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Hypercontractivity for local states of the quantized electromagnetic field.

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Abstract. — The quantized free electromagnetic field provides a good example of the structures that arise in the theory of quantized fields. There is a Gaussian measure on an infinite dimensional linear space along with a Dirichlet form on this space. Both are uniquely determined by special relativity. These will be described, along with the operators that represent the quantized electromagnetic field. Hypercontractivity of the operator associated to the Dirichlet form will be proved under the condition that observations made of the field take place in a bounded region of space.

Résumé. — Le champ électromagnétique libre quantifié est un bon exemple des structures qui apparaissent dans la théorie des champs quantifiés. On considère un espace vectoriel de dimension infinie équipé d'une mesure gaussienne et d'une forme de Dirichlet qui sont determinées par la théorie de la relativité restreinte. Nous décrivons ces objets ainsi que l'opérateur qui représente le champ électromagnétique quantifié. L'hypercontractivité de l'opérateur associé à cette forme de Dirichlet est obtenue sous la condition que les observations du champ sont effectuées dans une region bornée de l'espace.

1. Introduction

This note describes a Gaussian probability space occurring naturally in the quantum theory of the free electromagnetic field. Nominally, I intend to show how a Dirichlet form operator over this space with no spectral gap near zero can nevertheless generate a hypercontractive semigroup if one makes observations of the initial and final states of the field only in a fixed bounded region of space. This is distinct from, but related to, the fact that the Laplacian (on $\mathbb{R}^3$, say) has a discrete spectrum when restricted to bounded sets

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(with appropriate boundary conditions). Actually, I intend to use this example to explain structures which are prototypes of those for quantized Yang–Mills fields. There is evidence that some of these Gaussian structures will go over to the highly non-Gaussian setting associated with Yang–Mills fields. The corresponding hypercontractivity and logarithmic Sobolev inequality will undoubtedly require use of Dominique Bakry's Γ_2 techniques for proof.

Section 3 contains a more or less self contained exposition of the quantized free electromagnetic field. On the one hand, the computations in this section can be found in almost any book on quantum field theory. On the other hand, it is written in a form that emphasizes the concepts and structures familiar to participants in Dominique's 60th birthday conference. One could say that it is simply a translation from a standard part of the physics literature to a standard part of the mathematics literature. Dirichlet forms are well known to all of us.

Section 5 is aimed at sketching the changes that must be made in replacing the electromagnetic, linear configuration space by the corresponding Yang–Mills (infinite dimensional) Riemannian manifold. At the time of this writing there is significant progress (my opinion) on the construction of this manifold. But the construction of the non-Gaussian measure is still far off.

2. The classical electromagnetic field

The electromagnetic field is specified by a time dependent 1-form $E(x,t) = \sum_{j=1}^{3} E_j(x,t)\,\mathrm{d}x^j$ on $\mathbb{R}^3$ and a time dependent 2-form $B(x,t) = \sum B_i(x,t)\,\mathrm{d}x^j \wedge \mathrm{d}x^k$ on $\mathbb{R}^3$. The sum runs over the three cyclic permutations of $(1,2,3)$. The force on a particle of charge q located at x at time t and moving with a velocity v is given by $qE(x,t) + qv \lrcorner B(x,t)$, where $\lrcorner$ denotes interior product. Identify $\mathbb{R}^3$ with $(\mathbb{R}^3)^*$ to get the force. This is a velocity dependent force field, leading to all kinds of interesting problems in differential geometry. The time evolution of the electromagnetic field is most concisely expressed in terms of the 2-form $F = E\wedge \mathrm{d}t + B$ on $\mathbb{R}^4$. Denote by D the exterior derivative operator for forms over $\mathbb{R}^4$ and by D^* its adjoint with respect to the Lorentz invariant metric $\sum_{j=1}^{3} \mathrm{d}x^j \otimes \mathrm{d}x^j - \mathrm{d}t \otimes \mathrm{d}t$. In the presence of a charge and current given by a 1-form $J = \rho(x,t)\,\mathrm{d}t + \sum_{i=1}^{3} J_i\,\mathrm{d}x^i$ the equation of evolution of the fields is given by Maxwell's equations,

$$DF = 0, \quad D^*F = J. \tag{2.1}$$

To identify this with the form of Maxwell's equations most commonly seen on T-shirts on engineering campuses just write $D\omega = \mathrm{d}\omega + \mathrm{d}t \wedge (\partial/\partial t)\omega$ for a form ω on $\mathbb{R}^4$, where d is the exterior derivative operator for forms over $\mathbb{R}^3$.

