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Martial Agueh, Shirin Boroushaki and Nassif Ghoussoub

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A dual Moser–Onofri inequality and its extensions to higher dimensional spheres

Martial Agueh [(1)], Shirin Boroushaki [(2)] and Nassif Ghoussoub [(3)]

Abstract. — We use optimal mass transport to provide a new proof and a dual formula to the Moser–Onofri inequality on $\mathbb{S}^2$. This is in the same spirit as the approach of Cordero-Erausquin, Nazaret and Villani [5] to the Sobolev and Gagliardo–Nirenberg inequalities and the one of Agueh–Ghoussoub–Kang [1] to more general settings. There are however many hurdles to overcome once a stereographic projection on $\mathbb{R}^2$ is performed: Functions are not necessarily of compact support, hence boundary terms need to be evaluated. Moreover, the corresponding dual free energy of the reference probability density $\mu_2(x) = \frac{1}{\pi(1+|x|^2)^2}$ is not finite on the whole space, which requires the introduction of a renormalized free energy into the dual formula. We also extend this duality to higher dimensions and establish an extension of the Onofri inequality to spheres $\mathbb{S}^n$ with $n \geqslant 2$. What is remarkable is that the corresponding free energy is again given by $F(\rho) = -n\rho^{1-\frac{1}{n}}$, which means that both the *prescribed scalar curvature problem* and the *prescribed Gaussian curvature problem* lead essentially to the same dual problem whose extremals are stationary solutions of the fast diffusion equations.

Résumé. — Nous utilisons une méthode de transport optimal pour donner une nouvelle démonstration et une forme duale de l'inégalité de Moser-Onofri sur $\mathbb{S}^2$. Cette approche est dans le même esprit que celle des inégalités de Sobolev et de Gagliardi-Nirenberg par Cordero-Erausquin, Nazaret et Villani [5] ainsi que de leurs généralisations par Agueh–Ghoussoub–Kang [1]. Il y a néanmoins plusieurs difficultés nouvelles qui apparaissent une fois qu'on a effectué une projection stéréographique sur $\mathbb{R}^2$: les fonctions n'ont plus support compact, ce qui demande de tenir

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(1) Department of Mathematics and Statistics University of Victoria, Victoria, BC, V8W 3R4, Canada

Martial Agueh passed away on September 14, 2016. He was 47 years old.

(2) Department of Mathematics, 1984 Mathematics Road, University of British Columbia, BC, V6T 1Z2, Canada — shirinbr@math.ubc.ca

(3) Department of Mathematics, 1984 Mathematics Road, University of British Columbia, BC, V6T 1Z2, Canada — nassif@math.ubc.ca

compte de termes de bord. De plus, l'énergie libre duale de la densité probabilité de référence $\mu_2(x) = \frac{1}{\pi(1+|x|^2)^2}$ n'est pas finie sur l'espace entier ce qui demande d'introduire une énergie libre renormalisée dans la formule duale. Nous étendons aussi cette inégalité en dimensions supérieures et établissons une inégalité d'Onofri pour les sphères $\mathbb{S}^n$ quand $n \geqslant 2$. Il est remarquable que l'énergie libre correspondante est toujours donnée par $F(\rho) = -n\rho^{1-\frac{1}{n}}$, ce qui signifie que les problèmes de courbure scalaire prescrite et de courbure de Gauss prescrite conduisent essentiellement au même problème dual.

1. Introduction

One of the equivalent forms of Moser's inequality [11] on the 2-dimensional sphere $\mathbb{S}^2$ states that the functional

$$I(u) := \frac{1}{4}\int_{\mathbb{S}^2} |\nabla u|^2 \,\mathrm{d}\omega + \int_{\mathbb{S}^2} u \,\mathrm{d}\omega - \log\left(\int_{\mathbb{S}^2} e^u \,\mathrm{d}\omega\right) \tag{1.1}$$

is bounded below on $H^1(\mathbb{S}^2)$, where $\mathrm{d}\omega$ is the Lebesgue measure on $\mathbb{S}^2$, normalized so that $\int_{\mathbb{S}^2} \mathrm{d}\omega = 1$. Later, Onofri [12] showed that the infimum of (1.1) over $H^1(\mathbb{S}^2)$ is actually zero, and that modulo conformal transformations, $u = 0$ is the only optimal function. Note that this inequality is related to the "prescribed Gaussian curvature" problem on $\mathbb{S}^2$,

$$\Delta u + K(x)e^{2u} = 1 \quad \text{on } \mathbb{S}^2, \tag{1.2}$$

where $K(x)$ is the Gaussian curvature associated to the metric $g = e^{2u}g_0$ on $\mathbb{S}^2$, and $\Delta = \Delta_{g_0}$ is the Laplace–Beltrami operator corresponding to the standard metric g_0. Finding g for a given K leads to solving (1.2). Variationally, this reduces to finding the critical points of the functional

$$\mathcal{F}(u) = \int_{\mathbb{S}^2} |\nabla u|^2 \frac{\mathrm{d}V_0}{4\pi} + 2\int_{\mathbb{S}^2} u\frac{\mathrm{d}V_0}{4\pi} - \log\left(\int_{\mathbb{S}^2} K(x)e^{2u}\frac{\mathrm{d}V_0}{4\pi}\right) \quad \text{on } H^1(\mathbb{S}^2), \tag{1.3}$$

where the volume form is such that $\int_{\mathbb{S}^2} \mathrm{d}V_0 = 4\pi$. Onofri's result says that, modulo conformal transformations, $u \equiv 0$ is the only solution of the "prescribed Gaussian curvature" problem (1.2) for $K = 1$, i.e. $\frac{1}{2}\Delta u + e^u = 1$ on $\mathbb{S}^2$, which after rescaling, $u \mapsto 2u$, gives

$$\Delta u + e^{2u} = 1 \quad \text{on } \mathbb{S}^2. \tag{1.4}$$

The proof given by Onofri in [12] makes use of a constrained Moser inequality due to Aubin [2] combined with the invariance of the functional (1.1) under conformal transformations. Other proofs were given by Osgood–Philips–Sarnak [13] and by Hong [8]. See also Ghoussoub–Moradifam [7].

