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Distribution of zeroes of Rademacher Taylor series

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RÉSUMÉ. — Nous trouvons l'asymptotique de la fonction de comptage de zéros pour les fonctions entières aléatoires représentées par des séries de Taylor du type de Rademacher. Nous donnons aussi l'asymptotique pour la fonction de comptage à poids, qui prend en compte les arguments des zéros. Ces résultats répondent à certaines questions laissées ouvertes après le travail novateur de Littlewood et Offord en 1948.

Les preuves sont basées sur notre résultat récent sur l'intégrabilité logarithmique de séries de Fourier du type de Rademacher.

ABSTRACT. — We find the asymptotics of the counting function of zeroes of random entire functions represented by Rademacher Taylor series. We also give the asymptotics of the weighted counting function, which takes into account the arguments of zeroes. These results answer several questions left open after the pioneering work of Littlewood and Offord of 1948.

The proofs are based on our recent result on the logarithmic integrability of Rademacher Fourier series.

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1. Introduction and main results

In this work, we consider the zero distribution of random entire functions represented by the Rademacher Taylor series

$$F(z) = \sum_{k \geq 0} \xi_k a_k z^k,$$

where ξ_k are independent Rademacher (a.k.a. Bernoulli) random variables, which take the values ± 1 with probability $\frac{1}{2}$ each, and $\{a_k\}$ is a (non-random) sequence of complex numbers such that $\lim_k |a_k|^{1/k} = 0$ and $\#\{k: a_k \neq 0\} = \infty$.

1.1. Peculiarity of the Rademacher case. Rôle of the logarithmic integrability

Consider a more general class of random Taylor series with infinite radius of convergence:

$$F(z) = \sum_{k \geq 0} \chi_k a_k z^k,$$

in which the Rademacher random variables ξ_k are replaced with general independent identically distributed mean zero complex-valued random variables χ_k normalized by the condition $\mathcal{E}|\chi_k|^2 = 1$, and $\{a_k\}$ are as above. Let Z_F be the zero set of F (with multiplicities). Let us try to figure out how the asymptotics of the random counting function $n_F(r) = \#\{\zeta \in Z_F: |\zeta| \leq r\}$ should look as $r \rightarrow \infty$.

Put

$$\sigma_F(r)^2 = \mathcal{E}\{|F(z)|^2\} = \sum_{k \geq 0} |a_k|^2 r^{2k}.$$

To simplify the exposition, assume that $|a_0| = 1$. Denote by

$$N_F(r) = \int_0^r \frac{n_F(t)}{t} dt$$

the integrated counting function of the zero set Z_F . Then, by Jensen's formula,

$$\begin{aligned} N_F(r) &= \int_{-\pi}^{\pi} \log |F(re^{i\theta})| \frac{d\theta}{2\pi} - \log |F(0)| \\ &= \log \sigma_F(r) + \int_{-\pi}^{\pi} \log |\widehat{F}_r(\theta)| \frac{d\theta}{2\pi} - \log |\chi_0|, \end{aligned}$$

where $\widehat{F}_r(\theta) \stackrel{\text{def}}{=} F(re^{i\theta})/\sigma_F(r)$. Note that $\widehat{F}_r(\theta) = \sum_{k \geq 0} \chi_k \widehat{a}_k(r) e^{ik\theta}$ is a random Fourier series satisfying the condition $\sum_{k \geq 0} |\widehat{a}_k(r)|^2 = 1$.

First, assume that the χ_k 's are standard complex-valued Gaussian random variables. Then for every θ , the random variable $\widehat{F}_r(\theta)$ is again a standard complex-valued Gaussian random variable, and $\mathcal{E}|\log |\widehat{F}_r(\theta)||$ is a positive numerical constant. Therefore,

$$\sup_{r > 0} \mathcal{E}|N_F(r) - \log \sigma_F(r)| \leq C. \tag{1.1}$$

Since both $N_F(r)$ and $\log \sigma_F(r)$ are convex functions of $\log r$, we can derive from here that the functions

$$n_F(r) = \frac{dN_F(r)}{d \log r} \quad \text{and} \quad s_F(r) = \frac{d \log \sigma_F(r)}{d \log r} = \frac{\sum_{k \geq 1} k |a_k|^2 r^{2k}}{\sum_{k \geq 0} |a_k|^2 r^{2k}}$$

are also close for most values of r . If we are interested in the angular distribution of zeroes, the same idea works, we only need to replace Jensen's formula by its modification for angular sectors.

The same approach works in the Steinhaus case when $\chi_k = e^{2\pi i \gamma_k}$, where γ_k are independent and uniformly distributed on $[0, 1]$. In this case, one needs to estimate the expectation of the modulus of the logarithm of the absolute value of a normalized linear combination of independent Steinhaus variables. This was done by Offord in [11]; twenty years later, Ullrich [13, 14] and Favorov [2, 3] independently rediscovered his idea and applied it to various other problems. See also recent works by Mahola and Filevich [7, 8].

A linear combination of Rademacher random variables $x = \sum_k \xi_k a_k$ can vanish with positive probability. This leaves no hope to get a uniform lower bound for the logarithmic expectation $\mathcal{E}\{\log |x|\}$. In [6], Littlewood and Offord invented ingenious and formidable techniques to circumvent this obstacle. These techniques were further developed by Offord in [10, 12]. Apparently, the methods of these works were not sufficiently powerful to arrive at the same conclusions as for the Gaussian and the Steinhaus coefficients. Still, note that in order to estimate the error term in the Jensen formula we do not need to estimate $\mathcal{E}|\log |\widehat{F}_r(\theta)||$ uniformly in θ . Instead, we will be using the estimate

$$\mathcal{E}\left\{ \int_{-\pi}^{\pi} |\log |\widehat{F}_r(\theta)||^p \frac{d\theta}{2\pi} \right\} \leq (Cp)^{6p}, \quad p \geq 1, \tag{1.2}$$

proven in our recent work on the logarithmic integrability of Rademacher Fourier series [9, Corollary 1.2]. This will allow us to extend the results

known for the Gaussian and the Steinhaus coefficients to the Rademacher case.

Now, we describe the main results of this work. In what follows, we will use the notation $r\mathbb{D} = \{z: |z| < r\}$, $r\bar{\mathbb{D}} = \{z: |z| \leq r\}$, and $r\mathbb{T} = \{z: |z| = r\}$. By (Ω, \mathcal{P}) we always denote our probability space.

1.2. Asymptotics of the number of zeroes in disks of large radii

First, we address the asymptotics of the random counting function $n_F(r) = \#\{\zeta \in Z_F \cap r\bar{\mathbb{D}}\}$. Our asymptotics will hold when r tends to infinity outside an exceptional set $E \subset [1, \infty)$ of finite logarithmic length:

$$m_\ell(E) = \int_E \frac{dt}{t} < \infty.$$

Note that if the sequence $\{|a_k|\}$ is very irregular, the counting function $n_F(r)$ may exhibit a fast growth on short intervals, so the introduction of the set E is unavoidable.

THEOREM 1.1. — *There exists a set $E \subset [1, \infty)$ (depending on $|a_k|$ only) of finite logarithmic length such that*

(i) *for almost every $\omega \in \Omega$, there exists $r_0(\omega) \in [1, \infty)$ such that for every $r \in [r_0(\omega), \infty) \setminus E$ and every $\gamma > \frac{1}{2}$,*

$$|n_F(r) - s_F(r)| \leq C(\gamma)s_F(r)^\gamma;$$

(ii) *for every $r \in [1, \infty) \setminus E$, and every $\gamma > \frac{1}{2}$,*

$$\mathcal{E}|n_F(r) - s_F(r)| \leq C(\gamma)s_F(r)^\gamma.$$

1.3. Angular distribution of zeroes

To address the angular distribution of zeroes, we introduce the counting function

$$n_F(r, \varphi) = \sum_{\zeta \in (Z_F \setminus \{0\}) \cap r\bar{\mathbb{D}}} \varphi(\arg \zeta).$$

Here and below, φ is a 2π -periodic C^2 -function, $0 \leq \varphi \leq 1$.