The identities $DF = (\mathrm{d}E) \wedge \mathrm{d}t + \mathrm{d}B + \mathrm{d}t \wedge \dot{B}$ and $D^*F = (\mathrm{d}^*E)\,\mathrm{d}t - \dot{E} + \mathrm{d}^*B$ show that the equations (2.1) are equivalent to the four equations

$$\mathrm{d}E + \dot{B} = 0, \tag{2.2.a}$$

$$\mathrm{d}B = 0, \tag{2.2.b}$$

$$\mathrm{d}^*E = \rho, \tag{2.2.c}$$

$$-\dot{E} + \mathrm{d}^*B = \mathbf{J}. \tag{2.2.d}$$

Of course one must identify the 2-form B with a 1-form β via the Hodge star operator for engineers and use *curl* $\beta = \mathrm{d}^*(*\beta)$ on 1-forms β.

We are interested in the free electromagnetic field, which is specified by taking $J = 0$. Since $D^*D + DD^*$ is the d'Alembertian, $\Box$, the free electromagnetic field satisfies $\Box\, F = 0$. Hence each component of E and B satisfies the wave equation. Under very mild technical conditions the pair $\{E, B\}$ is determined for all time by its values at $t = 0$ because the time derivative of one is plus or minus the curl of the other, in accordance with the Maxwell equations (2.2.a) and (2.2.d).

The three dimensional Laplacian on forms is given by $-\Delta = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*$. It is a non-negative self-adjoint operator on k-forms in $L^2(\mathbb{R}^3; \Lambda^k)$ when the domain is chosen in the natural way (its closure on C_c^∞). The Laplacian has a zero nullspace in $L^2(\mathbb{R}^3; \Lambda^k)$. This allows us to define Sobolev spaces of forms easily. The $H_{-1/2}$ norm of a k-form ω on $\mathbb{R}^3$ is by definition

$$\|\omega\|_{-1/2} = \|(-\Delta)^{-1/4}\omega\|_{L^2(\mathbb{R}^3;\Lambda^k)}. \tag{2.3}$$

For a 1-form E and a 2-form B on $\mathbb{R}^3$ define a norm of this pair by

$$\|B, E\|^2 = \|B\|_{-1/2}^2 + \|E\|_{-1/2}^2. \tag{2.4}$$

Theorem 2.1 (Bargmann and Wigner, 1948 [1]). — *Suppose that $E(x,t)$, $B(x,t)$ is a solution to Maxwell's equations with zero charge and current. Then*

1. $\|B(\,\cdot\,,t), E(\,\cdot\,,t)\|$ *is independent of* t.
2. *It is also independent of which Lorentz frame one uses to define the $t = 0$ hyperplane used in the definition* (2.4).

Equivalently, if L is a linear transformation on $\mathbb{R}^4$ which preserves the Lorentz metric and if $F = E \wedge \mathrm{d}t + B$ then L^*F is another solution to Maxwell's equations and the pair $\hat{B}, \hat{E}$, defined by $L^*F = \hat{E} \wedge \mathrm{d}t + \hat{B}$, has the same norm.

