\text{-mon}}$ ,

$\widetilde{\mathcal{P}}^\lambda, \text{Eis}_!(\text{IC}_{\text{Bun}_T}), \text{Eis}_*(\text{IC}_{\text{Bun}_T})$  and  $\text{Eis}_{!*}(\text{IC}_{\text{Bun}_T}) \in \text{D-mod}(\text{Bun}_G)$ ,

respectively, under the above equivalences.

2.1.3. Hence, we can further reformulate Conjecture 1.4 as one about the existence of a canonical isomorphism

$$\text{Hom}(\widetilde{\mathcal{P}}_\kappa^\lambda, \text{Eis}_{\kappa,!*}(\text{IC}_{\text{Bun}_T})) \simeq \mathbb{C}^{\frac{\infty}{2}} \left( \mathfrak{u}_q(G), \text{Res}^{\text{big} \rightarrow \text{small}}(\mathfrak{T}_q^\lambda) \right). \quad (2.1)$$

## 2.2. Localization functors

In this subsection we will assume that the level  $\kappa$  is positive. Here and below by “positive” we mean that an each simple factor,  $\kappa = c \cdot \kappa_{\text{Kil}}$ , where  $(c + \frac{1}{2}) \notin \mathbb{Q}^{\leq 0}$ , while the restriction of  $\kappa$  to the center of  $\mathfrak{g}$  is *non-degenerate*.

We are going to introduce a crucial piece of structure, namely, the localization functors from Kac-Moody representations to (twisted) D-modules on  $\text{Bun}_G$ .

2.2.1. We now choose a point  $x_\infty \in X$ , different from the point  $x_0 \in X$  (the latter is one at which we are taking the reduction to  $N$ ). We consider the Kac-Moody Lie algebra  $\widehat{\mathfrak{g}}_{\kappa,x_\infty}$  at  $x_\infty$  at level  $\kappa$ , i.e., the central extension

$$0 \rightarrow \mathbb{C} \rightarrow \widehat{\mathfrak{g}}_{\kappa,x_\infty} \rightarrow \mathfrak{g}(\mathcal{K}_{x_\infty}) \rightarrow 0,$$

which is split over  $\mathfrak{g}(\mathcal{O}_{x_\infty}) \subset \mathfrak{g}(\mathcal{K}_{x_\infty})$ , and where the bracket is defined using  $\kappa$ .

Let  $\widehat{\mathfrak{g}}_{\kappa,x_\infty}\text{-mod}^{G(\mathcal{O}_{x_\infty})}$  denote the DG category of  $G(\mathcal{O}_{x_\infty})$ -integrable  $\widehat{\mathfrak{g}}_{\kappa,x_\infty}$ -modules, see [20, Sect. 2.3].

2.2.2. In what follows we will also consider the Kac-Moody algebra denoted  $\widehat{\mathfrak{g}}_\kappa$  that we think of as being attached to the standard formal disc  $\mathcal{O} \subset \mathcal{K} = \mathbb{C}[[t]] \subset \mathbb{C}((t))$ .

2.2.3. Consider the corresponding localization functors

$$\mathrm{Loc}_{G,\kappa,x_\infty} : \widehat{\mathfrak{g}}_{\kappa,x_\infty} \text{-mod}^{G(\mathcal{O}_{x_\infty})} \rightarrow \mathrm{D} \text{-mod}_\kappa(\mathrm{Bun}_G)$$

and

$$\mathrm{Loc}_{G,\kappa,x_\infty}^{N,x_0} : \widehat{\mathfrak{g}}_{\kappa,x_\infty} \text{-mod}^{G(\mathcal{O}_{x_\infty})} \rightarrow \mathrm{D} \text{-mod}_\kappa(\mathrm{Bun}_G^{N,x_0})^{T\text{-mon}},$$

see [20, Sect. 2.4].

2.2.4. Assume now that  $(X, x_0, x_\infty) = (\mathbb{P}^1, 0, \infty)$ .

Since  $\kappa$  was assumed positive, the theorem of Kashiwara-Tanisaki (see [23]) implies that the functor  $\mathrm{Loc}_{G,\kappa,x_\infty}^{N,x_0}$  defines a t-exact equivalence from the *regular block* of  $\widehat{\mathfrak{g}}_{\kappa,x_\infty} \text{-mod}^{G(\mathcal{O}_{x_\infty})}$  to  $\mathrm{D} \text{-mod}_\kappa(\mathrm{Bun}_G^{N,x_0})^{T\text{-mon}}$ .

Let

$$P_\kappa^\lambda \in (\widehat{\mathfrak{g}}_{\kappa,x_\infty} \text{-mod}^{G(\mathcal{O}_{x_\infty})})^\heartsuit$$

denote the object such that

$$\mathrm{Loc}_{G,\kappa,x_\infty}^{N,x_0}(P_\kappa^\lambda) \simeq \mathcal{P}_\kappa^\lambda.$$

2.2.5. Let

$$W_{\kappa,\mathrm{aff}} := W \ltimes \Lambda.$$

The datum of  $\kappa$  defines an action of  $W_{\kappa,\mathrm{aff}}$  on the weight lattice  $\check{\Lambda}$ , where  $\Lambda$  acts on  $\check{\Lambda}$  by translations via the map

$$\mathrm{Frob}_{\Lambda,\kappa} : \Lambda \rightarrow \check{\Lambda}, \quad \lambda \mapsto (\kappa - \kappa_{\mathrm{crit}})(\lambda, -),$$

where  $\kappa_{\mathrm{crit}} = -\frac{\kappa \mathrm{Kill}}{2}$ . (Again, we are considering the “dotted” action, so that the fixed point of the action of the finite Weyl group  $W$  is  $-\check{\rho}$ .)

Let  $\max_\lambda \in W_{\kappa,\mathrm{aff}}$  be the *longest* representative in the double coset of

$$\lambda \in \Lambda \subset W_{\kappa,\mathrm{aff}}$$

with respect to  $W \in W_{\kappa,\mathrm{aff}}$ .

Then the object  $P_\kappa^\lambda$  is the projective cover of the irreducible with highest weight  $\max_\lambda(0)$ .

2.2.6. We claim:

PROPOSITION 2.1. — *There exists a canonical isomorphism*

$$\mathrm{Loc}_{G,\kappa,x_\infty}(P_\kappa^\lambda) \simeq \widetilde{\mathcal{P}}_\kappa^\lambda$$

*of objects in  $\mathrm{D} \text{-mod}_\kappa(\mathrm{Bun}_G)$ .*

*Proof.* — Let

$$\Gamma_{\kappa, x_\infty} \quad \text{and} \quad \Gamma_{\kappa, x_\infty}^{N, x_0}$$

be the functors right adjoint to

$$\text{Loc}_{G, \kappa, x_\infty} \quad \text{and} \quad \text{Loc}_{G, \kappa, x_\infty}^{N, x_0},$$

respectively.

Interpreting the above functors as global sections on an appropriate scheme (it is the scheme classifying  $G$ -bundles with a full level structure at  $x_\infty$ ), one shows that

$$\Gamma_{\kappa, x_\infty} \simeq \Gamma_{\kappa, x_\infty}^{N, x_0} \circ \pi^![-\dim(G/N)].$$

Passing to the left adjoints, we obtain

$$\text{Loc}_{G, \kappa, x_\infty} \simeq \pi_! \circ \text{Loc}_{G, \kappa, x_\infty}^{N, x_0}[\dim(G/N)],$$

whence the assertion of the proposition.  $\square$

*2.2.7.* Thus, by Proposition 2.1, we can reformulate Conjecture 1.4 as the existence of a canonical isomorphism

$$\mathcal{H}om(\text{Loc}_{G, \kappa, x_\infty}(P_\kappa^\lambda), \text{Eis}_{\kappa, !*}(\text{IC}_{\text{Bun}_T})) \simeq C^{\frac{\infty}{2}}\left(\mathfrak{u}_q(G), \text{Res}^{\text{big} \rightarrow \text{small}}(\mathfrak{X}_q^\lambda)\right), \quad (2.2)$$

where  $\kappa$  is some positive integral level and  $(X, x_\infty) = (\mathbb{P}^1, \infty)$ .

*Remark 2.2.* — From now on we can “forget” about the point  $x_0$  and the stack  $\text{Bun}_G^{N, x_0}$ . It was only needed to reduce Conjecture 1.4 to Equation (2.2).

### 2.3. Duality on Kac-Moody representations

The goal of this and the next subsection is to replace  $\mathcal{H}om$  in the left-hand side in (2.2) by a *pairing*. I.e., we will rewrite the left-hand side in (2.2) as the value of a certain *covariant* functor. This interpretation will be important for our next series of manipulations.

We refer the reader to [10, Sect. 1.5] for a review of the general theory of duality in DG categories.

*2.3.1.* Recall (see [20, Sects. 2.2 and 2.3 or 4.3]) that the category  $\widehat{\mathfrak{g}}_{\kappa, x_\infty} \text{-mod}^{G(\mathcal{O}_{x_\infty})}$  is defined so that it is compactly generated by Weyl modules.

Let  $\kappa'$  be the *reflected* level, i.e.,  $\kappa' := -\kappa - \kappa_{\text{Kil}}$ . Note that we have  $(\kappa_{\text{crit}})' = \kappa_{\text{crit}}$ , where we remind that  $\kappa_{\text{crit}} = -\frac{\kappa_{\text{Kil}}}{2}$ .

We recall (see [20, Sect. 4.6]) that there exists a canonical equivalence

$$(\widehat{\mathfrak{g}}_{\kappa, x_\infty} - \text{mod}^{G(\mathcal{O}_{x_\infty})})^\vee \simeq \widehat{\mathfrak{g}}_{\kappa', x_\infty} - \text{mod}^{G(\mathcal{O}_{x_\infty})}.$$

This equivalence is uniquely characterized by the property that the corresponding pairing

$$\langle -, - \rangle_{\text{KM}} : \widehat{\mathfrak{g}}_{\kappa, x_\infty} - \text{mod}^{G(\mathcal{O}_{x_\infty})} \otimes \widehat{\mathfrak{g}}_{\kappa', x_\infty} - \text{mod}^{G(\mathcal{O}_{x_\infty})} \rightarrow \text{Vect}$$

is given by

$$\begin{aligned} & \widehat{\mathfrak{g}}_{\kappa, x_\infty} - \text{mod}^{G(\mathcal{O}_{x_\infty})} \otimes \widehat{\mathfrak{g}}_{\kappa', x_\infty} - \text{mod}^{G(\mathcal{O}_{x_\infty})} \rightarrow \\ & \rightarrow \widehat{\mathfrak{g}}_{\kappa, x_\infty} - \text{mod} \otimes \widehat{\mathfrak{g}}_{\kappa', x_\infty} - \text{mod} \xrightarrow{\otimes} \widehat{\mathfrak{g}}_{-\kappa_{\text{Kil}}, x_\infty} \rightarrow \text{Vect}, \end{aligned}$$

where

$$\widehat{\mathfrak{g}}_{-\kappa_{\text{Kil}}, x_\infty} \rightarrow \text{Vect}$$

is the functor of *semi-infinite cochains* with respect to  $\mathfrak{g}(\mathcal{K}_{x_\infty})$ , see [20, Sect. 4.5 and 4.6] or [3, Sect. 2.2].

2.3.2. We denote the resulting contravariant equivalence

$$(\widehat{\mathfrak{g}}_{\kappa, x_\infty} - \text{mod}^{G(\mathcal{O}_{x_\infty})})^c \simeq (\widehat{\mathfrak{g}}_{\kappa', x_\infty} - \text{mod}^{G(\mathcal{O}_{x_\infty})})^c$$

by  $\mathbb{D}^{\text{KM}}$ , see [10, Sect. 1.5.3].

It has the property that for an object  $M \in \widehat{\mathfrak{g}}_{\kappa, x_\infty} - \text{mod}^{G(\mathcal{O}_{x_\infty})}$ , induced from a compact (i.e., finite-dimensional) representation  $M_0$  of  $G(\mathcal{O}_{x_\infty})$ , the corresponding object

$$\mathbb{D}^{\text{KM}}(M) \in \widehat{\mathfrak{g}}_{\kappa', x_\infty} - \text{mod}^{G(\mathcal{O}_{x_\infty})}$$

is one induced from the dual representation  $M_0^\vee$ .

In particular, by taking  $M_0$  to be an irreducible representation of  $G$ , so that  $M$  is the Weyl module, we obtain that the functor  $\mathbb{D}^{\text{KM}}$  sends Weyl modules to Weyl modules.

2.3.3. Assume that  $\kappa$  (and hence  $\kappa'$  is integral). Let  $T_{\kappa'}^\lambda \in \widehat{\mathfrak{g}}_{\kappa', x_\infty} - \text{mod}^{G(\mathcal{O}_{x_\infty})}$  be the indecomposable tilting module with highest weight  $\min_\lambda(0)$ .

We claim:

PROPOSITION 2.3. — *Let  $\kappa$  be positive. Then  $\mathbb{D}^{\text{KM}}(P_\kappa^\lambda) \simeq T_{\kappa'}^\lambda$ .*

*Proof.* — Follows from the fact that the composition

$$(\widehat{\mathfrak{g}}_{\kappa', x_\infty} - \text{mod}^{G(\mathcal{O}_{x_\infty})})^c \rightarrow (\widehat{\mathfrak{g}}_{\kappa', x_\infty} - \text{mod}^{G(\mathcal{O}_{x_\infty})})^c \xrightarrow{\mathbb{D}^{\text{KM}}} (\widehat{\mathfrak{g}}_{\kappa, x_\infty} - \text{mod}^{G(\mathcal{O}_{x_\infty})})^c,$$

where the first arrow is the contragredient duality at the negative level, identifies with Arkhipov's functor (the longest intertwining operator), see [3, Theorem 9.2.4].  $\square$

## 2.4. Duality on $\text{Bun}_G$

Following [9, Sect. 4.3.3], in addition to  $\text{D-mod}(\text{Bun}_G)$ , one introduces another version of the category of  $\text{D-mod}$  on  $\text{Bun}_G$ , denoted  $\text{D-mod}(\text{Bun}_G)_{\text{co}}$ .

2.4.1. We will not give a detailed review of the definition of  $\text{D-mod}(\text{Bun}_G)_{\text{co}}$  here. Let us just say that the difference between  $\text{D-mod}(\text{Bun}_G)$  and  $\text{D-mod}(\text{Bun}_G)_{\text{co}}$  has to do with the fact that the stack  $\text{Bun}_G$  is not quasi-compact (rather, its connected components are not quasi-compact). So, when dealing with a fixed quasi-compact open  $U \subset \text{Bun}_G$ , there will not be any difference between the two categories.

One shows  $\text{D-mod}(\text{Bun}_G)$  is compactly generated by  $!$ -extensions of compact objects in  $\text{D-mod}(U)$  for  $U$  as above, whereas  $\text{D-mod}(\text{Bun}_G)_{\text{co}}$  is defined so that it is compactly generated by  $*$ -extensions of the same objects.

It follows from the construction of  $\text{D-mod}(\text{Bun}_G)_{\text{co}}$  that the  $\overset{!}{\otimes}$  tensor product defines a functor

$$\text{D-mod}(\text{Bun}_G) \otimes \text{D-mod}(\text{Bun}_G)_{\text{co}} \rightarrow \text{D-mod}(\text{Bun}_G)_{\text{co}}.$$

Again, by the construction of  $\text{D-mod}(\text{Bun}_G)_{\text{co}}$ , global de Rham cohomology<sup>12</sup> is a *continuous* functor

$$\Gamma_{\text{dr}}(\text{Bun}_G, -) : \text{D-mod}(\text{Bun}_G)_{\text{co}} \rightarrow \text{Vect}.$$

2.4.2. The usual Verdier duality for quasi-compact algebraic stacks implies that the category  $\text{D-mod}(\text{Bun}_G)_{\text{co}}$  identifies with the *dual* of  $\text{D-mod}(\text{Bun}_G)$ :

$$(\text{D-mod}(\text{Bun}_G))^{\vee} \simeq \text{D-mod}(\text{Bun}_G)_{\text{co}}. \quad (2.3)$$

We can describe the corresponding pairing

$$\langle -, - \rangle_{\text{Bun}_G} : \text{D-mod}(\text{Bun}_G)_{\text{co}} \otimes \text{D-mod}(\text{Bun}_G) \rightarrow \text{Vect}$$

explicitly using the functor  $\Gamma_{\text{dr}}(\text{Bun}_G, -)$ .

---

<sup>12</sup>Since  $\text{Bun}_G$  is a stack, when we talk about de Rham cohomology, we mean its renormalized version, see [9, Sect. 9.1]; this technical point will not be relevant for the sequel.

Namely,  $\langle -, - \rangle_{\text{Bun}_G}$  equals the composition

$$\text{D} - \text{mod}(\text{Bun}_G)_{\text{co}} \otimes \text{D} - \text{mod}(\text{Bun}_G) \xrightarrow{\overset{\dagger}{\otimes}} \text{D} - \text{mod}(\text{Bun}_G)_{\text{co}} \xrightarrow{\Gamma_{\text{dr}}(\text{Bun}_G, -)} \text{Vect}.$$

Equivalently, the functor *dual* to  $\Gamma_{\text{dr}}(\text{Bun}_G, -)$  is the functor

$$\text{Vect} \rightarrow \text{D} - \text{mod}(\text{Bun}_G), \quad \mathbb{C} \mapsto \omega_{\text{Bun}_G}.$$

2.4.3. A similar discussion applies in the twisted case, with the difference that the level gets reflected, i.e., we now have the canonical equivalence

$$(\text{D} - \text{mod}_{\kappa}(\text{Bun}_G))^{\vee} \simeq \text{D} - \text{mod}_{\kappa'}(\text{Bun}_G)_{\text{co}}. \quad (2.4)$$

The corresponding pairing

$$\langle -, - \rangle_{\text{Bun}_G} : \text{D} - \text{mod}_{\kappa'}(\text{Bun}_G)_{\text{co}} \otimes \text{D} - \text{mod}_{\kappa}(\text{Bun}_G) \rightarrow \text{Vect}$$

is equal to

$$\begin{aligned} \text{D} - \text{mod}_{\kappa'}(\text{Bun}_G)_{\text{co}} \otimes \text{D} - \text{mod}_{\kappa}(\text{Bun}_G) &\xrightarrow{\overset{\dagger}{\otimes}} \text{D} - \text{mod}_{-\kappa_{\text{Kil}}}(\text{Bun}_G)_{\text{co}} \simeq \\ &\simeq \text{D} - \text{mod}(\text{Bun}_G)_{\text{co}} \xrightarrow{\Gamma_{\text{dr}}(\text{Bun}_G, -)} \text{Vect}, \end{aligned} \quad (2.5)$$

where the equivalence  $\text{D} - \text{mod}_{-\kappa_{\text{Kil}}}(\text{Bun}_G)_{\text{co}} \simeq \text{D} - \text{mod}(\text{Bun}_G)_{\text{co}}$  is given by tensoring by the determinant line bundle  $\mathcal{L}_{G, \kappa_{\text{Kil}}}$ .

We denote the resulting contravariant equivalence

$$(\text{D} - \text{mod}_{\kappa}(\text{Bun}_G))^c \simeq (\text{D} - \text{mod}_{\kappa'}(\text{Bun}_G)_{\text{co}})^c$$

by  $\mathbb{D}^{\text{Verdier}}$ .

2.4.4. Note that when  $\kappa$  is integral, the equivalence (2.4) goes over to the non-twisted equivalence (2.3) under the identifications

$\text{D} - \text{mod}(\text{Bun}_G) \simeq \text{D} - \text{mod}_{\kappa}(\text{Bun}_G)$  and  $\text{D} - \text{mod}(\text{Bun}_G)_{\text{co}} \simeq \text{D} - \text{mod}_{\kappa'}(\text{Bun}_G)_{\text{co}}$ , given by tensoring by the corresponding line bundles, i.e.,  $\mathcal{L}_{G, \kappa}$  and  $\mathcal{L}_{G, \kappa'}$ , respectively.

## 2.5. Duality and localization

In this subsection we assume that the level  $\kappa$  is positive (see Sect. 2.2 for what this means).

We will review how the duality functor on the category of Kac-Moody representations interacts with Verdier duality on  $\text{D} - \text{mod}(\text{Bun}_G)$ .

2.5.1. The basic property of the functor

$$\mathrm{Loc}_{G,\kappa,x_\infty} : \widehat{\mathfrak{g}}_{\kappa,x_\infty} - \mathrm{mod}^{G(\mathcal{O}_{x_\infty})} \rightarrow \mathrm{D} - \mathrm{mod}_\kappa(\mathrm{Bun}_G)$$

is that sends compacts to compacts (this is established in [3, Theorem 6.1.8]).

In particular, we obtain that there exists a canonically defined continuous functor

$$\mathrm{Loc}_{G,\kappa',x_\infty} : \widehat{\mathfrak{g}}_{\kappa',x_\infty} - \mathrm{mod}^{G(\mathcal{O}_{x_\infty})} \rightarrow \mathrm{D} - \mathrm{mod}_{\kappa'}(\mathrm{Bun}_G)_{\mathrm{co}},$$

so that

$$\begin{aligned} \mathbb{D}^{\mathrm{Verdier}} \circ \mathrm{Loc}_{G,\kappa,x_\infty} \circ \mathbb{D}^{\mathrm{KM}} &\simeq \mathrm{Loc}_{G,\kappa',x_\infty}, \\ (\widehat{\mathfrak{g}}_{\kappa',x_\infty} - \mathrm{mod}^{G(\mathcal{O}_{x_\infty})})^c &\rightarrow (\mathrm{D} - \mathrm{mod}_{\kappa'}(\mathrm{Bun}_G)_{\mathrm{co}})^c. \end{aligned}$$

The functor  $\mathrm{Loc}_{G,\kappa',x_\infty}$  is *localization at the negative level*, and it is explicitly described in [3, Corollary 6.1.10].

*Remark 2.4.* — The functor  $\mathrm{Loc}_{G,\kappa',x_\infty}$  is closely related to the naive localization functor

$$\mathrm{Loc}_{G,\kappa',x_\infty}^{\mathrm{naive}} : \widehat{\mathfrak{g}}_{\kappa',x_\infty} - \mathrm{mod}^{G(\mathcal{O}_{x_\infty})} \rightarrow \mathrm{D} - \mathrm{mod}_{\kappa'}(\mathrm{Bun}_G)$$

(the difference is that the target of the latter is the usual category  $\mathrm{D} - \mathrm{mod}_{\kappa'}(\mathrm{Bun}_G)$  rather than  $\mathrm{D} - \mathrm{mod}_{\kappa'}(\mathrm{Bun}_G)_{\mathrm{co}}$ ).

Namely, for every quasi-compact open substack  $U \subset \mathrm{Bun}_G$ , the following diagram commutes:

$$\begin{array}{ccc} \widehat{\mathfrak{g}}_{\kappa',x_\infty} - \mathrm{mod}^{G(\mathcal{O}_{x_\infty})} & \xrightarrow{\mathrm{Id}} & \widehat{\mathfrak{g}}_{\kappa',x_\infty} - \mathrm{mod}^{G(\mathcal{O}_{x_\infty})} \\ \mathrm{Loc}_{G,\kappa',x_\infty}^{\mathrm{naive}} \downarrow & & \downarrow \mathrm{Loc}_{G,\kappa',x_\infty} \\ \mathrm{D} - \mathrm{mod}_{\kappa'}(\mathrm{Bun}_G)_{\mathrm{co}} & & \mathrm{D} - \mathrm{mod}_{\kappa'}(\mathrm{Bun}_G) \\ \downarrow & & \downarrow \\ \mathrm{D} - \mathrm{mod}_{\kappa'}(U) & \xrightarrow{\mathrm{Id}} & \mathrm{D} - \mathrm{mod}_{\kappa'}(U). \end{array}$$

2.5.2. Taking into account Proposition 2.3, we obtain that Conjecture 1.4 can be reformulated as the existence of a canonical isomorphism

$$\Gamma_{\mathrm{dr}}(\mathrm{Bun}_G, \mathrm{Loc}_{G,\kappa',x_\infty}(T_{\kappa'}^\lambda) \otimes^! \mathrm{Eis}_{\kappa,!}(\mathrm{IC}_{\mathrm{Bun}_T})) \simeq \mathrm{C}^{\frac{\infty}{2}} \left( \mathfrak{u}_q(G), \mathrm{Res}^{\mathrm{big} \rightarrow \mathrm{small}}(\mathfrak{T}_q^\lambda) \right), \quad (2.6)$$

where  $\kappa$  is some positive integral level and  $(X, x_\infty) = (\mathbb{P}^1, \infty)$ .

### 3. Duality and the Eisenstein functor

In this section we will perform a formal manipulation: we will rewrite the left-hand side in (2.6) so that instead of the functor  $\Gamma_{\text{dr}}(\text{Bun}_G, -)$  we will consider the functor  $\Gamma_{\text{dr}}(\text{Bun}_T, -)$ .