In this paper, we use the theory of mass transport to prove that 0 is the infimum of the functional (1.3) at least when $K = 1$. While this approach has by now become standard, there are many reasons why it has not been so far spelled out in the case of the Moser functional. The first is due to the fact that, unlike the case of $\mathbb{R}^n$, optimal mass transport on the sphere is harder to work with. To avoid this difficulty, we use an equivalent formulation of the Onofri inequality (1.1), which is obtained by projecting (1.1) on $\mathbb{R}^2$ via the stereographic projection with respect to the North pole $N = (0,0,1)$, i.e., $\Pi : \mathbb{S}^2 \to \mathbb{R}^2$, $\Pi(x) := \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\right)$ where $x = (x_1, x_2, x_3)$. The Moser–Onofri inequality becomes the *Euclidean Onofri inequality* on $\mathbb{R}^2$, namely

$$\frac{1}{16\pi}\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^2} u \,\mathrm{d}\mu_2 - \log\left(\int_{\mathbb{R}^2} e^u \,\mathrm{d}\mu_2\right) \geqslant 0 \quad \forall u \in H^1(\mathbb{R}^2). \quad (1.5)$$

Here μ_2 is the probability density on $\mathbb{R}^2$ defined by $\mu_2(x) = \frac{1}{\pi(1+|x|^2)^2}$, and $\mathrm{d}\mu_2 = \mu_2(x)\,\mathrm{d}x$.

One can then try to apply the Cordero–Nazaret–Villani [5] approach as generalized by Agueh–Ghoussoub–Kang [1] and write the *Energy-Entropy production duality* for functions that are of compact support in Ω,

$$\begin{aligned}\sup&\left\{-\int_\Omega (F(\rho) + \frac{1}{2}|x|^2\rho)\,\mathrm{d}x;\ \rho \in \mathcal{P}(\Omega)\right\}\\ &= \inf\left\{\int_\Omega \alpha|\nabla u|^2 - G(\psi\circ u)\,\mathrm{d}x;\ u \in H^1_0(\Omega), \int_\Omega \psi(u)\,\mathrm{d}x = 1\right\}, \quad (1.6)\end{aligned}$$

where $G(x) = (1-n)F(x) + nxF'(x)$ and where ψ and α are also computable from F. Here $\mathcal{P}(\Omega)$ denotes the set of probability densities on Ω.

By choosing $F(x) = -nx^{1-1/n}$ and $\psi(t) = |t|^{2^*}$ where $2^* = \frac{2n}{n-2}$ and $n > 2$, one obtains the following duality formula for the Sobolev inequality

$$\begin{aligned}\sup&\left\{n\int_{\mathbb{R}^n} \rho^{1-1/n}\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^n} |x|^2\rho\,\mathrm{d}x;\ \rho \in \mathcal{P}(\mathbb{R}^n)\right\}\\ &= \inf\left\{2\left(\frac{n-1}{n-2}\right)^2\int_{\mathbb{R}^n} |\nabla u|^2\,\mathrm{d}x; u \in D^{1,2}(\mathbb{R}^n), \int_{\mathbb{R}^n} |u|^{2^*}\,\mathrm{d}x = 1\right\}, \quad (1.7)\end{aligned}$$

where u and ρ have compact support in $\mathbb{R}^n$. The extremal u_∞ and ρ_∞ are then obtained as solutions of

$$\nabla\left(\frac{|x|^2}{2} - \frac{n-1}{\rho_\infty^{1/n}}\right) = 0, \quad \rho_\infty = u_\infty^{2^*} \in \mathcal{P}(\mathbb{R}^n). \quad (1.8)$$

The best constants are then obtained by computing ρ_∞ from (1.8) and inserting it into (1.7) in such a way that

$$\inf\left\{2\left(\frac{n-1}{n-2}\right)^2\int_{\mathbb{R}^n}|\nabla u|^2\,\mathrm{d}x;\ u\in D^{1,2}(\mathbb{R}^n), \|u\|_{2^*}=1\right\}$$
$$= n\int_{\mathbb{R}^n}\rho_\infty^{1-1/n}\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^n}|x|^2\rho_\infty\,\mathrm{d}x.$$

Note that this duality leads to a correspondence between a solution to the Yamabe equation

$$-\Delta u=|u|^{2^*-2}u\quad\text{on }\ \mathbb{R}^n \tag{1.9}$$

and stationary solution to the rescaled fast diffusion equation

$$\partial_t\rho=\Delta\rho^{1-\frac{1}{n}}+\operatorname{div}(x\rho)\quad\text{on }\ \mathbb{R}^n. \tag{1.10}$$

The above scheme does not however apply to inequality (1.5). For one, the functions $e^u\mu_2=\frac{e^{u(x)}}{\pi(1+|x|^2)^2}$ do not have compact support, and if one restricts them to bounded domains, we then need to take into consideration various boundary terms. What is remarkable is that a similar program can be carried out provided the dual formula involving the free energy

$$J_\Omega(\rho)=-\int_\Omega(F(\rho)+|x|^2\rho)\,\mathrm{d}x$$

is renormalized by substituting it with $J_\Omega(\rho)-J_\Omega(\mu_2)$.

Another remarkable fact is that the corresponding free energy turned out to be $F(\rho)=-2\rho^{\frac{1}{2}}$, which is the same as the one associated to the critical case of the Sobolev inequality $F(\rho)=-n\rho^{1-\frac{1}{n}}$ when $n\geqslant 3$. In other words, the Moser–Onofri inequality and the Sobolev inequality "dualize" in the same way, and both the Yamabe problem (1.9) and the prescribed Gaussian curvature problem (1.4) reduce to the study of the fast diffusion equation (1.10), with the caveat that in dimension $n=2$, the above equation needs to be considered only on bounded domains, with Neumann boundary conditions.