One has thereby a representation of the Lorentz group, including space-time translations, in the group of orthogonal transformations on the Hilbert space consisting of those solutions of Maxwell's equations for which the initial data norm (2.4) is finite. This was first proved in the classic paper by Bargmann and Wigner [1]. In that paper they classified all linear Lorentz invariant wave equations and established a one-to-one correspondence with certain unitary representations of the Lorentz group. Their description of these spaces is based on plane wave decompositions of solutions (i.e. Fourier transforms) and shows, in particular, that the real space of initial data $\{B(\,\cdot\,,0), E(\,\cdot\,,0)\}$ described above actually has a complex structure which commutes with the action of the Lorentz group and induces thereby a unitary representation of the Lorentz group. Most importantly, the representation is irreducible. This implies that there is no other inner product which is invariant under the Lorentz group. Thus, as soon as one says "special relativity" one is stuck with the $H_{-1/2}$ norm, as in (2.4), for better or worse. Many of the troubles and joys in relativistic quantum field theory can be traced back precisely to the central role of this norm. I have described the fundamental role of this norm because it dominates the rest of this note, not to mention all of relativistic quantum field theory.

Before turning to quantization we have to describe the Lorentz invariant norm in a more complicated way. One of the two Maxwell's equations (2.1) is $DF = 0$. Thus F is a closed 2-form on $\mathbb{R}^4$. Since $\mathbb{R}^4$ is cohomologically trivial there exists a (non-unique) 1-form A on $\mathbb{R}^4$ such that

$$F = DA. \tag{2.5}$$

If $DA = 0$ then A is itself closed, hence exact. There exists, then, a real valued function λ on $\mathbb{R}^4$ such that $A = \mathrm{d}\lambda$. Thus the set of 2−forms F for which $DF = 0$ is in one-to-one correspondence with the quotient space {1-forms} / {exact 1-forms}. It is useful and customary in various contexts to simplify this representation by choosing a subspace of {1-forms} which is, at least in part, complementary to {exact 1-forms}. Such a choice of subspace is called a gauge choice. For our purposes, and by way of example, consider the space of 1-forms whose fourth component is zero. For an arbitrary 1-form $A \equiv \sum_{j=1}^{3} A_j\,\mathrm{d}x^j + A_4\,\mathrm{d}t$ choose a real valued function λ on $\mathbb{R}^4$ such that $A_4(x,y,z,t) + \partial\lambda/\partial t = 0$ on all of $\mathbb{R}^4$. Then, clearly, $(A + \mathrm{d}\lambda)_4 = 0$ on $\mathbb{R}^4$. Thus every 1-form is "gauge equivalent" i.e. equivalent mod exact 1-forms to a 1-form with fourth component zero. There are lots of such functions λ. One could choose $\lambda(x,y,z,0)$ arbitrarily and solve the preceding ordinary differential equation for each x,y,z to find such a function λ. Hence {1-forms}/ {exact 1-forms} = {1-forms with $A_4 = 0$}/{t-independent functions λ}. A 1-form A with $A_4 = 0$ is said to be in the *temporal gauge*. For such a 1-form $\mathbf{A}(x,t) \equiv \sum_{j=1}^{3} A_j(x,t)\,\mathrm{d}x^j$ the definition (2.5) gives $E(x,t) = -\dot{\mathbf{A}}(x,t)$ and $B(x,t) = \mathrm{d}\mathbf{A}(x,t)$ on all of $\mathbb{R}^4$. Aside from being in temporal gauge there is still more gauge fixing that can be done. The equation $\mathrm{d}^*(\mathbf{A} + \mathrm{d}\lambda) = 0$

has a solution given by $\lambda = -(\mathrm{d}^*\mathrm{d})^{-1}\mathrm{d}^*\mathbf{A}$, where the Laplacian $-\mathrm{d}^*\mathrm{d}$ that appears here is the ordinary Laplacian on scalar functions. At the informal level that we are operating at right now we need not worry about things like regularity or behavior at infinity of any of the quantities in the last equation or in this paragraph. The argument shows that we can choose $\mathbf{A}$ to be not only in temporal gauge ($A_4 = 0$) but also in the so-called *Coulomb gauge* defined by $\mathrm{d}^*\mathbf{A} = 0$. Here are two neat consequences of use of the intersection of the temporal gauge and Coulomb gauge.

