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A Remark on Classical Pluecker's formulae

Vik.S. Kulikov[(1)]

Résumé. — Pour toute courbe réduite $C \subset \mathbb{P}^2$, on introduit la notion de nombre des points de rebroussement (cusps) virtuels c_v et celle de nombre des points doubles ordinaires (nodes) virtuels n_v. Ces deux nombres sont positifs ou nuls et ils coïncident avec le nombre des points singuliers du type respectif lorsque ce sont les seules singularités de la courbe. De plus, si $\hat{C}$ est la courbe duale d'une courbe irréducible C, et si $\hat{n}_v$ et $\hat{c}_v$ designent le nombre de singularités virtuelles de $\hat{C}$ du type respectif, alors les nombres entiers c_v, n_v, $\hat{c}_v$, $\hat{n}_v$ vérifient les formules de Plücker classiques.

Abstract. — For any reduced curve $C \subset \mathbb{P}^2$, we introduce the notions of the number of its virtual cusps c_v and the number of its virtual nodes n_v. We prove that the numbers c_v and n_v are non-negative and if C is a curve with only ordinary cusps and nodes as its singular points, then c_v is the number of its ordinary cusps and n_v is the number of its ordinary nodes. In addition, if $\hat{C}$ is the dual curve of an irreducible curve C and $\hat{n}_v$ and $\hat{c}_v$ are the numbers of its virtual nodes and virtual cusps, then the integers c_v, n_v, $\hat{c}_v$, $\hat{n}_v$ satisfy classical Plücker's formulae.

Introduction

Let $C \subset \mathbb{P}^2$ be a reduced curve defined over the field of complex numbers $\mathbb{C}$. A curve C is called *cuspidal* if the singular points of C are only the ordinary cusps and nodes.

In modern textbooks on algebraic geometry, classical Plücker's formulae are stated as follows (see, for example, [1], [3]).

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Classical Plücker's formulae. — *Let $C \subset \mathbb{P}^2$ be an irreducible cuspidal curve of genus g, degree $d \geqslant 2$, having c ordinary cusps and n nodes Assume that the dual curve $\hat{C}$ of C is also a cuspidal curve. Then*

$$\hat{d} = d(d-1) - 3c - 2n; \tag{0.1}$$

$$g = \frac{(d-1)(d-2)}{2} - c - n; \tag{0.2}$$

$$d = \hat{d}(\hat{d}-1) - 3\hat{c} - 2\hat{n}; \tag{0.3}$$

$$g = \frac{(\hat{d}-1)(\hat{d}-2)}{2} - \hat{c} - \hat{n}, \tag{0.4}$$

where $\hat{c}$ and $\hat{n}$ are the numbers of ordinary cusps and nodes of $\hat{C}$ and $\hat{d} = \deg \hat{C}$.

Denote by $V(d,c,n) \subset \mathbb{P}^{\frac{d(d+3)}{2}}$ the variety parametrizing the irreducible cuspidal curves of degree d with c ordinary cusps and n nodes. Very often, if for given d, c, and n one of the invariants $\hat{c}$ or $\hat{n}$, obtained as the solution of (0.1) – (0.4), is negative, then it is claimed that this is sufficient for the "proof" of the emptiness of $V(d,c,n)$. However, the correctness of the following statement is unknown: "*the dual curve $\hat{C}$ of a curve C corresponding to a generic point of $V(d,c,n)$ is cuspidal*". Therefore, in general case, it is impossible to conclude the non-existence of cuspidal curve C if $\hat{c}$ or $\hat{n}$ is negative. Of course, to avoid this problem, one can use generalized Plücker's formulae including the numbers of all possible types of singular points of $\hat{C}$. But, we again have a difficulty, namely, in this case we must take into account too many unknown variables.

To obviate the arising difficulty, in Section 1 for any reduced plane curve C we define the notions of the *number* of its *virtual cusps* c_v and the *number* of its *virtual nodes* n_v which are non-negative, coincide respectively with the numbers of ordinary cusps and nodes in the case of cuspidal curves, and if the dual curve $\hat{C}$ of an irreducible curve C has $\hat{n}_v$ virtual nodes and $\hat{c}_v$ virtual cusps, then the integers c_v, n_v, $\hat{c}_v$, and $\hat{n}_v$ satisfy Classical Plücker's formulae.

In Section 2, we investigate the behaviour of the Hessian curve H_C of a cuspidal curve C at cusps and nodes of C, and in Section 3, we generalize to the case of arbitrary irreducible plane curve the inequalities for the numbers of cusps and nodes of plane cuspidal curves of degree d which was obtained early in [5] under additional assumption that the dual curve of a generic cuspidal curve is also cuspidal.

