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Transverse nonlinear instability of Euler–Korteweg solitons (*)

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Abstract. — We show that solitary waves for the 2D Euler–Korteweg model for capillary fluids display nonlinear orbital instability when subjected to transverse perturbations, based on their linear instability.

Résumé. — On montre que les solitons de l'équation d'Euler–Korteweg 2D, un modèle pour les fluides avec capillarité, sont orbitalement instables lorsqu'ils sont soumis à des perturbations transverses, en partant de leur instabilité linéaire.

1. Introduction

We consider the motion of a compressible, inviscid and isentropic planar fluid, in which internal capillarity is taken into account. This latter phenomenon occurs for example at diffuse interfaces in liquid-vapour mixes [8]. In this model, the free energy of the fluid depends on both the density of the fluid, the scalar function ρ, and its gradient $\nabla\rho$ in the following way:

$$F(\rho, \nabla\rho) = F_0(\rho) + \frac{1}{2}K(\rho)|\nabla\rho|^2,$$

with K and F_0 two given smooth, positive functions for $\rho > 0$. We then derive the pressure from the free energy like so,

$$P(\rho, \nabla\rho) = \rho\frac{\partial F}{\partial \rho} - F = P_0(\rho) + \frac{1}{2}(\rho K'(\rho) - K(\rho))|\nabla\rho|^2,$$

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in which P_0 is the standard part of the pressure - in the physical context, one should think of a van der Waals-type law. The remainder of P models the capillarity effects.

Let $g_0(\rho)$ be the bulk chemical potential of the fluid, so that $\rho g_0'(\rho) = P_0'(\rho)$. Then, the principles of classical mechanics yield a system of two partial differential equations representing the conservation of mass and momentum, the Euler–Korteweg equation that we will study:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t u + (u \cdot \nabla) u = \nabla \left(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2 - g_0(\rho) \right) \\ \qquad (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases} \tag{1.1}$$

The variables of the system are $t \in \mathbb{R}^+$ and $(x, y) \in \mathbb{R}^2$; as is standard, the operators ∇, div and Δ contain only derivatives with respect to the space variables x and y. The unknowns of equation (1.1) are the density ρ and the velocity vector field $u : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$. The scalar functions g_0 and K are given, smooth and positive for $\rho > 0$. We recall that the fluid is isentropic, hence the temperature is assumed to be constant.

In this paper, we will be interested in the transverse stability of solitary wave solutions of (1.1). These are 1D travelling waves written as

$$Q_c(t, x) = \begin{pmatrix} \rho_c(t, x) \\ u_c(t, x) \end{pmatrix} = q_c(x - ct),$$

with u_c scalar (not a 2D vector field). Lower-case q_c designates the wave profile.

Based on a remark by T. Benjamin [4], S. Benzoni-Gavage, R. Danchin, S. Descombes and D. Jamet showed in [8] that the hamiltonian structure of the system led to the existence of travelling wave solutions for every $c \in \mathbb{R}$ and for any pair of endstates (limits at $+$ and $-\infty$) satisfying a Rankine–Hugoniot-type condition. We will consider waves with identical endstates, such that

$$\lim_{|z| \to +\infty} q_c(z) = q_\infty = (\rho_\infty, u_\infty),$$

and such a travelling wave solution is called a soliton. From now on, we set $c \in \mathbb{R}$ and Q_c a soliton such that the endstate satisfies

$$\rho_\infty g_0'(\rho_\infty) > (u_\infty - c)^2, \tag{1.2}$$

which means that q_∞ is a saddle point for the hamiltonian ODE solved by q_c. Under this condition, we have that ρ_c' vanishes only once, the density is a single bump, symmetric with respect to the extremum, and converges exponentially towards the endstate as $|x| \to +\infty$. Meanwhile, when endstates are different, travelling wave profiles are monotonous, and these solutions are called kinks. See [8] for details.

The standard Lyapunov stability notion is that if solutions of equation (1.1) have initial conditions close to $q_c(x)$, then they remain close to $q_c(x-ct)$ at all times. But this notion is not satisfactory in describing the stability of travelling waves. Indeed, let $c' \neq c$ be close to c. The function $q_{c'}$ is close to q_c, but, as the speeds are different, $q_{c'}(x-c't)$ and $q_c(x-ct)$ drift apart, despite their profiles remaining very similar. To see this, for a given t, compare $q_{c'}(x-c't)$ with the translated profile $q_c(x-c't) = q_c(x-ct+(c-c')t)$. The correct notion of stability therefore stems from taking the difference of solutions with all the translated versions of $q_c(x-ct)$. A travelling wave solution will be considered stable if it is *orbitally stable*: for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|(\rho_0, u_0) - q_c\|_E \leqslant \delta$ in a certain function space E, then a solution of the PDE with the initial condition (ρ_0, u_0) exists globally in time in another space F, and

$$\sup_{t\in\mathbb{R}^+} \inf_{a\in\mathbb{R}} \|(\rho(t), u(t)) - q_c(\cdot - a - ct)\|_F \leqslant \varepsilon.$$

It is worth noting that, for the Euler–Korteweg system, the first element of the stability criterion, global existence, is not guaranteed. We are looking at strong, smooth solutions, and in this framework, S. Benzoni-Gavage, R. Danchin and S. Descombes proved in [7] the local well-posedness of the Euler–Korteweg system in a neighbourhood in H^s of a reference solution with derivatives in H^{s+3}, as well as a blow-up criterion. In three dimensions and higher, global well-posedness for small irrotational data (potential velocity) was recently obtained by C. Audiard and B. Haspot [3], but, to our knowledge, the Benzoni-Gavage, Danchin and Descombes result is the best available in 2D. The approach by Audiard and Haspot is based on scattering methods, and, in [2], the authors cite the existence of low-energy travelling waves as one obstruction to their technique in 2D, meaning that global existence is still elusive.

On the subject of the Cauchy problem, we can also refer to D. Donatelli, E. Feireisl and P. Marcati [10] for a study of weak solutions to the Euler–Korteweg–Poisson system on a 3D torus. In their model, the fluid's velocity is coupled with the gradient of a function V such that $\Delta V = \rho$. They then prove that the initial-value problem has an infinite number of weak solutions.

The problem of orbital stability can be divided into two parts, depending on the type of perturbation we consider. 1D perturbations are perturbations of Q_c that depend only on x and satisfy $u_2(t,x) = 0$. The stability problem associated with these perturbations has been in part dealt with by Benzoni-Gavage et al. in [8], and improved upon by Benzoni-Gavage in [6]. A sufficient condition for orbital stability was obtained in the first paper using an argument by M. Grillakis, J. Shatah and W. Strauss [13], while the second article adds a sufficient condition for linear instability. Kinks were also studied in [8]; they were shown to be orbitally stable. See also J. Höwing [15, 16] for other stability results for the 1D Euler–Korteweg system.

The question of transverse stability deals with perturbations that also depend on the transverse variable y and have a 2D velocity field. So far, Benzoni-Gavage in [6] and F. Rousset and N. Tzvetkov in [23] have proved linear instability. This occurs when the linearised equation around Q_c has eigenvalues with positive real part. On one hand, Benzoni-Gavage used Evans functions computations to get that 1D-orbitally stable solitons are transversally linearly unstable. On the other hand, Rousset and Tzvetkov applied an abstract criterion to get instability for linearised PDEs with a hamiltonian structure in the case where the endstate of the soliton satisfies (1.2). We recall it in Theorem 2.2. This criterion was applied to other equations with solitary waves in the same article, namely KP-I and Gross–Pitaevskii.

The result of this paper is that the spectral instability mentioned above implies nonlinear instability of Euler–Korteweg solitons.

Theorem 1.1. — *Let $Q_c(t,x) = q_c(x-ct)$ be a soliton solution to* (1.1) *such that the endstate q_∞ satisfies* (1.2)*. Then there exist δ_0 and $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, there exists an initial condition $U_0 = (\rho_0, u_0)$ with*

$$\|U_0 - q_c\|_{H^s(\mathbb{R}^2)} \leqslant \varepsilon$$

for some $s > 0$, such that, for every $a \in \mathbb{R}$, the solution $U = (\rho, u)$ of (1.1) *with this initial condition satisfies, at a time $T^\varepsilon \sim \ln(\varepsilon^{-1})$,*

$$\|U(T^\varepsilon) - q_c(\cdot - cT^\varepsilon - a)\|_{L^2(\mathbb{R}^2)} \geqslant \delta_0.$$

Moreover, the velocity u can be chosen to be potential: $u = \nabla\varphi$.

The proof relies on an argument originally by E. Grenier [12], in which one constructs an approximate solution U^{ap} to the equation based on a WKB expansion starting with the reference solution plus ε times a wavepacket containing the most unstable eigenmodes of the linearised equation. One must then control the growth of the following terms so that, if there are enough terms in the expansion, U^{ap} does indeed approximate U and, for times lower than a certain T^ε, the linear instability is dominant. Primarily used to obtain nonlinear instability of boundary layers in numerous settings (unstable Euler shear flows and Prandtl layers [12], Ekman layers for rotating fluids [9, 20, 17], Ekman–Hartman layers in MHD [9], Navier–Stokes with a boundary-layer-scale slip condition [18]), the idea has been transposed to showing transverse nonlinear instability of solitary waves, when these can be shown to be linearly unstable. F. Rousset and N. Tzvetkov have thus obtained nonlinear instability of solitary waves in many models: KP-I and

NLS [22], multiple hamiltonian models including generalised KP-I and the Boussinesq equations [21], and the free-surface water-waves equation [24].