(1) If $B = \mathrm{d}\mathbf{A}$ and $E = -\dot{\mathbf{A}}$ and $\mathrm{d}^*\mathbf{A} = 0$ on all of $\mathbb{R}^4$ then the first three of the four Maxwell equations in line (2.2) are automatically satisfied. The fourth one may by written $\ddot{\mathbf{A}} + \mathrm{d}^*\mathrm{d}\mathbf{A} = 0$ (since $J = 0$), and since $\mathrm{d}\mathrm{d}^*\mathbf{A} = 0$ we find

$$\ddot{\mathbf{A}} = \Delta\mathbf{A} \quad \text{on all of } \mathbb{R}^4. \tag{2.6}$$

Maxwell's equations are thereby reduced to the wave equation for a divergence free, time dependent, 1-form on $\mathbb{R}^3$.

(2) Claim. — *If* $B = \mathrm{d}\mathbf{A}$ *and* $\mathrm{d}^*\mathbf{A} = 0$ *then*

$$\|B\|_{-1/2} = \|\mathbf{A}\|_{1/2}. \tag{2.7}$$

Proof of the Claim. — The identity $\mathrm{d}^*(\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*) = (\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)\mathrm{d}^*$ applied to 2-forms shows, by the functional calculus for the pair of self-adjoint operators $(\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)$ on 2-forms and 1-forms respectively, that $\mathrm{d}^*(\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)^{-1/2} = (\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)^{-1/2}\mathrm{d}^*$ and therefore

$$\begin{aligned}
\|B\|^2_{-1/2} &= ((\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)^{-1/2}B, B)_{L^2(\mathbb{R}^3)} \\
&= ((\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)^{-1/2}\mathrm{d}\mathbf{A}, \mathrm{d}\mathbf{A}) \\
&= (\mathrm{d}^*(\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)^{-1/2}\mathrm{d}\mathbf{A}, \mathbf{A}) \\
&= ((\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)^{-1/2}\mathrm{d}^*\mathrm{d}\mathbf{A}, \mathbf{A}) \\
&= ((\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)^{-1/2}(\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)\mathbf{A}, \mathbf{A}) \\
&= \|(\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)^{1/4}\mathbf{A}\|_{L^2}.
\end{aligned}$$

□

Therefore the Lorentz invariant norm, expressed in terms of $\mathbf{A}$ is given by

$$\|B\|^2_{-1/2} + \|E\|^2_{-1/2} = \|\mathbf{A}\|^2_{1/2} + \|\dot{\mathbf{A}}\|^2_{-1/2} \tag{2.8}$$

$$\text{if} \quad B = \mathrm{d}\mathbf{A}, \quad E = -\dot{\mathbf{A}} \quad \text{and} \quad \mathrm{d}^*\mathbf{A} = 0. \tag{2.9}$$

The heuristics in passing from a classical field to a quantized field involves some artistic devices which we are not accustomed to seeing in the mathematics literature. The wave equation (2.6) resembles the Newtonian equations for an assembly of infinitely many harmonic oscillators. In Remark 3.6 it will be superficially explained how this leads to Gaussian measures with the Lorentz invariant covariance. But first let us have some precise statements.

3. Quantization

Let $\mathcal{H}$ be the real Hilbert space defined by

$$\mathcal{H} = \{A \in H_{1/2}(\mathbb{R}^3; \Lambda^1) : \mathrm{d}^* A = 0\}. \tag{3.1}$$

We will identify the dual space as

$$\mathcal{H}^* = \{j \in H_{-1/2}(\mathbb{R}^3; \Lambda^1) : \mathrm{d}^* j = 0\}. \tag{3.2}$$

$\mathcal{H}^*$ consists of divergence free currents, which we will use to measure the potential A. $\mathcal{H}$ and $\mathcal{H}^*$ are dual in the pairing

$$\langle A, j\rangle \equiv \langle A, j\rangle_{L^2(\mathbb{R}^3;\Lambda^1)}. \tag{3.3}$$

