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Regular foliations on weak Fano manifolds (*)

Stéphane Druel (1)

Abstract. — In this paper we prove that a regular foliation on a complex weak Fano manifold is algebraically integrable.

Résumé. — Dans cette note, nous montrons que tout feuilletage régulier sur une variété de Fano faible est algébriquement intégrable.

1. Introduction

This paper is concerned with a sufficient criterion to guarantee that a given foliation has algebraic leaves. In [4], Bost proved an algebraicity criterion for leaves of algebraic foliations defined over a number field. The geometric counterpart of this result, independently obtained by Bogomolov and McQuillan, is the following.

Theorem 1.1 ([3, Theorem 0.1], [4, Theorem 3.5]). — Let $X$ be a complex projective manifold, and let $\mathcal{F}$ be a foliation on $X$. Let $C \subset X$ be a complete curve disjoint from the singular locus of $\mathcal{F}$. Suppose that the restriction $\mathcal{F}|_C$ is an ample vector bundle on $C$. Then the leaf of $\mathcal{F}$ through any point of $C$ is an algebraic variety.

We also would like to mention the recent paper of Campana and Păun [6] which present very interesting developments related to Theorem 1.1 above.

In this paper, we provide some evidence for the following conjecture.

Conjecture 1.2 (F. Touzet). — Let $X$ be a complex projective manifold, and let $\mathcal{F}$ be a regular foliation on $X$. Suppose that $X$ is rationally connected. Then the leaves of $\mathcal{F}$ are algebraic varieties.

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The statement is a tautology in the case of curves. For surfaces, it follows from the classification of foliation by curves on surfaces ([5]). It was also known to be true if $X$ is a rational homogeneous space (see [9] and [13]). Our result is the following. Recall that a weak Fano manifold is a complex projective manifold $X$ such that $-K_X$ is nef and big.

**Theorem 1.3.** — Let $X$ be a complex weak Fano manifold, and let $\mathcal{F} \subseteq T_X$ be a regular foliation. Then the foliation $\mathcal{F}$ is given by the fibers of a smooth morphism $X \rightarrow Y$ onto a projective manifold.

**Remark 1.4.** — In the setup of Theorem 1.3, $Y$ is a weak Fano manifold by [8, Theorem 1.1].

**Remark 1.5.** — Let $n \geq 2$ be an integer, and let $\mathcal{F}$ be a foliation on $\mathbb{P}^n$ induced by a general global holomorphic vector field. Then the leaf of $\mathcal{F}$ through a general point is not algebraic. This shows that Theorem 1.3 is wrong if one drops the regularity assumption on $\mathcal{F}$.

In order to prove Theorem 1.1, we consider the normal bundle $\mathcal{N} := T_X / \mathcal{F}$ of the foliation $\mathcal{F}$. We show first that $\det(\mathcal{N})$ is nef. This follows from a foliated version of the bend-and-break lemma (see also Proposition 3.7).

**Proposition 1.6.** — Let $X$ be a complex projective manifold, and let $\mathcal{F} \subseteq T_X$ be a regular foliation with normal bundle $\mathcal{N}$. Let $C \subset X$ be a rational curve with $\det(\mathcal{N}) \cdot C \neq 0$, and let $x$ be a point on $C$. If $\det(\mathcal{F}) \cdot C \geq 1$, then there exist a nonzero effective rational 1-cycle $Z$ passing through $x$, a rational curve $C_1$, and a positive integer $m$ such that $C \equiv mC_1 + Z$ and such that $\text{Supp}(Z)$ is tangent to $\mathcal{F}$.

From the base-point-free theorem, we conclude that $\det(\mathcal{N})$ is semiample. We then prove that the corresponding morphism $\varphi : X \rightarrow Y$ yields a first integral for $\mathcal{F}$ as follows. Let $F$ be a general fiber of $\varphi$. By the adjunction formula, $F$ is a weak Fano manifold. In particular, $F$ does not carry differential forms. This easily implies that $F$ is tangent to $\mathcal{F}$ (see Lemma 2.4). On the other hand, the Baum–Bott vanishing theorem yields $\dim Y \leq \dim X - \text{rank } \mathcal{F}$, and hence $\dim F = \text{rank } \mathcal{F}$, completing the proof of the claim.