To apply this method to the Euler–Korteweg system, we face the added difficulty that solutions are thus far not known to exist around 1-dimensional travelling waves in 2D: the result in Benzoni-Gavage, Danchin and Descombes's paper [7] deals with perturbations of a reference solution which has square-integrable derivatives. However, we will see that a small technical change in their proof allows to generalise it, and we will readily use their energy estimates to get that U^{ap} is effectively an approximate solution on a timescale $\mathcal{O}(|\ln(\varepsilon)|)$, also ensuring existence of the perturbed solution up to the time at which nonlinear instability is observed. We stress that this amplification is not related in any way to a possible blow-up; it is driven by the linear instability, thus this phenomenon can be observed even in contexts where solutions are global, and many aforementioned models fall in this category.

We end the introduction with a couple of remarks. First, our result is valid in some y-periodic settings, namely on the domain $\mathbb{R} \times \mathbb{T}_\tau$, where $\tau > 0$ is any period such that there exists $N \in \mathbb{N}$ with $k_N = 2N\pi/\tau$ in the set of unstable wavenumbers. The proof is technically a little simpler but essentially identical, and several analoguous solitary wave instability results are stated this way, for example F. Rousset and N. Tzvetkov's one on the KdV waves in KP-I [22]. The description of the set of unstable wavenumbers for Euler–Korteweg is not clear though, unlike KP-I for instance, for which explicit unstable eigenmodes were computed by J. Alexander, R. Pego and R. Sachs [1]. In fact, Rousset and Tzvetkov obtain in [25] a sharp transition from nonlinear instability to nonlinear stability when τ dips below $4\sqrt{3}$ (for a wave travelling at speed 1), due to the loss of linear instability when k is large. A similar nonlinear stability result for short periods in the Euler–Korteweg system would be an interesting development.

Secondly, we make a short note regarding propagating phase boundaries. S. Benzoni-Gavage showed in [5] that the linearised equation around this type of kink is weakly spectrally stable (it only has purely imaginary spectrum) no matter the dimension, and in fact, kinks are orbitally (nonlinearly) stable in 1D [8]. Since the method we employ relies on strict linear instability to drive the behaviour of the approximate solution, it cannot be used to show transverse nonlinear instability of kinks.

Outline of the proof. — The proof of Theorem 1.1 is in two parts. First, in section 2, we build on Rousset and Tzvetkov's linear instability theorem to obtain more necessary information on the Euler–Korteweg system linearised around the soliton Q_c. This will allow us, in section 3, to build an approximate solution U^{ap} with the appropriate behaviour of being predominantly unstable for $t \sim T^{\varepsilon}$. An energy estimate on $U - U^{\mathrm{ap}}$ will then be used to get that the time of existence of U is large enough to get the desired amplification. Combining the two will lead to the instability result. $\square$

2. Linear analysis

Considering that u is potential, we write the system on (ρ, φ), where $u = \nabla\varphi$:

$$\begin{cases} \partial_t \rho + \nabla\varphi \cdot \nabla\rho + \rho\Delta\varphi = 0 \\ \qquad \partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2 = K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2 - g_0(\rho), \end{cases}$$

which we linearise around $Q_c = (\rho_c, u_c)$. Having changed the space variable from x to $x - ct$ (which turns the solitary wave into a stationary solution), we are interested in

$$\begin{cases} \partial_t \rho = (c\partial_x - u_c\partial_x - u_c')\rho - (\rho_c'\partial_x + \rho_c\Delta)\varphi \\ \partial_t\varphi = (c\partial_x - u_c\partial_x)\varphi + (K(\rho_c)\Delta + K'(\rho_c)\rho_c'\partial_x - m)\rho, \end{cases} \tag{2.1}$$

with $m = g_0'(\rho_c) - K'(\rho_c)\rho_c'' - \frac{1}{2}K''(\rho_c)(\rho_c')^2$. We abbreviate the system by defining two operators

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\mathcal{L} = \begin{pmatrix} -\partial_x(K(\rho_c)\partial_x) - K(\rho_c)\partial_{yy}^2 + m & (u_c - c)\partial_x \\ -\partial_x((u_c - c)\cdot) & -\partial_x(\rho_c\partial_x) - \rho_c\partial_{yy}^2 \end{pmatrix},$$

thus the system (2.1) can be summed up as $\partial_t V = J\mathcal{L}V$, where $V = (\rho, \varphi)$.

The first part of the linear analysis involves finding unstable eigenmodes for (2.1). These are non-trivial solutions to the equation that can be written as $V(t,x,y) = e^{\sigma t}e^{iky}v(x)$ for $k \neq 0$ and $\mathrm{Re}(\sigma) > 0$. Rewriting (2.1) on V equates to using the Fourier transform on the transverse variable y, and the equation $\partial_t V = J\mathcal{L}V$ becomes $\sigma v = JL(k)v$ with

$$L(k) = \begin{pmatrix} -\partial_x(K(\rho_c)\partial_x) + K(\rho_c)k^2 + m & (u_c - c)\partial_x \\ -\partial_x((u_c - c)\cdot) & -\partial_x(\rho_c\partial_x) + \rho_c k^2 \end{pmatrix}.$$

We begin by examining the existence of eigenmodes and the behaviour of σ depending on k, and we follow up with an important resolvent estimate for $JL(k)$. The properties we need are summed up in the following proposition.

Proposition 2.1 (Properties of the linearised equation). —

(a) The linearised equation is unstable, that is there exist eigenmodes written as $V(t,x,y) = e^{\sigma t}e^{iky}v(x)$*, with* $v \in H^2(\mathbb{R})$ *and* $\mathrm{Re}(\sigma) > 0$*, that solve* (2.1)*. For each* k*, the dimension of the subspace of unstable solutions of* $\sigma v = JL(k)v$ *is at most 1.*

The instability is localised in the transverse Fourier space: there exists $k_{\max} > 0$ *such that, for* $|k| \geqslant k_{\max}$*, eigenvalues necessarily satisfy* $\mathrm{Re}(\sigma) \leqslant 0$*. Let* $\sigma(k)$ *be the eigenvalue of* $JL(k)$ *with highest real part. Then the even, continuous function* $k \mapsto \mathrm{Re}(\sigma(k))$ *has a global maximum* $\sigma_0 > 0$ *at a certain* $k_0 > 0$*.*

(b) If $V(t,x,y) = e^{iky}U(t,x)$*, we define the following semi-norm for* U*:*

$$\|U(t)\|^2_{X^j_k} = \|U_1(t)\|^2_{H^{j+1}(\mathbb{R})} + \|\partial_x U_2(t)\|^2_{H^j(\mathbb{R})} + |k|^2 \|U(t)\|^2_{H^j(\mathbb{R})}.$$

It is essentially the H^j *norm of* $|k|U(t)$ *plus the* H^{j+1} *norm of* $U(t)$*, omitting the* L^2 *norm of* $U_2(t)$*.*

Set $\gamma > \sigma_0$*,* $\tilde{k} > 0$*,* $n \in [0, +\infty[$*,* $s \in \mathbb{N}$ *and a function* F *satisfying, for every* $j \leqslant s$ *and* $|k| \leqslant \tilde{k}$

$$\left\|\partial_t^{s-j} F(k,t)\right\|_{H^{j+1}(\mathbb{R})} \leqslant M_s \frac{e^{\gamma t}}{(1+t)^n}, \tag{2.2}$$

for a constant M_s *which does not depend on* k*. Finally, let* U *solve*

$$\partial_t U(t,x) = JL(k)U(t,x) + F(t,x,k), \tag{2.3}$$

with $U(0,x) = 0$ *for a certain* $|k| \leqslant \tilde{k}$*. Then* U *satisfies similar bounds: there exists* $C_s > 0$*, depending on* s *and* $\tilde{k}$*, such that for every* $j \leqslant s$*, we have*

$$\left\|\partial_t^{s-j} U(t)\right\|_{X^j_k} \leqslant C_s \frac{e^{\gamma t}}{(1+t)^n}. \tag{2.4}$$

Remarks on part (b). — A quick energy estimate on the equation of U_2 yields that $U_2(t) \in L^2(\mathbb{R})$ (as $\varphi|_{t=0} = 0$), and this L^2 norm also satisfies (2.4). We will therefore subsequently consider that the result is valid in H^s, for any $s \geqslant 0$.

By the Parseval equality, this result also implies identical H^s bounds for finite Fourier sums or wavepackets written as $(\rho,\varphi) = \int_{\mathbb{R}} f(k)e^{iky}U(k,t,x)\,dk$, with $f \in \mathcal{C}_0^\infty(\mathbb{R})$. Indeed, norms of $|k|^2 U$ can be replaced, using equation (2.3), by derivatives on x and t that satisfy (2.4).

2.1. Proof of Proposition 2.1(a), properties of eigenmodes

2.1.1. Existence of unstable eigenmodes.

The existence of unstable eigenmodes was shown by F. Rousset and N. Tzvetkov [23] using a general criterion for detecting transverse linear instability of solitary waves in hamiltonian PDEs. We have seen that equation (2.1) for functions written as $V(t,x,y)=e^{\sigma t}e^{iky}v(x)$ becomes an eigenvalue problem, that is

$$\sigma v = JL(k)v \tag{2.5}$$

with J a skew-symmetric matrix and $L(k)$ a self-adjoint differential operator on $(L^2(\mathbb{R}))^2$ whose domain is seen to be $(H^2(\mathbb{R}))^2$. We have the following result for such systems.