#### 3.1. The functor of constant term

3.1.1. We consider the stack  $\overline{\text{Bun}}_B$ , and we note that it is *truncatable* in the sense of [10, Sect. 4]. In particular, it makes sense to talk about the category

$$\text{D} - \text{mod}(\overline{\text{Bun}}_B)_{\text{co}}.$$

Recall the canonical identifications given by Verdier duality

$$\begin{aligned} \text{D} - \text{mod}(\text{Bun}_G)_{\text{co}} &\simeq (\text{D} - \text{mod}(\text{Bun}_G))^{\vee} \\ \text{and } \text{D} - \text{mod}(\text{Bun}_T) &\simeq (\text{D} - \text{mod}(\text{Bun}_T))^{\vee}. \end{aligned}$$

We have a similar identification

$$\text{D} - \text{mod}(\overline{\text{Bun}}_B)_{\text{co}} \simeq (\text{D} - \text{mod}(\overline{\text{Bun}}_B))^{\vee}.$$

3.1.2. Under the above identifications the dual of the functor

$$\bar{q}^! : \text{D} - \text{mod}(\text{Bun}_T) \rightarrow \text{D} - \text{mod}(\overline{\text{Bun}}_B)$$

is the functor  $\bar{q}_* : \text{D} - \text{mod}(\overline{\text{Bun}}_B)_{\text{co}} \rightarrow \text{D} - \text{mod}(\text{Bun}_T)$ , and the dual of the functor

$$\bar{p}_* : \text{D} - \text{mod}(\overline{\text{Bun}}_B) \rightarrow \text{D} - \text{mod}(\text{Bun}_G),$$

is the functor  $\bar{p}^! : \text{D} - \text{mod}(\text{Bun}_G)_{\text{co}} \rightarrow \text{D} - \text{mod}(\overline{\text{Bun}}_B)_{\text{co}}$ .

3.1.3. Consider the functor

$$\text{D} - \text{mod}(\overline{\text{Bun}}_B)_{\text{co}} \otimes \text{D} - \text{mod}(\overline{\text{Bun}}_B) \rightarrow \text{D} - \text{mod}(\overline{\text{Bun}}_B)_{\text{co}}.$$

For a given  $\mathcal{T} \in \text{D} - \text{mod}(\overline{\text{Bun}}_B)$ , the resulting functor

$$\mathcal{S} \mapsto \mathcal{T} \overset{!}{\otimes} \mathcal{S} : \text{D} - \text{mod}(\overline{\text{Bun}}_B)_{\text{co}} \rightarrow (\overline{\text{Bun}}_B)_{\text{co}}$$

is the dual of the functor

$$\mathcal{S}' \mapsto \mathcal{T} \overset{!}{\otimes} \mathcal{S}' : \text{D} - \text{mod}(\overline{\text{Bun}}_B)_{\text{co}} \rightarrow \text{D} - \text{mod}(\overline{\text{Bun}}_B)_{\text{co}}.$$

Hence, we obtain that the dual of the (compactified) Eisenstein functor

$$\text{Eis}_{1*} : \text{D} - \text{mod}(\text{Bun}_T) \rightarrow \text{D} - \text{mod}(\text{Bun}_G)$$

is the functor  $\mathrm{CT}_{!*} : \mathrm{D} - \mathrm{mod}(\mathrm{Bun}_G)_{\mathrm{co}} \rightarrow \mathrm{D} - \mathrm{mod}(\mathrm{Bun}_T)$ , defined by

$$\mathrm{CT}_{!*}(\mathcal{F}') := \bar{q}_* \left( \bar{p}^! (\mathcal{F}') \otimes^{\dagger} \mathrm{IC}_{\overline{\mathrm{Bun}}_B} [\dim(\mathrm{Bun}_T)] \right).$$

### 3.2. Twistings on $\overline{\mathrm{Bun}}_B$

3.2.1. Recall the diagram

$$\begin{array}{ccc} & \overline{\mathrm{Bun}}_B & \\ \bar{q} \swarrow & & \searrow \bar{p} \\ \mathrm{Bun}_T & & \mathrm{Bun}_G \end{array} \quad (3.1)$$

Pulling back the  $\kappa$ -twisting on  $\mathrm{Bun}_G$  by means of  $\bar{p}$ , we obtain a twisting on  $\overline{\mathrm{Bun}}_B$ ; we denote the corresponding category of twisted D-modules  $\mathrm{D} - \mathrm{mod}_{\kappa,G}(\overline{\mathrm{Bun}}_B)$ .

Pulling back the  $\kappa$ -twisting on  $\mathrm{Bun}_T$  by means of  $\bar{q}$ , we obtain *another* twisting on  $\overline{\mathrm{Bun}}_B$ ; we denote the corresponding category of twisted D-modules  $\mathrm{D} - \mathrm{mod}_{\kappa,T}(\overline{\mathrm{Bun}}_B)$ .

We let  $\mathrm{D} - \mathrm{mod}_{\kappa,G/T}(\overline{\mathrm{Bun}}_B)$  the category of D-modules corresponding to the Baer difference of these two twistings.

In particular, tensor product gives rise to the functors

$$\mathrm{D} - \mathrm{mod}_{\kappa,T}(\overline{\mathrm{Bun}}_B) \otimes \mathrm{D} - \mathrm{mod}_{\kappa,G/T}(\overline{\mathrm{Bun}}_B) \rightarrow \mathrm{D} - \mathrm{mod}_{\kappa,G}(\overline{\mathrm{Bun}}_B) \quad (3.2)$$

$$\mathrm{D} - \mathrm{mod}_{\kappa',G}(\overline{\mathrm{Bun}}_B)_{\mathrm{co}} \otimes \mathrm{D} - \mathrm{mod}_{\kappa',G/T}(\overline{\mathrm{Bun}}_B) \rightarrow \mathrm{D} - \mathrm{mod}_{\kappa',T}(\overline{\mathrm{Bun}}_B)_{\mathrm{co}}. \quad (3.3)$$

3.2.2. Recall that  $j$  denotes the open embedding  $\mathrm{Bun}_B \rightarrow \overline{\mathrm{Bun}}_B$ .

We note that the category  $\mathrm{D} - \mathrm{mod}_{\kappa,G/T}(\overline{\mathrm{Bun}}_B)$  canonically identifies with the (untwisted)  $\mathrm{D} - \mathrm{mod}(\mathrm{Bun}_B)$ : indeed the pullback of the  $\kappa$ -twisting on  $\mathrm{Bun}_T$  by means of  $\bar{q}$  identifies canonically with the pullback of the  $\kappa$ -twisting on  $\mathrm{Bun}_G$  by means of  $\bar{p}$ , both giving rise to the category  $\mathrm{D} - \mathrm{mod}_{\kappa}(\mathrm{Bun}_B)$ .

Hence, it makes sense to speak of  $\mathrm{IC}_{\mathrm{Bun}_B}$  as an object of  $\mathrm{D} - \mathrm{mod}_{\kappa,G/T}(\overline{\mathrm{Bun}}_B)$ , and of

$$J_{\kappa,*}(\mathrm{IC}_{\mathrm{Bun}_B}), J_{\kappa,!}(\mathrm{IC}_{\mathrm{Bun}_B}) \text{ and } J_{\kappa,*}(\mathrm{IC}_{\mathrm{Bun}_B})$$

as objects of  $\mathrm{D} - \mathrm{mod}_{\kappa,G/T}(\overline{\mathrm{Bun}}_B)$ .

Note that when  $\kappa$  is integral, under the equivalence

$$D - \text{mod}_{\kappa, G/T}(\overline{\text{Bun}}_B) \simeq D - \text{mod}(\overline{\text{Bun}}_B),$$

given by tensoring by the corresponding line bundle, the above objects correspond to the objects

$$j_*(\text{IC}_{\text{Bun}}_B), j!(\text{IC}_{\text{Bun}}_B) \text{ and } j!_*(\text{IC}_{\text{Bun}}_B) \in D - \text{mod}(\overline{\text{Bun}}_B),$$

respectively.

3.2.3. Let us denote by

$$\text{CT}_{\kappa', !*} : D - \text{mod}_{\kappa'}(\text{Bun}_G)_{\text{co}} \rightarrow D - \text{mod}_{\kappa'}(\text{Bun}_T)$$

the functor

$$\mathcal{F}' \mapsto \bar{q}_* \left( \bar{p}'^!(\mathcal{F}') \overset{!}{\otimes} j_{\kappa, !*}(\text{IC}_{\text{Bun}}_B)[\dim(\text{Bun}_T)] \right).$$

Denote also by  $\text{CT}_{\kappa', *}$  (resp.,  $\text{CT}_{\kappa', !}$ ) the similarly defined functors

$$D - \text{mod}_{\kappa'}(\text{Bun}_G)_{\text{co}} \rightarrow D - \text{mod}_{\kappa'}(\text{Bun}_T),$$

where we replace  $j_{\kappa, !*}(\text{IC}_{\text{Bun}}_B)$  by  $j_{\kappa, *}(\text{IC}_{\text{Bun}}_B)$  (resp.,  $j_{\kappa, !}(\text{IC}_{\text{Bun}}_B)$ ).

3.2.4. Note that the object  $\text{Eis}_{\kappa, !*}(\text{IC}_{\text{Bun}}_T) \in D - \text{mod}_{\kappa}(\text{Bun}_G)$  that appears in (2.6) is the result of application of the functor

$$D - \text{mod}(\text{Bun}_T) \xrightarrow{\text{Eis}_{!*}} D - \text{mod}(\text{Bun}_G) \xrightarrow{-\otimes \mathcal{L}_{G, \kappa}} D - \text{mod}_{\kappa}(\text{Bun}_G)$$

to  $\text{IC}_{\text{Bun}}_T \simeq \omega_{\text{Bun}}_T[-\dim(\text{Bun}_T)]$ .

Hence, the functor

$$\mathcal{F} \mapsto \Gamma_{\text{dr}}(\text{Bun}_G, \mathcal{F} \overset{!}{\otimes} \text{Eis}_{\kappa, !*}(\text{IC}_{\text{Bun}}_T)) : D - \text{mod}_{\kappa'}(\text{Bun}_G)_{\text{co}} \rightarrow \text{Vect} \quad (3.4)$$

identifies with the composition

$$\begin{aligned} D - \text{mod}_{\kappa'}(\text{Bun}_G)_{\text{co}} &\xrightarrow{\text{CT}_{\kappa', !*}[-\dim(\text{Bun}_T)]} D - \text{mod}_{\kappa'}(\text{Bun}_T) \xrightarrow{-\otimes \mathcal{L}_{T, -\kappa'}} \\ &\rightarrow D - \text{mod}(\text{Bun}_T) \xrightarrow{\Gamma_{\text{dr}}(\text{Bun}_T, -)} \text{Vect}. \end{aligned} \quad (3.5)$$

3.2.5. Summarizing, we obtain that Conjecture 1.4 can be reformulated as the existence of a canonical isomorphism

$$\begin{aligned} \Gamma_{\text{dr}}(\text{Bun}_T, \text{CT}_{\kappa', !*} \circ \text{Loc}_{G, \kappa', x_{\infty}}(T_{\kappa'}^{\lambda}) \otimes \mathcal{L}_{T, -\kappa'})[-\dim(\text{Bun}_T)] &\simeq \\ &\simeq \mathbb{C}^{\frac{\infty}{2}} \left( \mathfrak{u}_q(G), \text{Res}^{\text{big} \rightarrow \text{small}}(\mathfrak{T}_q^{\lambda}) \right), \end{aligned} \quad (3.6)$$

where  $\kappa$  is some positive integral level and  $(X, x_{\infty}) = (\mathbb{P}^1, \infty)$ .

### 3.3. Anomalies

In the second half of the paper we will study the interaction between the Kac-Moody Lie algebra  $\widehat{\mathfrak{g}}_{\kappa'}$  and its counterpart when  $G$  is replaced by  $T$ . The point is that the passage from  $\mathfrak{g}$  to  $\mathfrak{t}$  introduces a *critical twist* and a  $\rho$ -shift.

In this subsection we will specify what we mean by this.

*3.3.1.* Let us denote by  $\widehat{\mathfrak{t}}_{\kappa'}$  the extension of  $\mathfrak{t}(\mathcal{K})$ , given by  $\kappa'|_{\mathfrak{t}}$ . I.e.,  $\widehat{\mathfrak{t}}_{\kappa'}$  is characterized by the property that

$$\mathfrak{b}(\mathcal{K}) \times_{\mathfrak{t}(\mathcal{K})} \widehat{\mathfrak{t}}_{\kappa'} \simeq \mathfrak{b}(\mathcal{K}) \times_{\mathfrak{g}(\mathcal{K})} \widehat{\mathfrak{g}}_{\kappa'},$$

as extensions of  $\mathfrak{b}(\mathcal{K})$ .

We let  $\widehat{\mathfrak{t}}_{\kappa'+\text{shift}}$  be the Baer sum of  $\widehat{\mathfrak{t}}_{\kappa'}$  with the *Tate* extension  $\widehat{\mathfrak{t}}_{\text{Tate}(\mathfrak{n})}$ , corresponding to  $\mathfrak{t}$ -representation  $\mathfrak{n}$  (equipped with the adjoint action).

We refer the reader to [6, Sect. 2.7-2.8], where the construction of the Tate extension is explained.

There are two essential points of difference that adding  $\widehat{\mathfrak{t}}_{\text{Tate}(\mathfrak{n})}$  introduces:

- (i) The *level* of the extension  $\widehat{\mathfrak{t}}_{\kappa'+\text{shift}}$  is no longer  $\kappa'|_{\mathfrak{t}}$ , but rather  $\kappa' - \kappa_{\text{crit}}|_{\mathfrak{t}}$ .
- (ii) The extension  $\widehat{\mathfrak{t}}_{\kappa'+\text{shift}}$  no longer splits over  $\mathfrak{t}(\mathcal{O})$  in a canonical way.

*3.3.2.* According to [6, Theorem 2.8.17], we can alternatively describe  $\widehat{\mathfrak{t}}_{\kappa'+\text{shift}}$  as follows. It is the Baer sum of  $\widehat{\mathfrak{t}}_{\kappa' - \kappa_{\text{crit}}}$  and the *abelian* extension  $\widehat{\mathfrak{t}}_{\check{\rho}(\omega)}$ . The latter is by definition the torsor over

$$(\mathfrak{t}(\mathcal{K}))^{\vee} \simeq \mathfrak{t}^{\vee} \otimes \omega_{\mathcal{K}}$$

equal to the push-out of  $\check{\rho}(\omega_{\mathcal{K}})$ , thought of as a  $\check{T}(\mathcal{K})$ -torsor, under the map

$$d\log : \check{T}(\mathcal{K}) \rightarrow \mathfrak{t}^{\vee} \otimes \omega_{\mathcal{K}}.$$

In the above formula,  $\check{T}$  denotes the torus dual to  $T$ .

*Remark 3.1.* The reason we need to make the modification  $\widehat{\mathfrak{t}}_{\kappa'} \rightsquigarrow \widehat{\mathfrak{t}}_{\kappa'+\text{shift}}$  will be explained in Remark 7.2: it has to do with the properties of the BRST reduction functor.

However, the short answer is that we have no choice:

For example, the fact that the level needs to be shifted is used in the matching of  $\kappa$  and  $q$  parameters (see Sect. 6.1.1). The fact that  $\widehat{\mathfrak{t}}_{\check{\rho}(\omega)}$  appears is reflected by the presence of the linear term in the definition of the gerbe  $\mathcal{G}_{g,\text{loc}}$  (see Sect. 4.1.4), while the latter is forced by the ribbon structure on the category of modules over the quantum group.

3.3.3. Corresponding to  $\widehat{\mathfrak{t}}_{\kappa'+\text{shift}}$  there is a canonically defined twisting on the stack  $\text{Bun}_T$ ; we denote the resulting category of twisted D-modules by  $\text{D} - \text{mod}_{\kappa'+\text{shift}}(\text{Bun}_T)$ .

According to the above description of  $\widehat{\mathfrak{t}}_{\kappa'+\text{shift}}$ , we can describe the twisting giving rise to  $\text{D} - \text{mod}_{\kappa'+\text{shift}}(\text{Bun}_T)$  as follows:

It is the Baer sum of the twisting corresponding to  $\text{D} - \text{mod}_{\kappa'}(\text{Bun}_T)$  (i.e., the twisting attached to the form  $\kappa'|_{\mathfrak{t}}$ , see Sect. 2.1.1) and the twisting corresponding to the line  $\mathcal{L}_{T,\text{Tate}(\mathfrak{n})}$  bundle on  $\text{Bun}_T$  that attaches to a  $T$ -bundle  $\mathcal{F}_T$  the line

$$\det R\Gamma(X, \mathfrak{n}_{\mathcal{F}_T})^{\otimes -1},$$

where  $\mathfrak{n}_{\mathcal{F}_T}$  is the vector bundle over  $X$  associated to the  $T$ -bundle  $\mathcal{F}_T$  and its representation  $\mathfrak{n}$ .

3.3.4. Note, however, that since the difference between the two twistings is given by the line bundle  $\mathcal{L}_{T,\text{Tate}(\mathfrak{n})}$ , the corresponding categories of twisted D-modules are canonically equivalent via the operation of tensor product with  $\mathcal{L}_{T,\text{Tate}(\mathfrak{n})}$ .

In what follows we will consider the equivalence

$$\text{D} - \text{mod}_{\kappa'}(\text{Bun}_T) \rightarrow \text{D} - \text{mod}_{\kappa'+\text{shift}}(\text{Bun}_T) \tag{3.7}$$

obtained by composing one given by tensoring by the line bundle  $\mathcal{L}_{T,\text{Tate}(\mathfrak{n})}$  with the cohomological shift  $[\chi(R\Gamma(X, \mathfrak{n}_{\mathcal{F}_T}))]$ .

*Remark 3.2.* — Note that  $\chi(R\Gamma(X, \mathfrak{n}_{\mathcal{F}_T})) = -\dim.\text{rel.}(\text{Bun}_B/\text{Bun}_T)$ . As we shall see below (see Remark 3.3), this is the source of the shift in the definition of the Eisenstein functors, see Sect. 1.1.2.

3.3.5. One can use Sect. 3.3.2, to give the following alternative description of the twisting giving rise to  $\text{D} - \text{mod}_{\kappa'+\text{shift}}(\text{Bun}_T)$ .

It is the Baer sum of the twisting giving rise to  $\text{D} - \text{mod}_{\kappa' - \kappa_{\text{crit}}}(\text{Bun}_T)$  and one corresponding to the line bundle on  $\text{Bun}_T$ , given by

$$\mathcal{F}_T \mapsto \text{Weil}(\check{\rho}(\mathcal{F}_T), \omega_X) \simeq \text{Weil}(\mathcal{F}_T, \check{\rho}(\omega_X)),$$

where in the left-hand side Weil denotes the pairing

$$\text{Pic} \times \text{Pic} \rightarrow \mathbb{G}_m$$

and in the right-hand side Weil denotes the induced pairing

$$\mathrm{Bun}_T \times \mathrm{Bun}_{\bar{T}} \rightarrow \mathbb{G}_m.$$

### 3.4. The level-shifted constant term functor

3.4.1. Let us denote by  $\mathrm{CT}_{\kappa'+\mathrm{shift},!*$  the functor

$$\mathrm{D} - \mathrm{mod}_{\kappa'}(\mathrm{Bun}_G)_{\mathrm{co}} \rightarrow \mathrm{D} - \mathrm{mod}_{\kappa'+\mathrm{shift}}(\mathrm{Bun}_T)$$

equal to the composition

$$\begin{aligned} \mathrm{D} - \mathrm{mod}_{\kappa'}(\mathrm{Bun}_G)_{\mathrm{co}} &\xrightarrow{\bar{p}!} \mathrm{D} - \mathrm{mod}_{\kappa',G}(\overline{\mathrm{Bun}}_B)_{\mathrm{co}} \rightarrow \\ &\xrightarrow{-\otimes_{\mathcal{J}_{\kappa,!*}(\mathrm{IC}_{\mathrm{Bun}}_B)}[\dim(\mathrm{Bun}_B)]} \mathrm{D} - \mathrm{mod}_{\kappa',T}(\overline{\mathrm{Bun}}_B)_{\mathrm{co}} \rightarrow \\ &\xrightarrow{\bar{q}_*} \mathrm{D} - \mathrm{mod}_{\kappa'}(\mathrm{Bun}_T) \rightarrow \mathrm{D} - \mathrm{mod}_{\kappa'+\mathrm{shift}}(\mathrm{Bun}_T), \end{aligned}$$

where the last arrow is the functor (3.7).

*Remark 3.3.* — Note that  $\mathcal{J}_{\kappa,!*}(\mathrm{IC}_{\mathrm{Bun}}_B)[\dim(\mathrm{Bun}_B)]$  is really  $\mathcal{J}_{\kappa,!*}(\omega_{\mathrm{Bun}}_B)$ ; the problem with the latter notation is that it is illegal to apply  $\mathcal{J}_{!*$  to objects that are not in the heart of the t-structure.

However, this shows that the functor  $\mathrm{CT}_{\kappa'+\mathrm{shift},!*$  (unlike its counterparts  $\mathrm{CT}_{\kappa',!*$  or  $\mathrm{CT}_{!*$ ) does not include any artificial cohomological shifts.

3.4.2. Denote by  $\mathrm{CT}_{\kappa'+\mathrm{shift},*$  (resp.,  $\mathrm{CT}_{\kappa'+\mathrm{shift},!}$ ) the similarly defined functor, where we replace  $\mathcal{J}_{\kappa,!*}(\mathrm{IC}_{\mathrm{Bun}}_B[\dim(\mathrm{Bun}_B)])$  by  $\mathcal{J}_{\kappa,*}(\omega_{\mathrm{Bun}}_B)$  (resp.,  $\mathcal{J}_{\kappa,!}(\omega_{\mathrm{Bun}}_B)$ ).

3.4.3. Note that the composition

$$\begin{aligned} \mathrm{D} - \mathrm{mod}_{\kappa'}(\mathrm{Bun}_G)_{\mathrm{co}} &\xrightarrow{\mathrm{CT}_{\kappa'+\mathrm{shift},!}*} \mathrm{D} - \mathrm{mod}_{\kappa'+\mathrm{shift}}(\mathrm{Bun}_T) \rightarrow \\ &\xrightarrow{-\otimes_{\mathcal{L}_{T,-\kappa'}-\mathrm{shift}}} \mathrm{D} - \mathrm{mod}(\mathrm{Bun}_T) \end{aligned}$$

identifies with the functor

$$\mathrm{D} - \mathrm{mod}_{\kappa'}(\mathrm{Bun}_G)_{\mathrm{co}} \xrightarrow{\mathrm{CT}_{\kappa',!}*} \mathrm{D} - \mathrm{mod}_{\kappa'}(\mathrm{Bun}_T) \xrightarrow{-\otimes_{\mathcal{L}_{T,-\kappa'}}} \mathrm{D} - \mathrm{mod}(\mathrm{Bun}_T),$$

where

$$\mathcal{L}_{T,-\kappa'} \simeq \mathcal{L}_{T,-\kappa'}-\mathrm{shift} \otimes \mathcal{L}_{T,\mathrm{Tate}(\mathfrak{n})}. \quad (3.8)$$

3.4.4. Summarizing, we obtain that Conjecture 1.4 can be reformulated as the existence of a canonical isomorphism

$$\begin{aligned} \Gamma_{\text{dr}}(\text{Bun}_T, \text{CT}_{\kappa'+\text{shift},!} \circ \text{Loc}_{G,\kappa',x_\infty}(T_{\kappa'}^\lambda) \otimes \mathcal{L}_{T,-\kappa'-\text{shift}}[-\dim(\text{Bun}_T)]) &\simeq \\ &\simeq \mathbb{C}^{\frac{\infty}{2}} \left( \mathfrak{u}_q(G), \text{Res}^{\text{big} \rightarrow \text{small}}(\mathfrak{T}_q^\lambda) \right), \end{aligned} \quad (3.9)$$

where  $\kappa$  is some positive integral level and  $(X, x_\infty) = (\mathbb{P}^1, \infty)$ .

#### 4. Digression: factorizable sheaves of [BFS]

Our next step in bringing the two sides of Conjecture 1.4 closer to one another is a geometric interpretation of the category  $\mathfrak{u}_q(G)\text{-mod}$ , and, crucially, of the functor  $\mathbb{C}^{\frac{\infty}{2}}(\mathfrak{u}_q(G), -)$ .

This interpretation is provided by the theory of *factorizable sheaves* of [12].

##### 4.1. Colored divisors

Our treatment of factorizable sheaves will be slightly different from that in [12], with the following two main points of difference:

(i) Instead of considering the various partially symmetrized powers of our curve  $X$ , we will assemble them into an (infinite-dimensional) algebro-geometric object, the colored Ran space  $\text{Ran}(X, \check{\Lambda})$  that parameterizes finite collections of points of  $X$  with elements of  $\check{\Lambda}$  assigned to them. The fact that we can consider the (DG) category of sheaves on such a space is a consequence of recent advances in higher category theory, see Sect. 0.7.7.

(ii) Instead of encoding the quantum parameter  $q$  by a local system, we let it be encoded by a  $\mathbb{C}^*$ -gerbe on  $\text{Ran}(X, \check{\Lambda})$ .

4.1.1. Let us recall that the Ran space of  $X$ , denoted  $\text{Ran}(X)$  is a prestack that associates to a test-scheme  $S$  the set of finite non-empty subsets in  $\text{Maps}(S, X)$ .

We let  $\text{Ran}(X, \check{\Lambda})$  be the prestack defined as follows. For a test-scheme  $S$ , we let

$$\text{Maps}(S, \text{Ran}(X, \check{\Lambda})) = \{I \subset \text{Maps}(S, X), \phi : I \rightarrow \check{\Lambda}\},$$

where  $I$  is a non-empty finite set.

4.1.2. Let

$$\text{Ran}(X, \check{\Lambda})^{\text{neg}} \subset \text{Ran}(X, \check{\Lambda}) \tag{4.1}$$

be the subfunctor, corresponding to the subset  $\check{\Lambda}^{\text{neg}} - 0 \subset \check{\Lambda}$  (here  $\check{\Lambda}^{\text{neg}}$  is the negative integral span of simple roots).