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### 2. Recollection: Foliations

In this section we recall the basic facts concerning foliations.
2.1. Foliations

**Definition 2.1.** — A foliation on a complex manifold $X$ is a coherent subsheaf $\mathcal{F} \subseteq T_X$ such that

- $\mathcal{F}$ is closed under the Lie bracket, and
- $\mathcal{F}$ is saturated in $T_X$. In other words, the quotient $T_X/\mathcal{F}$ is torsion free.

The rank $r$ of $\mathcal{F}$ is the generic rank of $\mathcal{F}$. The codimension of $\mathcal{F}$ is defined as $q := \dim X - r$. Let $X^\circ \subset X$ be the maximal open set where $\mathcal{F}$ is a subbundle of $T_X$. We say that $\mathcal{F}$ is regular if $X^\circ = X$.

A leaf of $\mathcal{F}$ is a connected, locally closed holomorphic submanifold $L \subset X^\circ$ such that $T_L = \mathcal{F}|_L$. A leaf is called algebraic if it is open in its Zariski closure.

The foliation $\mathcal{F}$ is said to be algebraically integrable if its leaves are algebraic.

**Definition 2.2.** — Let $\mathcal{F}$ be a foliation on a smooth variety $X$. The canonical class $K_\mathcal{F}$ of $\mathcal{F}$ is any Weil divisor on $X$ such that $O_X(-K_\mathcal{F}) \cong \det(\mathcal{N})$.

2.3 (Foliations defined by $q$-forms). — Let $q$ and $n$ be positive integers. Let $\mathcal{F}$ be a codimension $q$ foliation on an $n$-dimensional complex manifold $X$. The normal sheaf of $\mathcal{F}$ is $\mathcal{N} := (T_X/\mathcal{F})^\ast$. The $q$-th wedge product of the inclusion $\mathcal{N}^\ast \hookrightarrow (\Omega^1_X)^\ast$ gives rise to a nonzero global section $\omega \in H^0(X, \Omega^q_X \otimes \det(\mathcal{N}))$ whose zero locus has codimension at least 2 in $X$. Such $\omega$ is locally decomposable and integrable. To say that $\omega$ is locally decomposable means that, in a neighborhood of a general point of $X$, $\omega$ decomposes as the wedge product of $q$ local 1-forms $\omega = \omega_1 \wedge \cdots \wedge \omega_q$. To say that it is integrable means that for this local decomposition one has $d\omega_i \wedge \omega = 0$ for every $i \in \{1, \ldots, q\}$. The integrability condition for $\omega$ is equivalent to the condition that $\mathcal{F}$ is closed under the Lie bracket.

Conversely, let $\mathcal{L}$ be a line bundle on $X$, $q \geq 1$, and $\omega \in H^0(X, \Omega^q_X \otimes \mathcal{L})$ a global section whose zero locus has codimension at least 2 in $X$. Suppose that $\omega$ is locally decomposable and integrable. Then one defines a foliation of rank $r = n - q$ on $X$ as the kernel of the morphism $T_X \to \Omega^{q-1}_X \otimes \mathcal{L}$ given by the contraction with $\omega$. These constructions are inverse of each other.

We will need the following easy observation.

**Lemma 2.4.** — Let $q$ be a positive integer, and let $\mathcal{F}$ be a codimension $q$ foliation on a complex projective manifold $X$. Let $\varphi: X \to Y$ be a surjective
morphism with connected fibers onto a normal projective variety $Y$, with general fiber $F$. Set $\mathcal{N} := T_X / \mathcal{F}$ and $\mathcal{L} := \det(\mathcal{N})$. Suppose that $\mathcal{L}|_F \sim 0$ and that $h^0(F, \Omega^i_F) = 0$ for all $1 \leq i \leq \dim F$. Then $F$ is tangent to $\mathcal{F}$. In particular, we have $\dim Y \geq q$.