More precisely, we shall show that, when restricted to balls B_R of radius R in $\mathbb{R}^2$, there is a duality between the "Onofri functional"

$$I_R(u)=\frac{1}{16\pi}\int_{B_R}|\nabla u|^2\,\mathrm{d}x+\int_{B_R}u\,\mathrm{d}\mu_2$$
$$\text{on } X_R:=\left\{u\in H_0^1(B_R);\ \int_{\mathbb{R}^2}e^u\,\mathrm{d}\mu_2=1\right\},$$

and the free energy

$$J_R(\rho) = \frac{2}{\sqrt{\pi}} \int_{B_R} \sqrt{\rho}\, \mathrm{d}x - \int_{B_R} |x|^2 \rho \, \mathrm{d}x$$
$$\text{on } Y_R := \left\{ \rho \in L^1_+(B_R);\ \frac{1}{\mu_2(B_R)} \int_{B_R} \rho \, \mathrm{d}x = 1 \right\},$$

where

$$\mu_2(B_R) := \int_{B_R} \mathrm{d}\mu_2 = \frac{R^2}{1+R^2}.$$

Note that if u has its support in B_R, then

$$\int_{\mathbb{R}^2} e^u \, \mathrm{d}\mu_2 = 1 \quad \text{if and only if} \quad \frac{1}{\mu_2(B_R)} \int_{B_R} e^u \, \mathrm{d}\mu_2 = 1.$$

We show that once the free energy is re-normalized by subtracting the free energy of μ_2, we then have

$$\sup\{J_R(\rho) - J_R(\mu_2); \rho \in Y_R\} = 0 = \inf\{I_R(u);\ u \in X_R\}. \tag{1.11}$$

Note that when $R \to +\infty$, the right hand side yields the Onofri inequality

$$\inf \left\{ \frac{1}{16\pi} \int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x + \int_{\mathbb{R}^2} u \, \mathrm{d}\mu_2;\ u \in H^1_0(\mathbb{R}^2), \int_{\mathbb{R}^2} e^u \, \mathrm{d}\mu_2 = 1 \right\} = 0,$$

while the left-hand side doesn't yield a universal upper bound for $J_R(\rho)$ since

$$J_R(\mu_2) = \log(1+R^2) + \frac{R^2}{1+R^2} \to +\infty \quad \text{as } R \to +\infty.$$

We actually show that our approach extends to higher dimensions. More precisely, if B_R is a ball of radius R in $\mathbb{R}^n$ where $n \geqslant 2$, and if one considers the probability density μ_n on $\mathbb{R}^n$ defined by

$$\mu_n(x) = \frac{1}{\omega_n (1 + |x|^{\frac{n}{n-1}})^n}$$

(ω_n is the volume of the unit sphere in $\mathbb{R}^n$), and the operator $H_n(u, \mu_n)$ on $W^{1,n}(\mathbb{R}^n)$ by

$$H_n(u, \mu_n)$$
$$:= |\nabla u + \nabla(\log \mu_n)|^n - |\nabla(\log \mu_n))|^n - n |\nabla(\log \mu_n))|^{n-2} \nabla(\log \mu_n) \cdot \nabla u,$$

there is then a duality between the functional

$$I_R(u) = \frac{1}{\beta(n)} \int_{B_R} H_n(u, \mu_n) \, \mathrm{d}x + \int_{B_R} u \, \mathrm{d}\mu_n$$
$$\text{on } X_R := \left\{ u \in W^{1,n}_0(B_R);\ \int_{\mathbb{R}^n} e^u \, \mathrm{d}\mu_n = 1 \right\}$$

and the free energy — renormalized by again subtracting $J_R(\mu_n)$ —

$$J_R(\rho) = \alpha(n)\int_{B_R}\rho^{\frac{n-1}{n}}\,\mathrm{d}x - \int_{B_R}|x|^{\frac{n}{n-1}}\rho\,\mathrm{d}x$$
$$\text{on } Y_R := \left\{\rho\in L^1_+(B_R);\ \frac{1}{\mu_n(B_R)}\int_{B_R}\rho\,\mathrm{d}x = 1\right\},$$

where

$$\alpha(n) = \frac{n}{n-1}\omega_n{}^{-1/n},\quad \beta(n) = \omega_n\,(\frac{n}{n-1})^{n-1}n^{n+1}$$
$$\text{and}\quad \mu_n(B_R) := \int_{B_R}\mathrm{d}\mu_n = \frac{R^n}{(1+R^{\frac{n}{n-1}})^{n-1}}$$

We then deduce the following higher dimensional version of the Onofri inequality: For $n\geqslant 2$,

$$\frac{1}{\beta(n)}\int_{\mathbb{R}^n}H_n(u,\mu_n)\,\mathrm{d}x + \int_{\mathbb{R}^n}u\,\mathrm{d}\mu_n - \log\left(\int_{\mathbb{R}^n}e^u\,\mathrm{d}\mu_n\right)\geqslant 0$$
$$\text{for all } u\in W^{1,n}(\mathbb{R}^n). \quad (1.12)$$

We finish this introduction by mentioning that there was an attempt in [6] to use mass transport to establish the Euclidean Onofri inequality (1.5) in the radial case. In [9], Maggi and Villani also establish Sobolev-type inequalities involving boundary trace terms via mass transport methods. They actually deal with a family of Moser–Trudinger inequalities as a limiting case of Sobolev inequality when the power $p\to n$, in the presence of boundary terms on a Lipschitz domain in $\mathbb{R}^n$. However, to our knowledge, our duality result, the extensions of Onofri's inequality to higher dimensions, as well as the mass transport proof of the general (non-radial) Onofri inequality are new.

The paper is organized as follows. In Section 2, we recall the mass transport approach to sharp Sobolev inequalities and some consequences. In Section 3, we establish the n-dimensional mass transport duality principle, from which we could deduce the two dimensional Euclidean Onofri inequality (1.5). We thank an anonymous referee for valuable comments and suggestions.

2. Preliminaries

We start by briefly describing the mass transport approach to sharp Sobolev inequalities as proposed by [5]. We will follow here the framework of [1] as it clearly shows the correspondence between the Yamabe equation (1.9) and the rescaled fast diffision equation (1.10).

Let $\rho_0, \rho_1 \in \mathcal{P}(\mathbb{R}^n)$. If T is the optimal map pushing ρ_0 forward to ρ_1 (i.e. $T_\# \rho_0 = \rho_1$) in the mass transport problem for the quadratic cost $c(x-y) = \frac{|x-y|^2}{2}$ (see [14] for details), then $[0,1] \ni t \mapsto \rho_t = (T_t)_\# \rho_0$ is the geodesic joining ρ_0 and ρ_1 in $(\mathcal{P}(\mathbb{R}^n), d_2)$; here $T_t := (1-t)\mathrm{id} + tT$ and d_2 denotes the quadratic Wasserstein distance (see [14]). Moreover, given a function $F : [0,\infty) \to \mathbb{R}$ such that $F(0) = 0$ and $x \mapsto x^n F(x^{-n})$ is convex and non-increasing, the functional $H^F(\rho) := \int_{\mathbb{R}^n} F(\rho(x))\,\mathrm{d}x$ is displacement convex [10], in the sense that $[0,1] \ni t \mapsto H^F(\rho_t) \in \mathbb{R}$ is convex (in the usual sense), for all pairs (ρ_0, ρ_1) in $\mathcal{P}(\mathbb{R}^n)$. A direct consequence is the following convexity inequality, known as "*energy inequality*":