Theorem 2.2 (Rousset and Tzvetkov, [23]). — *If L has the following properties:*

- (H1) – *there exists $k_{\max}>0$ and $\alpha>0$ such that $L(k)\geqslant \alpha\mathrm{Id}$ for $|k|\geqslant k_{\max}$;*
- (H2) – *for every $k\neq 0$, the essential spectrum of $L(k)$ is included in $[\alpha_k,+\infty[$ with $\alpha_k>0$;*
- (H3) – *$L'(k)$ is a positive operator;*
- (H4) – *the spectrum of $L(0)$ consists of one isolated negative eigenvalue $-\lambda$ and a subset of $\mathbb{R}^+$;*

then there exist $\sigma>0$ and $k\neq 0$ such that (2.5) *has a non-trivial solution, and, for every unstable wavenumber k, such an eigenvalue σ is unique.*

This is shown by finding $k'>0$ such that $L(k')$ has a one-dimensional kernel, and by using the Lyapunov–Schmidt method in the vicinity of this point; we do not detail the proof of this theorem. Proof that the linearised Euler–Korteweg system satisfies the hypotheses of this theorem was also done in [23], but we shall briefly recall this, as it contains some useful arguments for the subsequent points of Proposition 2.1(a).

(H1). — Using Young's inequality,

$$ab\leqslant \frac{\delta}{2}a^2+\frac{1}{2\delta}b^2, \tag{2.6}$$

with $\delta=\frac{K(\rho_c)}{2}$, we quickly get that

$$(L(k)v,v)\geqslant \int_{\mathbb{R}} \frac{K(\rho_c)}{2}|\partial_x v_1|^2 + \left(k^2-\frac{1}{2}\right)|v_1|^2+\rho_c|\partial_x v_2|^2+\left(k^2-\frac{1}{K(\rho_c)}\right)|v_2|^2, \tag{2.7}$$

which is greater than $\alpha \|v\|_{L^2}^2$ for $|k|$ large enough (remember that ρ_c and $K(\rho_c)$ are positive).

(H2). — As $\lim_{x\to\pm\infty} q_c(x) = q_\infty = (\rho_\infty, u_\infty)$, with standard arguments [14], and using the fact that $L(k)$ is self-adjoint, the essential spectrum of $L(k)$ is given by that of

$$L_\infty(k) = \begin{pmatrix} K(\rho_\infty)(-\partial_{xx}^2 + k^2) + g_0'(\rho_\infty) & (u_\infty - c)\partial_x \\ (c - u_\infty)\partial_x & \rho_\infty(-\partial_{xx}^2 + k^2) \end{pmatrix},$$

whose essential spectrum can be determined by using the Fourier transform in the x-variable and explicitly writing the eigenvalues $\mu(\xi, k)$. We get that these are positive when $k \neq 0$. The essential spectrum of $L(k)$ is equal to that of $L_\infty(k)$, so (H2) is verified.

(H3). — We easily have $L'(k) = \operatorname{diag}(2kK(\rho_c), 2k\rho_c)$.

(H4). — We apply the following lemma to $L(0)$.

Lemma 2.3. — *Let L be a symmetric operator on a Hilbert space such that*

$$L = \begin{pmatrix} L_1 & A \\ A^* & L_2 \end{pmatrix}$$

with L_2 invertible. Then, we can write

$$(Lv, v) = \left((L_1 - AL_2^{-1}A^*)v_1, v_1\right) + \left(L_2(v_2 + L_2^{-1}A^*v_1), v_2 + L_2^{-1}A^*v_1\right).$$

As a result, we write

$$(L(0)v, v) = (Mv_1, v_1) + \int_{\mathbb{R}} \rho_c \left|\partial_x v_2 + \frac{1}{\rho_c}(u_c - c)v_1\right|^2 dx,$$

with $M = -\partial_x(K(\rho_c)\partial_x \cdot) + m - \frac{(u_c-c)^2}{\rho_c}$, which is a second-order differential operator on which we can perform Sturm–Liouville analysis [11]. First, the essential spectrum of M is included in $[\alpha, +\infty[$ with $\alpha > 0$; indeed M is a perturbation of $M_\infty = -K(\rho_\infty)\partial_{xx}^2 + g_0'(\rho_\infty) - \frac{(u_\infty-c)^2}{\rho_\infty}$, whose essential spectrum is positive under the assumption that $\rho_\infty g_0'(\rho_\infty) > (u_\infty - c)^2$. Next, the function ρ_c' is in the kernel of M, and it has one zero, so by Sturm–Liouville theory, M has a unique negative eigenvalue associated with an eigenfunction v_1^-. Setting v_2^- such that $\partial_x v_2^- = \frac{-1}{\rho_c}(u_c - c)v_1^-$, we have a generalised eigenfunction for $L(0)$ (the second component is not in L^2). By using H^2 approximations of v_2^-, we see that $(L(0)v, v)$ can be negative with $v \in H^2$, confirming that $L(0)$ has one negative eigenvalue.

2.1.2. Localisation of instability and boundedness of unstable eigenvalues.

We now prove the rest of Proposition 2.1(a). We start be taking the real part of the L^2 scalar product of the eigenvalue equation (2.5) by $L(k)v$: we get that $\mathrm{Re}(\sigma)(L(k)v,v)=0$, as J is skew-symmetric. The operator $L(k)$ satisfies (H1) of Theorem 2.2, so, if $\mathrm{Re}(\sigma)>0$, we must have $0=(L(k)v,v)\geqslant \alpha\left\|v\right\|_{L^2}^2$ for $|k|\geqslant k_{\max}$. Thus, the only function satisfying $\sigma v=JL(k)v$ with $\mathrm{Re}(\sigma)>0$ and $|k|\geqslant k_{\max}$ is $v=0$; there are no unstable eigenfunctions for $|k|$ large.

In order to get the boundedness of the unstable eigenvalues, we decompose $L(k)$ as follows: $L(k)=L_0(k)+L_1$ with

$$L_0(k):=\begin{pmatrix} -\partial_x(K(\rho_c)\partial_x)+K(\rho_c)k^2+m_0 & 0 \\ 0 & -\partial_x(\rho_c\partial_x)+\rho_c k^2 \end{pmatrix},$$

where $m_0(x)=\max\left(m(x),\frac{1}{2}g_0'(\rho_\infty)\right)$. We compute the scalar product of (2.5) and $L_0(k)v$, and take the real part, which gives us

$$\mathrm{Re}(\sigma)(L_0(k)v,v)=\mathrm{Re}(JL_1v,L_0(k)v). \tag{2.8}$$

It is quickly noticed that there exists $\alpha>0$ such that

$$\mathrm{Re}(\sigma)(L_0(k)v,v)\geqslant\alpha\mathrm{Re}(\sigma)\left(\left\|\partial_xv\right\|_{L^2}^2+k^2\left\|v\right\|_{L^2}^2+\left\|v_1\right\|_{L^2}^2\right). \tag{2.9}$$

We shall now bound $|\mathrm{Re}(JL_1v,L_0(k)v)|$ by the same norms as on the right. Note that

$$JL_1=\begin{pmatrix} -\partial_x((u_c-c)\cdot) & 0 \\ m_0-m & -(u_c-c)\partial_x \end{pmatrix}.$$

Let us illustrate what goes on in this scalar product with an example, $\int m_0(u_c-c)v_1\partial_xv_1$. Integrating this by parts, we notice some symmetry, thus

$$\int m_0(u_c-c)v_1\partial_xv_1=\frac{-1}{2}\int\partial_x(m_0(u_c-c))v_1^2.$$

This procedure allows us to reduce the expression of $(JL_1v,L_0(k)v)$, and get the bound

$$|\mathrm{Re}(JL_1v,L_0(k)v)|\leqslant C(\left\|\partial_xv\right\|_{L^2}^2+k^2\left\|v\right\|_{L^2}^2).$$

Combining with (2.8) and (2.9), there exists $C>0$ such that

$$\mathrm{Re}(\sigma)\left(\left\|\partial_xv\right\|_{L^2}^2+k^2\left\|v\right\|_{L^2}^2+\left\|v_1\right\|_{L^2}^2\right)\leqslant C\left(\left\|\partial_xv\right\|_{L^2}^2+k^2\left\|v\right\|_{L^2}^2+\left\|v_1\right\|_{L^2}^2\right),$$

which implies that $\mathrm{Re}(\sigma)$ cannot be unbounded when positive. As it was shown that positive eigenvalues are at most unique and simple for each k, the function $\sigma(k)$ which takes the highest real part of all eigenvalues of $JL(k)$ is analytic where it is non-negative, and does not explode. As the set where

$\sigma(k) \geqslant 0$ is compact, it has a global maximum at $k_0 > 0$ which will be denoted σ_0. This ends the proof of part (a) of Proposition 2.1.

2.2. Proof of Proposition 2.1(b), resolvent estimate

The proof of part (b) is split in two. First, we get the result for $s = 0$; we bound the X^0_k norm of U by similar norms of F by using the Laplace transform and spectral arguments. The case $s > 0$ is then obtained by induction on s, the number of total derivatives (time and space).