**Proof.** — Let $\omega \in H^0(X, \Omega^q_X \otimes \mathcal{L})$ be a twisted $q$-form defining $\mathcal{F}$ (see 2.3). The short exact sequence

$$0 \to \mathcal{N}^*_{F/X} \cong \mathcal{O}^{\dim X - \dim Y} \to \Omega^1_{X|F} \to \Omega^1_F \to 0$$

yields a filtration

$$\{0\} = \mathcal{E}_{q+1} \subseteq \mathcal{E}_q \subseteq \cdots \subseteq \mathcal{E}_0 = \Omega^q_{X|F}$$

with

$$\mathcal{E}_i/\mathcal{E}_{i+1} \cong \wedge^i(\mathcal{N}^*_{F/X}) \otimes \Omega^{q-i}_F.$$

Since $h^0(F, \Omega^i_F) = 0$ for all $0 \leq i \leq q - 1$, we conclude that

$$\omega|_F \in H^0(F, \mathcal{E}_q) = H^0(F, \wedge^q(\mathcal{N}^*_{F/X}) \subset H^0(F, \Omega^q_{X|F}).$$

This implies that $q \leq \dim Y$ and that $\mathcal{N}^*|_{F^o} \subset \mathcal{N}^*_{F/X|F^o} \subset \Omega^1_{X|F^o}$, where $X^o \subset X$ denotes the maximal open set where $\mathcal{F}$ is a subbundle of $T_X$, and $F^o := F \cap X^o$. Thus $T_{F^o} \subset \mathcal{F}|_{F^o}$, proving the lemma. □

### 2.2. Bott (partial) connection

2.5. — Let $X$ be a complex manifold, let $\mathcal{F} \subset T_X$ be a regular codimension $q$ foliation with $0 < q < \dim X$, and set $\mathcal{N} = T_X / \mathcal{F}$. Let $p: T_X \to \mathcal{N}$ denote the natural projection. For sections $U$ of $\mathcal{N}$, $T$ of $T_X$, and $V$ of $\mathcal{F}$ over some open subset of $X$ with $U = p(T)$, set $D_VU = p([V,U])$. This expression is well-defined, $\mathcal{O}_X$-linear in $V$ and satisfies the Leibnitz rule $D_V(fU) = fD_VU + (Vf)U$ so that $D$ is an $\mathcal{F}$-connection on $\mathcal{N}$ (see [2]).

**Lemma 2.6.** — Let $X$ be a complex manifold, and let $\mathcal{F} \subsetneq T_X$ be a regular foliation with normal bundle $\mathcal{N} = T_X / \mathcal{F}$. Let $f: Z \to X$ be a compact manifold, and suppose that $f(Z)$ is tangent to $\mathcal{F}$. Then $f^*\mathcal{N}$ admits a holomorphic flat connection. In particular, characteristic classes of $f^*\mathcal{N}$ vanish.

**Proof.** — This follows from 2.5 and [1]. □
3. Deformations of a morphism along a foliation

In this section, we provide a technical tool for the proof of the main result (see Corollary 3.9).

3.1. — Let $Z$, $Y$ and $X$ be normal complex projective varieties, and let $g: Z \to X$ be a morphism. Let $\text{Hom}(Y, X)$ denotes the space of morphisms $f: Y \to X$, and let $\text{Hom}(Y, X; g) \subset \text{Hom}(Y, X)$ denotes the Zariski closed subspace parametrizing morphisms $f: Y \to X$ such that $f|_Z = g$ (see [16, Proposition 1]).

Suppose now that $Z$, $Y$ and $X$ are complex projective manifolds, and consider a codimension $q$ regular foliation $\mathcal{F} \subseteq T_X$ on $X$ with $0 < q < \dim X$. Pick $[f] \in \text{Hom}(Y, X)$. Let $\text{Def}([f], \mathcal{F})$ denotes the germ of analytic space parametrizing small deformations of $[f]$ along $\mathcal{F}$. It is constructed as follows (see [15, Section 6], or [11, Corollary 5.6]). Choose an open cover $(U_i)_{i \in I}$ of $X$ with respect to the Euclidean topology such that, for each $i \in I$, $\mathcal{F}|_{U_i}$ is induced by a holomorphic submersion $\varphi_i: U_i \to W_i$ of complex analytic spaces. Let $(V_j)_{j \in J}$ be a finite open cover of $Y$. By replacing $(V_j)_{j \in J}$ with a refinement, we may assume that, for each $j \in J$, there exist $i_j \in I$ and an open neighborhood $H_j$ of $[f]$ such that $h(y) \in U_{i_j}$ for each $h \in H_j$ and each $y \in V_j$. Let $\text{Def}([f], \mathcal{F})$ be the connected component of the intersection