It is a crucial observation that  $\text{Ran}(X, \check{\Lambda})^{\text{neg}}$  is *essentially* a finite-dimensional algebraic variety.

For a given  $\check{\lambda} \in \check{\Lambda}^{\text{neg}}$ , let  $\text{Ran}(X, \check{\Lambda})^{\text{neg}, \lambda}$  be the connected component of  $\text{Ran}(X, \check{\Lambda})^{\text{neg}}$  corresponding to those  $S$ -points

$$\{I \subset \text{Maps}(S, X), \phi : I \rightarrow \check{\Lambda}\},$$

for which  $\sum_{i \in I} \phi(i) = \check{\lambda}$ . We have

$$\text{Ran}(X, \check{\Lambda})^{\text{neg}} = \bigsqcup_{\check{\lambda} \in \check{\Lambda}^{\text{neg}} - 0} \text{Ran}(X, \check{\Lambda})^{\text{neg}, \lambda}.$$

Denote

$$X^{\check{\lambda}} = \prod_s X^{(n_s)} \text{ if } \check{\lambda} = \sum n_s \cdot (-\check{\alpha}_s),$$

where the index  $s$  runs through the set of vertices of the Dynkin diagram, and  $\check{\alpha}_s$  denote the corresponding simple roots.

Note that we have a canonically defined map

$$\text{Ran}(X, \check{\Lambda})^{\text{neg}, \lambda} \rightarrow X^{\check{\lambda}}. \tag{4.2}$$

We have:

LEMMA 4.1. — *The map (4.2) induces an isomorphism of sheafifications in the topology generated by finite surjective maps.*

Let  $\overset{\circ}{X}^{\check{\lambda}} \subset X^{\check{\lambda}}$  be the open subscheme equal to the complement of the diagonal divisor. Let

$$\text{Ran}(X, \check{\Lambda})^{\text{neg}, \check{\lambda}} \hookrightarrow \text{Ran}(X, \check{\Lambda})^{\text{neg}}$$

be the subfunctor equal to the preimage of

$$\overset{\circ}{X}^{\check{\lambda}} \subset X^{\check{\lambda}}.$$

It is easy to see that the map

$$\text{Ran}(X, \check{\Lambda})^{\text{neg}, \check{\lambda}} \rightarrow \overset{\circ}{X}^{\check{\lambda}}$$

is actually an isomorphism.

4.1.3. We have a tautological projection

$$\mathrm{Ran}(X, \check{\Lambda}) \rightarrow \mathrm{Ran}(X),$$

that remembers the data of  $I$ .

The prestack  $\mathrm{Ran}(X, \check{\Lambda})$  has a natural factorization property with respect to the above projection. We refer the reader to [25, Sect. 1] for what this means.

4.1.4. The next to step is to associate to our choice of the quantum parameter  $q$  a certain canonical *factorizable*  $\mathbb{C}^*$ -gerbe on  $\mathrm{Ran}(X, \check{\Lambda})$ , denoted  $\mathcal{G}_{q,\mathrm{loc}}$ . This construction depends on an additional choice: we need to choose a  $W$ -invariant symmetric bilinear form  $b_q^{\frac{1}{2}}$  on  $\check{\Lambda}$  with coefficients in  $\mathbb{C}^*$  such that

$$q(\lambda) = b_q^{\frac{1}{2}}(\lambda, \lambda).$$

Note that by definition

$$(b_q^{\frac{1}{2}})^2 = b_q,$$

where  $b_q$  is as in Sect. 1.3.1.

4.1.5. Recall that given a line bundle  $\mathcal{L}$  over a space  $\mathcal{Y}$  and an element  $a \in \mathbb{C}^*$ , to this data we can attach a canonically defined  $\mathbb{C}^*$ -gerbe over  $\mathcal{Y}$ , denoted

$$\mathcal{L}^{\log(a)}.$$

Namely, the objects of  $\mathcal{L}^{\log(a)}$  are  $\mathbb{C}^*$ -local systems on the total space of  $\mathcal{L} - \{0\}$ , such that their monodromy along the fiber is given by  $a$ .

The gerbe  $\mathcal{G}_{q,\mathrm{loc}}$  is uniquely characterized by the following requirement. For an  $n$ -tuple  $\check{\lambda}_1, \dots, \check{\lambda}_n$  of elements of  $\check{\Lambda}$ , and the resulting map  $X^n \rightarrow \mathrm{Ran}(X, \check{\Lambda})$ , the pullback of  $\mathcal{G}_{q,\mathrm{loc}}$  to  $X^n$  is the gerbe

$$\left( \boxtimes_i \omega_X^{\log(b_q^{\frac{1}{2}}(\check{\lambda}_i, \check{\lambda}_i + 2\check{\rho}))} \right)^{\otimes -1} \otimes \left( \otimes_{i \neq j} \mathcal{O}(\Delta_{i,j})^{\log(b_q^{\frac{1}{2}}(\lambda_i, \lambda_j))} \right).$$

In other words, for a point of  $\mathrm{Ran}(X, \check{\Lambda})$  given by a collection of pairwise distinct points  $x_1, \dots, x_n$  with assigned weights  $\check{\lambda}_1, \dots, \check{\lambda}_n$ , the fiber of  $\mathcal{G}_{q,\mathrm{loc}}$  over this point is the tensor product

$$\bigotimes_{i=1, \dots, n} (\omega_{x_i}^{\otimes -1})^{\log(q(\check{\lambda}_i + \check{\rho})) - \log(q(\check{\rho}))}.$$

*Remark 4.2.* — As can be seen from the above formula, the individual fibers of the above gerbe do not depend on the additional datum of  $b_q^{\frac{1}{2}}$ ; the latter is needed in order to make the gerbe  $\mathcal{G}_{q,\text{loc}}$  well-defined on all of  $\text{Ran}(X, \check{\Lambda})$ .

*Remark 4.3.* — As can be observed from either of the above descriptions of  $\mathcal{G}_{q,\text{loc}}$ , it naturally arises as a tensor product of two gerbes: one comes from just the quadratic part  $\lambda \mapsto q(\check{\lambda})$ , and the other from the linear part  $\lambda \mapsto b_q(\check{\lambda}, \check{\rho})$ . The quadratic part encodes the “true” quantum parameter for  $\check{\Lambda}$ , whereas the linear part is the “ $\rho$ -shift” that we will comment on in Remark 7.5.

Note also that for each (local) trivialization of the canonical line bundle  $\omega_X$ , we obtain a trivialization of the linear part of the gerbe over the corresponding open sub-prestack of  $\text{Ran}(X, \check{\Lambda})$ .

4.1.6. Recall now that if  $\mathcal{Y}$  is a topological space (resp., prestack) equipped with a  $\mathbb{C}^*$ -gerbe  $\mathcal{G}$ , we can consider the category

$$\text{Shv}_{\mathcal{G}}(\mathcal{Y})$$

of sheaves on  $\mathcal{Y}$  twisted by  $\mathcal{G}$ .

We let

$$\text{Shv}_{\mathcal{G}_{q,\text{loc}}}(\text{Ran}(X, \check{\Lambda}))$$

the category of  $\mathcal{G}_{q,\text{loc}}$ -twisted sheaves on  $\text{Ran}(X, \check{\Lambda})$ .

The factorization property of  $\mathcal{G}_{q,\text{loc}}$  over  $\text{Ran}(X)$  implies that it makes sense to talk about factorization algebras in  $\text{Shv}_{\mathcal{G}_{q,\text{loc}}}(\text{Ran}(X, \check{\Lambda}))$ , and for a given factorization algebra, about factorization modules over it, see [27, Sect. 6].

## 4.2. The factorization algebra of BFS

4.2.1. The basic property of  $\mathcal{G}_{q,\text{loc}}$  is that its restriction to  $\text{Ran}(X, \check{\Lambda})^{\circ,\text{neg}}$  is canonically trivialized. This follows from the fact that for a simple root  $\check{\alpha}_i$ , we have

$$b_q^{\frac{1}{2}}(-\check{\alpha}_i, -\check{\alpha}_i + 2\check{\rho}) = \frac{q(\check{\rho} - \check{\alpha}_i)}{q(\check{\rho})} = 1 \in \mathbb{C}^*,$$

since  $s_i(\check{\rho}) = \check{\rho} - \check{\alpha}_i$  and  $q$  is  $W$ -invariant.

Therefore the category  $\text{Shv}_{\mathcal{G}_{q,\text{loc}}}(\text{Ran}(X, \check{\Lambda})^{\circ,\text{neg}})$  identifies canonically with the non-twisted category  $\text{Shv}(\text{Ran}(X, \check{\Lambda})^{\circ,\text{neg}})$ . In particular, we can

consider the sign local system  $sign$  on  $\mathring{\text{Ran}}(X, \check{\Lambda})^{\text{neg}}$  as an object of  $\text{Shv}_{\mathcal{G}_q, \text{loc}}(\mathring{\text{Ran}}(X, \check{\Lambda})^{\text{neg}})$ .

4.2.2. We define the object

$$\Omega_q^{\text{small}} \in \text{Shv}_{\mathcal{G}_q, \text{loc}}(\mathring{\text{Ran}}(X, \check{\Lambda})^{\text{neg}})$$

as follows.

First, we note that Lemma 4.1 implies that for the purposes of considering (twisted) sheaves, we can think that  $\mathring{\text{Ran}}(X, \check{\Lambda})^{\text{neg}}$  is an algebraic variety. Now, we let  $\Omega_q^{\text{small}}$  be the Goresky-MacPherson extension of  $sign \in \text{Shv}_{\mathcal{G}_q, \text{loc}}(\mathring{\text{Ran}}(X, \check{\Lambda})^{\text{neg}})$ , cohomologically shifted so that it lies in the heart of the perverse t-structure.

We shall regard  $\Omega_q^{\text{small}}$  as an object of  $\text{Shv}_{\mathcal{G}_q, \text{loc}}(\mathring{\text{Ran}}(X, \check{\Lambda}))$  via the embedding (4.1). It follows from the construction that  $\Omega_q^{\text{small}}$  has a natural structure of *factorization algebra* in  $\text{Shv}_{\mathcal{G}_q, \text{loc}}(\mathring{\text{Ran}}(X, \check{\Lambda}))$ .

4.2.3. Given points  $x_1, \dots, x_n \in X$ , we let

$$\Omega_q^{\text{small}}\text{-mod}_{x_1, \dots, x_n}$$

denote the category of *factorization  $\Omega_q^{\text{small}}$ -modules* at the above points.

*Remark 4.4.* In the terminology of [12], the category  $\Omega_q^{\text{small}}\text{-mod}_{x_1, \dots, x_n}$  is referred to as the category of *factorable sheaves*.

4.2.4. The main construction of [12] says that there is an equivalence

$$(\mathfrak{u}_q(G)\text{-mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{u}_q(G)\text{-mod})_{T_{x_n}(X)} \rightarrow \Omega_q^{\text{small}}\text{-mod}_{x_1, \dots, x_n}, \quad (4.3)$$

where for a one-dimensional  $\mathbb{C}$ -vector space  $\ell$ , we denote by

$$(\mathfrak{u}_q(G)\text{-mod})_{\ell}$$

the twist of  $\mathfrak{u}_q(G)\text{-mod}$  by  $\ell$  using the auto-equivalence, given by the ribbon structure. Here  $T_x(X)$  denotes the tangent line to  $X$  at  $x \in X$ .

We will reinterpret the construction of the functor (4.3) in Sect. 5.1.

4.2.5. We denote the resulting functor

$$\begin{aligned} (\mathfrak{u}_q(G)\text{-mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{u}_q(G)\text{-mod})_{T_{x_n}(X)} &\rightarrow \Omega_q^{\text{small}}\text{-mod}_{x_1, \dots, x_n} \\ &\rightarrow \text{Shv}_{\mathcal{G}_q, \text{loc}}(\mathring{\text{Ran}}(X, \check{\Lambda})), \end{aligned}$$

where the last arrow is a forgetful functor, by  $\text{BFS}_{\mathfrak{u}_q}^{\text{top}}$ .

### 4.3. Conformal blocks

In this section we will generalize a procedure from [12] that starts with  $n$  modules over  $\mathfrak{u}_q$ , thought of as placed at points  $x_1, \dots, x_n$  on  $X$ , and produces an object of Vect.

Unlike the functor (4.3), this construction will be of a *global* nature, in that it will involve taking cohomology over the stack  $\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}$ .

4.3.1. Let

$$\text{AJ} : \text{Ran}(X, \check{\Lambda}) \rightarrow \text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}$$

denote the Abel-Jacobi map

$$\{x_i, \check{\lambda}_i\} \mapsto \sum_i \mathcal{O}(-x_i) \otimes \check{\lambda}_i \in \text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}.$$

4.3.2. A basic property of  $\mathcal{G}_{q,\text{loc}}$  is that it canonically descends to a  $\mathbb{C}^*$ -gerbe on  $\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}$ . We shall denote the latter by  $\mathcal{G}_{q,\text{glob}}$ .

Specifically, this gerbe attaches to a point  $\sum_i \mathcal{L}_i \otimes \check{\lambda}_i \in \text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}$  (where  $\mathcal{L}_i$  are line bundles on  $X$ ) the  $\mathbb{C}^*$ -gerbe

$$\left( \otimes_i \text{Weil}(\mathcal{L}_i, \mathcal{L}_i)^{\log(b_q^{\frac{1}{2}}(\check{\lambda}_i, \check{\lambda}_i))} \right) \otimes \left( \otimes_{i \neq j} \text{Weil}(\mathcal{L}_i, \mathcal{L}_j)^{\log(b_q^{\frac{1}{2}}(\check{\lambda}_i, \check{\lambda}_j))} \right) \otimes \left( \otimes_i \text{Weil}(\mathcal{L}_i, \omega_X)^{\log(b_q^{\frac{1}{2}}(\check{\lambda}_i, 2\check{\rho}))} \right),$$

where

$$\text{Weil} : \text{Pic} \times \text{Pic} \rightarrow BG_m$$

is the Weil pairing.

4.3.3. There exists a canonically defined ( $\mathcal{G}_{q,\text{glob}}^{-1}$ -twisted) local system

$$\mathcal{E}_{q^{-1}} \in \text{Shv}_{\mathcal{G}_{q,\text{glob}}^{-1}}(\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}),$$

which is supported on the union of the connected components corresponding to

$$-(2g-2)\check{\rho} + \text{Im}(\text{Frob}_{\Lambda,q}) \subset \check{\Lambda}.$$

We will specify what  $\mathcal{E}_{q^{-1}}$  is in Sect. 6.3.3 in terms of the Fourier-Mukai transform.

*Remark 4.5.* One can show that the restriction of  $\mathcal{E}_{q^{-1}}$  to the connected component  $-(2g-2)\check{\rho}$  identifies with the Heisenberg local system of [12].

4.3.4. Let us specialize for a moment to the case when  $X = \mathbb{P}^1$ . Choosing  $x_\infty$  as our base point, we obtain an isomorphism

$$\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda} \simeq BG_m \times \check{\Lambda}.$$

In this case, the gerbe  $\mathcal{G}_{q,\mathrm{glob}}$  is trivial (and hence supports non-zero objects of the category  $\mathrm{Shv}_{\mathcal{G}_{q,\mathrm{glob}}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda})$ ) only on the connected components corresponding to

$$\check{\rho} + \mathrm{Im}(\mathrm{Frob}_{\Lambda,q}) \subset \check{\Lambda}.$$

Moreover, the choice of the base point  $x_\infty$  defines a preferred trivialization of  $\mathcal{G}_{q,\mathrm{glob}}$  on the above connected components.

The following be a corollary of the construction:

LEMMA 4.6. *With respect to the trivialization of the gerbe  $\mathcal{G}_{q,\mathrm{glob}}$  on each of the above connected components of  $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}$ , the twisted local system  $\mathcal{E}_{q-1}$  identifies with  $\omega_{BG_m}$ .*

4.3.5. For an  $n$ -tuple of points  $x_1, \dots, x_n \in X$  we consider the functor

$$(\mathbf{u}_q(G)\text{-mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathbf{u}_q(G)\text{-mod})_{T_{x_n}(X)} \rightarrow \mathrm{Vect}$$

equal to the composition

$$\begin{aligned} & (\mathbf{u}_q(G)\text{-mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathbf{u}_q(G)\text{-mod})_{T_{x_n}(X)} \xrightarrow{\mathrm{BFS}_{\mathbf{u}_q}^{\mathrm{top}}} \mathrm{Shv}_{\mathcal{G}_{q,\mathrm{loc}}}(\mathrm{Ran}(X, \check{\Lambda})) \xrightarrow{\mathrm{AJ}_1} \\ & \rightarrow \mathrm{Shv}_{\mathcal{G}_{q,\mathrm{glob}}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) \xrightarrow{-\otimes \mathcal{E}_{q-1}[-\dim(\mathrm{Bun}_T)]} \mathrm{Shv}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) \rightarrow \mathrm{Vect}, \end{aligned}$$

where the last arrow is the functor of sheaf cohomology.

We denote this functor by  $\mathrm{Conf}_{X;x_1,\dots,x_n}^{\mathbf{u}_q}$ .

Remark 4.7.— A version of this functor, when instead of all of  $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}$  we use its connected component corresponding to  $-(2g-2)\check{\rho}$  is the functor of conformal blocks of [12].

4.3.6. Assume now that  $X = \mathbb{P}^1$ ,  $n = 1$  and  $x_1 = \infty$ . We obtain a functor

$$\mathrm{Conf}_{\mathbb{P}^1;\infty}^{\mathbf{u}_q} : \mathbf{u}_q(G)\text{-mod} \rightarrow \mathrm{Vect}.$$

According to [12, Theorem IV.8.11], we have the following:

THEOREM 4.8. — *There exists a canonical isomorphism of functors  $\mathbf{u}_q(G)\text{-mod} \rightarrow \mathrm{Vect}$*

$$\mathrm{Conf}_{\mathbb{P}^1;\infty}^{\mathbf{u}_q} \simeq C^{\frac{\infty}{2}}.$$

## 5. Digression: quantum groups and configuration spaces

### 5.1. The construction of [12] via Koszul duality

In this subsection we will show how the functor (4.3) can be interpreted as the Koszul duality functor for the Hopf algebra  $u_q(N^+)$  in the braided monoidal category  $\text{Rep}_q(T)$ . Such an interpretation is crucial for strategy of the proof of the isomorphism (3.9).

5.1.1. For the material of this subsection we will need to recall the following constructions, essentially contained in [24, Sect. 5.5]:

(i) Given a braided monoidal category  $\mathcal{C}$ , we can canonically attach to it a category  $\mathcal{C}_{\text{Ran}(X)}$  over the Ran space of the curve  $X = \mathbb{A}^1$ .

(i') If  $\mathcal{C}$  is endowed with a *ribbon structure*, we can extend this construction and replace  $\mathbb{A}^1$  by an arbitrary algebraic curve  $X$ .

(ii) For a given monoidal category  $\mathcal{C}$  it make sense to talk about associative (a.k.a.  $\mathbb{E}_1$ ) algebras in  $\mathcal{C}$ . If  $\mathcal{C}$  is *braided monoidal*, we can talk about  $\mathbb{E}_2$ -algebras in  $\mathcal{C}$ .

One can take the following as a definition of the notion of  $\mathbb{E}_2$ -algebra in  $\mathcal{C}$ : the braided structure on  $\mathcal{C}$  makes the tensor product functor  $\mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C}$  into a monoidal functor. In particular, it induces a monoidal structure in the category  $\mathbb{E}_1\text{-alg}(\mathcal{C})$  of  $\mathbb{E}_1$ -algebras in  $\mathcal{C}$ . Now, the category  $\mathbb{E}_2\text{-alg}(\mathcal{C})$  is defined to be

$$\mathbb{E}_1\text{-alg}(\mathbb{E}_1\text{-alg}(\mathcal{C})).$$

Equivalently, for  $\mathcal{A} \in \mathbb{E}_1\text{-alg}(\mathcal{C})$ , to endow it with a structure of  $\mathbb{E}_2$ -algebra amounts to endowing the category  $\mathcal{A}\text{-mod}$  with a monoidal structure such that the forgetful functor

$$\mathcal{A}\text{-mod} \rightarrow \mathcal{C}$$

is monoidal.

(iii) Given an  $\mathbb{E}_2$ -algebra  $\mathcal{A}$  in  $\mathcal{C}$  we can canonically attach to it an object  $\mathcal{A}_{\text{Ran}(\mathbb{A}^1)} \in \mathcal{C}_{\text{Ran}(\mathbb{A}^1)}$  that is equipped with a structure of *factorization algebra*. Moreover, this construction is an equivalence between the category  $\mathbb{E}_2\text{-alg}(\mathcal{C})$  and the category of factorization algebras in  $\mathcal{C}_{\text{Ran}(\mathbb{A}^1)}$ .

(iii') If  $\mathcal{C}$  is as in (i') and if  $\mathcal{A}$  is equivariant with respect to the ribbon structure, we can attach to  $\mathcal{A}$  an object  $\mathcal{A}_{\text{Ran}(X)} \in \mathcal{C}_{\text{Ran}(X)}$  for any  $X$ .

(iv) For an  $\mathbb{E}_2$ -algebra  $\mathcal{A}$  in  $\mathcal{C}$  we can talk about the category of  $\mathbb{E}_2$ -modules over  $\mathcal{A}$ , denoted  $\mathcal{A}\text{-mod}_{\mathbb{E}_2}$ . The category  $\mathcal{A}\text{-mod}_{\mathbb{E}_2}$  is itself braided monoidal and we have a canonical identification

$$\mathcal{A}\text{-mod}_{\mathbb{E}_2} \simeq Z_{\text{Dr}, \mathcal{C}}(\mathcal{A}\text{-mod}), \quad (5.1)$$

where  $Z_{\text{Dr}, \mathcal{C}}(-)$  denotes the (relative to  $\mathcal{C}$ ) Drinfeld center of a given  $\mathcal{C}$ -linear monoidal category.

(v) We have a canonical equivalence between  $\mathcal{A}\text{-mod}_{\mathbb{E}_2}$  and the category of *factorization modules at  $0 \in \mathbb{A}^1$*  over  $\mathcal{A}_{\text{Ran}(\mathbb{A}^1)}$  in  $\mathcal{C}_{\text{Ran}(\mathbb{A}^1)}$ .

(v') In the situation of (iii'), given a point  $x \in X$ , let  $(\mathcal{A}\text{-mod}_{\mathbb{E}_2})_{T_x(X)}$  be the twist of the category  $\mathcal{A}\text{-mod}_{\mathbb{E}_2}$  by the tangent line of  $X$  at  $x$  (the ribbon structure allows to twist the category  $\mathcal{A}\text{-mod}_{\mathbb{E}_2}$  by a complex line). Then we have a canonical equivalence between  $(\mathcal{A}\text{-mod}_{\mathbb{E}_2})_{T_x(X)}$  and the category of *factorization modules at  $x \in X$*  over  $\mathcal{A}_{\text{Ran}(X)}$  in  $\mathcal{C}_{\text{Ran}(X)}$ .

5.1.2. Let  $\text{Rep}_q(T)$  denote the ribbon braided monoidal category, corresponding to  $(\check{\Lambda}, q)$ . Specifically, the braiding is defined by setting

$$R^{\check{\lambda}, \check{\mu}} : \mathbb{C}^{\check{\lambda}} \otimes \mathbb{C}^{\check{\mu}} \rightarrow \mathbb{C}^{\check{\mu}} \otimes \mathbb{C}^{\check{\lambda}}$$

to be the tautological map multiplied by  $b_q^{\frac{1}{2}}(\check{\lambda}, \check{\mu})$ .

We set the ribbon automorphism of  $\mathbb{C}^{\check{\lambda}}$  to be given by  $b_q^{\frac{1}{2}}(\check{\lambda}, \check{\lambda} + 2\check{\rho})$ .

The category  $\text{Shv}_{\mathcal{G}_{q, \text{loc}}}(\text{Ran}(X, \check{\Lambda}))$ , considered in Sect. 4.1.6, is the category over the Ran space of  $X$  corresponding to  $\text{Rep}_q(T)$  in the sense of Sect. 5.1.1(i').

*Remark 5.1.*— The extra linear term in the formula for the ribbon structure corresponds to the linear term in the definition of the gerbe  $\mathcal{G}_{q, \text{loc}}$ .

5.1.3. Consider  $\mathfrak{u}_q(N^+)$  as a Hopf algebra in  $\text{Rep}_q(T)$ . In particular, we can consider the monoidal category  $\mathfrak{u}_q(N^+)\text{-mod}$  of modules over  $\mathfrak{u}_q(N^+)$  in  $\text{Rep}_q(T)$ . We use a renormalized version of  $\mathfrak{u}_q(N^+)\text{-mod}$ , which is compactly generated by finite-dimensional modules. We consider the *lax* monoidal functor of  $\mathfrak{u}_q(N^+)\text{-invariants}$ <sup>13</sup>:

$$\text{Inv}_{\mathfrak{u}_q(N^+)} : \mathfrak{u}_q(N^+)\text{-mod} \rightarrow \text{Rep}_q(T). \quad (5.2)$$

The Hopf algebra structure on  $\mathfrak{u}_q(N^+)$  defines on  $\text{Inv}_{\mathfrak{u}_q(N^+)}(\mathbb{C})$  a natural structure of  $\mathbb{E}_2$ -algebra in  $\text{Rep}_q(T)$ , see Sect. 5.1.1(ii).

---

<sup>13</sup>Of course, the functor of invariants is understood in the derived sense.