2.2.1. The case $s = 0$

The proof of (2.4) for $s = 0$ relies on the Laplace transform, and is similar to the resolvent estimate proofs in [22, 24, 18]. Let $\sigma_0 < \gamma_0 < \gamma$. For a function $f(t)$ such that $e^{-\sigma_0 t} f(t) \in L^\infty$, we denote by $\tilde{f}(\tau)$ the following Laplace transform in time,

$$\tilde{f}(\tau) := \int_0^{+\infty} \exp(-(\gamma_0 + i\tau)t) f(t)\, dt.$$

Using the Laplace transform turns equation (2.3), $\partial_t U = JL(k)U + F$, in which the source term F is assumed to be such that $e^{-\sigma_0 t} F \in L^\infty(\mathbb{R}^+, H^s(\mathbb{R}))$, into an eigenvalue problem:

$$(\gamma_0 + i\tau)\tilde{U}(\tau) = JL(k)\tilde{U}(\tau) + \tilde{F}(\tau). \tag{2.10}$$

As $\gamma_0 > \sigma_0$, $\gamma_0 + i\tau$ is not in the spectrum of $JL(k)$. Indeed, we can use the strategy employed to prove that hypothesis (H2) of Theorem 2.2 is satisfied to show that the essential spectrum of $JL(k)$ is embedded in $i\mathbb{R}$. Once again using the argument from [14], we can examine the spectrum of the Fourier transform in x of $JL_\infty(k)$,

$$\mathcal{F}_x(JL_\infty)(\xi, k) = \begin{pmatrix} -i(u_\infty - c)\xi & \rho_\infty(\xi^2 + k^2) \\ -K(\rho_\infty)(\xi^2 + k^2) - g_0'(\rho_\infty) & -i(u_\infty - c)\xi \end{pmatrix},$$

which contains the solutions of the equation

$$X^2 + 2i\xi(u_\infty - c)X + \rho_\infty K(\rho_\infty)(\xi^2 + k^2)^2 + \rho_\infty g_0'(\rho_\infty)k^2 \\ + (\rho_\infty g_0'(\rho_\infty) - (u_\infty - c)^2)\xi^2 = 0,$$

which depend on (ξ, k). Using the positiveness of ρ_∞, $K(\rho_\infty)$ and condition (1.2), we get that the discriminant of this equation is negative for $(\xi, k) \neq 0$, and clearly the only eigenvalue at $(\xi, k) = (0, 0)$ is zero, so the essential spectrum of $JL(k)$ is imaginary.

As $\gamma_0 + i\tau$ is not in the spectrum of $JL(k)$ for any $\tau \in \mathbb{R}$, the norm of the resolvent $((\gamma_0 + i\tau)\mathrm{Id} - JL(k))^{-1}$ is uniformly bounded for (τ, k) in any compact subset of $\mathbb{R}^2$. It remains to show that, for $|k| \leqslant \tilde{k}$, there exists the following bound for $|\tau|$ large.

Lemma 2.4. — *If $\tilde{U}$ solves* (2.10)*, then there exist $C,\ M > 0$ such that, for $|\tau| \geqslant M$,*

$$||\tilde{U}(\tau)||_{X_k^0} \leqslant C||\tilde{F}(\tau)||_{H^1}. \tag{2.11}$$

Proof. — We consider the scalar product of the Laplace-transformed equation (2.10) with $L(k)\tilde{U}$, and write

$$(\gamma_0 + i\tau)(L(k)\tilde{U}, \tilde{U}) = (\tilde{F}, L(k)\tilde{U}). \tag{2.12}$$

Note that $(L(k)\tilde{U}, \tilde{U}) = (L(0)\tilde{U}, \tilde{U}) + K(\rho_c)k^2||\tilde{U}_1||_{L^2}^2 + \rho_c k^2||\tilde{U}_2||_{L^2}^2$, so let us concentrate on the term $(L(0)\tilde{U}, \tilde{U})$. Using Lemma 2.3, we know that it is equal to

$$(L(0)\tilde{U}, \tilde{U}) = (M\tilde{U}_1, \tilde{U}_1) + \int_{\mathbb{R}} \rho_c \left|\partial_x \tilde{U}_2 + \frac{1}{\rho_c}(u_c - c)\tilde{U}_1\right|^2\, dx,$$

The operator M, which we remind the reader is equal to $-\partial_x(K(\rho_c)\partial_x \cdot) + m - \frac{(u_c-c)^2}{\rho_c}$, and whose quadratic form is defined on H^1, has one simple negative eigenvalue, as well as a one-dimensional kernel containing ρ_c'.

Recall v^- the generalised eigenfunction corresponding to the negative eigenvalue of $L(0)$, defined in the verification of (H4) above. We do not have $v_2^- \in L^2$, so we set $U^- \propto (v_1^-, 0)$, renormalised so that $\|U^-\|_{L^2} = 1$. We denote $U^0 = \left(\frac{\rho_c'}{\|\rho_c'\|_{L^2}}, 0\right)$, which is in the kernel of $M \otimes \mathrm{Id}$. Let U^+ be orthogonal in L^2 to U^- and U^0. We show that, for some $\eta > 0$, we have

$$(L(0)U^+, U^+) \geqslant \eta \left\|U^+\right\|_{X_0^0}^2. \tag{2.13}$$

First, as U_1^+ is not in the kernel or the negative eigenspace of M, we have $(MU_1^+, U_1^+) \geqslant \alpha \left\|U_1^+\right\|_{H^1}^2$, since the essential spectrum of M is included in $[\alpha, +\infty[$. Thus, we already have

$$(L(0)U^+, U^+) \geqslant \alpha \left\|U_1^+\right\|_{H^1}^2. \tag{2.14}$$

But this does not suffice to get the X_0^0 norm; we also need to recover $\left\|\partial_x U_2^+\right\|_{L^2}$. Using the fact that ρ_c is positive, there is a positive β such that

$$\begin{aligned}\int_{\mathbb{R}} \rho_c \left|\partial_x U_2^+ + \frac{1}{\rho_c}(u_c - c)U_1^+\right|^2 dx \\ &\geqslant \beta \left\|\partial_x U_2^+ + \frac{1}{\rho_c}(u_c - c)U_1^+\right\|_{L^2}^2 \\ &\geqslant \beta \left\|\partial_x U_2^+\right\|_{L^2}^2 + C\beta \left\|U_1^+\right\|_{L^2}^2 - 2C|(\partial_x U_2^+, U_1^+)|\end{aligned}$$

for some $C > 0$. We use Young's inequality (2.6) on the final term with $\delta = \beta/2C$, thus there exists $C' \in \mathbb{R}$ such that

$$(L(0)U^+, U^+) \geqslant (\alpha + C') \left\|U_1^+\right\|_{H^1}^2 + \frac{\beta}{2} \left\|\partial_x U_2^+\right\|_{L^2}^2 .$$

If perchance C' is negative, we add $\frac{|C'|}{\alpha} \times$(2.14) to the above, and obtain that there does indeed exist $\eta > 0$ such that we have (2.13), and $(L(k)U^+, U^+) \geqslant \eta \left\|U^+\right\|_{X_k^0}^2$.

We now write the orthogonal decomposition in L^2 of the first component, $\tilde{U}_1 = aU_1^- + bU_1^0 + U_1^+$, and replace in (2.12). The eigenfunctions U^- and U^0 are fixed, so their H^1 norms are given constants. On the right-hand side, using integration by parts and basic estimates including Young's inequality (2.6) with an appropriate parameter δ, we have

$$|(\tilde{F}, L(k)\tilde{U})| \leqslant \frac{\eta}{4} \left\|U^+\right\|_{X_k^0}^2 + C(||\tilde{F}||_{H^1}^2 + a^2 + b^2),$$

while on the left-hand side, we have

$$(L(k)\tilde{U}, \tilde{U}) \geqslant \eta(|k|^2||U^+||_{L^2}^2 + \left\|U^+\right\|_{X_0^0}^2) - C(a^2 + (|a| + |b|) \left\|U^+\right\|_{X_0^0}^2).$$

Taking the real part of (2.12) and moving the negative part of the above to the right-hand side and once again applying Young's inequality to absorb $\|U^+\|_{X_0^0}$, we get

$$\left\|U^+\right\|_{X_k^0}^2 \leqslant C(||\tilde{F}||_{H^1}^2 + a^2 + b^2). \tag{2.15}$$

To finish off, we take the dot product of (2.10) with U^- and U^0. We quickly get

$$(\gamma_0 + i\tau)a = -(\tilde{U}, L(k)JU^-) + (\tilde{F}, U^-)$$

and

$$(\gamma_0 + i\tau)b = -(\tilde{U}, L(k)JU^0) + (\tilde{F}, U^0).$$

Integrating by parts as usual, we get

$$a^2 + b^2 \leqslant \frac{C}{\gamma_0^2 + |\tau|^2}(a^2 + b^2 + \left\|U^+\right\|_{X_k^0}^2 + ||\tilde{F}||_{H^1}^2).$$

Taking $C \geqslant 1$, we see that if $|\tau|$ is large enough, $\frac{C^2}{\gamma_0^2+|\tau|^2} \leqslant \frac{1}{2}$, and $\frac{1}{2}(a^2 + b^2 + \|U^+\|^2_{X^0_k})$ can be absorbed by the left-hand side when combining this last inequality with (2.15). This concludes the proof of the lemma. $\square$

For the end of the proof of Proposition 2.1 (b), we start by using the Parseval equality in the following,

$$\int_0^T e^{-2\gamma_0 t} \|U(t)\|^2_{X^0_k} \, dt = \int_0^T ||\tilde{U}(t)||^2_{X^0_k} \, dt$$
$$\leqslant C \int_0^T ||\tilde{F}(t)||^2_{H^1} \, dt = C \int_0^T e^{-2\gamma_0 t} \|F(t)\|^2_{H^1} \, dt.$$

Now recall the assumption on F, (2.2): we have

$$\int_0^T e^{-2\gamma_0 t} \|U(t)\|^2_{X^0_k} \, dt \leqslant CM_0 \int_0^T \frac{e^{2(\gamma-\gamma_0)t}}{(1+t)^n} \, dt \leqslant C_0 \frac{e^{2(\gamma-\gamma_0)T}}{(1+T)^n}.$$

We inject this in the energy estimate on (2.3), that is

$$\frac{d}{dt}(\|U(t)\|^2_{X^0_k}) \leqslant C(\|U(t)\|^2_{X^0_k} + \|F(t)\|^2_{H^1}),$$

and multiply the result by $e^{-2\gamma_0 t}$, integrate in time and we get the result.