$$\bigcap_{j \in J} \left\{ [h] \in H_j \mid \varphi_{i_j} \circ (h|_{V_j}) = \varphi_{i_j} \circ (f|_{V_j}) \right\}$$

which contains $[f]$. Notice that $\text{Def}([f], \mathcal{F})$ is a locally closed (possibly non-reduced) analytic subset. Set

$$\text{Def}([f], \mathcal{F}; g) = \text{Def}([f], \mathcal{F}) \cap \text{Hom}(Y, X; g).$$

Remark 3.2. — Let $\varphi: X \to Y$ be a surjective morphism with connected fibers of projective manifolds, let $Z$ be a projective manifold, and let $f: Z \to X$ be a morphism. Let $\mathcal{F}$ be the foliation on $X$ given by the fibers of $\varphi$. Recall that the space of deformations of $[f]$ over $Y$ are parametrized by the fiber $\text{Hom}_Y(Z, X)$ of $[\varphi \circ f]$ under the map

$$\text{Hom}(Z, X) \to \text{Hom}(Z, Y).$$

Suppose that $\mathcal{F}$ is regular. Then we have an embedding $\left(\text{Def}([f], \mathcal{F}), [f]\right) \subseteq \left(\text{Hom}_Y(Z, X), [f]\right)$ of pointed analytic spaces but they are not isomorphic in general. Indeed, suppose that $\dim Y = 1$. Let $y$ be a point on $Y$, and set $F := \varphi^{-1}(y)_{\text{red}}$. Suppose that the multiplicity $m$ of $F$ is $> 1$. Let $Z$ be a reduced point $\{z\}$, and suppose that $f(z) \subset F$. Then $\left(\text{Def}([f], \mathcal{F}), [f]\right)$ isomorphic to $(F, z) \cong \left(\text{Hom}_Y(Z, X)_{\text{red}}, [f]\right)$ while $\left(\text{Hom}_Y(Z, X), [f]\right) \cong (\varphi^{-1}(y), z)$. 

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The following observation will prove to be crucial. It is due to Loray, Pereira and Touzet (see proof of [14, Proposition 6.12]).

**Notation 3.3.** — Let \((A, a)\) be a pointed analytic space. We denote by \(\hat{A}\) the formal completion of \(A\) at \(a\). Given a morphism of pointed analytic spaces \(\lambda: (A, a) \to (B, b)\), we denote by \(\hat{\lambda}: \hat{A} \to \hat{B}\) the induced morphism of formal analytic spaces.

**Lemma 3.4.** — Let \(Y\) and \(X\) be complex projective manifolds, and let \(\mathcal{F} \subseteq T_X\) be a regular foliation. Let \(f: Y \to X\) be a morphism, and let \(y\) be a point on \(Y\). Then the Zariski closure \(T\) of \(\text{Def}([f], \mathcal{F}; f|_Y)\) parametrizes deformations of \([f]\) along \(\mathcal{F}\), i.e., for each \(y' \in Y\), \(ev(T \times \{y'\})\) is tangent to \(\mathcal{F}\), where \(ev: \text{Hom}(Y, X; f|_Y) \times Y \to X\) denotes the evaluation morphism.

**Proof.** — Set \(x := f(y)\), and let \(U\) be an open neighborhood of \(x\) in \(X\) with respect to the Euclidean topology such that \(\mathcal{F}|_U\) is induced by a submersion \(\varphi: U \to W\) of complex analytic spaces. Let \(\hat{T}\) be the connected component containing \([f]\) of the Zariski closed subset

\[
\left\{ [h] \in \text{Hom}(Y, X; f|_Y) \, | \, \varphi \circ h = \varphi \circ f: \hat{Y} \to \hat{W} \right\} \subset \text{Hom}(Y, X; f|_Y).
\]