$$H^F(\rho_1) - H^F(\rho_0) \geqslant \left[\frac{d}{dt} H^F(\rho_t)\right]_{t=0} = \int_{\mathbb{R}^n} \rho_0 \nabla \left(F'(\rho_0)\right) \cdot (T - \mathrm{id})\,\mathrm{d}x,$$

which, after integration by parts of the right hand side term, reads as

$$-H^F(\rho_1) \leqslant -H^{F+nP_F}(\rho_0) - \int_{\mathbb{R}^n} \rho_0 \nabla \left(F'(\rho_0)\right) \cdot T(x)\,\mathrm{d}x, \qquad (2.1)$$

where $P_F(x) = xF'(x) - F(x)$; here id denotes the identity function on $\mathbb{R}^n$. By the Young inequality

$$-\nabla \left(F'(\rho_0)\right) \cdot T(x) \leqslant \frac{|\nabla F'(\rho_0)|^p}{p} + \frac{|T(x)|^q}{q} \quad \forall p, q > 1 \quad \text{such that} \quad \frac{1}{p} + \frac{1}{q} = 1, \qquad (2.2)$$

(2.1) gives

$$-H^F(\rho_1) \leqslant -H^{F+nP_F}(\rho_0) + \frac{1}{p} \int_{\mathbb{R}^n} \rho_0 |\nabla F'(\rho_0)|^p\,\mathrm{d}x + \frac{1}{q} \int_{\mathbb{R}^n} \rho_0(x) |T(x)|^q\,\mathrm{d}x,$$

i.e.,

$$-H^F(\rho_1) - \frac{1}{q} \int_{\mathbb{R}^n} |y|^q \rho_1(y)\,\mathrm{d}y \leqslant -H^{F+nP_F}(\rho_0) + \frac{1}{p} \int_{\mathbb{R}^n} \rho_0 |\nabla F'(\rho_0)|^p\,\mathrm{d}x, \qquad (2.3)$$

where we use that $T_\# \rho_0 = \rho_1$. Furthermore, if $\rho_0 = \rho_1$, then $T = \mathrm{id}$ and equality holds in (2.1). Then equality holds in (2.3) if it holds in the Young inequality (2.2). This occurs when $\rho_0 = \rho_1$ satisfies $\nabla \left(F'(\rho_0(x)) + \frac{|x|^q}{q}\right) = 0$.

Therefore, we have established the following duality:

$$\sup\left\{-H^F(\rho_1)-\frac{1}{q}\int_{\mathbb{R}^n}|y|^q\rho_1(y)\,\mathrm{d}y;\quad \rho_1\in\mathcal{P}(\mathbb{R}^n)\right\}$$
$$=\inf\left\{-H^{F+nP_F}(\rho_0)+\frac{1}{p}\int_{\mathbb{R}^n}\rho_0|\nabla F'(\rho_0)|^p\,\mathrm{d}x;\ \rho_0\in\mathcal{P}(\mathbb{R}^n)\right\},\quad (2.4)$$

and an optimal function in both problems is $\rho_0=\rho_1:=\rho_\infty$ solution of

$$\nabla\left(F'\left(\rho_\infty(x)\right)+\frac{|x|^q}{q}\right)=0.\quad (2.5)$$

In particular, choosing $F(x)=-nx^{1-1/n}$ and $\rho_0=u^{2^*}$ where $2^*=\frac{2n}{n-2}$ and $n>2$, then $H^{F+nP_F}=0$, and (2.4)-(2.5) gives the duality formula for the Sobolev inequality (1.7).

Our goal now is to extend this mass transport proof of the Sobolev inequality to the Euclidean Onofri inequality (1.5). As already mentioned in the introduction, a first attempt on this issue was recently made by [6], but the result produced was only restricted to the radial case. Here we show in full generality (without restricting to radial functions u) that the Euclidean Onofri inequality (1.5) can be proved by mass transport techniques. More precisely, we establish an analogue of the duality (1.7) for Euclidean Onofri inequality (see Theorem 3.1), from which we deduce the Onofri inequality (1.5) (see Corollary 3.5). Furthermore, we obtain — as for the critical Sobolev inequality — a correspondence between the prescribed Gaussian curvature problem (1.4) and the rescaled fast diffusion equation (1.10). Finally, we extend our analysis to higher dimensions, and then produce a new version of the Onofri inequality in dimensions $n\geqslant 2$ (see Theorem 3.3).

We shall need the following general lemma from the theory of mass transport.

Lemma 2.1. — *Let $\rho_0,\rho_1\in\mathcal{P}(B_R)$, where $\mathcal{P}(B_R)$ denotes the set of probability densities on the ball $B_R\subset\mathbb{R}^n$. Let T be the optimal map pushing ρ_0 forward to ρ_1 (i.e. $T_\#\rho_0=\rho_1$) in the mass transport problem corresponding to the quadratic cost. Then*

$$\int_{B_R}\rho_1(y)^{1-\frac{1}{n}}\,dy\leqslant\frac{1}{n}\int_{B_R}\rho_0(x)^{1-\frac{1}{n}}\operatorname{div}(T(x))\,dx.\quad (2.6)$$

Proof. — By Brenier's theorem [4], there is a map $T:B_R\to B_R$ such that $T=\nabla\varphi$ where $\varphi:B_R\to\mathbb{R}$ is convex, and $T_\#\rho_0=\rho_1$. We therefore have the following Monge–Ampère equation,

$$\rho_0(x)=\rho_1(T(x))\det\nabla T(x)\quad (2.7)$$

or equivalently

$$\rho_1(T(x))=\rho_0(x)[\det\nabla T(x)]^{-1}.\quad (2.8)$$

By the arithmetic-geometric-mean inequality

$$[\det \nabla T(x)]^{\frac{1}{n}} \leqslant \frac{1}{n} \operatorname{div}(T(x)),$$

(2.8) gives

$$\rho_1(T(x))^{-\frac{1}{n}} \leqslant \frac{1}{n} \rho_0(x)^{-\frac{1}{n}} \operatorname{div}(T(x)). \tag{2.9}$$