1. The numbers of virtual cusps and nodes

Let $(C,p)\subset(\mathbb{P}^2,p)$ be a germ of a reduced plane singularity. It splits into several irreducible germs: $(C,p)=(C_1,p)\cup\ldots\cup(C_k,p)$. Denote by m_j the multiplicity of the singularity (C_j,p) at the point p and let δ_p be the δ-invariant of the singularity (C,p). By definition, the integers

$$c_{v,p}:=\sum_{i=1}^{k}(m_i-1)$$

and

$$n_{v,p}:=\delta_p-\sum_{i=1}^{k}(m_i-1)$$

are called respectively the *numbers of virtual cusps* and *virtual nodes* of the singularity (C,p). Note that in [2] it was shown that any reduced singular curve germ can be deformed into a germ with exactly $c_{v,p}$ cusps and $n_{v,p}$ nodes.

We have $\delta_p=c_{v,p}+n_{v,p}$.

Lemma 1.1. — *Let $(C,p)\subset(\mathbb{P}^2,p)$ be a germ of a reduced plane singularity, $c_{v,p}$ be the number of its virtual cusps and $n_{v,p}$ be the number of its virtual nodes. Then*

(i) $c_{v,p}\geqslant 0$, $n_{v,p}\geqslant 0$;

(ii) if (C,p) is an ordinary cusp, then $c_{v,p}=1$ and $n_{v,p}=0$;

(iii) if (C,p) is an ordinary node, then $c_{v,p}=0$ and $n_{v,p}=1$.

Proof. — We prove only the inequality $n_v\geqslant 0$, since all the other claims of Lemma 1.1 are obvious. Let $(C,p)=(C_1,p)\cup\ldots\cup(C_k,p)$ and m_i be the multiplicity of its irreducible branch (C_i,p). Then the multiplicity of (C,p) at p is equal to $m_p=\sum_{i=1}^{k}m_i$ and we have

$$\begin{aligned} n_{v,p}= \quad & \delta_p-\sum_{i=1}^{k}(m_i-1)\geqslant\delta_p-\sum_{i=1}^{k}m_i+1= \\ & \delta_p-(m_p-1)\geqslant\delta_p-\frac{m_p(m_p-1)}{2}\geqslant 0, \end{aligned}$$

since $m_p\geqslant 2$ for singular points and $\delta_p\geqslant\frac{m_p(m_p-1)}{2}$. Therefore, we have

$$n_v=\sum_{p\in\operatorname{Sing}C}n_{v,p}\geqslant 0.$$

Let $C \subset \mathbb{P}^2$ be a reduced curve. Denote by $\operatorname{Sing} C$ the set of its singular points. By definition, we put

$$c_v := \sum_{p \in \operatorname{Sing} C} c_{v,p},$$

$$n_v := \sum_{p \in \operatorname{Sing} C} n_{v,p}$$

and call these integers respectively the *number of virtual cusps* and the *number virtual nodes* of the curve C. If C is an irreducible curve of degree d and geometric genus g, then we have $g = \dfrac{(d-1)(d-2)}{2} - \delta_C$, where $\delta_C = \sum_{p \in \operatorname{Sing} C} \delta_p$ is the δ-invariant of C. Therefore, we have

$$g = \frac{(d-1)(d-2)}{2} - c_v - n_v. \tag{1.5}$$

The following proposition is a corollary of Lemma 1.1.

PROPOSITION 1.2. — *Let c_v be the number of virtual cusps and n_v be the number of virtual nodes of a reduced curve $C \subset \mathbb{P}^2$. We have*

(i) $c_v \geqslant 0$ and $n_v \geqslant 0$,

(ii) if C is a cuspidal curve, then c_v and n_v are equal respectively to the number c of cusps and the number n of nodes of C.