Notice that \(T \subset \hat{T}\). Let \([h] \in \hat{T}\), and consider an open neighborhood \(V\) of \(y\) and an open neighborhood \(H\) of \([h]\) in \(\hat{T}\) (with respect to the Euclidean topology) such that for each \([h'] \in H\) and each \(y' \in V\), we have \(h'(y') \in U\). If \([h'] \in H\), then

\[
\varphi \circ (h'|_V) = \varphi \circ (h|_V): V \to W \quad \text{since} \quad \varphi \circ h' = \varphi \circ f = \varphi \circ h: \hat{Y} \to \hat{W}.
\]

This implies that \(ev(H \times \{y'\})\) is tangent to \(\mathcal{F}\) for each \(y' \in V\), and hence so is \(ev(\hat{T} \times \{y'\})\). Since the set of points \(y' \in Y\) such that \(ev(\hat{T} \times \{y'\})\) is tangent to \(\mathcal{F}\) is Zariski closed in \(Y\), we conclude that \(ev(\hat{T} \times \{y'\})\) is tangent to \(\mathcal{F}\) for any \(y' \in Y\). This proves the lemma. \(\square\)

**Remark 3.5.** — One might ask whether Lemma 3.4 holds for a larger class of foliations. What we actually proved is the following. If \(\mathcal{F}\) is induced on an open neighborhood \(U\) of \(y\) (with respect to the Euclidean topology) by a holomorphic map \(U \to V\) of complex spaces, then the conclusion of Lemma 3.4 holds.

The following lemma provides a lower bound for the dimension of \(\text{Def}([f], \mathcal{F}; f|_B)\) at a point \([f]\), thereby allowing us in certain situations to produce many deformations of \(f\) (see Proposition 1.6).
Lemma 3.6. — Let $X$ be a complex projective manifold, and let $\mathcal{F} \subseteq T_X$ be a regular rank $r$ foliation on $X$. Let $f : C \to X$ be a smooth curve, and let $B$ be a finite subscheme of $C$. Then
\[
\dim_{[f]} \text{Def}([f], \mathcal{F}; f_{1[B]}) \geq -K_{\mathcal{F}} \cdot f_* C + (1 - g(C) - \ell(B)) \cdot r.
\]

Proof. — Let $(\mathcal{O}, m)$ be local ring of the germ of analytic space $\text{Def}([f], \mathcal{F}; f_{1[B]})$ at $[f]$, and let $\hat{\mathcal{O}}$ be its $m$-adic completion. Then $\hat{\mathcal{O}}$ represents the functor of infinitesimal deformations of $[f]$ along $\mathcal{F}$ with fixed subscheme $B$. We refer to [15, Section 6] for the definition of this functor. The lemma then follows from [15, Theorem 6.2] (see also [15, Corollary 6.6]).

The proof of Proposition 3.7 below is very similar to that of [7, Proposition 3.1] (see also [14, Proposition 6.13]), and so we leave some easy details to the reader.

Proposition 3.7. — Let $X$ be a complex projective manifold, and let $\mathcal{F} \subseteq T_X$ be a regular foliation. Let $f : C \to X$ be a smooth complete curve, and let $c$ be a point on $C$. If $C \cong \mathbb{P}^1$, suppose that $f(C)$ is transverse to $\mathcal{F}$ at a general point on $f(C)$. Suppose furthermore that $\dim_{[f]} \text{Def}([f], \mathcal{F}; f_{1[c]}) \geq 1$. There exist a morphism $g : C \to X$, a nonzero effective rational 1-cycle $Z$ on $X$ passing through $f(c)$ such that $f_* C \equiv g_* C + Z$ and such that $\text{Supp}(Z)$ is tangent to $\mathcal{F}$.