2.2.2. The induction for $s > 0$

The extension of Proposition 2.1(b) to every $s \geqslant 0$ is done with a double induction, double in the sense that one is embedded in the other.

The first induction is on s, the total number of derivatives. Set $s > 0$, and we assume that, for every $s' < s$ and $j \leqslant s'$, we have (2.4), that is

$$\left\|\partial_t^{s'-j} U(t)\right\|_{X^j_k} \leqslant C_{s'} \frac{e^{\gamma t}}{(1+t)^n}.$$

To get the wanted result, we must prove that, for every $0 \leqslant j \leqslant s$,

$$\|U\|^2_{\dot{X}_k} := \left\|\partial_t^{s-j} \partial_x^j U\right\|^2_{\dot{H}^1} + |k|^2 \left\|\partial_t^{s-j} \partial_x^j U\right\|^2_{L^2} \leqslant C_s \frac{e^{2\gamma t}}{(1+t)^{2n}}, \qquad (2.16)$$

where $\dot{H}^1$ is the usual homogeneous Sobolev norm on $\mathbb{R}$. The $\dot{X}_k$ norm (semi-norm if $k=0$) defined here is a sort of homogeneous Sobolev norm expressed in the Fourier space, and at rank s of the induction, we must get bounds for the L^2 norms of terms involving $s+1$ derivatives. This is done by induction on the number of space derivatives, j.

Starting with $j = 0$, we are interested in the X^0_k norm of $\partial_t^s U$. Simply differentiate equation (2.3) s times with respect to time, and notice that

$W(t) = \partial_t^s U(t) - \partial_t^s U(0)$ satisfies $\partial_t W = JL(k)W + G$, with $W|_{t=0} = 0$ and $G = \partial_t^s F - JL(k)\partial_t^s U(0)$. The source term satisfies $\|G(t)\|_{H^{s+1}} \leqslant 2M_s(1+t)^{-n}e^{\gamma t}$, and we can re-use the case $s = 0$ shown above.

Now, let $j > 0$. To lighten the notations, we will write $U_{s,j} = \partial_t^{s-j}\partial_x^j U$. We want to control the $\dot{X}_k$ norm of $U_{s,j}$, which means $s+1$ derivatives in total, of which $j+1$ are space or Fourier derivatives. We apply $\partial_t^{s-j}\partial_x^j$ to the equation. This time, the derivatives do not commute with $JL(k)$, hence we consider

$$\partial_t U_{s,j} = JL(k)U_{s,j} + J[\partial_x^j, L(k)]U_{s-j,0} + F_{s,j} := JM_{s,j}(k)U + F_{s,j}.$$

We take the real part of the scalar product of this equation with $M_{s,j}(k)U$, which yields

$$\frac{1}{2}\frac{d}{dt}(U_{s,j}, L(k)U_{s,j}) = -\mathrm{Re}(U_{s+1,j}, [\partial_x^j, L(k)]U_{s-j,0}) + \mathrm{Re}(F_{s,j}, M_{s,j}(k)U).$$

To bound the second part of the right-hand side, we look more closely at the commutator term in $M_{s,j}(k)U$. We notice that there exist two sets of L^∞ matrices $(m_i^1, m_i^2)_{0\leqslant i\leqslant j+1}$ such that

$$[\partial_x^j, L(k)]\partial_t^{s-j}U = \left(\sum_{i=0}^{j+1} m_i^1(x)U_{s-j+i,i}\right) + \left(\sum_{i=0}^{j-1} m_i^2(x)k^2 U_{s-j+i,i}\right). \quad (2.17)$$

We notice that all the terms, *except* the one with $i = j+1$, have a total of s derivatives or less, and thus, using $k^2 \leqslant \tilde{k}|k|$, they are controlled by our induction hypothesis on s. Integrating by parts in the terms of $(F_{s,j}, L(k)U_{s,j})$ involving $j+2$ space derivatives and using assumption (2.2) and Young's inequality with a parameter η to be chosen later, we obtain that the right-hand side is bounded by

$$|(F_{s,j}, M_{s,j}(k)U)| \leqslant \frac{\eta}{2}\|U_{s,j}\|_{\dot{X}_k}^2 + C\frac{e^{2\gamma t}}{(1+t)^{2n}}.$$

It remains to deal with the first term of the right-hand side, the one with $i = j+1$. We notice that $U_{s+1,j}$ has $s+1$ derivatives, of which j space derivatives, hence the L^2 norm of $U_{s+1,j}$ falls under our second induction hypothesis, the one on j. We can thus use Young's inequality and use decomposition (2.17) once again, and get

$$\frac{1}{2}\frac{d}{dt}(U_{s,j}, L(k)U_{s,j}) \leqslant \eta\|U_{s,j}\|_{\dot{X}_k}^2 + C\frac{e^{2\gamma t}}{(1+t)^{2n}}.$$

Finally, we integrate this in time, and recall (2.7) from the verification of the (H1) hypothesis of Theorem 2.2, which says that

$$(L(k)U_{s,j}, U_{s,j}) \geqslant \eta'\|U_{s,j}\|_{\dot{X}_k}^2 - C\|U_{s,j}\|_{L^2}^2,$$

in which the final term can be moved to the right-hand side and controlled by the induction hypothesis. In total, we therefore have

$$\|U_{s,j}(T)\|_{\dot{X}_k}^2 \leqslant \frac{\eta}{\eta'}\int_0^T \|U_{s,j}(t)\|_{\dot{X}_k}^2 \; dt + C\frac{e^{2\gamma t}}{(1+t)^{2n}}.$$

We choose η in the Young inequalities above so that $\eta/\eta' \leqslant 2\gamma$, and the Grönwall lemma gives us (2.16) for the couple (s,j). Both inductions are now complete.

3. Nonlinear instability

In this part, $U=(\rho,u)$. Obtaining Theorem 1.1 relies on the construction of an approximate solution U^{ap} built around a wavepacket of unstable eigenmodes for the linearised equation. In our case, this construction is classical and we will not write all the details of the calculations (see also, for instance, [12, 9, 21, 24]). Energy estimates must then be obtained on $U-U^{\mathrm{ap}}$ to ensure that the approximate solution is close enough to the exact solution for long enough to see the difference between U^{ap} and Q_c reach an amplitude $\mathcal{O}(1)$. This must also ensure that the solution U still exists when the instability appears, as we remind the reader that only local existence is guaranteed by the methods in [7], which will require a slight adaptation in order to work for perturbations of our travelling wave reference solution.

3.1. Construction and properties of the approximate solution

For a whole number N independent of ε to be chosen later, we will set

$$U^{\mathrm{ap}}(t,x,y) = \begin{pmatrix} \rho^{\mathrm{ap}}(t,x,y) \\ u^{\mathrm{ap}}(t,x,y) \end{pmatrix} = Q_c(t,x) + \sum_{j=1}^{N} \varepsilon^j U_j(t,x,y).$$

The velocity components in this expansion will be potential, so we define φ_j such that $u_j = \nabla\varphi_j$, φ^{ap} such that $u^{\mathrm{ap}} = \nabla\varphi^{\mathrm{ap}}$, and we denote $V_j = (\rho_j,\varphi_j)$ and $V^{\mathrm{ap}} = (\rho^{\mathrm{ap}},\varphi^{\mathrm{ap}})$. The construction V^{ap} is expected to solve the Euler–Korteweg system leaving an error of order ε^{N+1}, as follows,

$$\begin{cases} \partial_t\rho^{\mathrm{ap}} + \mathrm{div}(\rho^{\mathrm{ap}}\nabla\varphi^{\mathrm{ap}}) = \varepsilon^{N+1}R_1^{\mathrm{ap}} \\ \partial_t\varphi^{\mathrm{ap}} + \frac{1}{2}|\nabla\varphi^{\mathrm{ap}}|^2 + g_0(\rho^{\mathrm{ap}}) = K(\rho^{\mathrm{ap}})\Delta\rho^{\mathrm{ap}} + \frac{1}{2}K'(\rho^{\mathrm{ap}})|\nabla\rho^{\mathrm{ap}}|^2 + \varepsilon^{N+1}R_2^{\mathrm{ap}}, \end{cases}$$

thus, using the Taylor formula on K' and g_0, and isolating the terms of order ε^j, we see that V_j solves the linearised Euler–Korteweg equation around Q_c with a source term,

$$\partial_t V_j = J\mathcal{L}V_j + R_j, \tag{3.1}$$

in which R_j contains nonlinear interaction terms between the V_n with $n < j$, but with the sum on indices in each interaction term equal to j. For instance, while $R_1 = 0$, the second term of the expansion solves

$$\partial_t V_2 = J\mathcal{L}V_2 + \begin{pmatrix} -\operatorname{div}(\rho_1\nabla\varphi_1) \\ -|\nabla\varphi_1|^2 + K'(\rho_c)\left[\rho_1\Delta\rho_1 + \frac{1}{2}|\nabla\rho_1|^2\right] + K''(\rho_c)\partial_x\rho_c\rho_1\partial_x\rho_1 \end{pmatrix}.$$

For $j \geqslant 3$, terms involving the product of three lower-order elements also appear in the equation on φ_j. These stem from the nonlinearity $K'(\rho)|\nabla\rho|^2$; for example, the term $K''(\rho_c)\rho_1|\nabla\rho_1|^2$ appears in R_3.