In what follows, we denote by A_F various positive constants that may depend only on the sequence $\{|a_k|\}$ of the absolute values of the Taylor

coefficients of F . The symbol $\langle h \rangle$ will stand for the mean

$$\langle h \rangle = \int_{-\pi}^{\pi} h(\theta) \frac{d\theta}{2\pi}.$$

THEOREM 1.2. — *There exists a set $E \subset [1, \infty)$ (depending on $|a_k|$ only) of finite logarithmic length such that*

(i) *for almost every $\omega \in \Omega$, every $r \in [r_0(\omega), \infty) \setminus E$, every 2π -periodic C^2 -smooth function $\varphi: [-\pi, \pi] \rightarrow [0, 1]$, every $\gamma > \frac{1}{2}$, and every $q > 1$,*

$$|n_F(r, \varphi) - \mathcal{E}\{n_F(r, \varphi)\}| \leq C(\gamma, q) (1 + \|\varphi''\|_q) (s_F(r)^\gamma + \log^\gamma r + A_F);$$

(ii) *for every $r \in [1, \infty) \setminus E$, every 2π -periodic C^2 -smooth function $\varphi: [-\pi, \pi] \rightarrow [0, 1]$, every $\gamma > \frac{1}{2}$, and every $q > 1$,*

$$\mathcal{E}|n_F(r, \varphi) - \langle \varphi \rangle s_F(r)| \leq C(\gamma, q) (1 + \|\varphi''\|_q) (s_F(r)^\gamma + \log r + A_F).$$

Theorem 1.2 yields the angular equidistribution of zeroes of F provided that $s_F(r)$ does not grow too slowly:

$$\lim_{r \rightarrow \infty} \frac{s_F(r)}{\log r} = +\infty.$$

Taking into account that $\log \sigma_F(r)$ is a convex function of $\log r$, it is not difficult to see that this condition is equivalent to

$$\lim_{r \rightarrow \infty} \frac{\log \sigma_F(r)}{\log^2 r} = +\infty,$$

which in turn is equivalent to a more customary growth condition:

$$\lim_{r \rightarrow \infty} \frac{\log M_F(r)}{\log^2 r} = +\infty, \quad M_F(r) = \max_{r\bar{D}} |F|,$$

which often occurs in the theory of entire functions, cf. [4, Section 7.2].

It is also worth mentioning that the first statement of Theorem 1.2 remains meaningful as long as $s_F(r) > \log^\kappa r$ with some $\kappa > \frac{1}{2}$; i.e., beyond the $\log r$ -threshold.

1.4. Relation of our results to those by Littlewood and Offord

In [6], Littlewood and Offord studied the distribution of zeroes of random entire functions of finite positive order represented by Rademacher

Taylor series. They used the maximal term $\mu_F(r) = \max_{k \geq 0} (|a_k| r^k)$ of the Rademacher Taylor series F , which is basically equivalent to the quantity $\sigma_F(r)$ we are using here: obviously, $\mu_F(r) \leq \sigma_F(r)$ everywhere, while, for every $\gamma > \frac{1}{2}$, $\sigma_F(r) \leq \mu_F(r) \log^\gamma \mu_F(r)$ outside an exceptional set of r 's of finite logarithmic length (this is a classical result of Wiman and Valiron, see, for example, [4, Section 6.2]). Littlewood and Offord discovered that, for every $\varepsilon > 0$,

$$\log |F(re^{i\theta})| \geq \log \mu_F(r) - O_\varepsilon(r^\varepsilon) \tag{1.3}$$

everywhere in the complex plane outside a union of simply connected domains of small diameters. They called these domains “pits”. Littlewood and Offord provided a very detailed information about the sizes of the pits and the distribution of their locations. From this, they were able to obtain some upper and lower a.s. bounds for the random integrated counting functions $N_F(r)$ and $N_F(r, \varphi)$. However, these bounds differed by a positive constant factor and did not yield the leading term of the asymptotics.

Later, Offord [10, 12] extended the main results of [6] to random entire functions of positive or infinite order of growth represented by random Taylor series with more or less arbitrarily distributed sequence of independent random coefficients.

1.5. Regularly decaying sequences $\{|a_k|\}$

If the sequence of absolute values $\{|a_k|\}$ behaves very regularly:

$$|a_k| = (\Delta + o(1))^k e^{-\alpha k \log k}, \quad k \rightarrow \infty,$$

with some positive constants Δ and α , then combining (1.3) with some results from the Levin-Pfluger theory of entire functions of completely regular growth, one can obtain the leading term of the asymptotics provided by Theorems 1.1 and 1.2. It is also worth mentioning that recently Kabluchko and Zaporozhets [5, Corollary 2.6] found a new elegant approach to this special case, which is based on estimates for the concentration function combined with some tools from potential theory. Their approach works for a very general class of non-degenerate i.i.d. random variables χ_k (it needs only that $\mathcal{E}\{\log^+ |\chi_k|\} < \infty$). However, it seems that their approach should not work when $|a_k|$ does not have a very regular behavior.

Yet another approach was recently developed by Borichev, Nishry and Sodin in [1]. That approach works for certain correlated stationary sequences χ_k as well as for some pseudo-random sequences of arithmetic origin, but still requires a high regularity of the non-random sequence $\{|a_k|\}$.

1.6. Series with dominating central terms

We complete this introduction with a brief discussion of (deterministic) Taylor series

$$F(z) = \sum_{k \geq 0} \varepsilon_k a_k z^k, \quad \varepsilon_k \in \{\pm 1\}$$

in which each non-zero term dominates on some circumference centered at the origin, i.e., series such that for every k with $a_k \neq 0$, there exists $r_k > 0$ such that

$$|a_k| r_k^k > K \sum_{\ell: \ell \neq k} |a_\ell| r_k^\ell \quad \text{with some } K \geq 1.$$

Note that this condition does not depend on the choice of the signs ε_k , so the corresponding central term $\varepsilon_k a_k z^k$ dominates in all series simultaneously and, by Rouché's theorem,

$$n_F(r_k) = k \quad \text{regardless of } \{\varepsilon_\ell\}.$$

This can be used to check sharpness of our constructions.

1.6.1. First, we can give each power k a possibility to dominate, thus ensuring that each annulus $A_k = \{z: r_k < |z| < r_{k+1}\}$ contains exactly one zero of F . If K is sufficiently large, then the sum $\varepsilon_k a_k z^k + \varepsilon_{k+1} a_{k+1} z^{k+1}$ dominates the rest of the series in the whole annulus A_k except a small angle where the arguments of the two terms are nearly opposite. So we can guarantee that the argument of the unique zero of F in A_k is close to that of

$$-\frac{\varepsilon_k}{\varepsilon_{k+1}} \frac{a_k}{a_{k+1}}.$$

Since the first factor is just ± 1 , we can create almost as irregular angular distribution of arguments of zeroes as we want. For instance, if a_k 's are real, then all zeroes of F will be real as well. This does not contradict Theorem 1.2 because giving each index a possibility to dominate imposes a severe restriction on the growth of f and, thereby, on the growth of $n_F(r)$. It turns out that in this "totally irregular angular distribution case", we have $n_F(r)$ and $s_F(r)$ comparable to $\log r$, so the error term in part (ii) of Theorem 1.2 starts to exceed the main one.

1.6.2. Another possibility is to create a lacunary series

$$F(z) = \sum_{j \geq 0} \varepsilon_j a_j z^{\lambda_j}, \quad \varepsilon_j \in \{\pm 1\}$$

in which the positive integer indices $\{\lambda_j\}$, $\lambda_0 < \lambda_1 < \dots$, are sufficiently sparse. In this case, there are sharp jumps in the number of zeroes of F in

narrow annuli around the circumferences $C_j = \{z: |z| = \rho_j\}$ with radii given by

$$\rho_j^{\lambda_{j+1}-\lambda_j} = \frac{a_j}{a_{j+1}}$$

on which the subsequent non-zero terms of the series have equal absolute value. On the other hand, the function $s_F(r)$, being defined by a relatively nice formula, is necessarily rather smooth near the radii ρ_j , so it starts growing somewhat earlier and finishes growing somewhat later than $n_F(r)$. This creates large errors of opposite signs in the formula $n_F(r) \approx s_F(r)$ slightly to the left and slightly to the right of ρ_j , which shows that, in general, allowing an exceptional set E in Theorem 1.1 is inevitable.

2. Preliminaries

2.1. Notation

Throughout the paper we use the following notation:

- ◇ For a function $h: [-\pi, \pi] \rightarrow \mathbb{C}$, we write

$$\langle h \rangle = \int_{-\pi}^{\pi} h(\theta) \frac{d\theta}{2\pi} \quad \text{and} \quad \|h\|_q = \left(\int_{-\pi}^{\pi} |h(\theta)|^q \frac{d\theta}{2\pi} \right)^{\frac{1}{q}}.$$

- ◇ For a random variable Y with finite first moment, we write $\bar{Y} = Y - \mathcal{E}Y$.
- ◇ By F we denote a random entire function represented by a Rademacher Taylor series.
- ◇ We denote the variance of $F(z)$ by $\sigma_F(r)^2 = \mathcal{E}\{|F(z)|^2\}$, $r = |z|$, and put $\widehat{F}_r(\theta) = F(re^{i\theta})/\sigma_F(r)$.
- ◇ We often use the notation $X_r = X_r(\theta) = \log |\widehat{F}_r(\theta)|$.
- ◇ By Z_F we denote the zero set of F .
- ◇ By C, c we denote various positive numerical constants. Their values may change from line to line. If κ is a parameter, then $C(\kappa), c(\kappa)$ are positive expressions that depend only on κ .
- ◇ By A_F we denote various positive expressions that may depend only on the sequence $\{a_k\}$ of the absolute values of the Taylor coefficients of F .