Note also that  $\mathfrak{u}_q(N^+)$  is naturally equivariant with respect to the ribbon twist on  $\text{Rep}_q(T)$ , thus inducing an equivariant structure on the functor  $\text{Inv}_{\mathfrak{u}_q(N^+)}(\mathbb{C})$ . In particular, the braided monoidal category  $\text{Inv}_{\mathfrak{u}_q(N^+)}(\mathbb{C})\text{-mod}_{\mathbb{E}_2}$  carries a canonical ribbon structure.

By Sect. 5.1.1(iii'), we can attach to the  $\mathbb{E}_2$ -algebra  $\text{Inv}_{\mathfrak{u}_q(N^+)}(\mathbb{C})$  in the ribbon braided monoidal category  $\text{Rep}_q(T)$  a factorization algebra in the category over the Ran space of  $X$  corresponding to  $\text{Rep}_q(T)$ , i.e.,  $\text{Shv}_{\mathcal{G}_q, \text{loc}}(\text{Ran}(X, \check{\Lambda}))$ .

We have (see [18, Corollary 6.8]):

PROPOSITION 5.2. — *The factorization algebra in  $\text{Shv}_{\mathcal{G}_q, \text{loc}}(\text{Ran}(X, \check{\Lambda}))$  corresponding to the  $\mathbb{E}_2$ -algebra  $\text{Inv}_{\mathfrak{u}_q(N^+)}(\mathbb{C}) \in \text{Rep}_q(T)$  identifies canonically with  $\Omega_q^{\text{small}}$ .*

Hence, by Sect. 5.1.1(v'), we obtain a canonical equivalence

$$\begin{aligned} & (\text{Inv}_{\mathfrak{u}_q(N^+)}(\mathbb{C})\text{-mod}_{\mathbb{E}_2})_{T_{x_1}(X)} \otimes \\ & \dots \otimes (\text{Inv}_{\mathfrak{u}_q(N^+)}(\mathbb{C})\text{-mod}_{\mathbb{E}_2})_{T_{x_n}(X)} \simeq \Omega_q^{\text{small}}\text{-mod}_{x_1, \dots, x_n}. \end{aligned} \quad (5.3)$$

5.1.4. Note now that the *lax* monoidal functor

$$\text{Inv}_{\mathfrak{u}_q(N^+)} : \mathfrak{u}_q(N^+)\text{-mod} \rightarrow \text{Rep}_q(T) \quad (5.4)$$

upgrades to a monoidal equivalence

$$\text{Inv}_{\mathfrak{u}_q(N^+)}^{\text{enh}} : \mathfrak{u}_q(N^+)\text{-mod} \rightarrow \text{Inv}_{\mathfrak{u}_q(N^+)}(\mathbb{C})\text{-mod}, \quad (5.5)$$

and the latter induces a braided monoidal equivalence

$$Z_{\text{Dr}, \text{Rep}_q(T)}(\mathfrak{u}_q(N^+)\text{-mod}) \rightarrow Z_{\text{Dr}, \text{Rep}_q(T)}(\text{Inv}_{\mathfrak{u}_q(N^+)}(\mathbb{C})\text{-mod}). \quad (5.6)$$

Applying (5.1), we obtain a braided monoidal equivalence

$$Z_{\text{Dr}, \text{Rep}_q(T)}(\mathfrak{u}_q(N^+)\text{-mod}) \rightarrow \text{Inv}_{\mathfrak{u}_q(N^+)}(\mathbb{C})\text{-mod}_{\mathbb{E}_2}. \quad (5.7)$$

5.1.5. Finally, we recall that we have a canonical equivalence of ribbon braided monoidal categories

$$\mathfrak{u}_q(G)\text{-mod} \simeq Z_{\text{Dr}, \text{Rep}_q(T)}(\mathfrak{u}_q(N^+)\text{-mod}). \quad (5.8)$$

Combining, we obtain an equivalence

$$\begin{aligned}
 & (\mathfrak{u}_q(G)\text{-mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{u}_q(G)\text{-mod})_{T_{x_n}(X)} \stackrel{(5.8)}{\simeq} \\
 & \simeq (Z_{\text{Dr,Rep}_q(T)}(\mathfrak{u}_q(N^+)\text{-mod}))_{T_{x_1}(X)} \otimes \dots \otimes (Z_{\text{Dr,Rep}_q(T)}(\mathfrak{u}_q(N^+)\text{-mod}))_{T_{x_n}(X)} \stackrel{(5.6)}{\simeq} \\
 & \simeq (Z_{\text{Dr,Rep}_q(T)}(\text{Inv}_{\mathfrak{u}_q(N^+)}(\mathbb{C})\text{-mod}))_{T_{x_1}(X)} \otimes \dots \otimes (Z_{\text{Dr,Rep}_q(T)}(\text{Inv}_{\mathfrak{u}_q(N^+)}(\mathbb{C})\text{-mod}))_{T_{x_n}(X)} \stackrel{(5.1)}{\simeq} \\
 & \simeq \left( \text{Inv}_{\mathfrak{u}_q(N^+)}(\mathbb{C})\text{-mod}_{\mathbb{E}_2} \right)_{T_{x_1}(X)} \otimes \dots \otimes \left( \text{Inv}_{\mathfrak{u}_q(N^+)}(\mathbb{C})\text{-mod}_{\mathbb{E}_2} \right)_{T_{x_n}(X)} \stackrel{(5.3)}{\simeq} \\
 & \simeq \Omega_q^{\text{small}}\text{-mod}_{x_1, \dots, x_n}. \quad (5.9)
 \end{aligned}$$

The latter is the functor (4.3) from [12].

## 5.2. The Lusztig and Kac-De Concini versions of the quantum group

The contents of rest of this section are not needed for the proof of Conjecture 1.4.

5.2.1. In addition to  $\Omega_q^{\text{small}}$  one can consider (at least) two more factorization algebras associated to  $G$  in  $\text{Shv}_{\mathcal{G}_{q,\text{loc}}}(\text{Ran}(X, \check{\Lambda}))$ , denoted

$$\Omega_q^{\text{KD}} \text{ and } \Omega_q^{\text{Lus}}, \quad (5.10)$$

respectively.

These functors are defined as follows. We consider the Hopf algebras

$$\mathfrak{U}_q(N^+)^{\text{KD}} \text{ and } \mathfrak{U}_q(N^+)^{\text{Lus}}$$

in the braided monoidal category  $\text{Rep}_q(T)$ , corresponding to the Kac-De Concini and Lusztig versions of the quantum group, respectively.

Proceeding as in Sect. 5.1.2 we obtain  $\mathbb{E}_2$ -algebras in  $\text{Rep}_q(T)$ , denoted

$$\text{Inv}_{\mathfrak{U}_q(N^+)^{\text{KD}}}(\mathbb{C})\text{-mod}_{\mathbb{E}_2} \text{ and } \text{Inv}_{\mathfrak{U}_q(N^+)^{\text{Lus}}}(\mathbb{C})\text{-mod}_{\mathbb{E}_2},$$

respectively, equivariant with respect to the ribbon structure on  $\text{Rep}_q(T)$ . We let

$$\Omega_q^{\text{KD}} \text{ and } \Omega_q^{\text{Lus}}$$

be the corresponding factorization algebras in  $\text{Shv}_{\mathcal{G}_{q,\text{loc}}}(\text{Ran}(X, \check{\Lambda}))$ .

By construction we have canonical equivalences

$$\begin{aligned} (\mathrm{Inv}_{\mathfrak{U}_q(N^+) \mathrm{KD}}(\mathbb{C})\text{-mod}_{\mathbb{E}_2})_{T_{x_1}(X)} \otimes \dots \otimes (\mathrm{Inv}_{\mathfrak{U}_q(N^+) \mathrm{KD}}(\mathbb{C})\text{-mod}_{\mathbb{E}_2})_{T_{x_n}(X)} \\ \simeq \Omega_q^{\mathrm{KD}}\text{-mod}_{x_1, \dots, x_n} \end{aligned}$$

and

$$\begin{aligned} (\mathrm{Inv}_{\mathfrak{U}_q(N^+) \mathrm{Lus}}(\mathbb{C})\text{-mod}_{\mathbb{E}_2})_{T_{x_1}(X)} \otimes \dots \otimes (\mathrm{Inv}_{\mathfrak{U}_q(N^+) \mathrm{Lus}}(\mathbb{C})\text{-mod}_{\mathbb{E}_2})_{T_{x_n}(X)} \\ \simeq \Omega_q^{\mathrm{Lus}}\text{-mod}_{x_1, \dots, x_n}. \end{aligned}$$

5.2.2. As in Sect. 5.1.4 we have canonically defined braided monoidal equivalences

$$\mathrm{Inv}_{\mathfrak{U}_q(N^+) \mathrm{KD}}^{\mathrm{enh}} : Z_{\mathrm{Dr}, \mathrm{Rep}_q(T)}(\mathfrak{u}_q(N^+) \mathrm{KD}\text{-mod}) \rightarrow \mathrm{Inv}_{\mathfrak{U}_q(N^+) \mathrm{KD}}(\mathbb{C})\text{-mod}_{\mathbb{E}_2}$$

and

$$\mathrm{Inv}_{\mathfrak{U}_q(N^+) \mathrm{Lus}}^{\mathrm{enh}} : Z_{\mathrm{Dr}, \mathrm{Rep}_q(T)}(\mathfrak{u}_q(N^+) \mathrm{Lus}\text{-mod}) \rightarrow \mathrm{Inv}_{\mathfrak{U}_q(N^+) \mathrm{Lus}}(\mathbb{C})\text{-mod}_{\mathbb{E}_2},$$

respectively.

5.2.3. Consider the braided monoidal categories

$$Z_{\mathrm{Dr}, \mathrm{Rep}_q(T)}(\mathfrak{U}_q(N^+) \mathrm{KD}\text{-mod}) \text{ and } Z_{\mathrm{Dr}, \mathrm{Rep}_q(T)}(\mathfrak{U}_q(N^+) \mathrm{Lus}\text{-mod}).$$

They identify, respectively, with the categories of modules over the corresponding “lopsided” versions of the quantum group

$$\mathfrak{U}_q(G)^{+\mathrm{KD}, -\mathrm{Lus}}\text{-mod} \text{ and } \mathfrak{U}_q(G)^{+\mathrm{Lus}, -\mathrm{KD}}\text{-mod}.$$

This follows from the fact that the (graded and relative to  $\mathrm{Rep}_q(T)$ ) duals of the Hopf algebras  $\mathfrak{U}_q(N^+) \mathrm{KD}$  and  $\mathfrak{U}_q(N^+) \mathrm{Lus}$  are the Hopf algebras  $\mathfrak{U}_q(N^-) \mathrm{Lus}$  and  $\mathfrak{U}_q(N^-) \mathrm{KD}$ , respectively.

5.2.4. Composing, we obtain the functors

$$\begin{aligned} (\mathfrak{U}_q(G)^{+\mathrm{KD}, -\mathrm{Lus}}\text{-mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{U}_q(G)^{+\mathrm{KD}, -\mathrm{Lus}}\text{-mod})_{T_{x_n}(X)} \rightarrow \\ \rightarrow \Omega_q^{\mathrm{KD}}\text{-mod}_{x_1, \dots, x_n} \quad (5.11) \end{aligned}$$

and

$$\begin{aligned} (\mathfrak{U}_q(G)^{+\mathrm{Lus}, -\mathrm{KD}}\text{-mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{U}_q(G)^{+\mathrm{Lus}, -\mathrm{KD}}\text{-mod})_{T_{x_n}(X)} \rightarrow \\ \rightarrow \Omega_q^{\mathrm{Lus}}\text{-mod}_{x_1, \dots, x_n}. \quad (5.12) \end{aligned}$$

The functors (5.11) and (5.12) are the respective counterparts for  $\mathfrak{U}_q(N^+) \mathrm{KD}$  and  $\mathfrak{U}_q(N^+) \mathrm{Lus}$  of the functor (4.3).

5.2.5. Composing the functors (5.11) and (5.12) with the forgetful functors

$$\Omega_q^{\text{KD}}\text{-mod}_{x_1, \dots, x_n} \rightarrow \text{Shv}_{\mathcal{G}_{q, \text{loc}}}(\text{Ran}(X, \check{\Lambda}))$$

$$\text{and } \Omega_q^{\text{Lus}}\text{-mod}_{x_1, \dots, x_n} \rightarrow \text{Shv}_{\mathcal{G}_{q, \text{loc}}}(\text{Ran}(X, \check{\Lambda})),$$

we obtain the functors

$$\begin{aligned} & (\mathfrak{U}_q(G)^{+\text{KD}, -\text{Lus}}\text{-mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{U}_q(G)^{+\text{KD}, -\text{Lus}}\text{-mod})_{T_{x_n}(X)} \\ & \rightarrow \text{Shv}_{\mathcal{G}_{q, \text{loc}}}(\text{Ran}(X, \check{\Lambda})) \end{aligned}$$

and

$$\begin{aligned} & (\mathfrak{U}_q(G)^{+\text{Lus}, -\text{KD}}\text{-mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{U}_q(G)^{+\text{Lus}, -\text{KD}}\text{-mod})_{T_{x_n}(X)} \\ & \rightarrow \text{Shv}_{\mathcal{G}_{q, \text{loc}}}(\text{Ran}(X, \check{\Lambda})) \end{aligned}$$

that we denote by  $\text{BFS}_{\mathfrak{U}_q^{\text{KD}}}^{\text{top}}$  and  $\text{BFS}_{\mathfrak{U}_q^{\text{Lus}}}^{\text{top}}$ , respectively.

5.2.6. The functors

$$\text{BFS}_{\mathfrak{u}_q}^{\text{top}}, \text{BFS}_{\mathfrak{U}_q^{\text{KD}}}^{\text{top}} \text{ and } \text{BFS}_{\mathfrak{U}_q^{\text{Lus}}}^{\text{top}}$$

can each be viewed as coming from the corresponding *lax* braided monoidal functors

$$\text{Inv}_{\mathfrak{u}_q(N^+)} : \mathfrak{u}_q(G)\text{-mod} \rightarrow \text{Rep}_q(T),$$

$$\text{Inv}_{\mathfrak{U}_q(N^+)^{\text{KD}}} : \mathfrak{U}_q(G)^{+\text{KD}, -\text{Lus}}\text{-mod} \rightarrow \text{Rep}_q(T)$$

and

$$\text{Inv}_{\mathfrak{U}_q(N^+)^{\text{Lus}}} : \mathfrak{U}_q(G)^{+\text{Lus}, -\text{KD}}\text{-mod} \rightarrow \text{Rep}_q(T),$$

respectively.

The functors  $\text{BFS}_{\mathfrak{U}_q^{\text{KD}}}^{\text{top}}$  and  $\text{BFS}_{\mathfrak{U}_q^{\text{Lus}}}^{\text{top}}$  are the respective counterparts for  $\mathfrak{U}_q(N^+)^{\text{KD}}$  and  $\mathfrak{U}_q(N^+)^{\text{Lus}}$  of the functor  $\text{BFS}_{\mathfrak{u}_q}^{\text{top}}$  from Sect. 4.2.5.

### 5.3. Restriction functors and natural transformations

5.3.1. Note now that we have the homomorphisms of Hopf algebras in  $\text{Rep}_q(T)$ :

$$\mathfrak{U}_q(N^+)^{\text{KD}} \rightarrow \mathfrak{u}_q(N^+) \rightarrow \mathfrak{U}_q(N^+)^{\text{Lus}}. \quad (5.13)$$

In addition, the braided monoidal categories

$$\mathfrak{U}_q(G)^{+\text{KD}, -\text{small}}\text{-mod} \text{ and } \mathfrak{U}_q(G)^{+\text{small}, -\text{KD}}\text{-mod}$$

and the following commutative diagrams of braided monoidal functors

$$\begin{array}{ccc}
 \mathfrak{U}_q(G)\text{-mod} & \xrightarrow{\text{Res}^{\text{big}\rightarrow\text{KD}}} & \mathfrak{U}_q(G)^{+\text{KD},-\text{Lus}}\text{-mod} \\
 \text{Res}^{\text{big}\rightarrow\text{small}} \downarrow & & \downarrow \\
 \mathfrak{u}_q(G)\text{-mod} & \longrightarrow & \mathfrak{U}_q(G)^{+\text{KD},-\text{small}}\text{-mod}
 \end{array}$$

and

$$\begin{array}{ccc}
 \mathfrak{U}_q(G)\text{-mod} & \xrightarrow{\text{Res}^{\text{big}\rightarrow\text{Lus}}} & \mathfrak{U}_q(G)^{+\text{Lus},-\text{KD}}\text{-mod} \\
 \text{Res}^{\text{big}\rightarrow\text{small}} \downarrow & & \downarrow \\
 \mathfrak{u}_q(G)\text{-mod} & \longrightarrow & \mathfrak{U}_q(G)^{+\text{small},-\text{KD}}\text{-mod}.
 \end{array}$$

From here we obtain the natural transformations

$$\begin{array}{ccc}
 \mathfrak{U}_q(G)\text{-mod} & \xrightarrow{\text{Res}^{\text{big}\rightarrow\text{KD}}} & \mathfrak{U}_q(G)^{+\text{KD},-\text{Lus}}\text{-mod} \\
 \uparrow = & \swarrow & \uparrow \text{induction} \\
 \mathfrak{U}_q(G)\text{-mod} & \longrightarrow & \mathfrak{U}_q(G)^{+\text{KD},-\text{small}}\text{-mod} \\
 \uparrow = & \swarrow \sim & \uparrow \\
 \mathfrak{U}_q(G)\text{-mod} & \xrightarrow{\text{Res}^{\text{big}\rightarrow\text{small}}} & \mathfrak{u}_q(G)\text{-mod}
 \end{array}$$

and

$$\begin{array}{ccc}
 \mathfrak{U}_q(G)\text{-mod} & \xrightarrow{\text{Res}^{\text{big}\rightarrow\text{Lus}}} & \mathfrak{U}_q(G)^{+\text{Lus},-\text{KD}}\text{-mod} \\
 \downarrow = & \swarrow \sim & \downarrow \\
 \mathfrak{U}_q(G)\text{-mod} & \longrightarrow & \mathfrak{U}_q(G)^{+\text{small},-\text{KD}}\text{-mod} \\
 \downarrow = & \swarrow \sim & \downarrow \text{induction} \\
 \mathfrak{U}_q(G)\text{-mod} & \xrightarrow{\text{Res}^{\text{big}\rightarrow\text{small}}} & \mathfrak{u}_q(G)\text{-mod},
 \end{array}$$

where the induction functors

$$\mathfrak{U}_q(G)^{+\text{KD},-\text{small}}\text{-mod} \rightarrow \mathfrak{U}_q(G)^{+\text{KD},-\text{Lus}}\text{-mod}$$

and

$$\mathfrak{U}_q(G)^{+\text{small},-\text{KD}}\text{-mod} \rightarrow \mathfrak{U}_q(G)\text{-mod}$$

appearing in the above two diagrams are left adjoint to the restriction functors

$$\mathfrak{U}_q(G)^{+\text{KD}, -\text{Lus}}\text{-mod} \rightarrow \mathfrak{U}_q(G)^{+\text{KD}, -\text{small}}\text{-mod}$$

and

$$\mathfrak{U}_q(G)\text{-mod} \rightarrow \mathfrak{U}_q(G)^{+\text{small}, -\text{KD}}\text{-mod},$$

respectively.

5.3.2. In addition, by adjunction we obtain the natural transformations

$$\begin{array}{ccc}
 \mathfrak{U}_q(G)^{+\text{KD}, -\text{Lus}}\text{-mod} & \xrightarrow{\text{Inv}_{\mathfrak{U}_q(N^+)\text{KD}}} & \text{Rep}_q(T) \\
 \uparrow \text{induction} & \swarrow & \uparrow = \\
 \mathfrak{U}_q(G)^{+\text{KD}, -\text{small}}\text{-mod} & \xrightarrow{\text{Inv}_{\mathfrak{U}_q(N^+)\text{KD}}} & \text{Rep}_q(T) \\
 \uparrow & \swarrow & \uparrow = \\
 \mathfrak{u}_q(G)\text{-mod} & \xrightarrow{\text{Inv}_{\mathfrak{u}_q(N^+)}} & \text{Rep}_q(T)
 \end{array}$$

and

$$\begin{array}{ccc}
 \mathfrak{U}_q(G)^{+\text{Lus}, -\text{KD}}\text{-mod} & \xrightarrow{\text{Inv}_{\mathfrak{U}_q(N^+)\text{Lus}}} & \text{Rep}_q(T) \\
 \downarrow & \swarrow & \downarrow = \\
 \mathfrak{U}_q(G)^{+\text{Lus}, -\text{small}}\text{-mod} & \xrightarrow{\text{Inv}_{\mathfrak{u}_q(N^+)\text{Lus}}} & \text{Rep}_q(T) \\
 \downarrow \text{induction} & \swarrow & \downarrow = \\
 \mathfrak{u}_q(G)\text{-mod} & \xrightarrow{\text{Inv}_{\mathfrak{u}_q(N^+)}} & \text{Rep}_q(T).
 \end{array}$$

5.3.3. Composing, we obtain the natural transformations

$$\text{Inv}_{\mathfrak{U}_q(N^+)\text{Lus}} \circ \text{Res}^{\text{big} \rightarrow \text{Lus}} \rightarrow \text{Inv}_{\mathfrak{u}_q(N^+)} \circ \text{Res}^{\text{big} \rightarrow \text{small}} \rightarrow \text{Inv}_{\mathfrak{U}_q(N^+)\text{KD}} \circ \text{Res}^{\text{big} \rightarrow \text{KD}}$$

as braided monoidal functors

$$\mathfrak{U}_q(G)\text{-mod} \rightarrow \text{Rep}_q(T).$$

Hence we obtain the natural transformations

$$\mathrm{BFS}_{\mathfrak{u}_q^{\mathrm{Lus}}}^{\mathrm{top}} \circ \mathrm{Res}^{\mathrm{big} \rightarrow \mathrm{Lus}} \rightarrow \mathrm{BFS}_{\mathfrak{u}_q}^{\mathrm{top}} \circ \mathrm{Res}^{\mathrm{big} \rightarrow \mathrm{small}} \rightarrow \mathrm{BFS}_{\mathfrak{u}_q^{\mathrm{KD}}}^{\mathrm{top}} \circ \mathrm{Res}^{\mathrm{big} \rightarrow \mathrm{KD}} \quad (5.14)$$

as functors

$$(\mathfrak{U}_q(G))_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{U}_q(G))_{T_{x_n}(X)} \rightarrow \mathrm{Shv}_{\mathfrak{g}_{q,\mathrm{loc}}}(\mathrm{Ran}(X, \check{\Lambda})).$$

## 6. Passing from modules over quantum groups to Kac-Moody representations

In this section we let  $\kappa$  be a positive integral level and  $\kappa' = -\kappa - \kappa_{\mathrm{KI}}$  the reflected level.

Recall that we have reduced the statement of Conjecture 1.4 to the existence of the isomorphism (3.9). The bridge between between the two sides in (3.9) will be provided by the Kazhdan-Lusztig equivalence.

### 6.1. The Kazhdan-Lusztig equivalence

In this subsection we will finally explain what the tilting conjecture is “really about”<sup>14</sup>. Namely, we will replace it by a more general statement, in which the curve  $X$  will be arbitrary (rather than  $\mathbb{P}^1$ ), and instead of the tilting module we will consider an arbitrary collection of representations of the Kac-Moody algebra.

*6.1.1.* We take the data of  $q$  and  $\kappa'$  to match in the following sense.

Starting from the bilinear form  $\kappa'$ , consider the form  $\kappa' - \kappa_{\mathrm{crit}}|_{\mathfrak{t}}$ . Since the latter was assumed non-degenerate, we can consider its inverse, which is a symmetric bilinear form  $(\kappa' - \kappa_{\mathrm{crit}}|_{\mathfrak{t}})^{-1}$  on  $\mathfrak{t}^{\vee}$ , and can thus be regarded as a symmetric bilinear form on  $\check{\Lambda}$  with coefficients in  $\mathbb{C}$ .

Finally, we set

$$b_q = \exp\left(2\pi \cdot i \cdot \frac{(\kappa' - \kappa_{\mathrm{crit}}|_{\mathfrak{t}})^{-1}}{2}\right),$$

regarded as a symmetric bilinear form on  $\check{\Lambda}$  with coefficients in  $\mathbb{C}^*$ .

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<sup>14</sup>From the point of view taken in this paper.