We want to consider the Gaussian measure γ "on" $\mathcal{H}$ given informally by

$$\mathrm{d}\gamma(A) = Z^{-1} e^{-(1/2)\|A\|^2_{H_{1/2}}}\ \mathcal{D}A, \tag{3.4}$$

where $\mathcal{D}A$ is infinite dimensional Lebesque measure and Z is a normalization constant. We all understand that this expression must be interpreted as a measure on some considerably larger space than $\mathcal{H}$ itself. For example one could interpret it as a genuine Gaussian probability measure on the very big space $\mathcal{S}'(\mathbb{R}^3; \Lambda^1)$, or on some Banach space $\mathcal{W}$ for which $(\mathcal{H}, \mathcal{W}, \gamma)$ constitutes an abstract Wiener space. I will describe a Dirichlet form operator H on "$L^2(\mathcal{H}, \gamma)$" which implements the Schrodinger equation for the quantized electromagnetic field. We will then be ready to consider whether the semigroup $e^{-t\mathsf{H}}$ is hypercontractive or not. This, of course, is the connection with some of the subject matter of this conference. It is the reason for discussing the quantized electromagnetic field in these conference notes. I will sometimes write $L^2(\Omega, \gamma)$ to emphasize that γ is really a measure on some large space Ω and sometimes write $L^2(\mathcal{H}, \gamma)$ to emphasize that a computation in progress, e.g. an integration by parts, can be legally made using the informal expression (3.4). All of the computations in this paper will be made on polynomials over $\mathcal{H}$. These are dense in $L^2(\Omega, \gamma)$ and constitute a core for all of the operators of interest for us. The computations can therefore be made without technical concerns and identities extend automatically to the entire domain of the relevant closed operator.

First I want to use this Gaussian measure space to describe the quantized electromagnetic field itself. Let $j \in \mathcal{H}^*$. Define

$$q_j(A) = \langle A, j\rangle, \qquad A \in \mathcal{H}, \qquad j \in \mathcal{H}^*. \tag{3.5}$$

The pairing $\langle \cdot, \cdot \rangle$ is again the L^2 pairing, usually between an element in $H_{1/2}$ and an element in $H_{-1/2}$. q_j is a continuous linear functional on $\mathcal{H}$ and extends to a Gaussian random variable on Ω. We denote the extension by q_j also.

Define

$$\mathsf{A}_j = \text{multiplication by } \; q_j \;\; \text{on } L^2(\Omega, \gamma), \qquad j \in \mathcal{H}^* \tag{3.6}$$

Let $g \in \mathcal{H}$. Define

$$(\partial_g \psi)(A) = (\mathrm{d}/\mathrm{d}s)|_{s=0} \psi(A + sg). \tag{3.7}$$

The product rule shows that

$$[\partial_g, \mathsf{A}_j]\psi = \langle g, j \rangle \psi, \qquad j \in \mathcal{H}^*, \quad g \in \mathcal{H}. \tag{3.8}$$

Since A_j is self-adjoint we also have $-[\partial_g^*, \mathsf{A}_j] = \langle j, g \rangle Id$ on polynomial functions in $L^2(\mathcal{H}, \gamma)$. Define an operator

$$\mathsf{E}_g = \sqrt{-1}(\partial_g - \partial_g^*). \tag{3.9}$$

Then

$$[\mathsf{E}_g, \mathsf{A}_j] = 2i\langle g, j \rangle. \tag{3.10}$$

For $h \in H_{1/2}(\mathbb{R}^3; \Lambda^2)$ we have $\mathrm{d}^* h \in \mathcal{H}^*$ and we can therefore define an operator

$$\mathsf{B}_h = \mathsf{A}_{\mathrm{d}^* h}. \tag{3.11}$$

Then we have the commutation relations

$$[\mathsf{E}_g, \mathsf{B}_h] = 2i\langle g, \mathrm{d}^* h \rangle. \tag{3.12}$$

These are the standard commutation relations for the quantized electromagnetic field, [2, p. 72].

Why does a Gaussian measure appear in a theory that starts from a hyperbolic wave equation? And where did Maxwell's equations go? For a heuristic discussion explaining the appearance of the Gaussian measure, see Remark 3.6. Concerning Maxwell's equations, we will recover them now. Time evolution in quantum theories is determined by a Hamiltonian. For us, the Hamiltonian will be the operator associated to a Dirichlet form on $L^2(\mathcal{H}, \gamma)$.