2.2. Normalization

When proving Theorems 1.1 and 1.2 we assume that

$$F(z) = 1 + \sum_{k \geq 1} \xi_k a_k z^k$$

with

$$\sum_{k \geq 1} |a_k| \leq \frac{1}{2}$$

(as before, $\lim_k |a_k|^{1/k} = 0$, $\#\{k: a_k \neq 0\} = \infty$, and ξ_k are independent Rademacher random variables). To reduce the arbitrary Rademacher Taylor series F to this special form, first, we replace F by the function $F_1(z) = F(z)/(\xi_m a_m z^m)$, where m is the least index with $a_m \neq 0$. For this function, we have $n_{F_1}(r) = n_F(r) - m$, and $n_{F_1}(r, \varphi) = n_F(r, \varphi)$. Furthermore, $\log \sigma_{F_1}(r) = \log \sigma_F(r) - \log |a_m| - m \log r$, whence, $s_{F_1}(r) = s_F(r) - m$. Therefore, both assumptions and conclusions of Theorems 1.1 and 1.2 remain invariant under this normalization.

Then, we put $F_2(z) = F_1(A_F^{-1}z)$ with $A_F = \max\{2 \sum_{k \geq 1} |a_k|, 1\}$. This function already has the form we need, and both assumptions and conclusions of Theorems 1.1 and 1.2 remain invariant under the scaling $z \mapsto A_F^{-1}z$.

2.3. Main tools

Our main tool will be the following lemma:

LEMMA 2.1 (Log-integrability). — *For any $p \geq 1$ and $t > 0$,*

$$\mathcal{E} \|X_t\|_p^p \leq (Cp)^{6p}.$$

In particular, for $\lambda \geq 1$,

$$(\mathcal{P} \times m) \{(\omega, \theta) \in \Omega \times [-\pi, \pi]: |X_t(\theta)| > \lambda\} \leq C \exp(-c\lambda^{1/6}),$$

where m is the Lebesgue measure on $[-\pi, \pi]$.

The first statement of this lemma is our recent result from [9]. The second statement follows from the first one by Chebyshev's inequality.

Our second tool is a version of the classical Jensen formula. The standard version corresponds to the case $\varphi \equiv 1$.

LEMMA 2.2 (Jensen-type formula). — *Let F be an entire function with $F(0) \neq 0$. Then, for any 2π -periodic C^2 -function φ and every $R > 0$, we have*

$$\int_0^R \frac{n_F(t, \varphi)}{t} dt = \int_{-\pi}^{\pi} \varphi(\theta) [\log |F(Re^{i\theta})| - \log |F(0)|] \frac{d\theta}{2\pi} + \int_0^R \frac{dt}{t} \int_0^t \frac{ds}{s} \int_{-\pi}^{\pi} \varphi''(\theta) \log |F(se^{i\theta})| \frac{d\theta}{2\pi}.$$

Remark. — The repeated integral of the function

$$s \mapsto \int_{-\pi}^{\pi} \varphi''(\theta) \log |F(se^{i\theta})| \frac{d\theta}{2\pi}$$

on the RHS converges absolutely at $s = 0, t = 0$, since for $s \rightarrow 0$,

$$\int_{-\pi}^{\pi} \varphi''(\theta) \log |F(se^{i\theta})| \frac{d\theta}{2\pi} = \int_{-\pi}^{\pi} \varphi''(\theta) [\log |F(se^{i\theta})| - \log |F(0)|] \frac{d\theta}{2\pi} = O(s).$$

Proof of Lemma 2.2. — For C^2 -functions U, V on a bounded domain G with smooth boundary, Green’s identity states that

$$\iint_G (U \Delta V - V \Delta U) dA = \int_{\partial G} \left(U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n} \right) dS,$$

where A stands for the planar area measure and S for the length.

We set $U(r, \theta) = \frac{1}{2\pi} \log |F(re^{i\theta})|$ and $V(r, \theta) = \varphi(\theta) \log \frac{R}{r}$. These functions are not in C^2 , but their singularities can be handled by a standard device: first, we exclude from the disk $R\bar{\mathbb{D}}$ ε -neighbourhoods of zeroes of F and of the origin, then apply Green’s formula and let $\varepsilon \rightarrow 0$. The rest is a straightforward computation. \square

3. Proof of Theorem 1.1

3.1. By Jensen’s formula,

$$\begin{aligned} \int_{R_1}^{R_2} \frac{n_F(t)}{t} dt &= \int_{-\pi}^{\pi} [\log |F(R_2 e^{i\theta})| - \log |F(R_1 e^{i\theta})|] \frac{d\theta}{2\pi} \\ &= [\log \sigma_F(R_2) - \log \sigma_F(R_1)] + \int_{-\pi}^{\pi} [X_{R_2}(\theta) - X_{R_1}(\theta)] \frac{d\theta}{2\pi} \\ &= \int_{R_1}^{R_2} \frac{s_F(t)}{t} dt + \int_{-\pi}^{\pi} [X_{R_2}(\theta) - X_{R_1}(\theta)] \frac{d\theta}{2\pi}. \end{aligned}$$

We define the sequence $r_k \uparrow \infty$ so that $s_F(r_k) = k^2$ and put $\delta_k = k^{-1} \log^{-2} k$, $k \geq 2$. The set

$$E = [1, r_2 e^{\delta_2}] \cup \bigcup_{k \geq 3} [r_k e^{-\delta_{k-1}}, r_k e^{\delta_k}]$$

will be the exceptional set of finite logarithmic length. Note that we will be interested only in the intervals $[r_k, r_{k+1}]$ whose logarithmic length is not small: $\log(r_{k+1}/r_k) \geq 2\delta_k$. Otherwise, the whole interval $[r_k, r_{k+1}]$ is contained in the exceptional set E .

3.2. Given $r \in [1, \infty) \setminus E$, we choose k so that $r_k e^{\delta_k} \leq r \leq r_{k+1} e^{-\delta_k}$. Then

$$\begin{aligned} n_F(r) &\leq n_F(r_{k+1} e^{-\delta_k}) \leq \frac{1}{\delta_k} \int_{r_{k+1} e^{-\delta_k}}^{r_{k+1}} \frac{n_F(t)}{t} dt \\ &= \frac{1}{\delta_k} \int_{r_{k+1} e^{-\delta_k}}^{r_{k+1}} \frac{s_F(t)}{t} dt + \frac{1}{\delta_k} \int_{-\pi}^{\pi} [X_{r_{k+1}}(\theta) - X_{r_{k+1} e^{-\delta_k}}(\theta)] \frac{d\theta}{2\pi} \\ &\leq s_F(r_{k+1}) + \frac{1}{\delta_k} [\|X_{r_{k+1}}\|_1 + \|X_{r_{k+1} e^{-\delta_k}}\|_1]. \end{aligned}$$

Similarly,

$$n_F(r) \geq s_F(r_k) - \frac{1}{\delta_k} [\|X_{r_k}\|_1 + \|X_{r_k e^{\delta_k}}\|_1].$$

Combining these bounds and using the monotonicity of the function s_F , we get

$$\begin{aligned} |n_F(r) - s_F(r)| &\leq [s_F(r_{k+1}) - s_F(r_k)] \\ &\quad + \frac{1}{\delta_k} [\|X_{r_k}\|_1 + \|X_{r_k e^{\delta_k}}\|_1 + \|X_{r_{k+1}}\|_1 + \|X_{r_{k+1} e^{-\delta_k}}\|_1]. \end{aligned} \tag{3.1}$$

Since $s_F(r_k) = k^2$, we have $s_F(r_{k+1}) - s_F(r_k) = 2k + 1$. Applying Hölder's inequality and then Lemma 2.1, we see that, for any $r \geq 1$ and any $p < \infty$,

$$\mathcal{E}\{\|X_r\|_1^p\} \leq \mathcal{E}\{\|X_r\|_p^p\} \leq (Cp)^{6p},$$

whence

$$\mathcal{P}\{\|X_r\|_1 > t\} \leq t^{-p} \mathcal{E}\{\|X_r\|_1^p\} \leq (t^{-1} \cdot Cp^6)^p.$$

Letting $t = e \cdot Cp^6$ and $p = 2 \log k$, we get

$$\mathcal{P}\{\|X_r\|_1 > C \log^6 k\} \leq \frac{1}{k^2}.$$

Therefore, by the Borel-Cantelli lemma, for almost every $\omega \in \Omega$, there exists $k_0(\omega)$ such that, for $k \geq k_0(\omega)$,

$$\begin{aligned} &\frac{1}{\delta_k} [\|X_{r_k}\|_1 + \|X_{r_k e^{\delta_k}}\|_1 + \|X_{r_{k+1}}\|_1 + \|X_{r_{k+1} e^{-\delta_k}}\|_1] \\ &= k \log^2 k \cdot O(\log^6 k) = O(k \log^8 k). \end{aligned}$$

Hence,

$$|n_F(r) - s_F(r)| \leq O(k \log^8 k) = O_\gamma(s_F(r)^\gamma).$$

This proves the first part of Theorem 1.1.