6.1.2. According to [22], we have a canonical equivalence

$$\mathrm{KL}_G : \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} \rightarrow \mathfrak{U}_q(G)\text{-mod}.$$

Note that we have:

$$\mathrm{KL}_G(T_{\kappa'}^\lambda) \simeq \mathfrak{T}_q^\lambda. \quad (6.1)$$

6.1.3. For an  $n$ -tuple of points  $x_1, \dots, x_n \in X$  we consider the following two functors

$$\widehat{\mathfrak{g}}_{\kappa', x_1}\text{-mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa', x_n}\text{-mod}^{G(\mathcal{O}_{x_n})} \rightarrow \mathrm{Vect}.$$

One functor is

$$\begin{aligned} & \widehat{\mathfrak{g}}_{\kappa', x_1}\text{-mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa', x_n}\text{-mod}^{G(\mathcal{O}_{x_n})} \xrightarrow{\mathrm{Loc}_{G, \kappa', x_1, \dots, x_n}} \mathrm{D}\text{-mod}_{\kappa'}(\mathrm{Bun}_G)_{\mathrm{co}} \rightarrow \\ & \mathrm{CT}_{\kappa'+\mathrm{shift}, !*} \mathrm{D}\text{-mod}_{\kappa'+\mathrm{shift}}(\mathrm{Bun}_T) \xrightarrow{-\otimes \mathcal{L}_{T, -\kappa' - \mathrm{shift}}} \mathrm{D}\text{-mod}(\mathrm{Bun}_T) \xrightarrow{\Gamma_{\mathrm{dr}}(\mathrm{Bun}_T, -)} \mathrm{Vect}. \end{aligned} \quad (6.2)$$

The other functor is

$$\begin{aligned} & \widehat{\mathfrak{g}}_{\kappa', x_1}\text{-mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa', x_n}\text{-mod}^{G(\mathcal{O}_{x_n})} \xrightarrow{\mathrm{KL}_G} \\ & (\mathfrak{U}_q(G)\text{-mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{U}_q(G)\text{-mod})_{T_{x_n}(X)} \xrightarrow{\mathrm{BFS}_{\mathfrak{u}_q}^{\mathrm{top}} \circ \mathrm{Res}^{\mathrm{big} \rightarrow \mathrm{small}}} \mathrm{Shv}_{\mathcal{G}_{q, \mathrm{loc}}}(\mathrm{Ran}(X, \check{\Lambda})) \\ & \xrightarrow{\mathrm{AJ}_!} \mathrm{Shv}_{\mathcal{G}_{q, \mathrm{glob}}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) \xrightarrow{-\otimes \mathcal{E}_{q-1}} \mathrm{Shv}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) \rightarrow \mathrm{Vect}. \end{aligned} \quad (6.3)$$

6.1.4. Taking into account the reformulation of Conjecture 1.4 as the existence of an isomorphism (3.6), combining with Theorem 4.8 and the isomorphism (6.1), we obtain that we can reformulate Conjecture 1.4 as the existence of an isomorphism

$$\begin{aligned} & \Gamma_{\mathrm{dr}}(\mathrm{Bun}_T, \mathrm{CT}_{\kappa'+\mathrm{shift}, !*} \circ \mathrm{Loc}_{G, \kappa', x_\infty}(T_{\kappa'}^\lambda) \otimes \mathcal{L}_{T, -\kappa' - \mathrm{shift}}[-\dim(\mathrm{Bun}_T)]) \simeq \\ & \simeq \mathrm{Conf}_{\mathbb{P}^1; \infty}^{\mathfrak{u}_q}(\mathrm{Res}^{\mathrm{big} \rightarrow \mathrm{small}} \circ \mathrm{KL}_G(T_{\kappa'}^\lambda)) = \\ & = \Gamma \left( \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}, \mathrm{AJ}_! \left( \mathrm{BFS}_{\mathfrak{u}_q}^{\mathrm{top}} \circ \mathrm{Res}^{\mathrm{big} \rightarrow \mathrm{small}} \circ \mathrm{KL}_G(T_{\kappa'}^\lambda) \right) \otimes \mathcal{E}_{q-1}[-\dim(\mathrm{Bun}_T)] \right). \end{aligned} \quad (6.4)$$

Hence, Conjecture 1.4 follows from the following more general statement:

CONJECTURE 6.1. — *The functors (6.2) and (6.3) are canonically isomorphic. I.e., the diagram of functors*

$$\begin{array}{ccc}
 \widehat{\mathfrak{g}}_{\kappa', x_1} - \text{mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa', x_n} - \text{mod}^{G(\mathcal{O}_{x_n})} & \xrightarrow{\text{Loc}_{G, \kappa', x_1, \dots, x_n}} & \text{D} - \text{mod}_{\kappa'}(\text{Bun}_G)_{\text{co}} \\
 \text{KLG} \downarrow & & \downarrow \text{CT}_{\kappa' + \text{shift}, !*} \\
 (\mathfrak{U}_q(G) - \text{mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{U}_q(G) - \text{mod})_{T_{x_n}(X)} & & \text{D} - \text{mod}_{\kappa' + \text{shift}}(\text{Bun}_T) \\
 \text{BFS}_{\mathfrak{U}_q}^{\text{top}} \circ \text{Res}^{\text{big} \rightarrow \text{small}} \downarrow & & \downarrow \otimes \mathcal{L}_{T, -\kappa' - \text{shift}} \\
 \text{Shv}_{\mathfrak{g}_{q, \text{loc}}}(\text{Ran}(X, \check{\Lambda})) & & \text{D} - \text{mod}(\text{Bun}_T) \\
 \text{AJ}_1 \downarrow & & \\
 \text{Shv}_{\mathfrak{g}_{q, \text{glob}}}(\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) & & \downarrow \Gamma_{\text{dr}}(\text{Bun}_T, -) \\
 \overset{!}{\otimes} \mathcal{E}_{q-1} \downarrow & & \\
 \text{Shv}(\text{Ran}(X, \check{\Lambda})) & \xrightarrow{\Gamma(\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}, -)} & \text{Vect}
 \end{array}$$

*commutes.*

The rest of the paper is devoted to the sketch of a proof of Conjecture 6.1.

6.1.5. In addition to Conjecture 6.1 we will sketch the proof of the following two of its versions:

CONJECTURE 6.2. —

(a) *The following functors  $\widehat{\mathfrak{g}}_{\kappa', x_1} - \text{mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa', x_n} - \text{mod}^{G(\mathcal{O}_{x_n})} \rightarrow \text{Vect}$  are canonically isomorphic:*

$$\begin{array}{c}
 \widehat{\mathfrak{g}}_{\kappa', x_1} - \text{mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa', x_n} - \text{mod}^{G(\mathcal{O}_{x_n})} \xrightarrow{\text{Loc}_{G, \kappa', x_1, \dots, x_n}} \text{D} - \text{mod}_{\kappa'}(\text{Bun}_G)_{\text{co}} \rightarrow \\
 \text{CT}_{\kappa' + \text{shift}, *} \xrightarrow{\quad} \text{D} - \text{mod}_{\kappa' + \text{shift}}(\text{Bun}_T) \xrightarrow{-\otimes \mathcal{L}_{T, -\kappa' - \text{shift}}} \text{D} - \text{mod}(\text{Bun}_T) \xrightarrow{\Gamma_{\text{dr}}(\text{Bun}_T, -)} \text{Vect}
 \end{array}$$

and

$$\begin{array}{c}
 \widehat{\mathfrak{g}}_{\kappa', x_1} - \text{mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa', x_n} - \text{mod}^{G(\mathcal{O}_{x_n})} \xrightarrow{\text{KLG}} \\
 \rightarrow (\mathfrak{U}_q(G) - \text{mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{U}_q(G) - \text{mod})_{T_{x_n}(X)} \xrightarrow{\text{BFS}_{\mathfrak{U}_q^{\text{KD}}}^{\text{top}} \circ \text{Res}^{\text{big} \rightarrow \text{KD}}} \\
 \text{Shv}_{\mathfrak{g}_{q, \text{loc}}}(\text{Ran}(X, \check{\Lambda})) \xrightarrow{\text{AJ}_1} \text{Shv}_{\mathfrak{g}_{q, \text{glob}}}(\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) \xrightarrow{\overset{!}{\otimes} \mathcal{E}_{q-1}} \text{Shv}(\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) \rightarrow \text{Vect}.
 \end{array}$$

(b) *The following functors  $\widehat{\mathfrak{g}}_{\kappa',x_1}\text{-mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa',x_n}\text{-mod}^{G(\mathcal{O}_{x_n})} \rightarrow \text{Vect}$  are canonically isomorphic:*

$$\widehat{\mathfrak{g}}_{\kappa',x_1}\text{-mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa',x_n}\text{-mod}^{G(\mathcal{O}_{x_n})} \xrightarrow{\text{Loc}_{G,\kappa',x_1,\dots,x_n}} \text{D-mod}_{\kappa'}(\text{Bun}_G)_{\text{co}} \rightarrow \text{CT}_{\kappa'+\text{shift},!} \text{D-mod}_{\kappa'+\text{shift}}(\text{Bun}_T) \xrightarrow{-\otimes \mathcal{L}_{T,-\kappa'-\text{shift}}} \text{D-mod}(\text{Bun}_T) \xrightarrow{\Gamma_{\text{dr}}(\text{Bun}_T,-)} \text{Vect}$$

and

$$\begin{aligned} & \widehat{\mathfrak{g}}_{\kappa',x_1}\text{-mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa',x_n}\text{-mod}^{G(\mathcal{O}_{x_n})} \xrightarrow{\text{KLG}} \\ & \rightarrow (\mathfrak{U}_q(G)\text{-mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{U}_q(G)\text{-mod})_{T_{x_n}(X)} \xrightarrow{\text{BFS}_{\mathfrak{U}_q^{\text{Lus}}}^{\text{top}} \circ \text{Res}^{\text{big} \rightarrow \text{Lus}}} \\ & \text{Shv}_{\mathcal{G}_{q,\text{loc}}}(\text{Ran}(X, \check{\Lambda})) \xrightarrow{\text{AJ}_!} \text{Shv}_{\mathcal{G}_{q,\text{glob}}}(\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) \xrightarrow{\otimes \mathcal{E}_{q^{-1}}} \text{Shv}(\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) \rightarrow \text{Vect}. \end{aligned}$$

*Remark 6.3.* — As we shall see, the natural transformations between the left-hand sides in Conjectures 6.1 and 6.2, induced by the natural transformations

$$\text{Eis}_! \rightarrow \text{Eis}_{!*} \rightarrow \text{Eis}_*$$

correspond to the natural transformations between the right-hand sides in Conjectures 6.1 and 6.2, induced by the natural transformations

$$\text{BFS}_{\mathfrak{U}_q^{\text{Lus}}}^{\text{top}} \circ \text{Res}^{\text{big} \rightarrow \text{Lus}} \rightarrow \text{BFS}_{\mathfrak{u}_q}^{\text{top}} \circ \text{Res}^{\text{big} \rightarrow \text{small}} \rightarrow \text{BFS}_{\mathfrak{U}_q^{\text{KD}}}^{\text{top}} \circ \text{Res}^{\text{big} \rightarrow \text{KD}}$$

of (5.14).

## 6.2. Riemann-Hilbert correspondence

Let us observe that in Conjecture 6.1 the left-hand side, i.e., (6.2), is completely algebraic (i.e., is formulated in the language of twisted D-modules), while right-hand side deals with sheaves in the analytic topology.

In this subsection we will start the process of replacing sheaves by D-modules, by applying Riemann-Hilbert correspondence. In particular, we will introduce the D-module counterparts of the objects discussed in Sect. 4.

*6.2.1.* Let us return to the construction of the gerbe  $\mathcal{G}_{q,\text{loc}}$  in Sect. 4.1.4. If instead of the symmetric bilinear form

$$b_q : \check{\Lambda} \otimes \check{\Lambda} \rightarrow \mathbb{C}^*$$

we use

$$\frac{(\kappa' - \kappa_{\text{crit}}|_t)^{-1}}{2} : \check{\Lambda} \otimes \check{\Lambda} \rightarrow \mathbb{C},$$

the same constructing yields a twisting on the prestack  $\text{Ran}(X, \check{\Lambda})$ .

We denote the resulting category of twisted D-modules by

$$\mathrm{D} - \mathrm{mod}_{\kappa^{-1} + \mathrm{trans}}(\mathrm{Ran}(X, \check{\Lambda}))$$

(the reason for the choice of the notation “ $\kappa^{-1} + \mathrm{trans}$ ” in the subscript will become clear in Sect. 6.2.4).

By construction, Riemann-Hilbert correspondence defines a fully-faithful embedding

$$\mathrm{Shv}_{\mathcal{G}_{q,\mathrm{loc}}}(\mathrm{Ran}(X, \check{\Lambda})) \xrightarrow{\mathrm{RH}} \mathrm{D} - \mathrm{mod}_{\kappa^{-1} + \mathrm{trans}}(\mathrm{Ran}(X, \check{\Lambda})).$$

6.2.2. Thus, starting from the factorization algebra

$$\Omega_q^{\mathrm{small}} \in \mathrm{Shv}_{\mathcal{G}_{q,\mathrm{loc}}}(\mathrm{Ran}(X, \check{\Lambda})),$$

we obtain the factorization algebra

$$\Omega_{\kappa^{-1} + \mathrm{trans}}^{\mathrm{small}} \in \mathrm{D} - \mathrm{mod}_{\kappa^{-1} + \mathrm{trans}}(\mathrm{Ran}(X, \check{\Lambda}))$$

and the functor

$$(\mathfrak{u}_q(G) - \mathrm{mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{u}_q(G) - \mathrm{mod})_{T_{x_n}(X)} \rightarrow \Omega_{\kappa^{-1} + \mathrm{trans}}^{\mathrm{small}} - \mathrm{mod}_{x_1, \dots, x_n},$$

the latter being the D-module counterpart of the functor (4.3).

We denote the composition of this functor with the forgetful functor

$$\Omega_{\kappa^{-1} + \mathrm{trans}}^{\mathrm{small}} - \mathrm{mod}_{x_1, \dots, x_n} \rightarrow \mathrm{D} - \mathrm{mod}_{\kappa^{-1} + \mathrm{trans}}(\mathrm{Ran}(X, \check{\Lambda}))$$

by  $\mathrm{BFS}_{\mathfrak{u}_q}^{\mathrm{Dmod}}$ .

6.2.3. Consider now the stack

$$\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda} \simeq \mathrm{Bun}_{\check{T}}.$$

On it we will consider the twisting given by the bilinear form  $(\kappa - \kappa_{\mathrm{crit}})^{-1}$  on  $\mathfrak{t}^{\vee}$ ; denote the resulting category of twisted D-modules by

$$\mathrm{D} - \mathrm{mod}_{(\kappa - \kappa_{\mathrm{crit}})^{-1}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}).$$

We will now consider another twisting on  $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}$ , obtained from one above by *translation* by the point  $\omega_X \otimes \check{\rho} \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}$ . We denote the resulting category of D-modules by

$$\mathrm{D} - \mathrm{mod}_{\kappa^{-1} + \mathrm{trans}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}).$$

By definition, we have an equivalence of categories

$$\mathrm{D} - \mathrm{mod}_{(\kappa - \kappa_{\mathrm{crit}})^{-1}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) \rightarrow \mathrm{D} - \mathrm{mod}_{\kappa^{-1} + \mathrm{trans}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}),$$

given by translation by  $\omega_X \otimes -\check{\rho}$ .

6.2.4. It follows from the construction that the above twistings on  $\mathrm{Ran}(X, \check{\Lambda})$  and  $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}$  are compatible under the Abel-Jacobi map

$$\mathrm{AJ} : \mathrm{Ran}(X, \check{\Lambda}) \rightarrow \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}$$

(see Sect. 4.3.1).

In particular, we have a pair of mutually adjoint functors

$$\mathrm{AJ}^! : \mathrm{D} - \mathrm{mod}_{\kappa^{-1} + \mathrm{trans}}(\mathrm{Ran}(X, \check{\Lambda})) \rightleftarrows \mathrm{D} - \mathrm{mod}_{\kappa^{-1} + \mathrm{trans}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) : \mathrm{AJ}^!$$

6.2.5. In addition, by the construction of  $\mathrm{D} - \mathrm{mod}_{\kappa^{-1} + \mathrm{trans}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda})$ , the Riemann-Hilbert correspondence defines a fully-faithful embedding

$$\mathrm{Shv}_{\mathcal{G}_q, \mathrm{glob}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) \xrightarrow{\mathrm{RH}} \mathrm{D} - \mathrm{mod}_{\kappa^{-1} + \mathrm{trans}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda})$$

so that the diagram

$$\begin{array}{ccc} \mathrm{Shv}_{\mathcal{G}_q, \mathrm{loc}}(\mathrm{Ran}(X, \check{\Lambda})) & \xrightarrow{\mathrm{RH}} & \mathrm{D} - \mathrm{mod}_{\kappa^{-1} + \mathrm{trans}}(\mathrm{Ran}(X, \check{\Lambda})) \\ \mathrm{AJ}^! \downarrow & & \downarrow \mathrm{AJ}^! \\ \mathrm{Shv}_{\mathcal{G}_q, \mathrm{glob}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) & \xrightarrow{\mathrm{RH}} & \mathrm{D} - \mathrm{mod}_{\kappa^{-1} + \mathrm{trans}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) \end{array}$$

commutes.

6.2.6. Let  $\mathcal{E}_{-\kappa^{-1} - \mathrm{trans}}$  be the image under the Riemann-Hilbert correspondence (for the opposite twisting) of the local system

$$\mathcal{E}_{q-1} \in \mathrm{Shv}_{\mathcal{G}_q, \mathrm{glob}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}),$$

see Sect. 4.3.3.

Thus, we can rewrite the functor appearing in (6.3), i.e., the right-hand side of Conjecture 6.1, as

$$\begin{aligned} & \widehat{\mathfrak{g}}_{\kappa', x_1} - \mathrm{mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa', x_n} - \mathrm{mod}^{G(\mathcal{O}_{x_n})} \xrightarrow{\mathrm{KL}\mathcal{G}} \\ & \rightarrow (\mathfrak{U}_q(G) - \mathrm{mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{U}_q(G) - \mathrm{mod})_{T_{x_n}(X)} \xrightarrow{\mathrm{BFS}_{\mathfrak{U}_q}^{\mathrm{Dmod} \circ \mathrm{Res}^{\mathrm{big} \rightarrow \mathrm{small}}} \rightarrow} \\ & \rightarrow \mathrm{D} - \mathrm{mod}_{\kappa^{-1} + \mathrm{trans}}(\mathrm{Ran}(X, \check{\Lambda})) \xrightarrow{\mathrm{AJ}^!} \mathrm{D} - \mathrm{mod}_{\kappa^{-1} + \mathrm{trans}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) \xrightarrow{-\otimes \mathcal{E}_{-\kappa^{-1} - \mathrm{trans}}} \\ & \rightarrow \mathrm{D} - \mathrm{mod}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) \xrightarrow{\Gamma_{\mathrm{dir}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}, -)} \mathrm{Vect}. \quad (6.5) \end{aligned}$$

Thus, we obtain that Conjecture 6.1 is equivalent to the existence of an isomorphism between the functors (6.2) and (6.5).

*Remark 6.4.* Note that part of the functor in (6.5) is the composition

$$\begin{aligned} & \widehat{\mathfrak{g}}_{\kappa', x_1} - \text{mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa', x_n} - \text{mod}^{G(\mathcal{O}_{x_n})} \xrightarrow{\text{KLG}} \\ & \rightarrow (\mathfrak{U}_q(G) - \text{mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{U}_q(G) - \text{mod})_{T_{x_n}(X)} \xrightarrow{\text{BFS}_{u_q}^{\text{Dmod}} \circ \text{Res}^{\text{big} \rightarrow \text{small}}} \\ & \rightarrow \text{D} - \text{mod}_{\kappa^{-1} + \text{trans}}(\text{Ran}(X, \check{\Lambda})). \end{aligned}$$

This functor combines the Kazhdan-Lusztig functor with the Riemann-Hilbert functor. As we shall see, the transcendental aspects of both of these functors cancel each other out, so the above composition is actually of algebraic nature (in particular, it can be defined over an arbitrary ground field of characteristic zero).

### 6.3. Global Fourier-Mukai transform

Note that the assertion that the functors (6.2) and (6.5) are isomorphic is non-tautological even for  $G = T$  since in the case of the former we are dealing with the stack  $\text{Bun}_T$ , and in the case of the latter with  $\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda} \simeq \text{Bun}_{\check{T}}$ .

In this subsection we will apply the Fourier-Mukai transform in order to replace  $\check{T}$  by  $T$  throughout.

*6.3.1.* Note that the Fourier-Mukai transform defines an equivalence

$$\text{D} - \text{mod}_{(\kappa - \kappa_{\text{crit}})^{-1}}(\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) \simeq \text{D} - \text{mod}_{\kappa' - \kappa_{\text{crit}}}(\text{Bun}_T)$$

From here, using Sect. 3.3.5, we obtain the equivalence

$$\text{FM}_{\text{glob}} : \text{D} - \text{mod}_{\kappa^{-1} + \text{trans}}(\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) \simeq \text{D} - \text{mod}_{\kappa' + \text{shift}}(\text{Bun}_T). \quad (6.6)$$

*6.3.2.* Consider also the categories endowed with opposite twistings, denoted

$$\text{D} - \text{mod}_{-\kappa^{-1} - \text{trans}}(\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) \text{ and } \text{D} - \text{mod}_{-\kappa' - \text{shift}}(\text{Bun}_T),$$

respectively.

As in Sect. 2.4.3, we have the canonical identifications

$$\text{D} - \text{mod}_{-\kappa^{-1} - \text{trans}}(\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) \simeq (\text{D} - \text{mod}_{\kappa^{-1} + \text{trans}}(\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}))^\vee$$

and

$$\text{D} - \text{mod}_{-\kappa' - \text{shift}}(\text{Bun}_T) \simeq (\text{D} - \text{mod}_{\kappa' + \text{shift}}(\text{Bun}_T))^\vee.$$

Let  $\mathrm{FM}'_{\mathrm{glob}}$  denote the resulting dual equivalence

$$\mathrm{D} - \mathrm{mod}_{-\kappa' - \mathrm{shift}}(\mathrm{Bun}_T) \rightarrow \mathrm{D} - \mathrm{mod}_{-\kappa - 1 - \mathrm{trans}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}).$$

6.3.3. Since  $\kappa$  was assumed integral, the twisting  $\mathrm{D} - \mathrm{mod}_{-\kappa' - \mathrm{shift}}(\mathrm{Bun}_T)$  is also integral, i.e., we have a canonical equivalence

$$\mathrm{D} - \mathrm{mod}(\mathrm{Bun}_T) \rightarrow \mathrm{D} - \mathrm{mod}_{-\kappa' - \mathrm{shift}}(\mathrm{Bun}_T), \quad (6.7)$$

given by tensoring with the line bundle  $\mathcal{L}_{T, -\kappa' - \mathrm{shift}}$ , the latter being defined by (3.8).

We are finally able to give the definition of the object

$$\mathcal{E}_{q-1} \in \mathrm{Shv}_{\mathcal{G}_q, \mathrm{glob}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}).$$

Namely, it is defined so that its Riemann-Hilbert image

$$\mathcal{E}_{-\kappa - 1 - \mathrm{trans}} \in \mathrm{D} - \mathrm{mod}_{-\kappa' - \mathrm{shift}}(\mathrm{Bun}_T)$$

equals

$$\mathrm{FM}'_{\mathrm{glob}}(\omega_{\mathrm{Bun}_T} \otimes \mathcal{L}_{T, -\kappa' - \mathrm{shift}}).$$

6.3.4. Thus, we can rewrite the functor (6.5) as the composition

$$\begin{aligned} & \widehat{\mathfrak{g}}_{\kappa', x_1} - \mathrm{mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa', x_n} - \mathrm{mod}^{G(\mathcal{O}_{x_n})} \xrightarrow{\mathrm{KL}\mathcal{G}} \\ & \rightarrow (\mathfrak{U}_q(G) - \mathrm{mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{U}_q(G) - \mathrm{mod})_{T_{x_n}(X)} \xrightarrow{\mathrm{BFS}_{\mathfrak{u}_q}^{\mathrm{Dmod}} \circ \mathrm{Res}^{\mathrm{big} \rightarrow \mathrm{small}}} \\ & \rightarrow \mathrm{D} - \mathrm{mod}_{\kappa - 1 + \mathrm{trans}}(\mathrm{Ran}(X, \check{\Lambda})) \xrightarrow{\mathrm{AJ}_1} \mathrm{D} - \mathrm{mod}_{\kappa - 1 + \mathrm{trans}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) \xrightarrow{\mathrm{FM}'_{\mathrm{glob}}} \\ & \rightarrow \mathrm{D} - \mathrm{mod}_{\kappa' + \mathrm{shift}}(\mathrm{Bun}_T) \xrightarrow{- \otimes \mathcal{L}_{T, -\kappa' - \mathrm{shift}}} \mathrm{D} - \mathrm{mod}(\mathrm{Bun}_T) \xrightarrow{\Gamma_{\mathrm{dr}}(\mathrm{Bun}_T, -)} \mathrm{Vect}. \end{aligned} \quad (6.8)$$

Thus, we obtain that we can rewrite the statement of Conjecture 6.1 as saying that there exists a canonical isomorphism of functors between (6.2) and (6.8).