In total, the remainders R_j are

$$R_{j,1} = -\sum_{\substack{j_1+j_2=j \\ j_1,j_2>0}} \operatorname{div}(\rho_{j_1}\nabla\varphi_{j_2}),$$

$$\begin{aligned} R_{j,2} = &\sum_{\substack{j_1+j_2=j \\ j_1,j_2>0}} -(\nabla\varphi_{j_1}\cdot\nabla\varphi_{j_2}) + K'(\rho_c)\rho_{j_1}\Delta\rho_{j_2} + \frac{1}{2}K'(\rho_c)(\nabla\rho_{j_1}\cdot\nabla\rho_{j_2}) \\ &+ \sum_{\substack{j_1+j_2=j \\ j_1,j_2>0}} K''(\rho_c)\partial_x\rho_c\rho_{j_1}\partial_x\rho_{j_2} + \sum_{\substack{j_1+j_2+j_3=j \\ j_1,j_2,j_3>0}} \frac{1}{2}K''(\rho_c)\rho_{j_1}(\nabla\rho_{j_2}\cdot\nabla\rho_{j_3}). \end{aligned}$$

The remainder for V^{ap}, R^{ap}, contains all the interaction terms whose sum of indices is greater than N.

We now construct V_1 as a wavepacket of unstable eigenmodes of the linear equation (3.1) with $R_1 = 0$. Recall that $k_0 > 0$ is a point of global maximum for the function

$$\tilde{\sigma} : k \mapsto \max\{\operatorname{Re}(\lambda) \mid \lambda \in \sigma(JL(k))\},$$

where $\sigma(JL(k))$ is the spectrum of the operator $JL(k)$. We then define

$$V_1(t,x,y) = \int_{\mathbb{R}} f_1(k)e^{iky}e^{\tilde{\sigma}(k)t}v_1(k,x)\,dk,$$

with $f_1(k)$ smooth, even, equal to 1 in the vicinity of k_0 and supported in the set $\{k \mid \tilde{\sigma}(k) > 3\sigma_0/4\}$, and $w(k,t,x) = e^{\tilde{\sigma}(k)t}v_1(k,x)$ solving $\partial_t w = JL(k)w$. For any $s \geqslant 0$, using the Parseval equality, we need to get an equivalent for

$$\|V_1(t)\|^2_{H^s(\mathbb{R}^2)} = \sum_{s'=0}^{s}\int_{\mathbb{R}} f_1^2(k)|k|^{2s'}\,\|v_1(k)\|^2_{H^s(\mathbb{R})}\,e^{2\tilde{\sigma}(k)t}\,dk.$$

Since an eigenvalue of $JL(k)$ with positive real part is unique and simple, the function $\tilde{\sigma}$ is analytic and there exists $p \geqslant 1$ such that the critical point k_0 satisfies

$$\tilde{\sigma}'(k_0) = \cdots = \tilde{\sigma}^{(2p-1)}(k_0) = 0 \text{ and } \tilde{\sigma}^{(2p)}(k_0) < 0. \tag{3.2}$$

Hence we can use the stationary phase method around the critical point k_0 (see [26] for example) to get that $\|V_1(t)\|_{H^s}^2 \underset{t\to+\infty}{\sim} t^{-1/2p}e^{2\sigma_0 t}$, so, for some constant $C_{1,s}$, we have

$$\frac{1}{C_{1,s}}\frac{e^{\sigma_0 t}}{(1+t)^{1/4p}} \leqslant \|V_1(t)\|_{H^s(\mathbb{R}^2)} \leqslant C_{1,s}\frac{e^{\sigma_0 t}}{(1+t)^{1/4p}}. \tag{3.3}$$

We get estimates on V_j by induction. Assume that $I = \mathrm{supp}(f_1)$ is made up of two separate intervals around $\pm k_0$, and we set

$$V_j(t,x,y) = \int_I \cdots \int_I w_j(k_1,\cdots,k_j;t,x)e^{ik_1 y}\cdots e^{ik_j y}\ dk_1\cdots dk_j.$$

Assuming that, for every $n < j$,

$$\|w_n(k_1,\cdots,k_n;t)\|_{H^s} \leqslant C_n \exp[n(\tilde{\sigma}(k_1)+\cdots+\tilde{\sigma}(k_n))t], \tag{3.4}$$

we get that w_j solves the linearised Fourier-transformed equation

$$\partial_t w_j(k_1,\cdots,k_j) = JL(k_1+\cdots+k_j)w_j(k_1,\cdots,k_j) + r_j(k_1,\cdots,k_j), \tag{3.5}$$

in which $\|r_j(k_1,\cdots,k_j)\|_{H^s} \leqslant C\exp[j(\tilde{\sigma}(k_1)+\cdots+\tilde{\sigma}(k_j))t]$ by the structure of the remainder and (3.4). Then, since, for $k \in I$, $\tilde{\sigma}(k) > 3\sigma_0/4$, the sum in the exponential is greater than σ_0, and we can apply Proposition 2.1(b) to get that $w_j(k_1,\cdots,k_j;t)$, defined as the solution of (3.5) with $w_j|_{t=0} = 0$, satisfies (3.4). When the maximum k_0 is *non-degenerate*, we then use Parseval's equality and the Taylor expansion of $\tilde{\sigma}$ around the critical point to write that, for some $\beta > 0$,

$$\begin{aligned}
&\|V_j(t)\|_{H^s}^2\\
&= \int_{jk\in jI}\left\|\int_{k_1+\cdots+k_j=jk} w_j(k_1,\cdots,k_j;t,x)e^{ijky}\ dk_1\cdots dk_{j-1}\right\|_{H^s(x,y)}^2 dk\\
&\leqslant \int_{jk\in jI} Ce^{2(j\sigma_0 - j\beta(k-k_0)^2)t}\\
&\qquad \times \int_{\sum_{m=1}^j k_m = jk} e^{-2\beta\sum_{m=1}^j (k_m-k)^2 t}\ dk_1\cdots dk_{j-1}\ dk
\end{aligned}$$

Integrate these gaussian functions (remembering that $k_j = jk - \sum_{m=1}^{j-1} k_m$), and we get the desired inequality: for every $j \leqslant 1$,

$$\|V_j(t)\|_{H^s(\mathbb{R}^2)} \leqslant C_j\frac{e^{j\sigma_0 t}}{(1+t)^{j/4}}. \tag{3.6}$$

We now take a look at the remainder of the equation on V^{ap}, R^{ap}, which contains the interaction terms of the equation whose sum of indices is greater than N. Similarly to our proof of (3.9), we have

$$\left\|\varepsilon^{N+1} R^{\mathrm{ap}}(t)\right\|_{H^s} \leqslant \sum_{j=N+1}^{3N} C_j \frac{\varepsilon^j e^{j\sigma_0 t}}{(1+t)^{j/4}}, \tag{3.7}$$

and, in what follows, we will be interested in times for which the smaller powers of $\varepsilon e^{\sigma_0 t}(1+t)^{-1/4}$ are dominant. We set $T_\varepsilon^* = \mathcal{O}(|\ln(\varepsilon)|)$ such that

$$\frac{\varepsilon e^{\sigma_0 T_\varepsilon^*}}{(1+T_\varepsilon^*)^{1/4}} = \kappa,$$

for $0 < \kappa < 1$ to be chosen later. Replace t with $T_\varepsilon^* - \tau$ in (3.7), and we have

$$\begin{aligned} \left\|\varepsilon^{N+1} R^{\mathrm{ap}}(T_\varepsilon^* - \tau)\right\|_{H^s} &\leqslant \left(\max_{j\in\{N+1,\cdots,3N\}} C_j\right) \sum_{j=N+1}^{3N} \kappa^j e^{-j\sigma_0\tau} \\ &\leqslant C_R \kappa^{N+1} e^{-(N+1)\sigma_0\tau}, \end{aligned}$$

which, returning to the original time variable t, gives us, for $t \leqslant T_\varepsilon^*$,

$$\left\|\varepsilon^{N+1} R^{\mathrm{ap}}(t)\right\|_{H^s} \leqslant C_R \frac{\varepsilon^{N+1} e^{(N+1)\sigma_0 t}}{(1+t)^{(N+1)/4}}. \tag{3.8}$$

Let us finish this part by dealing with the case in which k_0 is *degenerate*, that is, p in (3.2) is strictly greater than 1. Indeed, when estimating V_j above, we strongly used the identity

$$\sum_{m=1}^{j} (k_m - k_0)^2 = j(k-k_0)^2 + \sum_{m=1}^{j} (k_m - k)^2,$$

which will not work at order $2p$. In order to get bounds on $\|V_j\|_{H^s}$, we bound $\tilde{\sigma}$ by a function which has a non-degenerate maximum at k_0 with a slightly higher value, say $(1+1/N)\sigma_0$, at k_0, and repeat the gaussian integrations. Ultimately, using that $(1+t)^{-1/2} \leqslant (1+t)^{-1/2p}$, we get

$$\|V_j(t)\|_{H^s(\mathbb{R}^2)} \leqslant C_j \frac{e^{(j+1)\sigma_0 t}}{(1+t)^{j/4p}}.$$