3.3. The proof of the second part (that is, the estimate for $\mathcal{E}|n_F(r) - s_F(r)|$) is similar. Averaging the upper bound (3.1) and then using Lemma 2.1, we get

$$\begin{aligned} \mathcal{E}|n_F(r) - s_F(r)| &\leq [s_F(r_{k+1}) - s_F(r_k)] \\ &\quad + \frac{1}{\delta_k} \mathcal{E}[\|X_{r_k}\|_1 + \|X_{r_k e^{\delta_k}}\|_1 + \|X_{r_{k+1}}\|_1 + \|X_{r_{k+1} e^{-\delta_k}}\|_1] \\ &\leq 2k + 1 + Ck \log^2 k = O_\gamma(s_F(r)^\gamma). \end{aligned}$$

This completes the proof of Theorem 1.1 □

3.4. Remark on the notion of “smallness” of an exceptional set E

While the notion of smallness we used (finite logarithmic measure) is standard and convenient for most applications, the proof shows a bit more. Namely, our exceptional set E can be covered by intervals whose logarithmic lengths form a fixed decreasing sequence with a finite sum $((k \log^2 k)^{-1}$ in our case). Replacing the particular choice of parameters used in the proof of Theorem 1.1 by a free one, we can fix an arbitrary increasing convex sequence (λ_k) , $\lambda_1 > 1$, and take the points r_k so that $s_F(r_k) = \lambda_k$. Put

$$\delta_k = \frac{\log^6(k+1)}{\lambda_{k+1} - \lambda_k}.$$

Then, with probability 1, we get

$$|n_F(r) - s_F(r)| \leq C(\lambda_{k+1} - \lambda_k) \quad \text{for } r_k e^{\delta_k} \leq r \leq r_{k+1} e^{-\delta_k} \text{ and large enough } k.$$

Choosing various sequences λ_k , we get statements similar to Theorem 1.1 in which better control of the exceptional set E can be achieved at the cost of worse control of the error term. Note that since we cannot control the sequence r_k without any a priori knowledge about the growth of F , a result of this type is meaningful only when $\sum_k \delta_k < \infty$ (otherwise, the exceptional intervals $[r_k e^{-\delta_{k-1}}, r_k e^{\delta_k}]$ may cover the whole ray $[r_1, +\infty)$). This forces us to take λ_k of order k^2 at the very least. So, Theorem 1.1, as stated, is, in a sense, an extremal case.

Also note that the considerations of Section 1.6 show that each result of this type is essentially sharp up to a factor $\log^6(k+1)$ in the definition of δ_k , which comes from the Borel-Cantelli estimate.

4. Several lemmas

Here, we collect several lemmas needed for the proof of Theorem 1.2. The first lemma is a straightforward corollary to the Jensen-type formula given in Lemma 2.2.

LEMMA 4.1. — *Let F be a random entire function represented by Rademacher Taylor series with $F(0) \neq 0$. Then, for any 2π -periodic C^2 -function φ and every $0 < R_1 < R_2 < \infty$, we have*

$$\int_{R_1}^{R_2} \frac{\mathcal{E}n_F(t, \varphi)}{t} dt = \langle \varphi \rangle \int_{R_1}^{R_2} \frac{s_F(t)}{t} dt + \mathcal{E} \langle \varphi \cdot (X_{R_2} - X_{R_1}) \rangle + \int_{R_1}^{R_2} \frac{dt}{t} \int_0^t \frac{\mathcal{E} \langle \varphi'' X_s \rangle ds}{s}, \quad (4.1)$$

and

$$\int_{R_1}^{R_2} \frac{n_F(t, \varphi)}{t} dt = \int_{R_1}^{R_2} \frac{\mathcal{E}n_F(t, \varphi)}{t} dt + \langle \varphi \cdot (\bar{X}_{R_2} - \bar{X}_{R_1}) \rangle + \int_{R_1}^{R_2} \frac{dt}{t} \int_0^t \frac{\langle \varphi'' \bar{X}_s \rangle ds}{s}. \quad (4.2)$$

The next lemma gives an approximation of the Taylor series F by “the central group” of its terms. We recall that the maximal term and the central index of the Taylor series F are defined as

$$\mu_F(r) = \max_{k \geq 0} \{|a_k| r^k\} \quad \text{and} \quad \nu_F(r) = \max\{k: |a_k| r^k = \mu_F(r)\}.$$

LEMMA 4.2. — *Given $r \geq 1$ and $\tau > 0$, we write $\nu_- = \nu_F(re^{-\tau})$, $\nu_+ = \nu_F(re^\tau)$. Then*

$$\left| F(z) - \sum_{\nu_- \leq k \leq \nu_+} \xi_k a_k z^k \right| \leq \frac{2\sigma_F(r)}{e^\tau - 1}, \quad r = |z|.$$

Proof. — By the definition of the indices ν_\pm , we have

$$|a_k| r^k = |a_k| (re^{-\tau})^k e^{\tau k} \leq |a_{\nu_-}| (re^{-\tau})^{\nu_-} e^{\tau k} = |a_{\nu_-}| r^{\nu_-} e^{-\tau(\nu_- - k)}$$

for $0 \leq k \leq \nu_-$, and similarly, $|a_k| r^k \leq |a_{\nu_+}| r^{\nu_+} e^{-\tau(k - \nu_+)}$ for $k \geq \nu_+$. Therefore,

$$\begin{aligned} \left(\sum_{0 \leq k < \nu_-} + \sum_{k > \nu_+} \right) |a_k| r^k &\leq (|a_{\nu_-}| r^{\nu_-} + |a_{\nu_+}| r^{\nu_+}) \frac{1}{e^\tau - 1} \\ &\leq \frac{\sqrt{2}}{e^\tau - 1} (|a_{\nu_-}|^2 r^{2\nu_-} + |a_{\nu_+}|^2 r^{2\nu_+})^{\frac{1}{2}} \leq \frac{2\sigma_F(r)}{e^\tau - 1} \end{aligned}$$

proving the lemma. \square

Our last lemma is a simple application of the Borel-Cantelli Lemma.

LEMMA 4.3. — *Let Y_k be a sequence of random variables such that, for every $p \geq p_0$,*

$$\mathcal{E} |Y_k|^p \leq (G_k(p))^p, \tag{4.3}$$

where $p \mapsto G_k(p)$ is a sequence of increasing functions on $[1, \infty)$. Then, almost surely,

$$\limsup_{k \rightarrow \infty} \frac{|Y_k|}{G_k(\log k)} \leq e.$$

Proof. — Let $\eta > 1, t > 0$. By Chebyshev's inequality,

$$\mathcal{P}\{|Y_k| > t\} \leq \frac{\mathcal{E} |Y_k|^p}{t^p} \leq \left(\frac{G_k(p)}{t} \right)^p.$$

Choosing $t = e^\eta G_k(p)$ and $p = \log k$, we see that

$$\mathcal{P}\{|Y_k| > e^\eta G_k(\log k)\} \leq k^{-\eta}.$$

Then, by the Borel-Cantelli Lemma, almost surely,

$$\limsup_{k \rightarrow \infty} \frac{|Y_k|}{G_k(\log k)} \leq e^\eta.$$

Letting $\eta \rightarrow 1$, we get the result. \square

5. Proof of Theorem 1.2

The idea of the proof is similar to the one for Theorem 1.1: we need to find a sufficiently dense sequence of “interpolation points” r_k where $n_F(r_k, \varphi)$ is well approximated by $\langle \varphi \rangle_{S_F(r_k)}$. The proof of the almost sure bound is significantly more complicated since we have to control the error term

$$\int_1^r \frac{\langle \varphi'' \overline{X}_s \rangle ds}{s},$$

which requires a new idea when the value $S_F(r)$ is comparable to or less than $\log r$.

We start by introducing two sequences $r_k \uparrow \infty, k \geq 3$ and $\delta_k \downarrow 0, \sum_k \delta_k < \infty$, to be chosen later. The set

$$E = [1, r_3] \cup \bigcup_{k \geq 3} [r_k e^{-\delta_{k-1}}, r_k e^{\delta_k}]$$

will serve as our exceptional set of finite logarithmic length. Below, we always assume that $\log \frac{r_{k+1}}{r_k} > 2\delta_k$; otherwise, the whole interval $[r_k, r_{k+1}]$ is contained in the exceptional set E .

Till the end of the proof, we fix some $q_0 > 1$ and put $p_0 = \frac{q_0}{q_0 - 1}$.

5.1. Preliminary estimates

We use the following notation:

$$Q(t) = Q(t; \varphi) \stackrel{\text{def}}{=} \int_1^t \langle \varphi'' X_s \rangle \frac{ds}{s}, \quad \bar{Q}(t) = Q(t) - \mathcal{E}Q(t) = \int_1^t \langle \varphi'' \bar{X}_s \rangle \frac{ds}{s}.$$

The next two claims approximate the functions n_F and $\mathcal{E}n_F$ outside the exceptional set.