Now using the change of variable $y = T(x)$, we have

$$\int_{B_R} \rho_1(y)^{1-\frac{1}{n}} \,\mathrm{d}y = \int_{B_R} \rho_1 (T(x))^{1-\frac{1}{n}} \det(\nabla T(x)) \,\mathrm{d}x,$$

which implies by (2.7) and (2.9), that

$$\begin{aligned}\int_{B_R} \rho_1(y)^{1-\frac{1}{n}} \,\mathrm{d}y &\leqslant \frac{1}{n} \int_{B_R} \rho_0(x)^{-\frac{1}{n}} \operatorname{div}(T(x)) \rho_0(x) \,\mathrm{d}x \\ &= \frac{1}{n} \int_{B_R} \rho_0(x)^{1-\frac{1}{n}} \operatorname{div}(T(x)) \,\mathrm{d}x,\end{aligned}$$

and we are done. □

3. Euclidean n-dimensional Onofri inequality: A duality formula

Consider the probability density on $\mathbb{R}^n$, $\mu_n(y) = \frac{1}{\omega_n (1+|y|^{\frac{n}{n-1}})^n}$, where ω_n is the volume of the unit sphere in $\mathbb{R}^n$, and set

$$\theta_R := \int_{B_R} \mu_n(y) \,\mathrm{d}y = \frac{R^n}{(1 + R^{\frac{n}{n-1}})^{n-1}}.$$

We shall establish the following duality formula.

THEOREM 3.1 (Duality for n-dimensional Euclidean Onofri inequality). *For each ball B_R in $\mathbb{R}^n$ with radius $R > 0$, we consider the following free functional*

$$J_R(\rho) = \alpha(n) \int_{B_R} \rho(y)^{\frac{n-1}{n}} \,dy - \int_{B_R} |y|^{\frac{n}{n-1}} \rho(y) \,dy \qquad \text{for } \rho \in L^1_+(B_R),$$

as well as the "entropy" functional

$$I_R(u) = \frac{1}{\beta(n)} \int_{B_R} H_n(u, \mu_n) \,dx + \int_{B_R} u(x) \,d\mu_n \qquad \text{for } u \in W^{1,n}_0(B_R),$$

where

$$\alpha(n) = \frac{n}{n-1} \left(\frac{1}{\omega_n}\right)^{1/n}, \qquad \beta(n) = \left(\frac{n}{n-1}\right)^{n-1} n^{(n+1)} \,\omega_n$$

and

$$H_n(u,\mu_n) \\ := |\nabla u + \nabla(\log\mu_n)|^n - |\nabla(\log\mu_n)|^n - n|\nabla(\log\mu_n)|^{n-2}\nabla(\log\mu_n)\cdot\nabla u.$$

The following duality formula then holds:

$$\sup\left\{J_R(\rho) - J_R(\mu_n); \rho\in L^1_+(\mathbb{R}^n),\ \int_{B_R}\rho\,dy = \theta_R\right\} \\ = \inf\left\{I_R(u);\ u\in W_0^{1,n}(B_R), \int_{B_R} e^u\,d\mu_n = \theta_R\right\} = 0. \quad (3.1)$$

Moreover, the maximum on the l.h.s. is attained only at $\rho_{max} = \mu_n$, and the minimum on the r.h.s. is attained only at $u_{min} = 0$.

Remark 3.2. — Before proving the theorem, we make a few remarks on the operator $H_n(u,\mu_n)$.

(1) Consider the function $c:\mathbb{R}^n\to\mathbb{R}$, $c(z) = |z|^n$, $n\geqslant 2$. Clearly c is strictly convex, and $\nabla c(z) = n|z|^{n-2}z$. So we have the convexity inequality

$$c(z_1) - c(z_0) - \nabla c(z_0)\cdot(z_1 - z_0) \geqslant 0 \quad \forall z_0, z_1\in\mathbb{R}^n. \quad (3.2)$$

Setting $z_0 = \nabla(\log\mu_n)$ and $z_1 = \nabla u + \nabla(\log\mu_n)$, we see that $H_n(u,\mu_n)$ is nothing but the l.h.s of (3.2); we then deduce that

$$H_n(u,\mu_n)\geqslant 0 \qquad \forall\, u,\mu_n.$$

(2) For all $u\in W_0^{1,n}(B_R)$, the integral of $H_n(u,\mu_n)$ over B_R involves a well-known operator, the n-Laplacian Δ_n, defined by

$$\Delta_n v := \operatorname{div}(|\nabla v|^{n-2}\nabla v). \quad (3.3)$$

Indeed, this can be seen after performing an integration by parts in the last term of $H_n(u,\mu_n)$,

$$\int_{B_R} H_n(u,\mu_n)\,\mathrm{d}y = \int_{B_R}|\nabla u + \nabla\log\mu_n|^n\,\mathrm{d}y - \int_{B_R}|\nabla(\log\mu_n))|^n\,\mathrm{d}y \\ + n\int_{B_R} u\,\Delta_n(\log\mu_n)\,\mathrm{d}y.$$

Proof. — By applying Lemma 2.1, and using an integration by parts and $\frac{1}{m} := 1 - \frac{1}{n}$, we have

$$n\int_{B_R}\rho_1^{1/m}\,\mathrm{d}y \leqslant -\int_{B_R}\nabla(\rho_0^{1/m})\cdot T(x)\,\mathrm{d}x + \int_{\partial B_R}\rho_0^{1/m}T(x)\cdot\nu\,\mathrm{d}S.$$

Use the elementary identity $\nabla(\rho_0^{1/m}) = \frac{1}{m}\rho_0^{1/m}\nabla(\log\rho_0)$ to obtain

$$mn\int_{B_R}\rho_1^{1/m}\,\mathrm{d}y \leqslant -\int_{B_R}\rho_0^{1/m}\nabla(\log\rho_0)\cdot T(x)\,\mathrm{d}x + m\int_{\partial B}\rho_0^{1/m}T(x)\cdot\nu\,\mathrm{d}S. \quad (3.4)$$

Set $\rho_0 = \frac{e^u\mu_n}{\theta_R}$, where $u\in W_0^{1,n}(B_R)$ satisfies $\int_{B_R}e^u d\mu_n = \theta_R$, and let ρ_1 be any probability density supported on B_R. By applying (3.4) to ρ_0 and ρ_1, we get

$$mn\int_{B_R}(\theta_R\rho_1)^{1/m}\,\mathrm{d}y \leqslant -\int_{B_R}(e^u\mu_n)^{1/m}\nabla(\log(e^u\mu_n))\cdot T(x)\,\mathrm{d}x + m\int_{\partial B_R}(e^u\mu_n)^{1/m}T(x)\cdot\nu\,\mathrm{d}S.$$