The exponential growth rate we obtain is controlled by $(j+1)\sigma_0$, but, on our time of study T_ε^*, we can bring this down to $j\sigma_0$ in the same way as we obtained (3.8). Define for the rest of the paper T_ε^* such that

$$\frac{\varepsilon e^{\sigma_0 T_\varepsilon^*}}{(1+T_\varepsilon^*)^{1/4p}} = \kappa,$$

then we get, more generally, that, for $t \leqslant T_\varepsilon^*$,

$$\|V_j(t)\|_{H^s(\mathbb{R}^2)} \leqslant C_j \frac{e^{(j+1)\sigma_0 t}}{(1+t)^{j/4p}}, \tag{3.9}$$

$$\text{and } \left\|\varepsilon^{N+1} R^{\mathrm{ap}}(t)\right\|_{H^s} \leqslant C_R \frac{\varepsilon^{N+1} e^{(N+1)\sigma_0 t}}{(1+t)^{(N+1)/4p}}. \tag{3.10}$$

3.2. Getting the instability

If U is the solution of the Euler–Korteweg system (1.1) with the initial condition $U(0) = U^{\mathrm{ap}}(0)$, we will observe the instability by studying

$$\|U(t) - Q_c(t)\|_{L^2(\mathbb{R}^2)} \geqslant \|U^{\mathrm{ap}}(t) - Q_c(t)\|_{L^2(\mathbb{R}^2)} - \|U(t) - U^{\mathrm{ap}}(t)\|_{L^2(\mathbb{R}^2)}.$$

On one hand, we have $U^{\mathrm{ap}} - Q_c = \sum_{j=1}^N \varepsilon^j U_j$, and

$$\begin{aligned}
\left\|\sum_{j=1}^N \varepsilon^j U_j\right\|_{L^2(\mathbb{R}^2)} &\geqslant \|\varepsilon U_1(t)\|_{L^2(\mathbb{R}^2)} - \sum_{j=2}^N \left\|\varepsilon^j U_j\right\|_{L^2(\mathbb{R}^2)} \\
&\geqslant C_1' \frac{\varepsilon e^{\sigma_0 t}}{(1+t)^{1/4p}} - \sum_{j=2}^N C_j \frac{\varepsilon^j e^{j\sigma_0 t}}{(1+t)^{j/4p}}
\end{aligned}$$

by (3.3) and (3.9). Taking times smaller than T_ε^*, we can consider that the sum on the right behaves like $\varepsilon^2 e^{2\sigma_0 t}(1+t)^{-1/2p}$, and, replacing t by $T_\varepsilon^* - \tau$, we write

$$\begin{aligned}
\|(U^{\mathrm{ap}} - Q_c)(T_\varepsilon^* - \tau)\|_{L^2} &\geqslant \kappa \left[C_1' e^{-\sigma_0 \tau} - \kappa C_2' e^{-2\sigma_0 \tau}\right] \\
&\geqslant \kappa C_1' e^{-\sigma_0 \tau} \left(1 - \frac{\kappa C_2'}{C_1'} e^{-\sigma_0 \tau}\right).
\end{aligned}$$

We notice that, for a given $C > 0$, there exists $\tau_C > 0$ such that, for $\tau \geqslant \tau_C$, $1 - Ce^{-\sigma_0 \tau} \geqslant 1/2$, so we set $\tau_1 > 0$, independent of ε, such that, for $\tau \geqslant \tau_1$,

$$\|(U^{\mathrm{ap}} - Q_c)(T_\varepsilon^* - \tau)\|_{L^2(\mathbb{R}^2)} \geqslant \frac{\kappa C_1'}{2} e^{-\sigma_0 \tau}. \tag{3.11}$$

On the other hand, we require energy estimates to ensure that $\|U(t) - U^{\mathrm{ap}}(t)\|_{L^2(\mathbb{R}^2)}$ is small. We would like to readily use those shown by S. Benzoni-Gavage, R. Danchin and S. Descombes in [7], which are obtained by considering the equation on $(G, z) = (G, u + iw)$, with G a primitive of the function $\rho \mapsto \sqrt{K(\rho)/\rho}$ and $w = \nabla(G(\rho))$. Let $a(\rho) = \sqrt{\rho K(\rho)}$, and notice that $w = \nabla A(\rho)$, with A a primitive of $a(\rho)/\rho$, which is an increasing function, thus we can define its inverse to link ρ and G.

At present, we need to assume that there is no vacuum: that there exist $\underline{\rho}$ and $\overline{\rho}$ such that

$$0 < \underline{\rho} \leqslant \rho(t,x,y) \leqslant \overline{\rho},$$

thus we also have a uniform bound assumption on the functions K, g_0 and A, and the new unknown G is in an interval $[\underline{G},\overline{G}]$. A bootstrap argument will recover the no-vacuum property. Setting

$$\mathcal{A}(G) = (a \circ A^{-1})(G) \text{ and } \mathcal{Q}(G) = \left[\left(\int \frac{-\rho g_0'(\rho)}{a(\rho)}\, d\rho\right) \circ A^{-1}\right](G),$$

the equation on z is a Schrödinger-type equation, written as

$$\partial_t z + u\cdot\nabla z + i\nabla z\cdot w + i\nabla(\mathcal{A}(G)\operatorname{div} z) = \nabla\mathcal{Q}(G), \tag{3.12}$$

while G satisfies

$$\partial_t G + (u\cdot\nabla)G + \mathcal{A}(G)\operatorname{div} u = 0. \tag{3.13}$$

The approximate solution satisfies a similar system with a remainder term which we will denote

$$\mathcal{R} := (\mathcal{R}_1,\mathcal{R}_2) = (R_1^{\mathrm{ap}} \times (A'\circ A^{-1})(G^{\mathrm{ap}}), \nabla(R_2^{\mathrm{ap}} + i\mathcal{R}_1)).$$

From now on, we use deltas to designate the difference between the exact and approximate terms in this system, e.g. $\delta u = u - u^{\mathrm{ap}}$, $\delta\mathcal{A} = \mathcal{A}(G) - \mathcal{A}(G^{\mathrm{ap}})$. The difference $(\delta G, \delta z)$ satisfies the equation

$$\begin{cases} \partial_t(\delta G) + u\cdot\nabla(\delta G) + (\delta u)\cdot\nabla G^{\mathrm{ap}} + \mathcal{A}(G)\operatorname{div}(\delta u) + (\delta\mathcal{A})\operatorname{div} u^{\mathrm{ap}} = -\mathcal{R}_1 \\ \partial_t(\delta z) + u\cdot\nabla(\delta z) + (\delta u)\cdot\nabla z^{\mathrm{ap}} + i\nabla(\delta z)\cdot w + i\nabla z^{\mathrm{ap}}\cdot(\delta w) \\ \qquad + i\nabla(\mathcal{A}(G)\operatorname{div}(\delta z)) + i\nabla((\delta\mathcal{A})\operatorname{div} z^{\mathrm{ap}}) = \nabla(\delta Q) - \mathcal{R}_2. \end{cases} \tag{3.14}$$

Lemma 3.1. — *Let $s > 2$ be a whole number, $(G^{\mathrm{ap}}, z^{\mathrm{ap}}) \in W^{s+2,\infty}$ and $(\delta G, \delta z)|_{t=0} \in H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$. Then there exists a time $T > 0$ such that equation* (3.14) *has a unique solution in $\mathcal{C}([0,T], H^{s+1}\times H^s)$, and we have the following energy estimate: setting $Z(t)^2 = \|\delta z(t)\|_{H^s}^2 + \|\delta G(t)\|_{L^2}^2$, we have*

$$\frac{d}{dt} Z(t)^2 \leqslant \tilde{M}(\|G^{\mathrm{ap}}, z^{\mathrm{ap}}\|_{W^{s+2,\infty}})(Z(t)^2 + Z(t)^3) + C\,\|\mathcal{R}\|_{H^s}^2 \tag{3.15}$$

for a certain positive increasing function $\tilde{M}$.

Proof. — The proof of this result is the same as in part 6 of [7] with a few minor differences, so we focus on these and send to the reference for details.

The main difference with Theorem 6.1 in [7] is that we lift the restriction on the reference solution, the role of which is played in our case by $(G^{\mathrm{ap}}, z^{\mathrm{ap}})$. Benzoni-Gavage, Danchin and Descombes solved this problem when the reference solution had derivatives in $H^{s+3}(\mathbb{R}^2)$, which is impossible for our perturbation of a 1D travelling wave. To be able to bypass this condition, we need to understand where it comes from. Indeed, the authors consider an order-4 mollified version of equations (3.13, 3.12) for which existence can be established with a fixed point method. This equation is (3.13, 3.12) $-\eta\Delta^2(G,z)$ (add $-\eta\Delta^2(G,z)$ on the right-hand-side of the equations), and as a result, the equation on the difference between their regularised solution and the reference solution is

$$\begin{cases} \partial_t(\delta G) + \eta\Delta^2\delta G + u\cdot\nabla(\delta G) + (\delta u)\cdot\nabla G^{\mathrm{ap}} \\ \qquad\qquad\qquad + \mathcal{A}(G)\operatorname{div}(\delta u) + (\delta\mathcal{A})\operatorname{div} u^{\mathrm{ap}} = -\eta\Delta^2 G^{\mathrm{ap}} \\ \partial_t(\delta z) + \eta\Delta^2\delta z + u\cdot\nabla(\delta z) + (\delta u)\cdot\nabla z^{\mathrm{ap}} + i\nabla(\delta z)\cdot w + i\nabla z^{\mathrm{ap}}\cdot(\delta w) \\ \qquad\qquad + i\nabla(\mathcal{A}(G)\operatorname{div}(\delta z)) + i\nabla((\delta\mathcal{A})\operatorname{div} z^{\mathrm{ap}}) = \nabla(\delta Q) - \eta\Delta^2 z^{\mathrm{ap}}. \end{cases}$$

When the reference solution has derivatives in H^{s+3}, one treats the bilaplacians on the right-hand side simply as a remainder term, using the Cauchy–Schwarz inequality. In our case, it is impossible to do this.