CLAIM 5.1. — *Suppose that $r \in [r_3, \infty] \setminus E$. Choose k so that $r_k e^{\delta_k} \leq r \leq r_{k+1} e^{-\delta_k}$. Then*

$$|n_F(r, \varphi) - \mathcal{E}n_F(r, \varphi)| \leq [s_F(r_{k+1}) - s_F(r_k)] + |\bar{Q}(r_k)| + |\bar{Q}(r_{k+1})| + \text{ET}_1 + \text{ET}_2,$$

where the error terms ET_1 and ET_2 are given by

$$\begin{aligned} \text{ET}_1 &= \frac{1}{\delta_k} [\|\bar{X}_{r_k}\|_1 + \|\bar{X}_{r_k e^{\delta_k}}\|_1 + \|\bar{X}_{r_{k+1} e^{-\delta_k}}\|_1 + \|\bar{X}_{r_{k+1}}\|_1] \\ &\quad + \|\varphi''\|_{q_0} \left(\int_{r_k}^{r_k e^{\delta_k}} + \int_{r_{k+1} e^{-\delta_k}}^{r_{k+1}} \right) \|\bar{X}_s\|_{p_0} \frac{ds}{s}, \end{aligned}$$

and

$$\text{ET}_2 = C \left(\frac{1}{\delta_k} + \|\varphi''\|_1 \right).$$

CLAIM 5.2. — *Under the assumptions of Claim 5.1, we have*

$$\mathcal{E}|n_F(r, \varphi) - \langle \varphi \rangle s_F(r)| \leq [s_F(r_{k+1}) - s_F(r_k)] + C(q_0) \|\varphi''\|_{q_0} \log r_{k+1} + \frac{C}{\delta_k}.$$

Proof of Claim 5.1. — By the monotonicity of the function $n_F(r, \varphi)$,

$$\begin{aligned} n_F(r, \varphi) &\leq n_F(r_{k+1}e^{-\delta_k}, \varphi) \leq \frac{1}{\delta_k} \int_{r_{k+1}e^{-\delta_k}}^{r_{k+1}} \frac{n_F(t, \varphi)}{t} dt \\ &\stackrel{(4.2)}{=} \frac{1}{\delta_k} \int_{r_{k+1}e^{-\delta_k}}^{r_{k+1}} \frac{\mathcal{E}n_F(t, \varphi)}{t} dt + \frac{1}{\delta_k} \left\langle \varphi \cdot \left(\overline{X}_{r_{k+1}} - \overline{X}_{r_{k+1}e^{-\delta_k}} \right) \right\rangle \\ &\quad + \frac{1}{\delta_k} \int_{r_{k+1}e^{-\delta_k}}^{r_{k+1}} \frac{dt}{t} \int_0^t \frac{ds}{s} \langle \varphi'' \overline{X}_s \rangle. \end{aligned}$$

Since $0 \leq \varphi \leq 1$, the second term on the RHS does not exceed

$$\frac{1}{\delta_k} \left(\|\overline{X}_{r_{k+1}}\|_1 + \|\overline{X}_{r_{k+1}e^{-\delta_k}}\|_1 \right).$$

The third term can be written as

$$\begin{aligned} &\left(\int_0^1 + \int_1^{r_{k+1}} \right) \langle \varphi'' \overline{X}_s \rangle \frac{ds}{s} - \frac{1}{\delta_k} \int_{r_{k+1}e^{-\delta_k}}^{r_{k+1}} \frac{dt}{t} \int_t^{r_{k+1}} \langle \varphi'' \overline{X}_s \rangle \frac{ds}{s} \\ &\leq |\overline{Q}(r_{k+1})| + \int_0^1 \|\varphi'' \overline{X}_s\|_1 \frac{ds}{s} + \frac{1}{\delta_k} \int_{r_{k+1}e^{-\delta_k}}^{r_{k+1}} \frac{dt}{t} \int_t^{r_{k+1}} \|\varphi'' \overline{X}_s\|_1 \frac{ds}{s} \\ &\leq |\overline{Q}(r_{k+1})| + \|\varphi''\|_1 \int_0^1 \|\overline{X}_s\|_\infty \frac{ds}{s} + \|\varphi''\|_{q_0} \int_{r_{k+1}e^{-\delta_k}}^{r_{k+1}} \|\overline{X}_s\|_{p_0} \frac{ds}{s} \end{aligned}$$

Due to our normalization of F , for $|z| \leq 1$, we have $1 - \frac{1}{2}|z| \leq |F(z)| \leq 1 + \frac{1}{2}|z|$, whence, $-|z| \leq \log |F(z)| \leq \frac{1}{2}|z|$. We also have $1 \leq \sigma_F(r) \leq 1 + r$, whence, $0 \leq \log \sigma_F(r) \leq r$. Thus, for $r = |z| \leq 1$, we get $-2r \leq X_r = \log |F| - \log \sigma_F \leq \frac{1}{2}r$, whence, $|\overline{X}_r| \leq 2.5r$. Therefore,

$$\int_0^1 \|\overline{X}_s\|_\infty \frac{ds}{s} \leq 2.5.$$

Putting these estimates together, we obtain

$$\begin{aligned} n_F(r, \varphi) &\leq \frac{1}{\delta_k} \int_{r_{k+1}e^{-\delta_k}}^{r_{k+1}} \frac{\mathcal{E}n_F(t, \varphi)}{t} dt + |\overline{Q}(r_{k+1})| \tag{5.3} \\ &\quad + \frac{1}{\delta_k} \left(\|\overline{X}_{r_{k+1}}\|_1 + \|\overline{X}_{r_{k+1}e^{-\delta_k}}\|_1 \right) + \|\varphi''\|_{q_0} \int_{r_{k+1}e^{-\delta_k}}^{r_{k+1}} \|\overline{X}_s\|_{p_0} \frac{ds}{s} + 2.5\|\varphi''\|_1. \end{aligned}$$

Next,

$$n_F(r, \varphi) - \mathcal{E}n_F(r, \varphi) \leq \frac{1}{\delta_k} \int_{r_{k+1}e^{-\delta_k}}^{r_{k+1}} \frac{\mathcal{E}[n_F(t, \varphi) - n_F(r, \varphi)]}{t} dt + |\overline{Q}(r_{k+1})|$$

$$+ \frac{1}{\delta_k} \left(\|\overline{X}_{r_{k+1}}\|_1 + \|\overline{X}_{r_{k+1}e^{-\delta_k}}\|_1 \right) + \|\varphi''\|_{q_0} \int_{r_{k+1}e^{-\delta_k}}^{r_{k+1}} \|\overline{X}_s\|_{p_0} \frac{ds}{s} + 2.5\|\varphi''\|_1.$$

Since $0 \leq \varphi \leq 1$, we have for $t \geq r$, $n_F(t, \varphi) - n_F(r, \varphi) \leq n_F(t) - n_F(r)$. Therefore,

$$\begin{aligned} \frac{1}{\delta_k} \int_{r_{k+1}e^{-\delta_k}}^{r_{k+1}} \frac{\mathcal{E}[n_F(t, \varphi) - n_F(r, \varphi)]}{t} dt &\leq \frac{1}{\delta_k} \int_{r_{k+1}e^{-\delta_k}}^{r_{k+1}} \frac{\mathcal{E}[n_F(t) - n_F(r)]}{t} dt \\ &\stackrel{r \geq r_k e^{\delta_k}}{\leq} \frac{1}{\delta_k} \left[\int_{r_{k+1}e^{-\delta_k}}^{r_{k+1}} \frac{\mathcal{E}n_F(t)}{t} dt - \int_{r_k}^{r_k e^{\delta_k}} \frac{\mathcal{E}n_F(t)}{t} dt \right]. \end{aligned}$$

Applying relation (4.2) in Lemma 4.1 (with $\varphi = 1$) and then Lemma 2.1, we see that the RHS equals

$$\begin{aligned} &\frac{1}{\delta_k} \left[\int_{r_{k+1}e^{-\delta_k}}^{r_{k+1}} - \int_{r_k}^{r_k e^{\delta_k}} \right] \frac{s_F(t)}{t} dt + \frac{1}{\delta_k} \mathcal{E}[\langle X_{r_{k+1}} - X_{r_{k+1}e^{-\delta_k}} \rangle - \langle X_{r_k e^{\delta_k}} - X_{r_k} \rangle] \\ &\leq [s_F(r_{k+1}) - s_F(r_k)] + \frac{1}{\delta_k} \mathcal{E}[\|X_{r_{k+1}}\|_1 + \|X_{r_{k+1}e^{-\delta_k}}\|_1 + \|X_{r_k e^{\delta_k}}\|_1 + \|X_{r_k}\|_1] \\ &\leq [s_F(r_{k+1}) - s_F(r_k)] + \frac{C}{\delta_k}, \end{aligned}$$

whence

$$\begin{aligned} n_F(r, \varphi) - \mathcal{E}n_F(r, \varphi) &\leq [s_F(r_{k+1}) - s_F(r_k)] + |\overline{Q}(r_{k+1})| \\ &+ \frac{1}{\delta_k} \left(\|\overline{X}_{r_{k+1}}\|_1 + \|\overline{X}_{r_{k+1}e^{-\delta_k}}\|_1 \right) + \|\varphi''\|_{q_0} \int_{r_{k+1}e^{-\delta_k}}^{r_{k+1}} \|\overline{X}_s\|_{p_0} \frac{ds}{s} + 2.5\|\varphi''\|_1 \\ &+ \frac{C}{\delta_k}. \end{aligned}$$