Using Young's inequality

$$-(e^u\mu_n)^{1/m}\,\nabla\left(\log(e^u\mu_n)\right)\cdot T(x) \leqslant \frac{1}{n\varepsilon}|\nabla\left(\log(e^u\mu_n)\right)|^n + \frac{\varepsilon^{m/n}}{m}\,e^u\mu_n|T(x)|^m \qquad \forall\,\varepsilon>0$$

and the fact that $T_{\#}\rho_0 = \rho_1$, we get

$$mn^2\varepsilon\int_{B_R}(\theta_R\rho_1)^{1/m}\,\mathrm{d}y - mn\varepsilon\int_{\partial B_R}\mu_n^{1/m}T(x)\cdot\nu\,\mathrm{d}S - \frac{n}{m}\varepsilon^m\int_{B_R}|y|^m\theta_R\rho_1\,\mathrm{d}y \leqslant \int_{B_R}|\nabla u+\nabla\log\mu_n|^n\,\mathrm{d}x. \quad (3.5)$$

We now estimate the boundary term. Since $T: B_R\to B_R$, then $|T(x)|\leqslant R$ for all $x\in B_R$

$$\begin{aligned}\int_{\partial B_R}\mu_n^{1/m}T(x)\cdot\nu\,\mathrm{d}S &= \left(\frac{1}{\omega_n}\right)^{1/m}\frac{1}{(1+R^m)^{n/m}}\int_{\partial B_R}T(x)\cdot\frac{x}{|x|}\,\mathrm{d}S\\ &\leqslant n\omega_n^{1/n}\frac{R^n}{(1+R^m)^{n/m}} = n\omega_n^{1/n}\,\theta_R.\end{aligned} \quad (3.6)$$

where we used the fact that $\nu = \frac{x}{|x|}$.

Inserting (3.6) into (3.5), and setting $\rho := \theta_R\rho_1$, we get for all $\varepsilon>0$,

$$\varepsilon\left[n^2m\int_{B_R}\rho^{1/m}\,\mathrm{d}y - n^2m\omega_n^{1/n}\theta_R\right] - \varepsilon^m\frac{n}{m}\int_{B_R}|y|^m\rho\,\mathrm{d}y \leqslant \int_{B_R}|\nabla u+\nabla\log\mu_n|^n\,\mathrm{d}x. \quad (3.7)$$

Now, we introduce the operator $H_n(u,\mu_n)$ in the r.h.s of (3.7). We have

$$|\nabla u+\nabla\log\mu_n|^n \\ = H_n(u,\mu_n)+|\nabla(\log\mu_n))|^n+n|\nabla(\log\mu_n))|^{n-2}\nabla(\log\mu_n)\cdot\nabla u,$$

which, after an integration by parts, yields

$$\int_{B_R}|\nabla u+\nabla\log\mu_n|^n\,\mathrm{d}x \\ =\int_{B_R}H_n(u,\mu_n)\,\mathrm{d}x+\int_{B_R}|\nabla(\log\mu_n))|^n\,\mathrm{d}x-n\int_{B_R}u\,\Delta_n(\log\mu_n)\,\mathrm{d}x,$$

where Δ_n is the n-Laplacian operator defined by (3.3). By a direct computation, we note that

$$\Delta_n(\log\mu_n)=-n^n m^{n-1}\omega_n\mu_n.$$

It follows that

$$\int_{B_R}|\nabla u+\nabla\log\mu_n|^n\,\mathrm{d}x \\ =\int_{B_R}H_n(u,\mu_n)\,\mathrm{d}x+n^{n+1}m^{n-1}\omega_n\int_{B_R}u\,\mathrm{d}\mu_n+\int_{B_R}|\nabla(\log\mu_n))|^n\,\mathrm{d}x,$$

and so (3.7) becomes for all $\varepsilon>0$,

$$\varepsilon\left[n^2m\int_{B_R}\rho^{1/m}\,\mathrm{d}y-n^2m\omega_n^{1/n}\theta_R\right]-\varepsilon^m\frac{n}{m}\int_{B_R}|y|^m\rho\,\mathrm{d}y \\ \leqslant\int_{B_R}H_n(u,\mu_n)\,\mathrm{d}x+n^{n+1}m^{n-1}\omega_n\int_{B_R}u\,\mathrm{d}\mu_n+\int_{B_R}|\nabla(\log\mu_n)|^n\,\mathrm{d}x. \quad (3.8)$$

Next, we focus on the l.h.s of (3.8). For convenience, we denote

$$A_\rho:=n^2m\int_{B_R}\rho^{1/m}\,\mathrm{d}y-n^2m\omega_n^{1/n}\theta_R,\quad B_\rho:=\frac{n}{m}\int_{B_R}|y|^m\rho\,\mathrm{d}y,$$

and

$$G_\rho(\varepsilon):=\varepsilon A_\rho-\varepsilon^m B_\rho.$$

Then (3.8) reads as

$$G_\rho(\varepsilon)\leqslant\int_{B_R}H_n(u,\mu_n)\,\mathrm{d}y+n^{n+1}m^{n-1}\omega_n\int_{B_R}u\,\mathrm{d}\mu_n \\ +\int_{B_R}|\nabla(\log\mu_n))|^n\,\mathrm{d}y\qquad\forall\varepsilon>0. \quad (3.9)$$

Clearly, $G'_\rho(\varepsilon)=A_\rho-m\varepsilon^{m-1}B_\rho$, so $\max_{\varepsilon>0}[G_\varepsilon(\rho)]$ is attained at

$$\varepsilon_{max}(\rho):=\left(\frac{A_\rho}{mB_\rho}\right)^{1/(m-1)}. \quad (3.10)$$

In particular, if $\rho = \mu_n$, we have

$$\varepsilon_{max}(\mu_n) := \left(\frac{A_{\mu_n}}{mB_{\mu_n}}\right)^{1/(m-1)},$$

where

$$A_{\mu_n} = n^2 m \left(\int_{B_R} \mu_n^{1/m}\,\mathrm{d}y - \omega_n^{1/n}\theta_R\right),$$

and

$$B_{\mu_n} = \frac{n}{m\omega_n^{1/n}} \left(\int_{B_R} \mu_n^{1/m}\,\mathrm{d}y - \omega_n^{1/n}\theta_R\right).$$