6.3.5. In particular, we obtain Conjecture 6.1 follows from the following stronger statement, namely, that the following two functors

$$\widehat{\mathfrak{g}}_{\kappa', x_1} - \mathrm{mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa', x_n} - \mathrm{mod}^{G(\mathcal{O}_{x_n})} \rightarrow \mathrm{D} - \mathrm{mod}_{\kappa' + \mathrm{shift}}(\mathrm{Bun}_T)$$

are canonically isomorphic:

$$\begin{aligned} & \widehat{\mathfrak{g}}_{\kappa', x_1} - \mathrm{mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa', x_n} - \mathrm{mod}^{G(\mathcal{O}_{x_n})} \xrightarrow{\mathrm{Loc}_{G, \kappa', x_1, \dots, x_n}} \mathrm{D} - \mathrm{mod}_{\kappa'}(\mathrm{Bun}_G)_{\mathrm{co}} \\ & \xrightarrow{\mathrm{CT}_{\kappa' + \mathrm{shift}, !*}} \mathrm{D} - \mathrm{mod}_{\kappa' + \mathrm{shift}}(\mathrm{Bun}_T) \end{aligned} \quad (6.9)$$

and

$$\begin{aligned}
 & \widehat{\mathfrak{g}}_{\kappa', x_1} - \text{mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa', x_n} - \text{mod}^{G(\mathcal{O}_{x_n})} \xrightarrow{\text{KL}_{\mathcal{G}}} \\
 & \rightarrow (\mathfrak{U}_q(G) - \text{mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{U}_q(G) - \text{mod})_{T_{x_n}(X)} \xrightarrow{\text{BFS}_{\mathfrak{u}_q}^{\text{Dmod}} \circ \text{Res}^{\text{big} \rightarrow \text{small}}} \\
 & \rightarrow \text{D} - \text{mod}_{\kappa^{-1} + \text{trans}}(\text{Ran}(X, \check{\Lambda})) \xrightarrow{\text{AJ}_!} \text{D} - \text{mod}_{\kappa^{-1} + \text{trans}}(\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) \xrightarrow{\text{FM}_{\text{glob}}} \\
 & \rightarrow \text{D} - \text{mod}_{\kappa' + \text{shift}}(\text{Bun}_T). \quad (6.10)
 \end{aligned}$$

I.e., that the diagram

$$\begin{array}{ccc}
 \widehat{\mathfrak{g}}_{\kappa', x_1} - \text{mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa', x_n} - \text{mod}^{G(\mathcal{O}_{x_n})} & \xrightarrow{\text{Loc}_{G, \kappa', x_1, \dots, x_n}} & \text{D} - \text{mod}_{\kappa'}(\text{Bun}_G)_{\text{co}} \\
 \text{KL}_G \downarrow & & \downarrow \text{CT}_{\kappa' + \text{shift}, !*} \\
 (\mathfrak{U}_q(G) - \text{mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{U}_q(G) - \text{mod})_{T_{x_n}(X)} & & \text{D} - \text{mod}_{\kappa' + \text{shift}}(\text{Bun}_T) \\
 \text{BFS}_{\mathfrak{u}_q}^{\text{Dmod}} \circ \text{Res}^{\text{big} \rightarrow \text{small}} \downarrow & & \uparrow \text{FM}_{\text{glob}} \\
 \text{D} - \text{mod}_{\kappa^{-1} + \text{trans}}(\text{Ran}(X, \check{\Lambda})) & \xrightarrow{\text{AJ}_!} & \text{D} - \text{mod}_{\kappa^{-1} + \text{trans}}(\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda})
 \end{array}$$

commutes.

## 6.4. Local Fourier-Mukai transform

Our next step is to interpret the composition

$$\begin{aligned}
 & \widehat{\mathfrak{g}}_{\kappa', x_1} - \text{mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa', x_n} - \text{mod}^{G(\mathcal{O}_{x_n})} \xrightarrow{\text{KL}_{\mathcal{G}}} \\
 & \rightarrow (\mathfrak{U}_q(G) - \text{mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{U}_q(G) - \text{mod})_{T_{x_n}(X)} \xrightarrow{\text{BFS}_{\mathfrak{u}_q}^{\text{Dmod}} \circ \text{Res}^{\text{big} \rightarrow \text{small}}} \\
 & \rightarrow \text{D} - \text{mod}_{\kappa^{-1} + \text{trans}}(\text{Ran}(X, \check{\Lambda})) \xrightarrow{\text{AJ}_!} \text{D} - \text{mod}_{\kappa^{-1} + \text{trans}}(\text{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) \xrightarrow{\text{FM}_{\text{glob}}} \\
 & \rightarrow \text{D} - \text{mod}_{\kappa' + \text{shift}}(\text{Bun}_T)
 \end{aligned}$$

appearing as part of the functor (6.8) in terms of the localization functor for the Kac-Moody Lie algebra associated to the group  $T$ , and the corresponding functor  $\text{KL}_T$ .

The version of the Kac-Moody algebra for  $T$  that we will consider is  $\widehat{\mathfrak{t}}_{\kappa' + \text{shift}}$ , introduced in Sect. 3.3.1.

*6.4.1.* We consider the category  $\widehat{\mathfrak{t}}_{\kappa' + \text{shift}} - \text{mod}^{T^{(\mathcal{O})}}$  as a *factorization category*. In particular, we can consider the corresponding category over the Ran space

$$(\widehat{\mathfrak{t}}_{\kappa' + \text{shift}} - \text{mod}^{T^{(\mathcal{O})}})_{\text{Ran}(X)},$$

and the localization functor

$$\mathrm{Loc}_{T, \kappa' + \mathrm{shift}, \mathrm{Ran}(X)} : (\widehat{\mathfrak{t}}_{\kappa' + \mathrm{shift}} - \mathrm{mod}^{T(\mathcal{O})})_{\mathrm{Ran}(X)} \rightarrow \mathrm{D} - \mathrm{mod}_{\kappa' + \mathrm{shift}}(\mathrm{Bun}_T).$$

6.4.2. The key observation is that we have the following (nearly tautological) equivalence of categories

$$\mathrm{FM}_{\mathrm{loc}} : \mathrm{D} - \mathrm{mod}_{\kappa^{-1} + \mathrm{trans}}(\mathrm{Ran}(X, \check{\Lambda})) \simeq (\widehat{\mathfrak{t}}_{\kappa' + \mathrm{shift}} - \mathrm{mod}^{T(\mathcal{O})})_{\mathrm{Ran}(X)},$$

which is compatible with the factorization structure, and makes the following diagram commute:

$$\begin{array}{ccc} \mathrm{D} - \mathrm{mod}_{\kappa^{-1} + \mathrm{trans}}(\mathrm{Ran}(X, \check{\Lambda})) & \xrightarrow{\mathrm{FM}_{\mathrm{loc}}} & (\widehat{\mathfrak{t}}_{\kappa' + \mathrm{shift}} - \mathrm{mod}^{T(\mathcal{O})})_{\mathrm{Ran}(X)} \\ \mathrm{AJ}_! \downarrow & & \downarrow \mathrm{Loc}_{T, \kappa' + \mathrm{shift}, \mathrm{Ran}(X)} \\ \mathrm{D} - \mathrm{mod}_{\kappa^{-1} + \mathrm{trans}}(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \check{\Lambda}) & \xrightarrow{\mathrm{FM}_{\mathrm{glob}}} & \mathrm{D} - \mathrm{mod}_{\kappa' + \mathrm{shift}}(\mathrm{Bun}_T). \end{array}$$

6.4.3. We now consider the Kazhdan-Lusztig equivalence for the group  $T$ :

$$\mathrm{KL}_T : \widehat{\mathfrak{t}}_{\kappa' + \mathrm{shift}} - \mathrm{mod}^{T(\mathcal{O})} \simeq \mathrm{Rep}_q(T).$$

Let  $\mathrm{Inv}_{\mathfrak{n}(\mathcal{X}), !*}$  denote the *factorizable* functor

$$\widehat{\mathfrak{g}}_{\kappa'} - \mathrm{mod}^{G(\mathcal{O})} \rightarrow \widehat{\mathfrak{t}}_{\kappa' + \mathrm{shift}} - \mathrm{mod}^{T(\mathcal{O})}$$

that makes the diagram

$$\begin{array}{ccc} \widehat{\mathfrak{g}}_{\kappa'} - \mathrm{mod}^{G(\mathcal{O})} & \xrightarrow{\mathrm{Inv}_{\mathfrak{n}(\mathcal{X}), !*}} & \widehat{\mathfrak{t}}_{\kappa' + \mathrm{shift}} - \mathrm{mod}^{T(\mathcal{O})} \\ \mathrm{KL}_G \downarrow & & \downarrow \mathrm{KL}_T \\ \mathfrak{U}_q(G) - \mathrm{mod} & \xrightarrow{\mathrm{Inv}_{\mathfrak{u}_q(N^+)} \circ \mathrm{Res}^{\mathrm{big} \rightarrow \mathrm{small}}} & \mathrm{Rep}_q(T) \end{array}$$

commute, where  $\mathrm{Inv}_{\mathfrak{u}_q(N^+)}$  is as in (5.2).

6.4.4. Let

$$((\mathrm{Inv}_{\mathfrak{n}(\mathcal{X}), !*})_{\mathrm{Ran}(X)} : (\widehat{\mathfrak{g}}_{\kappa'} - \mathrm{mod}^{G(\mathcal{O})})_{\mathrm{Ran}(X)} \rightarrow (\widehat{\mathfrak{t}}_{\kappa' + \mathrm{shift}} - \mathrm{mod}^{T(\mathcal{O})})_{\mathrm{Ran}(X)})$$

denote the resulting functor between the corresponding categories over the Ran space.

Using the interpretation of the functor  $\text{BFS}_{u_q}^{\text{top}}$ , given in Sect. 5.1, we obtain the following commutative diagram

$$\begin{array}{ccc}
 \widehat{\mathfrak{g}}_{\kappa', x_1} - \text{mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa', x_n} - \text{mod}^{G(\mathcal{O}_{x_n})} & \longrightarrow & (\widehat{\mathfrak{g}}_{\kappa'} - \text{mod}^{G(\mathcal{O})})_{\text{Ran}(X)} \\
 \text{KL}_G \downarrow & & \\
 (\mathfrak{U}_q(G) - \text{mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{U}_q(G) - \text{mod})_{T_{x_n}(X)} & & \downarrow ((\text{Inv}_{\mathfrak{n}(X), !*})_{\text{Ran}(X)}) \\
 \text{BFS}_{u_q}^{\text{Dmod}} \circ \text{Res}^{\text{big} \rightarrow \text{small}} \downarrow & & \\
 \text{D} - \text{mod}_{\kappa^{-1} + \text{trans}}(\text{Ran}(X, \check{\Lambda})) & \xrightarrow{\text{FM}_{\text{loc}}} & (\widehat{\mathfrak{t}}_{\kappa' + \text{shift}} - \text{mod}^{T(\mathcal{O})})_{\text{Ran}(X)}.
 \end{array}$$

6.4.5. Taking into account the fact that the functor

$\widehat{\mathfrak{g}}_{\kappa', x_1} - \text{mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa', x_n} - \text{mod}^{G(\mathcal{O}_{x_n})} \xrightarrow{\text{Loc}_{G, \kappa', x_1, \dots, x_n}} \text{D} - \text{mod}_{\kappa'}(\text{Bun}_G)_{\text{co}}$   
 is isomorphic to the composition

$$\begin{array}{ccc}
 \widehat{\mathfrak{g}}_{\kappa', x_1} - \text{mod}^{G(\mathcal{O}_{x_1})} \otimes \dots \otimes \widehat{\mathfrak{g}}_{\kappa', x_n} - \text{mod}^{G(\mathcal{O}_{x_n})} & \rightarrow & (\widehat{\mathfrak{g}}_{\kappa'} - \text{mod}^{G(\mathcal{O})})_{\text{Ran}(X)} \\
 & & \downarrow \text{Loc}_{G, \kappa', \text{Ran}(X)} \\
 & & \text{D} - \text{mod}_{\kappa'}(\text{Bun}_G)_{\text{co}},
 \end{array}$$

and using Sect. 6.3.5, we obtain that Conjecture 6.1 follows from the following one:

CONJECTURE 6.5. *The following diagram of functors commutes*

$$\begin{array}{ccc}
 (\widehat{\mathfrak{g}}_{\kappa'} - \text{mod}^{G(\mathcal{O})})_{\text{Ran}(X)} & \xrightarrow{((\text{Inv}_{\mathfrak{n}(X), !*})_{\text{Ran}(X)})} & (\widehat{\mathfrak{t}}_{\kappa' + \text{shift}} - \text{mod}^{T(\mathcal{O})})_{\text{Ran}(X)} \\
 \text{Loc}_{G, \kappa', \text{Ran}(X)} \downarrow & & \downarrow \text{Loc}_{T, \kappa' + \text{shift}, \text{Ran}(X)} \\
 \text{D} - \text{mod}_{\kappa'}(\text{Bun}_G)_{\text{co}} & \xrightarrow{\text{CT}_{\kappa' + \text{shift}, !*}} & \text{D} - \text{mod}_{\kappa' + \text{shift}}(\text{Bun}_T).
 \end{array}$$

*Remark 6.6.* All we have done so far was push the content of Conjecture 6.1 into the understanding of the functor  $\text{Inv}_{\mathfrak{n}(X), !*}$ : on the one hand, it was defined via the Kazhdan-Lusztig functors  $\text{KL}_G$  and  $\text{KL}_T$ , and on the other hand we must relate it to the functor of Eisenstein series.

As we shall see (see Sect. 7.4), even though the definition of  $\text{Inv}_{\mathfrak{n}(X), !*}$  involves a transcendental procedure (the functors  $\text{KL}_G$  and  $\text{KL}_T$ ), it is actually algebraic in nature.

## 6.5. Other versions of the functor of invariants

6.5.1. Parallel to  $\Omega_{\kappa^{-1} + \text{trans}}^{\text{small}}$ , we also have the factorization algebras

$$\Omega_{\kappa^{-1} + \text{trans}}^{\text{KD}} \text{ and } \Omega_{\kappa^{-1} + \text{trans}}^{\text{Lus}} \in \text{D} - \text{mod}_{\kappa^{-1} + \text{trans}}(\text{Ran}(X, \check{\Lambda})),$$

and the functors

$$\mathrm{BFS}_{\mathfrak{M}_{\mathrm{KD}}}^{\mathrm{Dmod}} \circ \mathrm{Res}^{\mathrm{big} \rightarrow \mathrm{KD}} \quad \text{and} \quad \mathrm{BFS}_{\mathfrak{M}_{\mathrm{Lus}}}^{\mathrm{Dmod}} \circ \mathrm{Res}^{\mathrm{big} \rightarrow \mathrm{Lus}}$$

both mapping

$$(\mathfrak{U}_q(G)\text{-mod})_{T_{x_1}(X)} \otimes \dots \otimes (\mathfrak{U}_q(G)\text{-mod})_{T_{x_n}(X)} \rightarrow \mathrm{D}\text{-mod}_{\kappa^{-1} + \mathrm{trans}}(\mathrm{Ran}(X, \check{\Lambda})).$$

6.5.2. Let  $\mathrm{Inv}_{\mathfrak{n}(\mathcal{X}),*}$  denote the factorizable functor

$$\widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} \rightarrow \widehat{\mathfrak{t}}_{\kappa'+\mathrm{shift}}\text{-mod}^{T(\mathcal{O})}$$

that makes the diagram

$$\begin{array}{ccc} \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} & \xrightarrow{\mathrm{Inv}_{\mathfrak{n}(\mathcal{X}),*}} & \widehat{\mathfrak{t}}_{\kappa'+\mathrm{shift}}\text{-mod}^{T(\mathcal{O})} \\ \mathrm{KL}_G \downarrow & & \downarrow \mathrm{KL}_T \\ \mathfrak{U}_q(G)\text{-mod} & \xrightarrow{\mathrm{Inv}_{\mathfrak{U}_q(N^+)\mathrm{KD}} \circ \mathrm{Res}^{\mathrm{big} \rightarrow \mathrm{KD}}} & \mathrm{Rep}_q(T) \end{array}$$

commute.

Let  $\mathrm{Inv}_{\mathfrak{n}(\mathcal{X}),!}$  denote the factorizable functor

$$\widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} \rightarrow \widehat{\mathfrak{t}}_{\kappa'+\mathrm{shift}}\text{-mod}^{T(\mathcal{O})}$$

that makes the diagram

$$\begin{array}{ccc} \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} & \xrightarrow{\mathrm{Inv}_{\mathfrak{n}(\mathcal{X}),!}} & \widehat{\mathfrak{t}}_{\kappa'+\mathrm{shift}}\text{-mod}^{T(\mathcal{O})} \\ \mathrm{KL}_G \downarrow & & \downarrow \mathrm{KL}_T \\ \mathfrak{U}_q(G)\text{-mod} & \xrightarrow{\mathrm{Inv}_{\mathfrak{U}_q(N^+)\mathrm{Lus}} \circ \mathrm{Res}^{\mathrm{big} \rightarrow \mathrm{Lus}}} & \mathrm{Rep}_q(T) \end{array}$$

commute.

6.5.3. We obtain the corresponding functors

$$(\mathrm{Inv}_{\mathfrak{n}(\mathcal{X}),*})_{\mathrm{Ran}(X)} \quad \text{and} \quad (\mathrm{Inv}_{\mathfrak{n}(\mathcal{X}),!})_{\mathrm{Ran}(X)},$$

both of which map

$$(\widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})})_{\mathrm{Ran}(X)} \rightarrow (\widehat{\mathfrak{t}}_{\kappa'+\mathrm{shift}}\text{-mod}^{T(\mathcal{O})})_{\mathrm{Ran}(X)}.$$

We obtain that Conjecture 6.2 follows from the next one:

CONJECTURE 6.7. —

(a) *The following diagram of functors commutes:*

$$\begin{array}{ccc}
 (\widehat{\mathfrak{g}}_{\kappa'} - \text{mod}^{G(\mathcal{O})})_{\text{Ran}(X)} & \xrightarrow{((\text{Inv}_{\mathfrak{n}(\mathcal{K}),*})_{\text{Ran}(X)})} & (\widehat{\mathfrak{t}}_{\kappa'+\text{shift}} - \text{mod}^{T(\mathcal{O})})_{\text{Ran}(X)} \\
 \text{Loc}_{G,\kappa',\text{Ran}(X)} \downarrow & & \downarrow \text{Loc}_{T,\kappa'+\text{shift},\text{Ran}(X)} \\
 \text{D} - \text{mod}_{\kappa'}(\text{Bun}_G)_{\text{co}} & \xrightarrow{\text{CT}_{\kappa'+\text{shift},*}} & \text{D} - \text{mod}_{\kappa'+\text{shift}}(\text{Bun}_T).
 \end{array}$$

(b) *The following diagram of functors commutes:*

$$\begin{array}{ccc}
 (\widehat{\mathfrak{g}}_{\kappa'} - \text{mod}^{G(\mathcal{O})})_{\text{Ran}(X)} & \xrightarrow{((\text{Inv}_{\mathfrak{n}(\mathcal{K}),!})_{\text{Ran}(X)})} & (\widehat{\mathfrak{t}}_{\kappa'+\text{shift}} - \text{mod}^{T(\mathcal{O})})_{\text{Ran}(X)} \\
 \text{Loc}_{G,\kappa',\text{Ran}(X)} \downarrow & & \downarrow \text{Loc}_{T,\kappa'+\text{shift},\text{Ran}(X)} \\
 \text{D} - \text{mod}_{\kappa'}(\text{Bun}_G)_{\text{co}} & \xrightarrow{\text{CT}_{\kappa'+\text{shift},!}} & \text{D} - \text{mod}_{\kappa'+\text{shift}}(\text{Bun}_T).
 \end{array}$$

*Remark 6.8.* The natural transformations (5.14) induce natural transformations

$$\text{BFS}_{\mathfrak{u}_q^{\text{Lus}}}^{\text{Dmod}} \circ \text{Res}^{\text{big} \rightarrow \text{Lus}} \rightarrow \text{BFS}_{\mathfrak{u}_q}^{\text{Dmod}} \circ \text{Res}^{\text{big} \rightarrow \text{small}} \rightarrow \text{BFS}_{\mathfrak{u}_q^{\text{KD}}}^{\text{Dmod}} \circ \text{Res}^{\text{big} \rightarrow \text{KD}}$$

and also natural transformations

$$\text{Inv}_{\mathfrak{n}(\mathcal{K}),!} \rightarrow \text{Inv}_{\mathfrak{n}(\mathcal{K}),!*} \rightarrow \text{Inv}_{\mathfrak{n}(\mathcal{K}),*},$$

the latter as factorizable functors

$$\widehat{\mathfrak{g}}_{\kappa'} - \text{mod}^{G(\mathcal{O})} \rightarrow \widehat{\mathfrak{t}}_{\kappa'+\text{shift}} - \text{mod}^{T(\mathcal{O})}.$$

We will see that the above natural transformations are compatible with the natural transformations

$$\text{Eis}_! \rightarrow \text{Eis}_!* \rightarrow \text{Eis}_*$$

via the isomorphisms of functors of Conjectures 6.5 and 6.7.

## 7. The semi-infinite flag space

In the previous section we replaced the Tilting Conjecture (Conjecture 1.4) by a more general statement, Conjecture 6.1, and subsequently reduced the latter to Conjecture 6.5.

In order to tackle Conjecture 6.5, we need to understand the functor

$$\text{Inv}_{\mathfrak{n}(\mathcal{K}),!*} : \widehat{\mathfrak{g}}_{\kappa'} - \text{mod}^{G(\mathcal{O})} \rightarrow \widehat{\mathfrak{t}}_{\kappa'+\text{shift}} - \text{mod}^{T(\mathcal{O})}.$$

In this section we will show how to produce functors  $\widehat{\mathfrak{g}}_{\kappa'} - \text{mod}^{G(\mathcal{O})} \rightarrow \widehat{\mathfrak{t}}_{\kappa'+\text{shift}} - \text{mod}^{T(\mathcal{O})}$  starting from objects of the category of D-modules on *the semi-infinite flag space* of  $G$ . Our functor  $\text{Inv}_{\mathfrak{n}(\mathcal{K}),!,*}$  will correspond to some particular object of this category (as will do the functors  $\text{Inv}_{\mathfrak{n}(\mathcal{K}),*}$  and  $\text{Inv}_{\mathfrak{n}(\mathcal{K}),!}$ ).

### 7.1. The category of D-modules on the semi-infinite flag space

Morally, the semi-infinite flag space of  $G$  is the quotient  $N(\mathcal{K}) \backslash G(\mathcal{K})$ , and  $G(\mathcal{O})$ -equivariant D-modules on this space should be D-modules on the double quotient  $N(\mathcal{K}) \backslash G(\mathcal{K}) / G(\mathcal{O})$ .

Unfortunately, we still do not know how to make sense of  $N(\mathcal{K}) \backslash G(\mathcal{K})$  as an algebro-geometric object so that the category of D-modules on it is defined a priori. Instead, we will define spherical D-modules on it by first considering D-modules on the affine Grassmannian

$$\text{Gr}_G := G(\mathcal{K}) / G(\mathcal{O}),$$

and then imposing an equivariance condition with respect to  $N(\mathcal{K})$ .

7.1.1. Consider the affine Grassmannian  $\text{Gr}_G$ , the category  $\text{D} - \text{mod}(\text{Gr}_G)$  and its twisted version

$$\text{D} - \text{mod}_{\kappa'}(\text{Gr}_G).$$

The latter category is equipped with an action of the group  $G(\mathcal{K})$  (at level  $\kappa'$ ), and in particular, of the group  $N(\mathcal{K})$ .

For the purposes of this paper we will be interested in the category

$$\mathcal{C}_{\kappa'} := \text{D} - \text{mod}_{\kappa'}(\text{Gr}_G)_{N(\mathcal{K})}$$

of  $N(\mathcal{K})$ -*coinvariants* on  $\text{D} - \text{mod}(\text{Gr}_G)_{\kappa'}$ .

Note that  $\mathcal{C}_{\kappa'}$  carries an action of  $T(\mathcal{O})$  and we will also consider the category of  $T(\mathcal{O})$ -equivariant objects

$$\mathcal{C}_{\kappa'}^{T(\mathcal{O})}.$$

7.1.2. By definition,

$$\text{D} - \text{mod}_{\kappa'}(\text{Gr}_G)_{N(\mathcal{K})} := \text{colim}_i \text{D} - \text{mod}_{\kappa'}(\text{Gr}_G)^{N_i}, \quad (7.1)$$

where  $N_i$  is a family of *group-schemes* such that  $N(\mathcal{K}) = \text{colim}_i N_i$ , and where in (7.1) the transition functors

$$D - \text{mod}_{\kappa'}(\text{Gr}_G)^{N_i} \rightarrow D - \text{mod}_{\kappa'}(\text{Gr}_G)^{N_j}, \quad N_i \subset N_j$$

are given by \*-averaging over  $N_j/N_i$ .

In other words, since each  $N_i$  is a group-scheme (rather than a group ind-scheme), we can think of  $D - \text{mod}_{\kappa'}(\text{Gr}_G)^{N_i}$  as  $D - \text{mod}_{\kappa'}(\text{Gr}_G)_{N_i}$  and in this interpretation the transition functors are the tautological projections

$$D - \text{mod}_{\kappa'}(\text{Gr}_G)_{N_i} \rightarrow D - \text{mod}_{\kappa'}(\text{Gr}_G)_{N_j}.$$

*Remark 7.1.* Another version of the category of  $(G(\mathcal{O})$ -equivariant)  $D$ -modules on the semi-infinite flag space is

$$D - \text{mod}_{\kappa'}(\text{Gr}_G)^{N(\mathcal{K})} := \varinjlim_i D - \text{mod}_{\kappa'}(\text{Gr}_G)^{N_i},$$

where the transition functors

$$D - \text{mod}_{\kappa'}(\text{Gr}_G)^{N_i} \leftarrow D - \text{mod}_{\kappa'}(\text{Gr}_G)^{N_j}$$

are the forgetful functors.

It is not difficult to see that  $D - \text{mod}_{\kappa'}(\text{Gr}_G)_{N(\mathcal{K})}$  and  $D - \text{mod}_{\kappa'}(\text{Gr}_G)^{N(\mathcal{K})}$  are duals of each other.

*7.1.3.* For any  $x \in X$  we can consider the version of  $\mathcal{C}_{\kappa'}$  with  $\mathcal{O}$  replaced by  $\mathcal{O}_x$ ; we denote the resulting category by  $\mathcal{C}_{\kappa',x}$ . We can view  $\mathcal{C}_{\kappa'}$  as a *unital* factorization category (see [27, Sect. 6] for what this means); let

$$\mathcal{C}_{\kappa',\text{Ran}(X)}$$

denote the corresponding category over the Ran space.