Proof. — Denote by $\text{Def}([f], \mathcal{F}; f_{1[c]})_{\text{red}}$ the Zariski closure of $\text{Def}([f], \mathcal{F}; f_{1[c]})_{\text{red}}$. Let $T \to \text{Def}([f], \mathcal{F}; f_{1[c]})_{\text{red}}$ be the normalization of a 1-dimensional subvariety passing through $[f]$, and let $\overline{T}$ be a smooth compactification. Let $\epsilon : S \xrightarrow{\epsilon} C \times \overline{T} \xrightarrow{\text{ev}} X$ be a resolution of the indeterminacies of the rational map $\text{ev} : C \times \overline{T} \dashrightarrow X$ coming from $T \to \text{Hom}(C, X; f_{1[c]})$, where $\epsilon : S \to C \times \overline{T}$ is obtained by blowing-up points. From the rigidity lemma, we conclude that there exists a point $t_0 \in \overline{T}$ such that $\epsilon$ is not defined at $(c, t_0)$. The fiber of $t_0$ under the projection $S \to \overline{T}$ is the union of the strict transform of $C \times \{t_0\}$ and a (connected) exceptional rational 1-cycle $E$ which is not entirely contracted by $\epsilon$ and meets the strict transform of $\{c\} \times T$. Since the latter is contracted by $\epsilon$ to the point $f(c)$, the rational 1-cycle $Z := \epsilon_* E$ passes through $f(c)$.

By Lemma 3.4, $\text{Def}([f], \mathcal{F}; f_{1[c]})_{\text{red}}$ parametrizes deformations of $[f]$ along $\mathcal{F}$. Therefore, if $C$ is transverse to $\mathcal{F}$ at a general point on $C$, $\text{Aut}(C, c) \cdot [f]$ and $\text{Def}([f], \mathcal{F}; f_{1[c]})_{\text{red}}$ intersect at finitely many points in $\text{Hom}(C, X; f_{1[c]})$. If $C$ is irrational, then the orbit $\text{Aut}(C, c) \cdot [f]$ is finite because the group $\text{Aut}(C, c)$ is. In either case, we conclude that $\dim \epsilon(S) = 2$.

Let $\mathcal{G} \subseteq T_{C \times T}$ be the foliation on $C \times \overline{T}$ induced by $\epsilon^* \mathcal{F} \cap T_{C \times T}$, and set $\mathcal{G}_S := \epsilon^{-1}(\mathcal{G})$. If $C$ is tangent to $\mathcal{F}$, then $\mathcal{G} = T_{C \times T}$ (and hence $\mathcal{G}_S = T_S$).
If $C$ is transverse to $\mathcal{F}$ at a general point on $C$, then $\mathcal{G}$ is induced by the projection $C \times \overline{\mathcal{F}} \to C$. In either case, any $\varepsilon$-exceptional curve is tangent to $\mathcal{G}_S$. Hence $\text{Supp}(Z)$ is tangent to $\mathcal{F}$. This completes the proof of the proposition. 

**Proof of Proposition 1.6.** — Let $X$ be a complex projective manifold, and let $\mathcal{F} \subset T_X$ be a regular foliation with normal bundle $\mathcal{N}$. Let $C \subset X$ be a rational curve with $\det(\mathcal{N}) \cdot C \neq 0$, and let $x$ be a point on $C$. Suppose that $-K_{\mathcal{F}} \cdot C \geq 1$. Let $f : \mathbb{P}^1 \to C \subset X$ be the normalization morphism, and let $p \in \mathbb{P}^1$ such that $f(p) = x$. Notice that $C$ is transverse to $\mathcal{F}$ at a general point on $C$ by Lemma 2.6. By Lemma 3.6, we have \[
\dim([f], \mathcal{F}; f_{\{p\}}) \geq -K_{\mathcal{F}} \cdot C \geq 1
\] so that Proposition 3.7 applies. There exist a morphism $g : \mathbb{P}^1 \to X$ and a nonzero effective rational 1-cycle $Z$ on $X$ such that $f^*\mathbb{P}^1 \equiv g_*\mathbb{P}^1 + Z$, and such that $\text{Supp}(Z)$ is tangent to $\mathcal{F}$. From Lemma 2.6 again, we deduce that $\det(\mathcal{N}) \cdot Z = 0$. Thus \[
0 \neq \det(\mathcal{N}) \cdot f_*\mathbb{P}^1 = \det(\mathcal{N}) \cdot g_*\mathbb{P}^1 + \det(\mathcal{N}) \cdot Z = \det(\mathcal{N}) \cdot g_*\mathbb{P}^1.
\] In particular, $g$ is a nonconstant morphism. Set $C_1 := g(\mathbb{P}^1)$ and $m := \deg(g)$. Then $C \equiv mC_1 + Z$, completing the proof of the proposition. 