Note that we have used above the relation

$$\int_{B_R} |y|^m \mu_n\,\mathrm{d}x = \left(\frac{1}{\omega_n}\right)^{1/n} \int_{B_R} \mu_n^{1/m}\,\mathrm{d}x - \theta_R. \tag{3.11}$$

Then

$$\varepsilon_{max}(\mu_n) = (nm\,\omega_n^{1/n})^{1/(m-1)}. \tag{3.12}$$

Choosing $\varepsilon = \varepsilon_{max}(\mu_n)$ in (3.9), we have

$$G_\rho\left(\varepsilon_{max}(\mu_n)\right) - \int_{B_R} |\nabla(\log \mu_n))|^n\,\mathrm{d}x \\ \leqslant \int_{B_R} H_n(u,\mu_n)\,\mathrm{d}x + n^{n+1}m^{n-1}\omega_n \int_{B_R} u\,\mathrm{d}\mu_n,$$

that is, after dividing by $\beta(n) = n^{n+1}m^{n-1}\omega_n$,

$$\frac{n^2 m\,\varepsilon_{max}(\mu_n)}{\beta(n)} \int_{B_R} \rho^{1/m}\,\mathrm{d}y - \frac{n(\varepsilon_{max}(\mu_n))^m}{m\beta(n)} \int_{B_R} |y|^m \rho\,\mathrm{d}y \\ - \frac{1}{\beta(n)}\left[\int_{B_R} |\nabla(\log \mu_n))|^n\,\mathrm{d}x + n^2 m\omega_n^{1/n}\,\varepsilon_{max}(\mu_n)\,\theta_R\right] \\ \leqslant \frac{1}{\beta(n)} \int_{B_R} H_n(u,\mu_n)\,\mathrm{d}x + \int_{B_R} u\,\mathrm{d}\mu_n = I_R(u). \tag{3.13}$$

We now simplify the l.h.s of (3.13) by using the following basic identities which can be checked by direct computations:

$$\frac{n^2 m\,\varepsilon_{max}(\mu_n)}{\beta(n)} = m(1/\omega_n)^{1/n} = \alpha(n),$$

$$\frac{n(\varepsilon_{max}(\mu_n))^m}{m\beta(n)} = 1,$$

$$n^2 m\omega_n^{1/n}\,\varepsilon_{max}(\mu_n) = m^n n^{n+1}\omega_n,$$

$$\int_{B_R} |\nabla(\log\mu_n)|^n \,\mathrm{d}x = n^n m^n\,\omega_n \int_{B_R} |y|^m \mu_n \,\mathrm{d}y,$$

$$\theta_R = \left(\frac{1}{\omega_n}\right)^{1/n} \int_{B_R} \mu_n^{1/m}\,\mathrm{d}y - \int_{B_R} |y|^m \mu_n \,\mathrm{d}y.$$

Then (3.13) yields

$$J_R(\rho) - J_R(\mu_n) \leqslant I_R(u)$$

for all functions u and ρ such that $u \in W_0^{1,n}(B_R)$, $\int_{B_R} e^u \,\mathrm{d}\mu_n = \theta_R$ and $\int_{B_R} \rho(y)\,\mathrm{d}y = \theta_R$.

We conclude the proof by noting that the left-hand side is equal to 0 for $\rho \equiv \mu_n$, while the right-hand side is equal to 0 for $u \equiv 0$. □

From Theorem 3.1, we obtain the following n-dimensional Onofri inequality.

Theorem 3.3 (n-dimensional Euclidean Onofri inequality). — *For any $n \geqslant 2$, the following holds for any $u \in W^{1,n}(\mathbb{R}^n)$,*

$$\frac{1}{\beta(n)} \int_{\mathbb{R}^n} H_n(u,\mu_n)\,dx + \int_{\mathbb{R}^n} u\,d\mu_n - \log\left(\int_{\mathbb{R}^n} e^u\,d\mu_n\right) \geqslant 0, \tag{3.14}$$

hence the infimum is attained at $u \equiv 0$.

Proof. — Take $u \in C_c^1(\mathbb{R}^n)$ such that it has its support in a ball B_R. Let $v = u - C$ on B_R and 0 elsewhere, where C is chosen so that $\int_{B_R} e^v \,\mathrm{d}\mu_n = \mu_n(B_R)$. It follows that $\int_{\mathbb{R}^n} e^v \,\mathrm{d}\mu_n = 1$, hence applying Theorem (3.1) we get that

$$I_R(v) = \frac{1}{\beta(n)} \int_{B_R} H_n(v,\mu_n)\,\mathrm{d}x + \int_{B_R} v\,\mathrm{d}\mu_n(x) - \log\int_{\mathbb{R}^n} e^v\,\mathrm{d}\mu_n \geqslant 0. \tag{3.15}$$

Since $H_n(v,\mu_n) = H_n(u,\mu_n)$, then (3.15) gives

$$\frac{1}{\beta(n)} \int_{\mathbb{R}^n} H_n(u,\mu_n)\,\mathrm{d}x + \int_{\mathbb{R}^n} u(x)\,\mathrm{d}\mu_n(x) - \log\left(\int_{\mathbb{R}^n} e^u\,\mathrm{d}\mu_n\right) \geqslant 0. \quad □$$

From the proof of Theorem 3.1, we can also derive the following inequality.

Corollary 3.4. — *Let $n \geqslant 2$ be an integer. For $v \in C_c^1(\mathbb{R}^n)$ with compact support in $B_R \subset \mathbb{R}^n$ for some $R > 0$, we have*

$$\left(\frac{1}{\omega_n}\right)^{\frac{n-1}{n}} \frac{1}{(1+R^{\frac{n}{n-1}})^n} \int_{B_R} e^v \, dx + \frac{n-1}{n^2} \int_{B_R} |\nabla v|^n \, dx \geqslant \int_{B_R} \mu_n^{\frac{n-1}{n}} \, dy. \tag{3.16}$$

In particular, if $n = 2$, then (3.16) *gives*

$$\int_{B_R} e^v \, dx + \frac{(1+R^2)^2\sqrt{\pi}}{4} \int_{B_R} |\nabla v|^2 \, dx \geqslant \pi(1+R^2)^2 \log(1+R^2). \tag{3.17}$$