We let

$$\mathbf{1}_{\mathcal{C}_{\kappa'}} \in \mathcal{C}_{\kappa'} \quad \text{and} \quad \mathbf{1}_{\mathcal{C}_{\kappa',\text{Ran}(X)}} \in \mathcal{C}_{\kappa',\text{Ran}(X)}$$

denote the corresponding unit objects.

By definition,  $\mathbf{1}_{\mathcal{C}_{\kappa'}}$  (resp.,  $\mathbf{1}_{\mathcal{C}_{\kappa',\text{Ran}(X)}}$ ) is the image of the  $\delta$ -function under the canonical projections

$$D - \text{mod}_{\kappa'}(\text{Gr}_G) \rightarrow \mathcal{C}_{\kappa'} \quad \text{and} \quad D - \text{mod}_{\kappa'}((\text{Gr}_G)_{\text{Ran}(X)}) \rightarrow \mathcal{C}_{\kappa',\text{Ran}(X)},$$

respectively.

A similar discussion applies to the  $T(\mathcal{O})$ -equivariant version.

*7.1.4.* Note that when  $\kappa'$  is integral, the category  $\mathcal{C}_{\kappa'}$  along with all its variants, identifies with the corresponding non-twisted version, denoted simply by  $\mathcal{C}$ .

## 7.2. The completion

7.2.1. Recall that  $\mathrm{Gr}_G$  is stratified by  $N(\mathcal{K})$ -orbits, and the latter are parameterized by elements of  $\Lambda$ . We let  $(\mathrm{Gr}_G)^{\leq \lambda}$  denote the (closed) union of orbits with parameters  $\leq \lambda$ .

For  $\lambda \in \Lambda$  we let  $(\mathcal{C}_{\kappa'})^{\leq \lambda}$  denote the full subcategory that consists of objects supported on  $(\mathrm{Gr}_G)^{\leq \lambda}$ .

We have  $\mathbf{1}_{\mathcal{C}_{\kappa'}} \in (\mathcal{C}_{\kappa'})^{\leq 0}$ .

7.2.2. The inclusion

$$(\mathcal{C}_{\kappa'})^{\leq \lambda} \hookrightarrow \mathcal{C}_{\kappa'}$$

admits a continuous right adjoint. Thus, we obtain a localization sequence

$$(\mathcal{C}_{\kappa'})^{\leq \lambda} \rightleftarrows \mathcal{C}_{\kappa'} \rightleftarrows \mathcal{C}_{\kappa'} / (\mathcal{C}_{\kappa'})^{\leq \lambda}.$$

We let  $\bar{\mathcal{C}}_{\kappa'}$  denote the category

$$\lim_{\Lambda} \mathcal{C}_{\kappa'} / (\mathcal{C}_{\kappa'})^{\leq -\lambda},$$

where we regard  $\Lambda$  as a poset with respect to the usual order relation.

In other words, an object of  $\bar{\mathcal{C}}_{\kappa'}$  is a system of objects  $\mathcal{F}^\lambda \in \mathcal{C}_{\kappa'} / (\mathcal{C}_{\kappa'})^{\leq -\lambda}$  that are compatible in the sense that for  $\lambda_1 \leq \lambda_2$ , the image of  $\mathcal{F}^{\lambda_2}$  under the projection

$$\mathcal{C}_{\kappa'} / (\mathcal{C}_{\kappa'})^{\leq -\lambda_2} \rightarrow \mathcal{C}_{\kappa'} / (\mathcal{C}_{\kappa'})^{\leq -\lambda_1}$$

identifies with  $\mathcal{F}^{\lambda_1}$ .

7.2.3. For every  $\lambda$  we have a full subcategory

$$(\bar{\mathcal{C}}_{\kappa'})^{\leq \lambda} \subset \bar{\mathcal{C}}_{\kappa'}$$

and a localization sequence

$$(\bar{\mathcal{C}}_{\kappa'})^{\leq \lambda} \rightleftarrows \bar{\mathcal{C}}_{\kappa'} \rightleftarrows \bar{\mathcal{C}}_{\kappa'} / (\bar{\mathcal{C}}_{\kappa'})^{\leq \lambda},$$

where the tautological functor

$$\mathcal{C}_{\kappa'} / (\mathcal{C}_{\kappa'})^{\leq \lambda} \rightarrow \bar{\mathcal{C}}_{\kappa'} / (\bar{\mathcal{C}}_{\kappa'})^{\leq \lambda}$$

is an equivalence.

7.2.4. As in Sect. 7.1.3, for  $x \in X$  we have the corresponding categories

$$\bar{\mathcal{C}}_{\kappa',x} \text{ and } x \mapsto (\bar{\mathcal{C}}_{\kappa',x})^{\leq 0}.$$

Moreover  $\bar{\mathcal{C}}_{\kappa'}$  and  $(\bar{\mathcal{C}}_{\kappa'})^{\leq 0}$  are unital factorization categories, and we let

$$\bar{\mathcal{C}}_{\kappa',\text{Ran}(X)} \text{ and } (\bar{\mathcal{C}}_{\kappa',\text{Ran}(X)})^{\leq 0}$$

denote the corresponding categories over the Ran space.

Furthermore, the above discussion extends to the  $T(\mathcal{O})$ -equivariant case. I.e., we have the categories

$$(\mathcal{C}_{\kappa'}^{T(\mathcal{O})})^{\leq \lambda} \Leftrightarrow \mathcal{C}_{\kappa'}^{T(\mathcal{O})} \Leftrightarrow \mathcal{C}_{\kappa'}^{T(\mathcal{O})} / (\mathcal{C}_{\kappa'}^{T(\mathcal{O})})^{\leq \lambda}$$

and

$$(\bar{\mathcal{C}}_{\kappa'}^{T(\mathcal{O})})^{\leq \lambda} \Leftrightarrow \bar{\mathcal{C}}_{\kappa'}^{T(\mathcal{O})} \Leftrightarrow \bar{\mathcal{C}}_{\kappa'}^{T(\mathcal{O})} / (\bar{\mathcal{C}}_{\kappa'}^{T(\mathcal{O})})^{\leq \lambda},$$

and the categories over the Ran space

$$(\mathcal{C}_{\kappa',\text{Ran}(X)}^{T(\mathcal{O})})^{\leq 0}, \bar{\mathcal{C}}_{\kappa',\text{Ran}(X)}^{T(\mathcal{O})} \text{ and } (\bar{\mathcal{C}}_{\kappa',\text{Ran}(X)}^{T(\mathcal{O})})^{\leq 0}.$$

### 7.3. The functor of BRST reduction

A key ingredient in understanding the functor

$$\text{Inv}_{\mathfrak{n}(\mathcal{K}),!*} : \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} \rightarrow \widehat{\mathfrak{t}}_{\kappa'+\text{shift}}\text{-mod}^{T(\mathcal{O})}$$

is a canonically defined functor

$$\text{BRST}_{\mathfrak{n}}^{\text{conv}} : \mathcal{C}_{\kappa'}^{T(\mathcal{O})} \otimes \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} \rightarrow \widehat{\mathfrak{t}}_{\kappa'+\text{shift}}\text{-mod}^{T(\mathcal{O})}.$$

In this subsection we will describe the construction of this functor.

7.3.1. The action of  $G(\mathcal{K})$  on  $\widehat{\mathfrak{g}}_{\kappa'}\text{-mod}$  defines a functor

$$\text{D-mod}_{\kappa'}(\text{Gr}_G) \otimes \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} \rightarrow \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}.$$

This functor respects the actions of  $G(\mathcal{K})$  (at level  $\kappa'$ ), where the  $G(\mathcal{K})$ -action on the source is via the first factor, i.e.,  $\text{D-mod}_{\kappa'}(\text{Gr}_G)$ . In particular, it gives rise to a functor

$$(\text{D-mod}_{\kappa'}(\text{Gr}_G)_{N(\mathcal{K})})^{T(\mathcal{O})} \otimes \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} \rightarrow (\widehat{\mathfrak{g}}_{\kappa'}\text{-mod}_{N(\mathcal{K})})^{T(\mathcal{O})}. \quad (7.2)$$

7.3.2. Consider now the functor of BRST reduction

$$\mathrm{BRST}_{\mathfrak{n}} : \widehat{\mathfrak{g}}_{\kappa'}\text{-mod} \rightarrow \widehat{\mathfrak{t}}_{\kappa'+\mathrm{shift}}\text{-mod}, \quad (7.3)$$

see [6, Sect. 3.8].

*Remark 7.2.* The fact that the target of the functor is the Kac-Moody extension  $\widehat{\mathfrak{t}}_{\kappa'+\mathrm{shift}}$  rather than  $\widehat{\mathfrak{t}}_{\kappa'-\kappa_{\mathrm{crit}}}$  (or even more naively  $\widehat{\mathfrak{t}}_{\kappa'}$ ) is the reason we needed to add the Tate extension  $\widehat{\mathfrak{t}}_{\mathrm{Tate}(\mathfrak{n})}$  in Sect. 3.3

7.3.3. The functor (7.3) is invariant with respect to the  $N(\mathcal{K})$ -action on  $\widehat{\mathfrak{g}}_{\kappa'}$ , and hence, gives rise to a functor

$$\widehat{\mathfrak{g}}_{\kappa'}\text{-mod}_{N(\mathcal{K})} \rightarrow \widehat{\mathfrak{t}}_{\kappa'+\mathrm{shift}}\text{-mod}.$$

The latter functor respects the action of  $T(\mathcal{O})$ , and thus gives rise to a functor

$$(\widehat{\mathfrak{g}}_{\kappa'}\text{-mod}_{N(\mathcal{K})})^{T(\mathcal{O})} \rightarrow \widehat{\mathfrak{t}}_{\kappa'+\mathrm{shift}}\text{-mod}^{T(\mathcal{O})}. \quad (7.4)$$

Finally, composing (7.2) and (7.4), we obtain the desired functor  $\mathrm{BRST}_{\mathfrak{n}}^{\mathrm{conv}}$

$$\begin{aligned} \mathcal{C}_{\kappa'}^{T(\mathcal{O})} \otimes \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} &= (\mathrm{D} - \mathrm{mod}_{\kappa'}(\mathrm{Gr}_G)_{N(\mathcal{K})})^{T(\mathcal{O})} \otimes \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} \\ &\rightarrow \widehat{\mathfrak{t}}_{\kappa'+\mathrm{shift}}\text{-mod}^{T(\mathcal{O})}. \end{aligned}$$

7.3.4. We have the following basic assertion:

**LEMMA 7.3.** *Assume that  $\kappa'$  is negative. Then the functor  $\mathrm{BRST}_{\mathfrak{n}}^{\mathrm{conv}}$  canonically extends to a functor*

$$\mathrm{BRST}_{\mathfrak{n}}^{\mathrm{conv}} : \widehat{\mathcal{C}}_{\kappa'}^{T(\mathcal{O})} \otimes \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} \rightarrow \widehat{\mathfrak{t}}_{\kappa'+\mathrm{shift}}\text{-mod}^{T(\mathcal{O})}.$$

*Remark 7.4.* This lemma amounts to the following observation. For  $\kappa'$  negative, the restriction of the functor  $\mathrm{BRST}_{\mathfrak{n}}^{\mathrm{conv}}$  to

$$(\mathcal{C}_{\kappa'}^{T(\mathcal{O})})^{\leq -\lambda} \otimes M,$$

where  $M \in \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})}$  is a given *compact* object has the property that it maps to a subcategory of  $\widehat{\mathfrak{t}}_{\kappa'+\mathrm{shift}}\text{-mod}^{T(\mathcal{O})}$ , consisting of objects whose  $\mathfrak{t}$ -weights are of the form

$$\check{\mu}_i(M) + \mathrm{Frob}_{\Lambda, \kappa}(\lambda - \Lambda^{\mathrm{pos}}),$$

where  $\check{\mu}_i(M) \in \mathfrak{t}^{\vee}$  is a finite collection of weights only depends on  $M$ .

7.3.5. The functor  $\text{BRST}_n^{\text{conv}}$  is factorizable. We shall denote by  $(\text{BRST}_n^{\text{conv}})_{\text{Ran}(X)}$  the corresponding functor

$$\mathcal{C}_{\kappa', \text{Ran}(X)}^{T(\emptyset)} \otimes_{\text{D-mod}(\text{Ran}(X))} (\widehat{\mathfrak{g}}_{\kappa'} - \text{mod}^{G(\emptyset)})_{\text{Ran}(X)} \rightarrow (\widehat{\mathfrak{t}}_{\kappa' + \text{shift}} - \text{mod}^{T(\emptyset)})_{\text{Ran}(X)}.$$

Similarly, if  $\kappa'$  is negative, we will denote by the same symbol the resulting functor

$$\overline{\mathcal{C}}_{\kappa', \text{Ran}(X)}^{T(\emptyset)} \otimes_{\text{D-mod}(\text{Ran}(X))} (\widehat{\mathfrak{g}}_{\kappa'} - \text{mod}^{G(\emptyset)})_{\text{Ran}(X)} \rightarrow (\widehat{\mathfrak{t}}_{\kappa' + \text{shift}} - \text{mod}^{T(\emptyset)})_{\text{Ran}(X)}.$$

*Remark 7.5.* — We can now explain the presence of the linear term in the gerbe  $\mathcal{G}_{q, \text{loc}}$  from Sect. 4.1.4. The actual source is the fact that the target of the functor (7.3) is the category of modules over  $\widehat{\mathfrak{t}}_{\kappa' + \text{shift}}$ , rather than  $\widehat{\mathfrak{t}}_{\kappa' - \kappa_{\text{crit}}}$ , the difference being the abelian extension of  $\mathfrak{t}(\mathcal{K})$  described in Sect. 3.3.2.

Since we are dealing with  $\widehat{\mathfrak{t}}_{\kappa' + \text{shift}}$ , the local Fourier-Mukai transforms implies that over  $\text{Ran}(X, \check{\Lambda})$  we need to consider the category  $\text{D-mod}_{\kappa^{-1} + \text{trans}}(\text{Ran}(X, \check{\Lambda}))$ , rather than  $\text{D-mod}_{(\kappa - \kappa_{\text{crit}})^{-1}}(\text{Ran}(X, \check{\Lambda}))$ .

The category  $\text{D-mod}_{\kappa^{-1} + \text{trans}}(\text{Ran}(X, \check{\Lambda}))$  is related via Riemann-Hilbert to the category of sheaves on  $\text{Ran}(X, \check{\Lambda})$ , twisted by the gerbe  $\mathcal{G}_{q, \text{loc}}$ . This is while  $\text{D-mod}_{(\kappa - \kappa_{\text{crit}})^{-1}}(\text{Ran}(X, \check{\Lambda}))$  corresponds to the category of sheaves twisted by the gerbe that only has the quadratic part.

## 7.4. Relation to the Kac-Moody equivalence

In this subsection we will formulate a crucial statement, Quasi-Theorem 7.9 that will express the functor  $\text{Inv}_{\mathfrak{n}(\mathcal{K}), !*}$  in terms of the functor  $\text{BRST}_n^{\text{conv}}$ .

In particular, this will show that the functor  $\text{Inv}_{\mathfrak{n}(\mathcal{K}), !*}$  is of algebraic nature, as was promised in Remark 6.6.

7.4.1. For any level  $\kappa'$  we consider the object

$$J_{\kappa', 0, *} := \mathbf{1}_{\overline{\mathcal{C}}_{\kappa'}^{G(\emptyset)}} \in \overline{\mathcal{C}}_{\kappa'}^{T(\emptyset)},$$

which is equal to the image of  $\mathbf{1}_{\mathcal{C}_{\kappa'}^{G(\emptyset)}}$  under the tautological projection

$$\mathcal{C}_{\kappa', \text{Ran}(X)}^{T(\emptyset)} \rightarrow \overline{\mathcal{C}}_{\kappa'}^{T(\emptyset)}.$$

Being the unit of  $\overline{\mathcal{C}}_{\kappa'}^{T(\mathcal{O})}$ , the object  $J_{\kappa',0,*}$  has a natural structure of factorization algebra in  $\overline{\mathcal{C}}_{\kappa'}^{T(\mathcal{O})}$ , and hence gives rise to an object

$$(J_{\kappa',0,*})_{\text{Ran}(X)} \in \overline{\mathcal{C}}_{\kappa',\text{Ran}(X)}^{T(\mathcal{O})},$$

which identifies tautologically with  $\mathbf{1}_{\overline{\mathcal{C}}_{\kappa',\text{Ran}(X)}^{G(\mathcal{O})}}$ .

7.4.2. The following result (along with Quasi-Theorem 7.8) can be viewed as a characterization of the Kazhdan-Lusztig equivalence:

QUASI-THEOREM 7.6. — *Let  $\kappa'$  be negative. Then the (factorizable) functor*

$$\text{Inv}_{\mathfrak{n}(\mathcal{K}),*} : \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} \rightarrow \widehat{\mathfrak{t}}_{\kappa'+\text{shift}}\text{-mod}^{T(\mathcal{O})}$$

of Sect. 6.5.2 identifies canonically with the (factorizable) functor

$$\text{BRST}_{\mathfrak{n}}^{\text{conv}}(J_{\kappa',0,*}, -) : \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} \rightarrow \widehat{\mathfrak{t}}_{\kappa'+\text{shift}}\text{-mod}^{T(\mathcal{O})}.$$

7.4.3. In what follows we will denote the above functor  $\text{BRST}_{\mathfrak{n}}^{\text{conv}}(J_{\kappa',0,*}, -)$  by  $\text{BRST}_{\mathfrak{n},*}$ . Note that this is essentially the functor  $\text{BRST}_{\mathfrak{n}}$  of (7.3) in the sense that we have a commutative diagram

$$\begin{array}{ccc} \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} & \longrightarrow & \widehat{\mathfrak{g}}_{\kappa'}\text{-mod} \\ \text{BRST}_{\mathfrak{n},*} \downarrow & & \downarrow \text{BRST}_{\mathfrak{n}} \\ \widehat{\mathfrak{t}}_{\kappa'+\text{shift}}\text{-mod}^{T(\mathcal{O})} & \longrightarrow & \widehat{\mathfrak{t}}_{\kappa'+\text{shift}}\text{-mod}. \end{array}$$

We shall denote by  $(\text{BRST}_{\mathfrak{n},*})_{\text{Ran}(X)}$  the corresponding functor

$$(\widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})})_{\text{Ran}(X)} \rightarrow (\widehat{\mathfrak{t}}_{\kappa'+\text{shift}}\text{-mod}^{T(\mathcal{O})})_{\text{Ran}(X)}.$$

7.4.4. From now on, until the end of this subsection we will assume that  $\kappa'$  is integral. In Sect. 8 we will describe two more factorization algebras in  $\overline{\mathcal{C}}_{\kappa'}^{T(\mathcal{O})}$ , denoted

$$J_{\kappa',0,!} \text{ and } J_{\kappa',0,!*},$$

respectively. We let

$$(J_{\kappa',0,!})_{\text{Ran}(X)} \text{ and } (J_{\kappa',0,!*})_{\text{Ran}(X)}$$

denote the resulting objects of  $\overline{\mathcal{C}}_{\kappa',\text{Ran}(X)}^{T(\mathcal{O})}$ .

Remark 7.7. — Unlike  $J_{\kappa',0,*}$ , the objects  $J_{\kappa',0,!}$  and  $J_{\kappa',0,!*}$  do not belong to the image of the functor  $\mathcal{C}_{\kappa'}^{T(\mathcal{O})} \rightarrow \overline{\mathcal{C}}_{\kappa'}^{T(\mathcal{O})}$ .

7.4.5. We shall denote the resulting (factorizable) functors  $\widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G^{(0)}} \rightarrow \widehat{\mathfrak{t}}_{\kappa'+\text{shift}}\text{-mod}^{T^{(0)}}$

$$\text{BRST}_{\mathfrak{n}}^{\text{conv}}(J_{\kappa',0,!}, -) \text{ and } \text{BRST}_{\mathfrak{n}}^{\text{conv}}(J_{\kappa',0,!*}, -)$$

by  $\text{BRST}_{\mathfrak{n},!}$  and  $\text{BRST}_{\mathfrak{n},!*}$ , respectively.

We have the following counterparts of Quasi-Theorem 7.6:

QUASI-THEOREM 7.8. — *Let  $\kappa'$  be negative. Then the (factorizable) functor*

$$\text{Inv}_{\mathfrak{n}(\mathcal{X}),!} : \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G^{(0)}} \rightarrow \widehat{\mathfrak{t}}_{\kappa'+\text{shift}}\text{-mod}^{T^{(0)}}$$

of Sect. 6.5.2 identifies canonically with the (factorizable) functor

$$\text{BRST}_{\mathfrak{n},!} : \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G^{(0)}} \rightarrow \widehat{\mathfrak{t}}_{\kappa'+\text{shift}}\text{-mod}^{T^{(0)}}.$$

QUASI-THEOREM 7.9. — *Let  $\kappa'$  be negative. Then the (factorizable) functor*

$$\text{Inv}_{\mathfrak{n}(\mathcal{X}),!*} : \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G^{(0)}} \rightarrow \widehat{\mathfrak{t}}_{\kappa'+\text{shift}}\text{-mod}^{T^{(0)}}$$

of Sect. 6.4.3 identifies canonically with the (factorizable) functor

$$\text{BRST}_{\mathfrak{n},!*} : \widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G^{(0)}} \rightarrow \widehat{\mathfrak{t}}_{\kappa'+\text{shift}}\text{-mod}^{T^{(0)}}.$$

7.4.6. Let  $(\text{BRST}_{\mathfrak{n},!})_{\text{Ran}(X)}$  and  $(\text{BRST}_{\mathfrak{n},!*})_{\text{Ran}(X)}$  denote the resulting functors

$$(\widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G^{(0)}})_{\text{Ran}(X)} \rightarrow (\widehat{\mathfrak{t}}_{\kappa'+\text{shift}}\text{-mod}^{T^{(0)}})_{\text{Ran}(X)}.$$

In view of Quasi-Theorem 7.9, we can reformulate Conjecture 6.5 as follows:

CONJECTURE 7.10. — *Let  $\kappa'$  be a negative integral level. Then the following diagram of functors commutes*

$$\begin{array}{ccc} (\widehat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G^{(0)}})_{\text{Ran}(X)} & \xrightarrow{(\text{BRST}_{\mathfrak{n},!*})_{\text{Ran}(X)}} & (\widehat{\mathfrak{t}}_{\kappa'+\text{shift}}\text{-mod}^{T^{(0)}})_{\text{Ran}(X)} \\ \text{Loc}_{G,\kappa',\text{Ran}(X)} \downarrow & & \downarrow \text{Loc}_{T,\kappa'+\text{shift},\text{Ran}(X)} \\ \text{D} - \text{mod}_{\kappa'}(\text{Bun}_G)_{\text{co}} & \xrightarrow{\text{CT}_{\kappa'+\text{shift},!*}} & \text{D} - \text{mod}_{\kappa'+\text{shift}}(\text{Bun}_T). \end{array}$$

In a similar way, we can use Quasi-Theorems 7.6 and Quasi-Theorem 7.8 to reformulate Conjecture 6.7 by substituting

$$((\text{Inv}_{\mathfrak{n}(\mathcal{X}),*})_{\text{Ran}(X)} \text{ and } ((\text{Inv}_{\mathfrak{n}(\mathcal{X}),!})_{\text{Ran}(X)})$$

by

$$(\text{BRST}_{\mathfrak{n},*})_{\text{Ran}(X)} \text{ and } (\text{BRST}_{\mathfrak{n},!})_{\text{Ran}(X)},$$

respectively.

*Remark 7.11.* — It will follow from the construction of the objects  $J_{\kappa',0,!}$  and  $J_{\kappa',0,!*}$  that we have the following canonical maps of factorization algebras

$$J_{\kappa',0,!} \rightarrow J_{\kappa',0,!*} \rightarrow J_{\kappa',0,*}. \quad (7.5)$$

In terms of the isomorphisms of Quasi-Theorems 7.6, 7.8 and 7.9, these maps correspond to the natural transformations

$$\mathrm{Inv}_{\mathfrak{n}(\mathcal{K}),!} \rightarrow \mathrm{Inv}_{\mathfrak{n}(\mathcal{K}),!*} \rightarrow \mathrm{Inv}_{\mathfrak{n}(\mathcal{K}),*}.$$

## 7.5. The !-extension

The contents of this subsection will not be used in the sequel.

The general construction of the object  $J_{\kappa',0,!}$  will be explained in Sect. 8. Here we indicate an alternative construction (that works for any  $\kappa$ , i.e., one that is not necessarily integral). Specifically, we will describe the object  $(J_{\kappa',0,!})_{\mathrm{Ran}(X)}$ .

7.5.1. Consider the functor

$$\begin{aligned} (\overline{\mathcal{C}}_{\kappa',\mathrm{Ran}(X)}^{T(\vartheta)})^{\leq 0} / (\overline{\mathcal{C}}_{\kappa',\mathrm{Ran}(X)}^{T(\vartheta)})^{< 0} &\simeq (\mathcal{C}_{\kappa',\mathrm{Ran}(X)}^{T(\vartheta)})^{\leq 0} / (\mathcal{C}_{\kappa',\mathrm{Ran}(X)}^{T(\vartheta)})^{< 0} \\ &\simeq \mathrm{D} - \mathrm{mod}(\mathrm{Ran}(X)). \end{aligned} \quad (7.6)$$

We have:

LEMMA 7.12. — *The functor in (7.6) admits a left adjoint.*

*Remark 7.13.* — The existence of the left adjoint in Lemma 7.12 would be *false* if we worked with the uncompleted category  $\mathcal{C}_{\kappa',\mathrm{Ran}(X)}^{T(\vartheta)}$  instead of  $\overline{\mathcal{C}}_{\kappa',\mathrm{Ran}(X)}^{T(\vartheta)}$ .