So we change the set-up slightly. Instead of considering the difference between a regularised solution and the reference solution, we mollify the difference equation (3.14) directly, which leads us to studying

$$\begin{cases} \partial_t(\delta G) + \eta\Delta^2\delta G + u\cdot\nabla(\delta G) + (\delta u)\cdot\nabla G^{\mathrm{ap}} \\ \qquad\qquad\qquad + \mathcal{A}(G)\operatorname{div}(\delta u) + (\delta\mathcal{A})\operatorname{div} u^{\mathrm{ap}} = -\mathcal{R}_1 \\ \partial_t(\delta z) + \eta\Delta^2\delta z + u\cdot\nabla(\delta z) + (\delta u)\cdot\nabla z^{\mathrm{ap}} + i\nabla(\delta z)\cdot w + i\nabla z^{\mathrm{ap}}\cdot(\delta w) \\ \qquad\qquad + i\nabla(\mathcal{A}(G)\operatorname{div}(\delta z)) + i\nabla((\delta\mathcal{A})\operatorname{div} z^{\mathrm{ap}}) = \nabla(\delta Q) - \mathcal{R}_2, \end{cases}$$

with the same regularised initial data as in [7]. Then, the functional we need to find a fixed point for is the same, bar changes to the source term (constant with respect to the variable of the functional), so the contraction property, existence and uniqueness for the mollified system are established in exactly the same way.

In [7], energy estimates are proved for general velocities, including ones that are not potential. Ours, (3.15), are obtained following the lines of their proof, but with a simpler expression for $Z(t)$ (when the velocity is potential, their weighted norm is equivalent to the standard Sobolev norm), and, as we use a whole number of derivatives, we can put all the appearances of $(G^{\mathrm{ap}}, z^{\mathrm{ap}})$ in L^∞. We do not detail this further. □

We now translate (3.15) into an estimate on the original variables. This can mostly be done by following the lines of Appendix B of [7], but we take extra care to control $\|\rho-\rho^{\mathrm{ap}}\|_{H^s}$. Indeed, $\rho = A^{-1}(G)$, so this is a difference between nonlinearly changed variables. We begin as in Corollary B.8 of [7], that is

$$\|\rho-\rho^{\mathrm{ap}}\|_{H^s} = \left\|A^{-1}(G) - A^{-1}(G^{\mathrm{ap}})\right\|_{H^s} \leqslant \int_0^1 \left\|\delta G (A^{-1})'(G^{\mathrm{ap}} + \tau\delta G)\right\|_{H^s} \, d\tau.$$

We then apply the Leibniz and Faà di Bruno formulae, which yields a long sum of products of derivatives of δG and of G^{ap}, which we do not venture to write in detail. The important note at this stage is that, in each term, only one element can bear more than $[s/2]$ derivatives, where $[x]$ designates the whole part of the number x. We thus put the terms involving G^{ap} and all but one of those involving δG (which must be the one with more than $[s/2]$ derivatives if there is one) into L^∞, hence there exists an increasing polynomial function M of degree $\leqslant s$ such that

$$\|\rho-\rho^{\mathrm{ap}}\|_{H^s} \leqslant C(s,A) M(\|G^{\mathrm{ap}}\|_{W^{s,\infty}} + \|\delta G\|_{W^{[s/2],\infty}}) \|\delta G\|_{H^s}.$$

If we choose s so that $[s/2]+2 < s$, that is $s > 4$, we can use the Sobolev embedding $H^s \hookrightarrow W^{[s/2],\infty}$ to bound the $W^{[s/2],\infty}$ norm of δG, so, in total, we have

$$\|\rho-\rho^{\mathrm{ap}}\|_{H^s} \leqslant C(s,A) M(\|G^{\mathrm{ap}}\|_{W^{s,\infty}}) \left(\|\delta G\|_{H^s} + \|\delta G\|_{H^s}^{s+1}\right). \tag{3.16}$$

A similar estimate exists to bound δG by $\rho - \rho^{\mathrm{ap}}$.

Setting $W = U - U^{\mathrm{ap}}$, we get that (3.15) and (3.16) imply that there exists an increasing polynomial function M and a power $r > 2$ such that, for $s > 4$,

$$\|W(t)\|_{H^s}^2 \leqslant \int_0^t M(\|Q_c\|_{W^{s+2,\infty}} + \|U^{\mathrm{ap}} - Q_c\|_{H^{s+4}}) \times (\|W\|_{H^s}^2 + \|W\|_{H^s}^r + \|R^{\mathrm{ap}}\|_{H^s}^2).$$

Here, we have chosen κ small enough so that $\|R^{\mathrm{ap}}(t)\|_{H^s} \leqslant 1$ for $t \leqslant T_\varepsilon^*$, hence the small powers of $\|R^{\mathrm{ap}}\|_{H^s}$ are dominant. We will now choose N to get the right growth in time for W, as well as κ to get the existence up to T_ε^* of the exact solution U. First of all, in the same way that we get (3.11), we note that

$$\|U^{\mathrm{ap}}(t) - Q_c\|_{H^{s+4}} \leqslant \sum_{j=1}^N C_j \frac{\varepsilon^j e^{j\sigma_0 t}}{(1+t)^{j/4p}} \leqslant 2\kappa$$

when $t \leqslant T_\varepsilon^* - \tau_2$, with $\tau_2 \geqslant \tau_1$ independent of ε. We consider times $t \leqslant T_W$ so that $\|W(t)\|_{H^s} \leqslant 1$ and $\rho(t,x,y) > 0$ (no vacuum on the exact solution), and choose N so that, for $t \leqslant T_W$,

$$2N\sigma_0 > M(\|Q_c\|_{W^{s+2,\infty}} + 2\kappa).$$

A variant of the Grönwall inequality from [19] then provides us with

$$\|W(t)\|_{H^s}^2 \leqslant C\frac{\varepsilon^{N+1}e^{2(N+1)\sigma_0 t}}{(1+t)^{(N+1)/2p}} \tag{3.17}$$

for $t \leqslant T_W$. Now, take $t = T_\varepsilon^* - \tau$: we notice that the right-hand side is smaller than $C(N)\kappa^{2(N+1)}$, which is therefore smaller than κ if $\kappa < 1$ is small enough. We now choose κ so that $2\kappa < \min \rho_c$, and this ensures that there is no vacuum on $[0, T^\varepsilon] \times \mathbb{R}^2$. We therefore have $T_W \geqslant T_\varepsilon^*$ by a bootstrap argument. So, (3.17) is valid for $t = T_\varepsilon^* - \tau$ with $\tau \geqslant \tau_2$, and we have

$$\|W(T_\varepsilon^* - \tau)\|_{H^s} \leqslant C_0'\kappa^{N+1}e^{-(N+1)\sigma_0\tau}. \tag{3.18}$$

Before we conclude, let us deal with the translations of Q_c. Recall f_1, the smooth function such that $f_1(k) = 1$ for $|k|$ in a neighbourhood I of k_0, not including 0. Now set f, another smooth compactly-supported function such that $f = 1$ on I and $f(0) = 0$, and define Π, a Fourier projector on frequencies in I, by

$$\mathcal{F}_y(\Pi u)(x,k) = f(k)(\mathcal{F}_y u)(x,k).$$

As $U(0,x,y) = q_c(x) + \varepsilon U_1(0,x,y)$, $\Pi U|_{t=0} = \varepsilon U_1|_{t=0}$. Moreover, for any $a \in \mathbb{R}$, the difference $q_c(x-ct-a) - q_c(x-ct)$ does not depend on y, hence

$$\Pi(q_c(\cdot - ct - a) - q_c(\cdot - ct)) = 0.$$

We can now combine (3.11) and (3.18) to show the instability: we have, for any a,

$$\begin{aligned}\|U(t) - q_c(\,\cdot - a - ct)\|_{L^2(\mathbb{R}^2)}\,|_{t=T_\varepsilon^*-\tau} &\geqslant \|\Pi(U(t) - q_c(\,\cdot - ct))\|_{L^2(\mathbb{R}^2)}\,|_{t=T_\varepsilon^*-\tau} \\ &\geqslant \frac{\kappa C_1'}{2}e^{-\sigma_0\tau}\left[1 - \frac{2\kappa^N C_0'}{C_1'}e^{-N\sigma_0\tau}\right],\end{aligned}$$

For $\tau \geqslant \tau_3 \geqslant \tau_2$, we have the last exponential on the right smaller than $1/2$, and, as a result, letting $\tau' \geqslant \tau_3$ be fixed, independent of ε,

$$\|(U - Q_c)(T_\varepsilon^* - \tau')\|_{L^2} \geqslant \frac{\kappa C_1'}{4}e^{-\sigma_0\tau'} := \delta_0.$$

The number δ_0 we have found is positive and does not depend on ε: Theorem 1.1 is proved.

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