THEOREM 1.3. — (Plücker's formulae). *Let C and $\hat{C}$ be irreducible dual curves of genus g, $\deg C = d \geqslant 2$, $\deg \hat{C} = \hat{d}$, and c_v, n_v, $\hat{c}_v$, $\hat{n}_v$ are the numbers of their virtual cusps and nodes, respectively. Then we have the following equalities:*

$$\hat{d} = d(d-1) - 3c_v - 2n_v; \tag{1.6}$$

$$2g = (d-1)(d-2) - 2c_v - 2n_v; \tag{1.7}$$

$$d = \hat{d}(\hat{d}-1) - 3\hat{c}_v - 2\hat{n}_v; \tag{1.8}$$

$$2g = (\hat{d}-1)(\hat{d}-2) - 2\hat{c}_v - 2\hat{n}_v. \tag{1.9}$$

Proof. — To prove Plücker's formulae, we need the following

Lemma 1.4. — *For an irreducible plane curve C we have*

$$\hat{d} = 2d + 2(g-1) - c_v, \tag{1.10}$$

$$\hat{c}_v = 3d + 6(g-1) - 2c_v, \tag{1.11}$$

$$d = 2\hat{d} + 2(g-1) - \hat{c}_v, \tag{1.12}$$

$$c_v = 3\hat{d} + 6(g-1) - 2\hat{c}_v. \tag{1.13}$$

Proof. — Denote by $\nu : \overline{C} \to C$ and $\hat{\nu} : \overline{C} \to \hat{C}$ the normalization morphisms, consider generic (with respect to C and $\hat{C}$) linear projections $\mathrm{pr} : \mathbb{P}^2 \to \mathbb{P}^1$ and $\hat{\mathrm{pr}} : \hat{\mathbb{P}}^2 \to \mathbb{P}^1$, and put $\pi = \mathrm{pr} \circ \nu$ and $\hat{\pi} = \hat{\mathrm{pr}} \circ \hat{\nu}$. We have $\deg \pi = d$ and $\deg \hat{\pi} = \hat{d}$.

Let $\nu^{-1}(x_i) = \{y_{i,1}, \dots, y_{i,m_i}\}$ for $x_i \in \operatorname{Sing} C$. For each point $y_{i,j}$ denote by $r_{i,j}$ the ramification index of π at $y_{i,j}$. It is easy to see that $r_{i,j}$ coincides with the multiplicity $m_{i,j}$ at x_i of the irreducible germ $(C_{i,j}, x_i) \subset (C, x_i)$ corresponding to the point $y_{i,j}$. Therefore, we have

$$c_v = \sum_{i,j} (r_{i,j} - 1).$$

Applying Hurwitz formula to π and $\hat{\pi}$, we obtain

$$2(g-1) = -2d + c_v + \hat{d} \tag{1.14}$$

and

$$2(g-1) = -2\hat{d} + \hat{c}_v + d \tag{1.15}$$

which give formulae (1.10) and (1.12).

To prove (1.11), note that $\hat{c}_v = 2\hat{d} + 2(g-1) - d$ by (1.12). Therefore

$$\hat{c}_v = 2(2d + 2(g-1) - c_v) + 2(g-1) - d$$

by (1.10), that is, $\hat{c}_v = 3d + 6(g-1) - 2c_v$. Formula (1.13) is obtained similarly. □

It follows from (1.5) that

$$2(g-1) + 2c_v + 2n_v = d(d-3), \qquad 2(g-1) + 2\hat{c}_v + 2\hat{n}_v = \hat{d}(\hat{d}-3) \tag{1.16}$$

which are equivalent to (1.7) and (1.9). To complete the proof of Plücker's formulae, notice that formulae (1.6) and (1.8) easily follow from equations (1.10) – (1.13) and (1.16). □

2. On the Hessian curve of a cuspidal curve

Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree d with c_v virtual cusps and n_v virtual nodes. It follows from (1.7) and (1.11) that

$$8c_v + 6n_v + \hat{c}_v = 3d(d-2). \tag{2.1}$$

If C is a cuspidal curve then quality (2.1) has a natural geometric meaning. To explain it, let the curve C is given by equation $F(x_0, x_1, x_2) = 0$, where x_0, x_1, x_2 are homogeneous coordinates in $\mathbb{P}^2$. Consider the Hessian curve $H_C \subset \mathbb{P}^2$ of the curve C. It is given by equation $\det(\frac{\partial^2 F}{\partial x_i \partial x_j}) = 0$. We have $\deg H_C = 3(d-2)$. Therefore the intersection number $(C, H_C)_{\mathbb{P}^2}$ is equal to $3d(d-2)$. On the other hand, it is well-known (see, for example, [1]) that the curves C and H_C meet at the singular points and at the inflection points of the curve C. Therefore we have

$$\Sigma'(C, H_C)_p + \Sigma''(C, H_C)_p + \Sigma'''(C, H_C)_p = (C, H_C)_{\mathbb{P}^2} = 3d(d-2), \tag{2.2}$$

where $(C, H_C)_p$ is the intersection number of the curves C and H_C at a point $p \in C$ and the sum $\sum'$ is taken over all cusps of C, the sum $\sum''$ is taken over all nodes of C, and the sum $\sum'''$ is taken over all inflection points of C.