The proof of the matching lower bound

$$\begin{aligned} n_F(r, \varphi) - \mathcal{E}n_F(r, \varphi) &\geq -[s_F(r_{k+1}) - s_F(r_k)] - |\overline{Q}(r_k)| \\ &- \frac{1}{\delta_k} (\|\overline{X}_{r_k}\|_1 + \|\overline{X}_{r_k e^{\delta_k}}\|_1) - \|\varphi''\|_{q_0} \int_{r_k}^{r_k e^{\delta_k}} \|\overline{X}_s\|_{p_0} \frac{ds}{s} - 2.5\|\varphi''\|_1 - \frac{C}{\delta_k} \end{aligned}$$

is very similar and we skip it. \square

Proof of Claim 5.2. — The proof is similar to the previous one. We estimate the first term on the RHS of the upper bound (5.3) applying Lemmas 4.1 and 2.1:

$$\frac{1}{\delta_k} \int_{r_{k+1}e^{-\delta_k}}^{r_{k+1}} \frac{\mathcal{E}n_F(t, \varphi)}{t} dt \leq \langle \varphi \rangle \frac{1}{\delta_k} \int_{r_{k+1}e^{-\delta_k}}^{r_{k+1}} \frac{s_F(t)}{t} dt$$

$$\begin{aligned}
 & + \frac{1}{\delta_k} \mathcal{E} [\|X_{r_{k+1}e^{-\delta_k}}\|_1 + \|X_{r_{k+1}}\|_1] + \frac{1}{\delta_k} \int_{r_{k+1}e^{-\delta_k}}^{r_{k+1}} \frac{dt}{t} \int_0^t \mathcal{E} \|\varphi'' X_s\|_1 \frac{ds}{s} \\
 & \leq \langle \varphi \rangle_{s_F(r_{k+1})} + \frac{C}{\delta_k} + \|\varphi''\|_{q_0} \left[\int_0^1 + \int_1^{r_{k+1}} \right] \mathcal{E} \|X_s\|_{p_0} \frac{ds}{s} \\
 & \leq \langle \varphi \rangle_{s_F(r_{k+1})} + \frac{C}{\delta_k} + C(q_0) \|\varphi''\|_{q_0} \log r_{k+1}.
 \end{aligned}$$

Plugging this estimate into (5.3), we obtain

$$\begin{aligned}
 n_F(r, \varphi) - \langle \varphi \rangle_{s_F(r)} & \leq [s_F(r_{k+1}) - s_F(r_k)] + \frac{C}{\delta_k} + \|\varphi''\|_{q_0} C(q_0) \log r_{k+1} \\
 & \quad + |\overline{Q}(r_{k+1})| + \frac{1}{\delta_k} \left(\|\overline{X}_{r_{k+1}}\|_1 + \|\overline{X}_{r_{k+1}e^{-\delta_k}}\|_1 \right) \\
 & \quad + \|\varphi''\|_{q_0} \int_{r_{k+1}e^{-\delta_k}}^{r_{k+1}} \|\overline{X}_s\|_{p_0} \frac{ds}{s} + 2.5 \|\varphi''\|_1.
 \end{aligned}$$

Combining with the matching lower bound and taking the expectation, we get

$$\mathcal{E} |n_F(r, \varphi) - \langle \varphi \rangle_{s_F(r)}| \leq [s_F(r_{k+1}) - s_F(r_k)] + \frac{C}{\delta_k} + C(q_0) \|\varphi''\|_{q_0} \log r_{k+1},$$

proving Claim 5.2. \square

5.2. Estimate of $\mathcal{E} |n_F(r, \varphi) - \langle \varphi \rangle_{s_F(r)}|$

Using Claim 5.2, we readily prove assertion (ii) of Theorem 1.2.

Proof. — We need to estimate the expression

$$[s_F(r_{k+1}) - s_F(r_k)] + C(q_0) \|\varphi''\|_{q_0} \log r_{k+1} + \frac{C}{\delta_k},$$

which appears on the RHS of the bound given in Claim 5.2. We choose the sequence r_k so that $s_F(r_k) + \log r_k = k^2$. Then

$$s_F(r_{k+1}) - s_F(r_k) \leq 3k \leq 3(s_F(r_k))^{\frac{1}{2}} + \log^{\frac{1}{2}} r_k < 3(s_F(r))^{\frac{1}{2}} + \log^{\frac{1}{2}} r,$$

and

$$\log r_{k+1} < 3k + \log r_k < 3s_F(r)^{\frac{1}{2}} + 4 \log r + 3.$$

Put $\delta_k = (k \log^2 k)^{-1}$. Then for $\gamma > \frac{1}{2}$, we have

$$\delta_k^{-1} = k \log^2 k < C(\gamma) (s_F(r_k))^\gamma + \log^\gamma r_k < C(\gamma) (s_F(r))^\gamma + \log r,$$

whence

$$\mathcal{E}|n_F(r, \varphi) - \langle \varphi \rangle_{s_F(r)}| < C(q_0, \gamma)(1 + \|\varphi''\|_{q_0})(s_F(r)^\gamma + \log r),$$

completing the proof. \square

Now, we start proving the more difficult part (i) of Theorem 1.2, that is, the almost sure estimate for $|n_F(r, \varphi) - \mathcal{E}n_F(r, \varphi)|$. For this, we need to estimate the RHS of the bound given in Claim 5.1.

5.3. Easy error terms

Here we give an almost sure bound for the error terms ET_1 in Claim 5.1. Recall that

$$\begin{aligned} \text{ET}_1 = \frac{1}{\delta_k} & \left(\|\overline{X}_{r_k}\|_1 + \|\overline{X}_{r_k e^{\delta_k}}\|_1 + \|\overline{X}_{r_{k+1} e^{-\delta_k}}\|_1 + \|\overline{X}_{r_{k+1}}\|_1 \right) \\ & + \|\varphi''\|_{q_0} \left(\int_{r_k}^{r_k e^{\delta_k}} + \int_{r_{k+1} e^{-\delta_k}}^{r_{k+1}} \right) \|\overline{X}_s\|_{p_0} \frac{ds}{s}. \end{aligned}$$

CLAIM 5.4. — *For almost every $\omega \in \Omega$, there exists $k_0 = k_0(\omega)$ such that*

$$\text{ET}_1 \leq \frac{C}{\delta_k} \log^6 k + C(q_0) \|\varphi''\|_{q_0} \quad \text{for all } k \geq k_0.$$

Proof. — Let ρ_k be one of the values $r_k, r_k e^{\delta_k}, r_{k+1} e^{-\delta_k}, r_{k+1}$. Then, by Lemma 2.1,

$$\mathcal{E}\|\overline{X}_{\rho_k}\|_1^p \leq 2^p \mathcal{E}\|X_{\rho_k}\|_p^p \leq (Cp)^{6p}.$$

Hence, by Lemma 4.3, for almost every $\omega \in \Omega$ and for every $k \geq k_0(\omega)$, $\|\overline{X}_{\rho_k}\|_1 \leq C \log^6 k$. Next, by Lemma 2.1,

$$\mathcal{E}\left\{ \left(\int_{r_k}^{r_k e^{\delta_k}} + \int_{r_{k+1} e^{-\delta_k}}^{r_{k+1}} \right) \|\overline{X}_s\|_{p_0} \frac{ds}{s} \right\} \leq C(q_0) \delta_k.$$

Recalling that $\sum_k \delta_k < \infty$ and applying Chebyshev's inequality and the Borel-Cantelli Lemma, we see that for almost every $\omega \in \Omega$, these two integrals do not exceed $C(q_0)$ for every $k \geq k_0(\omega)$. This proves the claim. \square

5.4. Crude estimate of the integral $\overline{Q}(r_{k+1})$

It remains to estimate the integral

$$\overline{Q} = \overline{Q}(r_{k+1}) = \int_1^{r_{k+1}} \frac{\langle \varphi'' \overline{X}_t \rangle}{t} dt.$$

We start with a crude bound.