We now provide a technical tool for the proof of the main result.

**Corollary 3.8.** — Let $X$ be a complex projective manifold, and let $\mathcal{F} \subset T_X$ be a regular foliation with normal bundle $\mathcal{N}$. Suppose that $-K_X$ is nef. If $C \subset X$ is a rational curve, then $\det(\mathcal{N}) \cdot C \geq 0$.

**Proof.** — Set $\mathcal{L} := \det(\mathcal{N})$, and pick an ample divisor $H$ on $X$. We argue by contradiction, and assume that $\mathcal{L} \cdot C < 0$ for some rational curve $C$. We have $-K_{\mathcal{F}} \cdot C = -K_X \cdot C - \mathcal{L} \cdot C \geq 1$ so that Proposition 1.6 applies. There exist a nonzero effective rational 1-cyle $Z$, a rational curve $C_1$, and a positive integer $m$ such that $C \equiv mC_1 + Z$ and such that $\text{Supp}(Z)$ is tangent to $\mathcal{F}$. Notice that $H \cdot C_1 < H \cdot C$. By Lemma 2.6, we have \[
\mathcal{L} \cdot C_1 = \frac{1}{m} \mathcal{L} \cdot (mC_1 + Z) = \frac{1}{m} \mathcal{L} \cdot C < 0.
\] This construction yields an infinite sequence of rational curves on $X$ with decreasing $H$-degrees. This is absurd and the corollary is proved. 

Let $X$ be a complex projective manifold and consider the finite dimensional $\mathbb{R}$-vector space \[
N_1(X) = (\{1 - \text{cycles}\}/\equiv) \otimes \mathbb{R},
\] where $\equiv$ denotes numerical equivalence. Recall that the *Mori cone* of $X$ is the closure $\overline{\text{NE}}(X) \subset N_1(X)$ of the cone spanned by classes of effective...
curves. An extremal ray is a subcone $R \subset \overline{\text{NE}}(X)$ of dimension 1 such that any two elements of $\overline{\text{NE}}(X)$ whose sum is in $R$ are both in $R$.

We believe that the following result will be useful when considering regular foliations on arbitrary projective manifold. Its proof is similar to that of Corollary 3.8 above.

**Corollary 3.9.** — Let $X$ be a complex projective manifold, and let $\mathcal{F} \subset T_X$ be a regular foliation with normal bundle $\mathcal{N}$. Let $C \subset X$ be a rational curve with $\det(\mathcal{N}) \cdot C \neq 0$. If $[C] \in \overline{\text{NE}}(X)$ generates an extremal ray, then $K_{\mathcal{F}} \cdot C \geq 0$.

**Proof.** — Pick an ample divisor $H$ on $X$. Let us assume to the contrary that $-K_{\mathcal{F}} \cdot C \geq 1$. By Proposition 1.6, $C$ is numerically equivalent to a connected nonintegral effective rational 1-cycle. Thus, there exists a rational curve $C_1$ on $X$ with $[C_1] \in \mathbb{R}^+[C]$ and such that $H \cdot C_1 < H \cdot C$. Since $[C_1] \in \mathbb{R}^+[C]$, we must have $-K_{\mathcal{F}} \cdot C_1 \geq 1$. This construction yields an infinite sequence of rational curves on $X$ with decreasing $H$-degrees. This is absurd, proving the corollary.

\[ \square \]

**4. Proof of Theorem 1.3**

We are now in position to prove our main result.

**Proof of Theorem 1.3.** — Set $\mathcal{N} = T_X / \mathcal{F}$, and denote by $q$ its rank. Suppose that $0 < q < \dim X$, and set $\mathcal{L} = \det(\mathcal{N})$.

By the cone theorem, there exist finitely many rational curves $C_1, \ldots, C_m$ such that

\[ \overline{\text{NE}}(X) = \mathbb{R}^+[C_1] + \cdots + \mathbb{R}^+[C_m] \]

where the $\mathbb{R}^+[C_i]$ are the extremal rays of $\overline{\text{NE}}(X)$ ([12, Theorem 3.7]). By Corollary 3.8, $\mathcal{L} \cdot C_i \geq 0$ for any $1 \leq i \leq m$, and thus $\mathcal{L}$ is nef. By the base-point-free theorem (see [12, Theorem 3.3]), the line bundle $\mathcal{L}^\otimes m$ is globally generated for all integers $m$ sufficiently large. Let $\varphi: X \to Y$ be the induced morphism.