Let us show that the coefficients involving in equation (2.1) have the following geometric meaning: *equality* (2.1) *is the same as equality* (2.2), that is, the coefficient 8 in (2.1) is the intersection number $(C, H_C)_p$ at a cusp $p \in C$, the coefficient 6 is the intersection number $(C, H_C)_p$ at a node $p \in C$, and $\hat{c}_v = \sum'''(C, H_C)_p$. Note that the following computations are classical (see, for example, [6]), but we give them here briefly.

Let p be a cusp of C. Without loss of generality, we can assume that $p = (0, 0, 1)$ and

$$F(x_0, x_1, x_2) = x_0^2 U(x_0, x_1, x_2) + x_0 x_1^2 V(x_0, x_1, x_2) + x_1^3 W(x_0, x_1, x_2),$$

where U is a homogeneous polynomial of degree $d-2$ such that $U(0, 0, 1) = 1$ and V and W are homogeneous polynomials of degree $d-3$ such that $W(0, 0, 1) = 1$. Put $a = V(0, 0, 1)$, then in non-homogeneous coordinates $x = \frac{x_0}{x_2}, y = \frac{x_1}{x_2}$ we have $p = (0, 0)$, the curve C is given by equation of the form

$$x^2 + y^3 + axy^2 + bx^2 y + cx^3 + \text{terms of higher degree} = 0,$$

and the curve H_C is given by equation of the form

$$x^2(6y + 2ax) + \text{terms of higher degree} = 0.$$

Easy computation (applying σ-process with center at p) gives the following inequality:

$$(C, H_C)_p \geqslant 8 \tag{2.3}$$

if p is a cusp of C.

Let p be a node of C. Again, without loss of generality, we can assume that $p = (0, 0, 1)$ and

$$F(x_0, x_1, x_2) = x_0 x_1 U(x_0, x_1, x_2) + V(x_0, x_1) W(x_0, x_1, x_2),$$

where U is a homogeneous polynomial of degree $d-2$ such that $U(0,0,1) = 1$, V is a homogeneous polynomial of degree 3, and W is a homogeneous polynomial of degree $d-3$. In non-homogeneous coordinates $x = \frac{x_0}{x_2}, y = \frac{x_1}{x_2}$ we have $p = (0, 0)$, the curve C is given by equation of the form

$$xy + \text{terms of higher degree} = 0,$$

and the curve H_C is given by equation of the same form

$$xy + \text{terms of higher degree} = 0.$$

Easy computation (applying σ-process with center at p) gives the following inequality:

$$(C, H_C)_p \geqslant 6 \tag{2.4}$$

if p is a node of C.

If p is an r-tuple inflection point of C (that is, $(C, L_p)_p = r + 2$, where the line L_p is tangent to C at p), then by Theorem 1 on page 289 in [1], we have $(C, H_C)_p = r$. On the other hand, the branch $(\hat{C}_i, \hat{p})$ of the dual curve $\hat{C}$, corresponding to an irreducible branch $(C_i, p) \subset (C, p)$ at a point p of a cuspidal curve C, is singular if and only if p is an inflection point of C; and the branch $(\hat{C}, \hat{p})$, corresponding to the branch (C, p) at r-tuple inflection point $p \in C$, has a singularity of type $u^{r+1} - v^{r+2} = 0$. The multiplicity $m_{\hat{p}}$ of this singularity is equal to $r + 1$. Therefore, we have

$$\Sigma'''(C, H_C)_p = \sum_{(\hat{C}, \hat{p})} (m_{\hat{p}} - 1) = \hat{c}_v. \tag{2.5}$$

Finally, it follows from (2.1) – (2.5) that inequalities (2.3) and (2.4) are the equalities in the case of cuspidal curves.

3. Lefschetz's inequalities

In [5], assuming that for a generic cuspidal curve with given numerical invariants the dual curve is also cuspidal, Lefschetz proved the following inequalities

$$c \leqslant \frac{3}{2}d + 3(g-1) \tag{3.1}$$

if d is even and

$$c \leqslant \frac{3d-1}{2} + 3(g-1) \tag{3.2}$$

if d is odd. It follows from (1.11) that these inequalities occur for any irreducible plane curve, since $\hat{c}_v$ is a non-negative integer. Namely, for any irreducible plane curve we have

$$c_v \leqslant \frac{3}{2}d + 3(g-1) \tag{3.3}$$

if d is even and

$$c_v \leqslant \frac{3d-1}{2} + 3(g-1) \tag{3.4}$$

if d is odd, and equality (2.1) gives rise also to the following inequality:

$$8c_v + 6n_v \leqslant 3d(d-2) - \frac{1-(-1)^d}{2}. \tag{3.5}$$

Remark 3.1. — One can show that *for any $d = 2k$, $k \geqslant 3$, and for any $g \geqslant 0$ such that $2 \leqslant 3g \leqslant k-4$ or $g \leqslant 1$, there exist a cuspidal curve of degree d having $c = 3(k+g-1)$ cusps and $n = 2(k-1)(k-2) - 4g$ nodes for which inequality* (3.5) *becomes the equality. If $d = 2k+1$, $k \geqslant 3$, then for any g such that $2 \leqslant 3g \leqslant k-4$ or $g \leqslant 1$, there exist a cuspidal curve of degree d having $c = 3(k+g) - 2$ cusps, $n = 2(k-1)^2 - 4g$ nodes, and for which inequality* (3.5) *becomes the equality.* The proof of these statements follows from the fact that the genus of such curves C is equal to g and for these curves the dual curves $\hat{C}$ have degree $\hat{d} = 2(g-1) + 7 + \frac{1-(-1)^d}{2}$ and the number of virtual cusps $\hat{c}_v = \frac{1-(-1)^d}{2}$. Therefore in the case of even d (resp., odd d) such curves can be obtained as the images of generic (resp., as the images of almost generic, that is, belonging to a codimension one variety in the space of linear projections) linear projections to $\mathbb{P}^2$ of a smooth curve $\overline{C} \subset \mathbb{P}^{\hat{d}-g}$ of degree $\hat{d}$ birationally isomorphic to C. Standard computations of codimension of the locus of "bad" projections (which we leave to the reader) show that in these cases there are linear projections $\mathrm{pr} : \mathbb{P}^{\hat{d}-g} \to \mathbb{P}^2$ such that $\mathrm{pr}(\overline{C}) = \hat{C}$ are cuspidal curves with $\hat{c} = \frac{1-(-1)^d}{2}$ and their dual curves C are also cuspidal.

For completeness, let me notice that there are also the following inequalities (well-known in the case of cuspidal curves) which we have for any plane irreducible curve:

$$3c_v + 2n_v < d(d-1) - \sqrt{d}, \tag{3.6}$$

$$2c_v + 2n_v \leqslant (d-1)(d-2), \tag{3.7}$$

$$d(d-2)(d^2-9) + (3c_v + 2n_v)^2 + 27c_v + 20n_v \geqslant 2d(d-1)(3c_v + 2n_v) \tag{3.8}$$

and which are consequences of equalities (1.6) – (1.9) and the inequalities $\hat{d} \geqslant \sqrt{d}$, $g \geqslant 0$, $\hat{n}_v \geqslant 0$.

Also, let me mention that the following inequality ([4]):

$$16c + 9n \leqslant d(5d-6) \tag{3.9}$$

holds for any (not necessary irreducible) cuspidal curve C of even degree d. It follows from inequality (3.9) that

$$16c + 9n \leqslant 5d(d-1) - 1 \tag{3.10}$$

for any cuspidal curve C of odd degree $d > 1$. To show this, it suffices to add a line L in general position with respect to C and apply inequality (3.9) to $C \cup L$. But, up to now it is unknown whether inequality (3.9) (and respectively (3.10)) holds for any reduced plane curve C of even degree (respectively, of odd degree) if we substitute n_v and c_v in (3.9) instead of n and c.

Bibliography

[1] Brieskorn (E.), Knörrer (H.). — Plane algebraic curves, Birkhäuser-Verlag, Basel-Boston-Stuttgart, (1986).

[2] Diaz (D.) and Haris (J.). — Ideals associated to deformations of singular plane curves, Transactions of Am. Math. Soc. 309, p. 433-468 (1988).

[3] Griffiths (P.) and Harris (J.). — Principles of algebraic geometry. A Wiley-Interscience Publication, John Wiley & Sons, New York Chichester Brisbane Toronto (1978).

[4] Hirzebruch (F.). — Singularities of algebraic surfaces and characteristic numbers. Algebraic geometry, Proc. Lefschetz Centen. Conf., Mexico City/Mex. 1984, Part I, Contemp. Math., 58, I, p. 141-155 (1986).

[5] Lefschetz (S.). — On the existence of loci with given singularities, Transactions of Am. Math. Soc. 14, p. 23-41 (1913).

[6] Van der Waerden (B.L.). — Einführung in die algebraische Geometrie, Springer (1939).