Proof. — Choosing $\rho = \mu_n$ and $\varepsilon = \varepsilon_{max}(\mu_n)$ in (3.7), we have

$$G_{\mu_n}(\varepsilon_{max}(\mu_n)) \leqslant \int_{B_R} |\nabla(u + \log \mu_n)|^n \, \mathrm{d}x, \tag{3.18}$$

for any u such that $\int_{B_R} e^u \, \mathrm{d}\mu_n = \theta_R$ and $u|_{\partial B_R} = 0$. Using the computations in the proof of Theorem 3.1 and setting $m := \frac{n}{n-1}$, we have

$$G_{\mu_n}(\varepsilon_{max}(\mu_n)) = \frac{A_{\mu_n}^n}{n(mB_{\mu_n})^{n-1}} = mn\left(\int_{B_R} \mu_n^{1/m} \, \mathrm{d}y - \omega_n^{1/n}\, \theta_R\right).$$

This gives

$$mn\left(\int_{B_R} \mu_n^{1/m} \, \mathrm{d}y - \omega_n^{1/n}\, \theta_R\right) \leqslant \int_{B_R} |\nabla(u + \log \mu_n)|^n \, \mathrm{d}x.$$

Set $v := u + \log \mu_n - \log(\mu_n|_{\partial B_R})$. We have

$$\nabla v = \nabla(u + \log \mu_n), \quad v|_{\partial B_R} = 0, \quad \theta_R = \int_{B_R} e^u \, \mathrm{d}\mu_n = \mu_n|_{\partial B_R} \int_{B_R} e^v \, \mathrm{d}x,$$

where $\mu_n|_{\partial B_R} = \frac{1}{\omega_n(1+R^m)^n}$. Then (3.18) reads as

$$mn\left(\int_{B_R} \mu_n^{1/m} \, \mathrm{d}y - \omega_n^{1/n} \frac{1}{\omega_n(1+R^m)^n} \int_{B_R} e^v \, \mathrm{d}x\right) \leqslant \int_{B_R} |\nabla v|^n \, \mathrm{d}x. \tag{3.19}$$

This gives (3.16) after simplification. Using $\int_{B_R} \sqrt{\mu_n} = \sqrt{\pi} \log(1+R^2)$ where $B_R \subset \mathbb{R}^2$, we get (3.17). □

In dimension $n = 2$, the operator H_n becomes $H_2(u, \mu_2) = |\nabla u|^2$, and Theorem (3.1) then yields the 2-dimensional Onofri inequality.

Corollary 3.5 (Duality for the 2-dimensional Euclidean Onofri inequality). — *For any ball B_R of radius $R > 0$ in $\mathbb{R}^2$, consider the functionals*

$$I_R(u) = \frac{1}{16\pi} \int_{B_R} |\nabla u(x)|^2 \, dx + \int_{B_R} u(x) \, d\mu_2(x) \quad \text{on } H_0^1(B_R),$$

and

$$J_R(\rho) = \frac{2}{\sqrt{\pi}} \int_{B_R} \sqrt{\rho(y)} \, dy - \int_{B_R} |y|^2 \rho(y) \, dy \quad \text{on } L_+^1(\mathbb{R}^2).$$

(1) *The following duality formula then holds:*

$$\sup\left\{J_R(\rho)-J_R(\mu_2);\ \rho\in L^1_+(\mathbb{R}^2),\ \frac{1}{\mu_2(B_R)}\int_{B_R}\rho\,dy=1\right\}$$
$$=\inf\left\{I_R(u);\ u\in H^1_0(B_R),\ \int_{\mathbb{R}^2}e^u\,d\mu_2=1\right\}=0,\quad (3.20)$$

and the maximum on the l.h.s. (resp. the minimum on the r.h.s.) is only attained at $\rho_{max}=\mu_2$ *(resp., at* $u_{min}=0$*).*

(2) *Consequently, the Euclidean Moser–Onofri inequality also holds:*

$$\log\left(\int_{\mathbb{R}^2}e^u\,d\mu_2\right)-\int_{\mathbb{R}^2}u\,d\mu_2\leqslant\frac{1}{16\pi}\int_{\mathbb{R}^2}|\nabla u|^2dx\quad\forall u\in H^1(\mathbb{R}^2).\quad (3.21)$$

Proof. — The duality is a direct consequence of Theorem (3.1). Note that this gives a correspondence between solutions to the prescribed Gaussian curvature problem and stationary solutions to a rescaled fast diffusion equation. Indeed, it is known from the theory of mass transport (see [14] for example) that a maximizer ρ to the supremum problem in (3.20) is a solution to the pde:

$$\frac{1}{2\sqrt{\pi}}\Delta(\sqrt{\rho})+\operatorname{div}(x\rho)=0\quad\text{in}\quad B_R,$$

(with Neumann boundary condition) that is the unique stationary solution of the rescaled fast diffusion equation

$$\partial_t\rho=\frac{1}{2\sqrt{\pi}}\Delta(\sqrt{\rho})+\operatorname{div}(x\rho)\quad\text{in}\quad B_R,\quad (3.22)$$

which is necessarily μ_2.

On the other hand, a minimizer u to (3.20) solves — up to an additive constant — the Dirichlet BVP:

$$\frac{1}{8\pi}\Delta u+\mu_2\,e^u=\mu_2\quad\text{in}\quad B_R,\quad u=0\text{ on }\partial B_R.\quad (3.23)$$

But u could be lifted through the stereographic projection to a function v on $\mathbb{S}^2$, that satisfies the equation

$$\Delta v+2e^v=2\quad\text{on }\mathbb{S}^2.\quad (3.24)$$

But it is known (see [3]) that all such solutions are conformal images of 0, hence of the form $v(x)=-2\log(1-x\cdot\zeta)$ where $\zeta\in\mathbb{R}^3,|\zeta|\leqslant 1$. By transferring such a v back to u via stereographic projection, we see that it cannot have a compact support on $\mathbb{R}^2$, unless u is the zero solution. □

Remark 3.6. — It would be interesting to show that $u\equiv 0$ is the only function such that $I_R(u)=0$, by considering the optimal transport map T that maps the probability measure $\frac{1}{\mu_2(B_R)}e^u\mu_2$ on B_R to $\frac{\mu_2}{\mu_2(B_R)}$, and arguing — by chasing back the inequalities in the proof of the duality — that $I_R(u) = 0$ implies that T is necessarily the identity map. This was the approach used in [5] to find the extremal in the Sobolev inequality.

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