We will show that $\mathcal{F}$ is induced by $\varphi$. By [2, Corollary 3.4], we have $\mathcal{L}^{q+1} \equiv 0$, and hence $\dim Y \leq q$. Let $F$ be a general fiber of $\varphi$. Notice that $F$ is a smooth projective variety with $-K_F = (-K_X)|_F$ nef and big by the adjunction formula, and that $\mathcal{L}|_F \equiv 0$. By [17], $F$ is simply connected and $h^0(F, \Omega_F) = 0$ for all $1 \leq i \leq \dim F$, so that Lemma 2.4 applies. We have $\dim Y \geq q$, and $F$ is tangent to $\mathcal{F}$. This in turn implies that $\dim Y = q$, and that $\mathcal{F}$ is induced by $\varphi$. By Lemma 4.1 below, we infer that $\varphi$ is a smooth morphism, completing the proof of the theorem.

\[ \square \]
Lemma 4.1. — Let $X$ be a complex projective manifold, and let $\varphi : X \to Y$ be a surjective morphism with connected fibers onto a normal projective variety $Y$. Suppose that $-K_X$ is $\varphi$-nef and $\varphi$-big. Suppose furthermore that the foliation $\mathcal{F}$ on $X$ induced by $\varphi$ is regular. Then $\varphi$ is a smooth morphism.

Proof. — Pick $x \in X$, and set $y := \varphi(x)$ and $F_0 := \varphi^{-1}(y)_{\text{red}}$. By [10, Proposition 2.5], $F_0$ has finite holonomy group $G$. By the holomorphic version of Reeb stability theorem (see [10, Theorem 2.4]), there exist a saturated open neighborhood $U$ of $F_0$ in $X$ with respect to the Euclidean topology, a (local) transversal section $S$ at $x$ with a $G$-action, an unramified Galois cover $\hat{U} \to U$ with group $G$, a smooth proper $G$-equivariant morphism $\hat{U} \to S$, an isomorphism $S/G \cong \varphi(U)$, and a commutative diagram:

$$
\begin{array}{ccc}
\hat{U} & \xrightarrow{p} & U, \\
\downarrow{\hat{\varphi}} & & \downarrow{\varphi} \\
S & \xrightarrow{q} & S/G \cong \varphi(U).
\end{array}
$$

Recall that $G$ is given by the holonomy representation

$$
\pi_1(F_0, x) \to \text{Diff}(S, x).
$$

Set $\widehat{F}_0 := \hat{\varphi}^{-1}(x)_{\text{red}}$, and consider a general fiber $\widehat{F}$ of $\hat{\varphi}$. Notice that $-K_{\widehat{U}} \cong -p^*K_U$ is $\hat{\varphi}$-nef and $\hat{\varphi}$-big. It follows that $-K_{\widehat{F}}$ is nef and big. Since $K_{\widehat{F}}^{\dim \widehat{F}_0} = K_{\widehat{F}}^{\dim \widehat{F}}$, we infer that $-K_{\widehat{F}_0}$ is nef and big as well. Since the restriction of $q$ to $\widehat{F}_0$ induces an étale morphism $q|_{\widehat{F}_0} : \widehat{F}_0 \to F_0$ of projective manifolds, we conclude that $-K_{F_0}$ is also nef and big. By [17], we must have $\pi_1(F_0, x) = \{1\}$. Therefore, the holonomy group $G$ is trivial, and $\varphi$ is a smooth morphism. This proves the lemma. \qed

Question 4.2. — Let $X$ be a complex projective manifold, and let $\mathcal{F}$ be a regular foliation on $X$. Suppose that $h^1(X, \mathcal{O}_X) = 0$, and that $-K_X$ is nef. Is $\mathcal{F}$ algebraically integrable?

Bibliography

Regular foliations on weak Fano manifolds