CLAIM 5.5. — *For almost every $\omega \in \Omega$, there exists $k_0 = k_0(q_0, \omega)$ such that*

$$|\overline{Q}(r_{k+1})| \leq C \|\varphi''\|_{q_0} \log^6 k \cdot \log r_{k+1}, \quad k \geq k_0.$$

Proof. — For any $p \geq p_0$ we have $|\langle \varphi'' X_t \rangle| \leq \|\varphi''\|_{q_0} \|X_t\|_p$. Thus, applying first Hölder's inequality and then Lemma 2.1, we get

$$\mathcal{E} |\overline{Q}|^p \leq \|\varphi''\|_{q_0}^p \cdot \int_1^{r_{k+1}} \mathcal{E} \|X_t\|_p^p \frac{dt}{t} \cdot (\log r_{k+1})^{p-1} \leq \|\varphi''\|_{q_0}^p \cdot (Cp)^{6p} \cdot \log^p r_{k+1}.$$

Now Lemma 4.3 yields the required result. \square

5.5. Refined estimate of the integral $\overline{Q}(r_{k+1})$

Here, we will present a more delicate estimate for \overline{Q} , which refines the previous one. The idea is to partition the interval $[1, r_{k+1}]$ into intervals of equal logarithmic length τ_k with $1 \ll \tau_k \ll \log r_{k+1}$ and represent \overline{Q} as a sum of integrals over these intervals. It turns out that these integrals can be well approximated by independent bounded random variables with zero mean. Then the natural cancellation in their sum yields an improved bound for $\overline{Q}(r_{k+1})$.

Put $Z = Z(t) \stackrel{\text{def}}{=} \langle \varphi'' X_t \rangle$ and $\overline{Z} = Z - \mathcal{E}Z$. Then

$$\overline{Q} = \overline{Q}(r_{k+1}) = \int_1^{r_{k+1}} \frac{\overline{Z}(t)}{t} dt.$$

We are going to estimate $\mathcal{E} |\overline{Q}|^p$. This will be done in several steps.

5.5.1. Truncation of the logarithm. Fix k and $\Lambda = \Lambda(k, r_{k+1})$, and put

$$\log_\Lambda x = \begin{cases} \log x, & |\log x| \leq \Lambda^6, \\ -\Lambda^6, & \log x < -\Lambda^6, \\ \Lambda^6, & \log x > \Lambda^6, \end{cases}$$

and $Z_\Lambda(t) = \langle \varphi'' \log_\Lambda |\widehat{F}_t| \rangle$. Then,

$$|Z(t) - Z_\Lambda(t)| \leq \|\varphi''\|_{q_0} \cdot \|\log |\widehat{F}_t| - \log_\Lambda |\widehat{F}_t|\|_{p_0},$$

and, for $p \geq p_0$,

$$\begin{aligned} \mathcal{E}\{|Z(t) - Z_\Lambda(t)|^p\} &\leq \|\varphi''\|_{q_0}^p \mathcal{E}\{\|\log |\widehat{F}_t| - \log_\Lambda |\widehat{F}_t|\|_p^p\} \\ &\leq \|\varphi''\|_{q_0}^p \mathcal{E}\{\langle \mathbb{1}_{E_\Lambda(t)} |X_t|^p \rangle\}, \end{aligned}$$

where $E_\Lambda(t) = \{\theta \in [-\pi, \pi] : |X_t(\theta)| > \Lambda^6\}$, and $\mathbb{1}_{E_\Lambda(t)}$ is the indicator function of the set $E_\Lambda(t)$. Using the Cauchy-Schwarz inequality and Lemma 2.1, we get

$$\begin{aligned} \mathcal{E}\{|\overline{Z}(t) - \overline{Z}_\Lambda(t)|^p\} &\leq \mathcal{E}\{|Z(t) - Z_\Lambda(t)|^p\} \\ &\leq \|\varphi''\|_{q_0}^p \sqrt{\mathcal{E}\{m_\theta(E_\Lambda(t))\}} \cdot \mathcal{E}\|X_t\|_{2p}^{2p} \\ &\leq \|\varphi''\|_{q_0}^p \cdot C \exp(-c\Lambda) \cdot (Cp^6)^p, \end{aligned}$$

where $\overline{Z}_\Lambda = Z_\Lambda - \mathcal{E}Z_\Lambda$ and m_θ is Lebesgue measure on $[-\pi, \pi]$. Then

$$\begin{aligned} \mathcal{E}\{|\overline{Q} - \overline{Q}_\Lambda|^p\} &= \mathcal{E}\left\{\left|\int_1^{r_{k+1}} (\overline{Z}(t) - \overline{Z}_\Lambda(t)) \frac{dt}{t}\right|^p\right\} \\ &\leq (Cp^6 \|\varphi''\|_{q_0} \log r_{k+1})^p e^{-c\Lambda}. \end{aligned} \tag{5.6}$$

5.5.2. Replacing the Taylor series F by a group of its central terms. Let $\tau = \tau(k, r_{k+1})$ be a large parameter (to be chosen later). Let

$$\widehat{P}(z) \stackrel{\text{def}}{=} \frac{1}{\sigma_F(r)} \sum_{\ell=\nu_F(re^{-\tau})}^{\nu_F(re^\tau)} \xi_\ell a_\ell z^\ell, \quad r = |z|,$$

$\widehat{P}_r(\theta) = \widehat{P}(re^{i\theta})$, $\overline{Z}_\Lambda^{\text{c.t.}}(t) = \langle \varphi''(\log_\Lambda |\widehat{P}_t| - \mathcal{E} \log_\Lambda |\widehat{P}_t|) \rangle$, and

$$\overline{Q}_\Lambda^{\text{c.t.}} = \overline{Q}_\Lambda^{\text{c.t.}}(r_{k+1}) = \int_1^{r_{k+1}} \overline{Z}_\Lambda^{\text{c.t.}}(t) \frac{dt}{t}.$$

As before, $\nu_F(r)$ denotes the central index of the Taylor series F .

Applying Lemma 4.2 and using the fact that

$$|\log_\Lambda x - \log_\Lambda y| \leq e^{\Lambda^6} |x - y| \quad x, y > 0,$$

we get

$$\sup_{t \geq 1} |\overline{Z}_\Lambda(t) - \overline{Z}_\Lambda^{\text{c.t.}}(t)| \leq C e^{\Lambda^6 - \tau} \|\varphi''\|_1,$$

whence, for every $\omega \in \Omega$,

$$|\overline{Q}_\Lambda - \overline{Q}_\Lambda^{\text{c.t.}}| = \left| \int_1^{r_{k+1}} \left(\overline{Z}_\Lambda(t) - \overline{Z}_\Lambda^{\text{c.t.}}(t) \right) \frac{dt}{t} \right| \leq C \|\varphi''\|_1 \exp(\Lambda^6 - \tau) \cdot \log r_{k+1}. \quad (5.7)$$

5.5.3. Fast and slow intervals. From now on, we assume that $1 \ll \tau \ll \log r_{k+1}$ and that $L \stackrel{\text{def}}{=} \tau^{-1} \log r_{k+1}$ is an integer. For any integer j , consider the intervals $J_j = [e^{j\tau}, e^{(j+1)\tau}]$ of equal logarithmic length τ . We call the interval J_j taken from this collection *slow* if the central index ν_F remains constant on J_j as well as on its two neighbouring intervals, that is, if $\nu_F(e^{(j-1)\tau}) = \nu_F(e^{(j+2)\tau})$. Otherwise, the interval J_j is called *fast*.

On every slow interval J_j the sum \widehat{P} consists of a single term

$$\widehat{P}(z) = \frac{\xi_{\nu_j} a_{\nu_j} z^{\nu_j}}{\sigma_F(|z|)},$$

where ν_j is the common value of ν_F on J_j , and therefore

$$|\widehat{P}_t| = \frac{|a_{\nu_j}| t^{\nu_j}}{\sigma_F(t)}$$

is non-random. Hence, for such t 's, $\overline{Z}_\Lambda^{\text{c.t.}}(t) = 0$; i.e., slow intervals do not contribute to the integral $\overline{Q}_\Lambda^{\text{c.t.}}$. Thus,

$$\overline{Q}_\Lambda^{\text{c.t.}} = \int_0^\tau \left(\sum_{j \in \mathfrak{J}} \overline{Z}_\Lambda^{\text{c.t.}}(e^{j\tau+s}) \right) ds, \quad (5.8)$$

where \mathfrak{J} is the set of indices j such that $J_j \subset [1, \log r_{k+1}]$ and J_j is fast.