7.5.2. Now, we claim that the object  $(J_{\kappa',0,!})_{\mathrm{Ran}(X)}$  is the value of the above left adjoint on

$$\omega_{\mathrm{Ran}(X)} \simeq \mathbf{1}_{\mathrm{Ran}(X)} \in \mathrm{D} - \mathrm{mod}(\mathrm{Ran}(X)).$$

## 8. The IC object on the semi-infinite flag space

In this section we will give the construction of the objects  $J_{\kappa',0,!}$  and  $J_{\kappa',0,!*}$  in  $\overline{\mathcal{C}}_{\kappa'}^{T(\vartheta)}$  for an integral level  $\kappa'$ .

Since  $\kappa'$  is assumed integral, the category  $\overline{\mathcal{C}}_{\kappa'}^{T(\vartheta)}$  is equivalent to  $\overline{\mathcal{C}}^{T(\vartheta)}$  (see Sect. 7.1.4), so we will consider the latter.

## 8.1. The spherical Hecke category for $T$

8.1.1. We consider the affine Grassmannian  $\mathrm{Gr}_T$  of the group  $T$ , and the category

$$\mathrm{Sph}_T := \mathrm{D} - \mathrm{mod}(\mathrm{Gr}_T)^{T(0)}.$$

This category acquires a monoidal structure given by convolution, and a compatible structure of factorization category.

We let  $(\mathrm{Sph}_T)_{\mathrm{Ran}(X)}$  denote the corresponding category over the Ran space.

*Remark 8.1.*— The *derived* geometric Satake equivalence gives a description of this category in terms of the Langlands dual torus, see [4, Theorem 12.5.3].

8.1.2. Consider now the category  $\mathrm{D} - \mathrm{mod}(\mathrm{Gr}_T)$ . Note that it identifies canonically with  $\mathrm{Vect}^\Lambda$ ; this identification is the *naive* (i.e., non-derived) geometric Satake for the group  $T$ .

The corresponding category over the Ran space  $(\mathrm{D} - \mathrm{mod}(\mathrm{Gr}_T))_{\mathrm{Ran}(X)}$  identifies canonically with  $\mathrm{D} - \mathrm{mod}(\mathrm{Ran}(X, \Lambda))$ .

We have the natural forgetful functors

$$f : \mathrm{Sph}_T \rightarrow \mathrm{D} - \mathrm{mod}(\mathrm{Gr}_T)$$

$$\text{and } f_{\mathrm{Ran}(X)} : (\mathrm{Sph}_T)_{\mathrm{Ran}(X)} \rightarrow (\mathrm{D} - \mathrm{mod}(\mathrm{Gr}_T))_{\mathrm{Ran}(X)}. \quad (8.1)$$

In particular, it makes sense to talk about objects of  $\mathrm{Sph}_T$  (resp.,  $(\mathrm{Sph}_T)_{\mathrm{Ran}(X)}$ ) supported over  $\Lambda^{\mathrm{neg}}$  (resp.,  $\mathrm{Ran}(X, \Lambda)^{\mathrm{neg}}$ ), see Sect. 4.1.2 for the notation. We denote the corresponding full subcategory by  $\mathrm{Sph}_T^{\mathrm{neg}}$  (resp.,  $(\mathrm{Sph}_T)_{\mathrm{Ran}(X)}^{\mathrm{neg}}$ ).

8.1.3. Since the group  $T$  is commutative, the category  $\mathrm{D} - \mathrm{mod}(\mathrm{Gr}_T)$  itself has a natural (symmetric) monoidal structure, and we have a naturally defined monoidal functor

$$g : \mathrm{D} - \mathrm{mod}(\mathrm{Gr}_T) \rightarrow \mathrm{Sph}_T, \quad (8.2)$$

compatible with the factorization structures.

The functor  $g$  is a right inverse of the functor  $f$  (but note that the latter does *not* have a natural monoidal structure).

## 8.2. The Hecke action on the semi-infinite flag space

8.2.1. Since the category  $\overline{\mathcal{C}}^{T(\mathcal{O})}$  is obtained by taking  $T(\mathcal{O})$ -invariants in the category  $\overline{\mathcal{C}}$  acted on by  $T(\mathcal{K})$ , the category  $\mathrm{Sph}_T$  naturally acts on  $\overline{\mathcal{C}}^{T(\mathcal{O})}$  by convolution.

We denote this action by

$$\mathcal{S}, \mathcal{T} \mapsto \mathcal{S} * \mathcal{T}.$$

8.2.2. Recall the object

$$j_{0,*} \in \overline{\mathcal{C}}^{T(\mathcal{O})}.$$

Let  $\mathcal{A} \in \mathrm{Sph}_T$  be the *universal* algebra object that acts on  $j_{0,*}$ . In particular, we have a canonical action map

$$\mathcal{A} * j_{0,*} \rightarrow \mathcal{A}.$$

The object  $\mathcal{A}$  has a natural structure of factorization algebra; we denote by  $\mathcal{A}_{\mathrm{Ran}(X)}$  the corresponding object in  $(\mathrm{Sph}_T)_{\mathrm{Ran}(X)}$ .

8.2.3. It is easy to see that  $\mathcal{A}$  is naturally augmented and its augmentation ideal  $\mathcal{A}^+$  (resp.,  $\mathcal{A}_{\mathrm{Ran}(X)}^+$ ) belongs in fact to  $\mathrm{Sph}_T^{\mathrm{neg}}$  (resp.,  $(\mathrm{Sph}_T)_{\mathrm{Ran}(X)}^{\mathrm{neg}}$ ).

*Remark 8.2.* — The object  $\mathcal{A}_{\mathrm{Ran}(X)}^+$  was introduced in [19, Sect. 6.1.2] under the name  $\widetilde{\Omega}(\check{\mathfrak{n}})$ , and a description of this object is given in *loc.cit.*, Conjecture 10.3.4 in terms of the geometric Satake equivalence. Proving this description is work-in-progress by S. Raskin.

8.2.4. *Construction of the !-extension.* We are finally able to define the sought-for object

$$j_{0,!} \in \overline{\mathcal{C}}^{T(\mathcal{O})}.$$

Namely, it is defined to be

$$\mathrm{coBar}(\mathcal{A}^+, j_{0,*}),$$

where  $\mathrm{coBar}$  stands for the co-Bar construction for  $\mathcal{A}^+$  (i.e., the co-Bar construction for  $\mathcal{A}$  relative to its augmentation).

Note that  $\mathrm{coBar}(\mathcal{A}^+, -)$  involves the procedure of taking the (inverse) limit. Now, one shows that this inverse limit is equivalent to one over a finite index category when projected to each  $\overline{\mathcal{C}}^{T(\mathcal{O})} / (\overline{\mathcal{C}}^{T(\mathcal{O})})_{\leq -\lambda}$ , and hence gives rise to a well-defined object of  $\overline{\mathcal{C}}^{T(\mathcal{O})}$ .

*Remark 8.3.* — If we worked with  $\mathcal{C}$  instead of  $\overline{\mathcal{C}}$ , the inverse limit involved in the definition of  $\mathrm{coBar}(\mathcal{A}^+, -)$  would be something unmanageable.

### 8.3. Definition of the IC object

8.3.1. Consider  $f_{\text{Ran}(X)}(\mathcal{A}_{\text{Ran}(X)}^+)$  as an object of  $\mathbf{D} - \text{mod}(\text{Ran}(X, \Lambda)^{\text{neg}})$ .

It follows from [19, Sect. 6.1] that it belongs to  $\mathbf{D} - \text{mod}(\text{Ran}(X, \Lambda)^{\text{neg}})^{\geq 0}$ , with respect to the natural  $t$ -structure; moreover

$$\mathcal{A}_{0, \text{Ran}(X)}^+ := \tau^{\leq 0}(f(\mathcal{A}_{\text{Ran}(X)}^+))$$

is the object in  $\mathbf{D} - \text{mod}(\text{Ran}(X, \Lambda)^{\text{neg}})$  associated to a canonically defined factorization algebra

$$\mathcal{A}_0^+ \in \mathbf{D} - \text{mod}(\text{Gr}_T).$$

Furthermore,  $\mathcal{A}_0^+$  is the augmentation ideal of a canonically defined (commutative) algebra object  $\mathcal{A}_0$  in the (symmetric) monoidal category  $\mathbf{D} - \text{mod}(\text{Gr}_T)$ .

Finally, the map  $\mathcal{A}_0^+ \rightarrow f(\mathcal{A}^+)$  canonically comes from a homomorphism of algebras

$$\mathfrak{g}(\mathcal{A}_0^+) \rightarrow \mathcal{A}^+,$$

compatible with the factorization structures.

*Remark 8.4.* — The object

$$(\mathcal{A}_0^+)_{\text{Ran}(X)} \in \mathbf{D} - \text{mod}(\text{Ran}(X, \Lambda)^{\text{neg}}) \subset \mathbf{D} - \text{mod}(\text{Ran}(X, \Lambda))$$

in fact identifies canonically with the object  $\Omega^{\text{Lus}}$  for the Langlands dual group  $\check{G}$  and the critical level for  $\check{G}$  (so that the corresponding twisting on  $\text{Ran}(X, \Lambda)^{\text{neg}}$  is trivial); see [8, Sects. 3 and 4].

8.3.2. We are finally able to define the object  $j_{0,!*} \in \overline{\mathcal{C}}^{T(0)}$ . Namely, it is defined to be

$$\text{coBar}(\mathcal{A}_0^+, j_{0,*}).$$

*Remark 8.5.* — Note that it follows from the construction that we have the canonical maps

$$j_{0,!} \rightarrow j_{0,!*} \rightarrow j_{0,*},$$

and hence the maps

$$(j_{0,!})_{\text{Ran}(X)} \rightarrow (j_{0,!*})_{\text{Ran}(X)} \rightarrow (j_{0,*})_{\text{Ran}(X)}, \quad (8.3)$$

as promised in Remark 7.11.

*Remark 8.6.* — The object  $j_{0,!*}$  plays the following role: the category of factorization modules over  $(j_{0,!*})_{\text{Ran}(X)}$  in  $\overline{\mathcal{C}}^{T(0)}$  is closely related to the version of the category of  $D$ -modules on the semi-infinite flag manifold, expected by Feigin-Frenkel, and whose global incarnation was the subject of [14].

When we consider this situation at the critical level, the above category is related by a localization functor to the category of Kac-Moody representations at the critical level.

*Remark 8.7.* — Note that Quasi-Theorems 7.8 and 7.9 imply that the functor  $\mathrm{Inv}_{\mathfrak{n}(\mathcal{K}),!}$  can be expressed through the functor  $\mathrm{Inv}_{\mathfrak{n}(\mathcal{K}),!*}$  via the factorization algebra  $\mathcal{A}_0$ .

Let us observe that this is natural from the point of view of quantum groups. Indeed, according to Remark 8.4, the factorization algebra  $\mathcal{A}_0$  encodes the Chevalley complex of  $\mathfrak{n}$ . Now, the precise statement at the level of quantum groups is that for  $\mathcal{M} \in \mathfrak{U}_q(G)\text{-mod}$ , the object

$$\mathrm{Inv}_{\mathfrak{u}_q(N^+)}(\mathcal{M})$$

carries an action of  $U(\mathfrak{n})$  via the quantum Frobenius, and

$$\mathrm{Inv}_{\mathfrak{U}_q^{\mathrm{Lus}}(N^+)}(\mathcal{M}) \simeq \mathrm{Inv}_{U(\mathfrak{n})}(\mathrm{Inv}_{\mathfrak{u}_q(N^+)}(\mathcal{M})).$$

## 9. The semi-infinite flag space vs. Drinfeld’s compactification

Our goal in this section is to deduce Conjecture 7.10 from another statement, Quasi-Theorem 9.7.

### 9.1. The local-to-global map (case of $G/N$ )

*9.1.1.* Consider the stack  $\overline{\mathrm{Bun}}_N$ . We let  $\mathrm{D} - \mathrm{mod}_{\kappa}(\overline{\mathrm{Bun}}_N)$  the category of twisted D-modules on it, where the twisting is the pullback from one on  $\mathrm{Bun}_G$  under the natural projection  $\overline{\mathrm{Bun}}_N \rightarrow \mathrm{Bun}_G$ .

For a point  $x \in X$  we have a naturally defined map

$$\phi_x : ((\mathrm{Gr}_G)_x)^{\leq 0} \rightarrow \overline{\mathrm{Bun}}_N$$

that remembers the reduction of our  $G$ -bundle to  $N$  on  $X - x$ .

Consider the functor

$$(\phi_x)_* : \mathrm{D} - \mathrm{mod}_{\kappa'}(((\mathrm{Gr}_G)_x)^{\leq 0}) \rightarrow \mathrm{D} - \mathrm{mod}_{\kappa}(\overline{\mathrm{Bun}}_N).$$

*Remark 9.1.* — The exchange of levels  $\kappa \mapsto \kappa'$  is due to the fact that we are thinking about  $(\mathrm{Gr}_G)_x$  as  $G(\mathcal{K}_x)/G(\mathcal{O}_x)$  (the quotient by  $G(\mathcal{O}_x)$  in the right), while  $\mathrm{Bun}_G$  is the quotient of  $\mathrm{Bun}_G^{\mathrm{level}_x}$  by  $G(\mathcal{O}_x)$  with the left action.

Indeed that according to [2], the  $\kappa$ -level on  $G(\mathcal{K})$  with respect to the left action corresponds to the level  $\kappa' := -\kappa - \kappa_{\mathrm{Kil}}$  with respect to the right action.

9.1.2. For a group-scheme  $N_i \subset N(\mathcal{K}_x)$  consider the composed functor

$$(\phi_x)_* \circ \text{Av}_*^{N_i} : \text{D} - \text{mod}_{\kappa'}(((\text{Gr}_G)_x)^{\leq 0}) \rightarrow \text{D} - \text{mod}_{\kappa}(\overline{\text{Bun}}_N).$$

These functors form an inverse family:

$$N_i \subset N_j \quad \rightsquigarrow \quad (\phi_x)_* \circ \text{Av}_*^{N_j} \rightarrow (\phi_x)_* \circ \text{Av}_*^{N_i}.$$

LEMMA 9.2. *For every compact object  $\mathcal{F} \in \text{D} - \text{mod}_{\kappa'}(((\text{Gr}_G)_x)^{\leq 0})$  the family*

$$i \mapsto (\phi_x)_* \circ \text{Av}_*^{N_i}(\mathcal{F}) \in \text{D} - \text{mod}_{\kappa}(\overline{\text{Bun}}_N)$$

*stabilizes.*

9.1.3. Hence, we obtain that the assignment

$$\mathcal{F} \in \text{D} - \text{mod}_{\kappa'}(((\text{Gr}_G)_x)^{\leq 0}) \text{ compact}$$

$$\rightsquigarrow \text{eventual value of } (\phi_x)_* \circ \text{Av}_*^{N_i}(\mathcal{F}) \in \text{D} - \text{mod}_{\kappa}(\overline{\text{Bun}}_N)$$

gives rise to a continuous  $N(\mathcal{K}_x)$ -invariant functor

$$\text{D} - \text{mod}_{\kappa'}(((\text{Gr}_G)_x)^{\leq 0}) \rightarrow \text{D} - \text{mod}_{\kappa}(\overline{\text{Bun}}_N),$$

i.e., a functor

$$(\mathcal{C}_{\kappa',x})^{\leq 0} \rightarrow \text{D} - \text{mod}_{\kappa}(\overline{\text{Bun}}_N). \tag{9.1}$$

The following results from the definitions:

LEMMA 9.3. — *The functor (9.1) canonically factors through a functor*

$$(\overline{\mathcal{C}}_{\kappa',x})^{\leq 0} \rightarrow \text{D} - \text{mod}_{\kappa}(\overline{\text{Bun}}_N).$$

9.1.4. We denote the resulting functor

$$(\overline{\mathcal{C}}_{\kappa',x})^{\leq 0} \rightarrow \text{D} - \text{mod}_{\kappa}(\overline{\text{Bun}}_N)$$

by  $\Phi_x$ .

By the same token we obtain a functor

$$\Phi_{\text{Ran}(X)} : (\overline{\mathcal{C}}_{\kappa',\text{Ran}(X)})^{\leq 0} \rightarrow \text{D} - \text{mod}_{\kappa}(\overline{\text{Bun}}_N).$$

## 9.2. The local-to-global map (case of $G/B$ )

We shall now discuss a variant of the functors  $\Phi_x$  and  $\Phi_{\text{Ran}(X)}$  above for  $\overline{\text{Bun}}_B$  instead of  $\overline{\text{Bun}}_N$ .

9.2.1. Note that for  $x \in X$  we have the following version of the map  $\phi_x$ :

$$\phi_x^{T(\mathcal{O}_x)} : ((\mathrm{Gr}_G)_x)^{\leq 0}/T(\mathcal{O}_x) \times_{\mathrm{pt}/T(\mathcal{O}_x)} \mathrm{Bun}_T \rightarrow \overline{\mathrm{Bun}}_B,$$

where the map  $\mathrm{Bun}_T \rightarrow \mathrm{pt}/T(\mathcal{O}_x)$  is given by restricting a  $T$ -bundle to the formal disc around the point  $x$ .

Recall the category  $\mathrm{D} - \mathrm{mod}_{\kappa, G/T}(\overline{\mathrm{Bun}}_B)$ , see Sect. 3.2. Repeating the construction of Sect. 9.1 we now obtain a functor

$$\Phi_x^{T(\mathcal{O}_x)} : (\overline{\mathcal{C}}_{\kappa', x}^{T(\mathcal{O}_x)})^{\leq 0} \rightarrow \mathrm{D} - \mathrm{mod}_{\kappa, G/T}(\overline{\mathrm{Bun}}_B),$$

and its Ran version

$$\Phi_{\mathrm{Ran}(X)}^{T(\mathcal{O})} : (\overline{\mathcal{C}}_{\kappa', \mathrm{Ran}(X)}^{T(\mathcal{O})})^{\leq 0} \rightarrow \mathrm{D} - \mathrm{mod}_{\kappa, G/T}(\overline{\mathrm{Bun}}_B).$$

9.2.2. The following is tautological:

LEMMA 9.4. —

$$\Phi_{\mathrm{Ran}(X)}^{T(\mathcal{O})}((\mathcal{J}_{\kappa', 0, *})_{\mathrm{Ran}(X)}) \simeq \mathcal{J}_{\kappa, *}(\omega_{\mathrm{Bun}_B}) \in \mathrm{D} - \mathrm{mod}_{\kappa, G/T}(\overline{\mathrm{Bun}}_B).$$

In addition, we have the following statement that essentially follows from [19, Sect. 6.1]:

PROPOSITION 9.5. — *Assume that  $\kappa$  is integral.*

(a) *There exists a canonical isomorphism*

$$\Phi_{\mathrm{Ran}(X)}^{T(\mathcal{O})}((\mathcal{J}_{\kappa', 0, !})_{\mathrm{Ran}(X)}) \simeq \mathcal{J}_{\kappa, !}(\omega_{\mathrm{Bun}_B}) \in \mathrm{D} - \mathrm{mod}_{\kappa, G/T}(\overline{\mathrm{Bun}}_B).$$

(b) *There exists a canonical isomorphism*

$$\Phi_{\mathrm{Ran}(X)}^{T(\mathcal{O})}((\mathcal{J}_{\kappa', 0, !*})_{\mathrm{Ran}(X)}) \simeq \mathcal{J}_{\kappa, !*}(\mathrm{IC}_{\mathrm{Bun}_B})[\dim(\mathrm{Bun}_B)] \in \mathrm{D} - \mathrm{mod}_{\kappa, G/T}(\overline{\mathrm{Bun}}_B).$$

*Remark 9.6.* One can show that the maps

$$(\mathcal{J}_{0, !})_{\mathrm{Ran}(X)} \rightarrow (\mathcal{J}_{0, !*})_{\mathrm{Ran}(X)} \rightarrow (\mathcal{J}_{0, *})_{\mathrm{Ran}(X)}$$

of (8.3) induce the natural maps

$$\mathcal{J}_{\kappa, !}(\omega_{\mathrm{Bun}_B}) \rightarrow \mathcal{J}_{\kappa, !*}(\mathrm{IC}_{\mathrm{Bun}_B})[\dim(\mathrm{Bun}_B)] \rightarrow \mathcal{J}_{\kappa, *}(\omega_{\mathrm{Bun}_B}).$$

### 9.3. Interaction of the BRST functor with localization

In the previous sections we have reduced Conjecture 6.1 (and hence Conjecture 1.4) to Conjecture 7.10 (and similarly for Conjecture 6.2).

In this subsection we will show how Conjecture 7.10 follows from a certain general statement, Quasi-Theorem 9.7, that describes the interaction of the functor  $\text{BRST}_{\mathfrak{n}}^{\text{conv}}$  with the localization functors  $\text{Loc}_G$  and  $\text{Loc}_T$ , respectively.

9.3.1. Namely, we claim:

QUASI-THEOREM 9.7. — *Let  $\kappa'$  be a negative level. Then the following diagram of functors commutes:*

$$\begin{array}{ccc}
 (\overline{\mathbb{C}}_{\kappa', \text{Ran}(X)}^{T(\emptyset)})^{\leq 0} \otimes_{\text{D-mod}(\text{Ran}(X))} (\widehat{\mathfrak{g}}_{\kappa'} - \text{mod}^{G(\emptyset)})_{\text{Ran}(X)} & \xrightarrow{\text{BRST}_{\mathfrak{n}}^{\text{conv}}} & (\widehat{\mathfrak{g}}_{\kappa', x_\infty} - \text{mod}^{T(\emptyset)})_{\text{Ran}(X)} \\
 \Phi_{\text{Ran}(X)}^{T(\emptyset)} \otimes_{\text{Loc}_{G, \kappa', \text{Ran}(X)}} \downarrow & & \downarrow \text{Loc}_{T, \kappa', \text{Ran}(X)} \\
 \text{D-mod}_{\kappa, G/T}(\overline{\text{Bun}}_B) \otimes \text{D-mod}_{\kappa'}(\text{Bun}_G)_{\text{co}} & & \text{D-mod}_{\kappa'}(\text{Bun}_T) \\
 \text{Id} \otimes \mathbb{P}^! \downarrow & & \uparrow \overline{\mathfrak{q}}_* \\
 \text{D-mod}_{\kappa, G/T}(\overline{\text{Bun}}_B) \otimes \text{D-mod}_{\kappa', G}(\overline{\text{Bun}}_B)_{\text{co}} & \xrightarrow{\overset{\circ}{\otimes}} & \text{D-mod}_{\kappa', T}(\overline{\text{Bun}}_B)_{\text{co}}.
 \end{array}$$

9.3.2. Let us show how Quasi-Theorem 9.7 implies Conjecture 7.10 (the situation with Conjecture 6.2 will be similar):

Let us evaluate the two circuits in the commutative diagram in Quasi-Theorem 9.7 on

$$(J_{\kappa', 0, !*})_{\text{Ran}(X)} \otimes M \in (\overline{\mathbb{C}}_{\kappa', \text{Ran}(X)}^{T(\emptyset)})^{\leq 0} \otimes_{\text{D-mod}(\text{Ran}(X))} (\widehat{\mathfrak{g}}_{\kappa'} - \text{mod}^{G(\emptyset)})_{\text{Ran}(X)}$$

for  $M \in (\widehat{\mathfrak{g}}_{\kappa'} - \text{mod}^{G(\emptyset)})_{\text{Ran}(X)}$ .

On the one hand, the clockwise circuit gives  $\text{Loc}_{G, \kappa', \text{Ran}(X)}(\text{BRST}_{\mathfrak{n}, !*}(M))$ , by the definition of  $\text{BRST}_{\mathfrak{n}, !*}$ .

On the other hand, applying Proposition 9.5(b), we obtain that the anti-clockwise circuit gives  $\text{CT}_{\kappa, !*}(\text{Loc}_{G, \kappa', \text{Ran}(X)}(M))$ , as required.

9.3.3. Note also that Quasi-Theorem 9.7, coupled with Remark 9.6, implies that the natural transformations

$$\text{Loc}_{G, \kappa', \text{Ran}(X)} \circ \text{BRST}_{\mathfrak{n}, !} \rightarrow \text{Loc}_{G, \kappa', \text{Ran}(X)} \circ \text{BRST}_{\mathfrak{n}, !*} \rightarrow \text{Loc}_{G, \kappa', \text{Ran}(X)} \circ \text{BRST}_{\mathfrak{n}, *}$$

that come from the maps (8.3) correspond to the natural transformations

$$\text{CT}_{\kappa, !} \circ \text{Loc}_{G, \kappa', \text{Ran}(X)} \rightarrow \text{CT}_{\kappa, !*} \circ \text{Loc}_{G, \kappa', \text{Ran}(X)} \rightarrow \text{CT}_{\kappa, *} \circ \text{Loc}_{G, \kappa', \text{Ran}(X)}$$

that come from the maps

$$J_{\kappa,!}(\omega_{\text{Bun}_B}) \rightarrow J_{\kappa,!*}(\text{IC}_{\text{Bun}_B})[\dim(\text{Bun}_B)] \rightarrow J_{\kappa,*}(\omega_{\text{Bun}_B}),$$

as expected (see Remarks 6.3 and 6.8).

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