5.5.4. Contribution of fast intervals. We split the set \mathfrak{J} into a bounded number of disjoint subsets $\mathfrak{J}' \subset \mathfrak{J}$ so that, for $j_1, j_2 \in \mathfrak{J}'$ and $j_1 \neq j_2$, the intervals

$$[\nu_F(e^{(j_1-1)\tau}), \nu_F(e^{(j_1+2)\tau})], \quad [\nu_F(e^{(j_2-1)\tau}), \nu_F(e^{(j_2+2)\tau})]$$

are disjoint (it is easy to see that six subsets \mathfrak{J}' suffice). Given $s \in [0, \tau]$, the random variable $\overline{Z}_\Lambda^{\text{c.t.}}(\exp(j\tau+s))$ may depend only on ξ_ℓ with $\nu_F(e^{(j-1)\tau}) \leq \ell \leq \nu_F(e^{(j+2)\tau})$. Therefore, given a subset \mathfrak{J}' and a value $s \in [0, \tau]$, the random variables $\{\overline{Z}_\Lambda^{\text{c.t.}}(\exp(j\tau+s))\}_{j \in \mathfrak{J}'}$ are independent. This observation

allows us to estimate $\mathcal{E}|K(s, \mathfrak{J}')|^p$, where

$$K(s, \mathfrak{J}') \stackrel{\text{def}}{=} \sum_{j \in \mathfrak{J}'} \overline{Z}_\Lambda^{\text{c.t.}}(\exp(j\tau + s)).$$

Indeed, recalling that $\mathcal{E}\overline{Z}_\Lambda^{\text{c.t.}}(t) = 0$ and that $|\overline{Z}_\Lambda^{\text{c.t.}}(t)| \leq 2\Lambda^6 \|\varphi''\|_1$, and applying the classical Khinchin-Marcinkiewicz-Zygmund inequality, we get

$$\mathcal{E}|K(s, \mathfrak{J}')|^p \leq (Cp)^{p/2} (2\Lambda^6 \|\varphi''\|_1)^p \cdot |\mathfrak{J}'|^{p/2}.$$

Since $|\mathfrak{J}'| \leq L = \tau^{-1} \log r_{k+1}$, the RHS does not exceed

$$(C\Lambda^6 \|\varphi''\|_1 \sqrt{p \cdot \tau^{-1} \log r_{k+1}})^p.$$

At last, using Minkowski's integral inequality and recalling that we use only a bounded number of subsets \mathfrak{J}' , we obtain

$$\begin{aligned} \mathcal{E}|\overline{Q}_\Lambda^{\text{c.t.}}|^p &= \mathcal{E}\left|\int_0^\tau \sum_{\mathfrak{J}'} K(s, \mathfrak{J}') \, ds\right|^p \leq C^p \left(\int_0^\tau \sum_{\mathfrak{J}'} (\mathcal{E}|K(s, \mathfrak{J}')|^p)^{1/p} \, ds\right)^p \\ &\leq (C\tau)^p \cdot (C\Lambda^6 \|\varphi''\|_1 \sqrt{p \cdot \tau^{-1} \log r_{k+1}})^p = (C\|\varphi''\|_1 \Lambda^6 \sqrt{p \cdot \tau \log r_{k+1}})^p. \end{aligned} \quad (5.9)$$

5.5.4. Final estimate of \overline{Q} . Here, we prove the following estimate:

CLAIM 5.10. — *For a.e. $\omega \in \Omega$, every $k \geq k_0(\omega)$, and every $\varepsilon > 0$, we have*

$$|\overline{Q}| \leq C(\varepsilon) \|\varphi''\|_{q_0} ((\log r_{k+1})^{\frac{1}{2} + \varepsilon} + k^\varepsilon). \quad (5.11)$$

Proof. — We assume that $\log r_{k+1} \gg \log^7 k$. Otherwise, the crude bound from Claim 5.5 yields

$$|\overline{Q}| \leq C\|\varphi''\|_{q_0} \log^{13} k,$$

which immediately gives us (5.11).

Combining our estimates (5.6), (5.7), and (5.9), we get

$$\begin{aligned} (\mathcal{E}|\overline{Q}|^p)^{1/p} &\leq [(\mathcal{E}|\overline{Q} - \overline{Q}_\Lambda|^p)^{1/p} + (\mathcal{E}|\overline{Q}_\Lambda - \overline{Q}_\Lambda^{\text{c.t.}}|^p)^{1/p} + (\mathcal{E}|\overline{Q}_\Lambda^{\text{c.t.}}|^p)^{1/p}] \\ &\leq C\|\varphi''\|_{q_0} \left[e^{-c\Lambda/p} p^6 \log r_{k+1} + e^{\Lambda^6 - \tau} \log r_{k+1} + \Lambda^6 \sqrt{p \cdot \tau \log r_{k+1}} \right]. \end{aligned}$$

Then, applying Lemma 4.3, we see that, for almost every $\omega \in \Omega$ and every $k \geq k_0(\omega)$,

$$|\overline{Q}| \leq C \|\varphi''\|_{q_0} \left[e^{-c\Lambda/\log k} \cdot \log^6 k \cdot \log r_{k+1} + e^{\Lambda^6 - \tau} \cdot \log r_{k+1} + \Lambda^6 \sqrt{\tau \log r_{k+1} \cdot \log k} \right].$$

Now it is time to choose the values of the parameters Λ and τ . We put

$$\Lambda = C_1 \log k (\log \log r_{k+1} + \log \log k) \quad \text{and then} \quad \tau = \Lambda^6 + \log \log r_{k+1}$$

with a sufficiently large constant C_1 . Recall that our derivation of the bound for $\mathcal{E}|\overline{Q}_\Lambda^{c.t.}|^p$ used the condition $1 \ll \tau \ll \log r_{k+1}$ which is guaranteed by the assumption $\log^7 k \ll \log r_{k+1}$.

The choice of the parameters Λ and τ yields boundedness of the terms

$$e^{-c\Lambda/\log k} \cdot \log^6 k \cdot \log r_{k+1}, \quad e^{\Lambda^6 - \tau} \cdot \log r_{k+1}.$$

Thus, it remains to estimate the term $\Lambda^6 \sqrt{\tau \log r_{k+1} \cdot \log k}$. Observe that, for sufficiently large k , both Λ^6 and τ do not exceed $(\log k)^C + (\log \log r_{k+1})^C$. This yields the estimate

$$\begin{aligned} \Lambda^6 \sqrt{\tau \log r_{k+1} \cdot \log k} &\leq C(\varepsilon) \left((\log r_{k+1})^{\frac{1}{2} + \varepsilon} + (\log k)^{C(\varepsilon)} \right) \\ &\leq C(\varepsilon) \left((\log r_{k+1})^{\frac{1}{2} + \varepsilon} + k^\varepsilon \right), \end{aligned}$$

proving the claim. □

5.6. Completing the proof of Theorem 1.2

We need to prove the almost sure part (i) of the theorem. Returning to Claim 5.1, and plugging in the estimates of all error terms, for $r_k e^{\delta_k} \leq r \leq r_{k+1} e^{-\delta_k}$, $k \geq k_0(\omega)$, we get

$$\begin{aligned} |n_F(r, \varphi) - \mathcal{E}n_F(r, \varphi)| &\leq (s_F(r_{k+1}) - s_F(r_k)) + \frac{C}{\delta_k} \log^6 k + \\ &\quad + C(q_0, \varepsilon) \|\varphi''\|_{q_0} \left((\log r_{k+1})^{\frac{1}{2} + \varepsilon} + k^\varepsilon \right). \end{aligned}$$

It remains to show that with the same choice of the parameters δ_k and r_k as in Section 5.2, we get the desired result. First, the choice $\delta_k = (k \log^2 k)^{-1}$ yields that the RHS of the previous estimate is

$$\leq (s_F(r_{k+1}) - s_F(r_k)) + C(\varepsilon) k^{1+\varepsilon} + C(q_0, \varepsilon) \|\varphi''\|_{q_0} \left((\log r_{k+1})^{\frac{1}{2} + \varepsilon} + k^\varepsilon \right).$$

At last, we take r_k so that $s_F(r_k) + \log r_k = k^2$. Repeating the estimates from Section 5.2, we have

$$s_F(r_{k+1}) - s_F(r_k) \leq 3(s_F(r)^{\frac{1}{2}} + (\log r)^{\frac{1}{2}})$$

and

$$k^\varepsilon \leq s_F(r)^{\frac{1}{2}\varepsilon} + (\log r)^{\frac{1}{2}\varepsilon}, \quad k^{1+\varepsilon} \leq s_F(r)^{\frac{1}{2}(1+\varepsilon)} + (\log r)^{\frac{1}{2}(1+\varepsilon)}.$$

In addition,

$$(\log r_{k+1})^{\frac{1}{2}+\varepsilon} \leq (k+1)^{1+2\varepsilon} < 4(s_F(r)^{\frac{1}{2}+\varepsilon} + (\log r)^{\frac{1}{2}+\varepsilon}).$$

Therefore, for $k > k_0(\omega)$, we have

$$|n_F(r, \varphi) - \mathcal{E}n_F(r, \varphi)| \leq C(q_0, \varepsilon) (1 + \|\varphi''\|_{q_0}) (s_F(r)^{\frac{1}{2}+\varepsilon} + (\log r)^{\frac{1}{2}+\varepsilon}).$$

Taking $0 < \varepsilon < \gamma - \frac{1}{2}$, we finish off the proof of Theorem 1.2. □

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