Notation 3.1 (Differential). — The differential of a function $\psi : \mathcal{H} \to \mathbb{R}$ at a point A is the linear functional

$$g \mapsto (D\psi)(A)\langle g \rangle \equiv (\partial_g \psi)(A), \quad g \in \mathcal{H}. \tag{3.13}$$

In order to assign a norm to the differential we have to identify it as an element of the appropriate Hilbert space. Physicists tell us that it must be done as follows. Suppose that there is an element $v \in L^2(\mathbb{R}^3;\Lambda^1)$ with $\mathrm{d}^* v = 0$ such that

$$(D\psi)(A)\langle g\rangle = \langle g, v\rangle_{L^2}. \tag{3.14}$$

Then we define

$$|(D\psi)(A)|_{L^2(\mathbb{R}^3;\Lambda^1)} = \|v\|_{L^2(\mathbb{R}^3;\Lambda^1)}. \tag{3.15}$$

Given that the Hilbert space on which ψ is defined is $H_{1/2}$ and not L^2, you may be wondering why we are using the L^2 norm in (3.15) instead of the $H_{-1/2}$ norm. At a conceptual level this issue goes back to the question of what exactly is the configuration space for the classical electromagnetic field. For the Yang–Mills theory this will become a serious technical issue. But for us, it suffices to know that this is the one that works for producing Maxwell's equations as per below. The norm of the differential that equals the Hilbert space gradient is $|(D\psi)(A)|_{-1/2} = \|v\|_{H_{-1/2}}$, when v is given by (3.14). The logarithmic Sobolev inequality that I want to prove hinges on the relation between these two norms of the differential.

Notation 3.2 (Q and H). — The quadratic form for the Hamiltonian of the free electromagnetic field is defined by

$$Q(\psi) = \int_\Omega |D\psi(A)|^2_{L^2(\mathbb{R}^3;\Lambda^1)}\,\mathrm{d}\gamma(A). \tag{3.16}$$

Since $H_{1/2}(\mathbb{R}^3)$ neither contains nor is contained in $L^2(\mathbb{R}^3)$ the integrand could be infinite at some points even if ψ is just polynomial. But, when defined on polynomial functions ψ for which Q is finite, Q has a closed extension and its closure defines in the usual way a non-negative self-adjoint operator H on $L^2(\mathcal{H},\gamma)$ such that

$$\langle \mathsf{H}\psi, \psi\rangle_{L^2(\mathcal{H},\gamma)} = Q(\psi). \tag{3.17}$$

H is the Hamiltonian for the free electromagnetic field.

Unlike in the classical case, where time propagation is determined by solving a partial differential equation for the fields, in the quantum case propagation is determined by the Hamiltonian H. Define

$$\mathsf{A}_j(t) = e^{it\mathsf{H}}\mathsf{A}_j e^{-it\mathsf{H}} \tag{3.18}$$

$$\mathsf{B}_h(t) = e^{it\mathsf{H}}\mathsf{B}_h e^{-it\mathsf{H}} \tag{3.19}$$

$$\mathsf{E}_g(t) = e^{it\mathsf{H}}\mathsf{E}_g e^{-it\mathsf{H}}. \tag{3.20}$$

These should be regarded as defining operator valued distributions $\mathsf{A}(x,t)$, $\mathsf{B}(x,t)$ and $\mathsf{E}(x,t)$ by

$$\sum_{i=1}^{3}\int_{\mathbb{R}^3} A_i(x,t)j_i(x)\,\mathrm{d}x = \mathsf{A}_j(t) \qquad \forall\, j \in \mathcal{H}^*, \tag{3.21}$$

$$\sum_{ik}\int_{\mathbb{R}^3} \mathsf{B}_{ik}(x,t)h_{ik}(x)\,\mathrm{d}x = \mathsf{B}_h(t) \qquad \forall\, h \in H_{1/2}(\mathbb{R}^3;\Lambda^2), \tag{3.22}$$

$$\sum_{i=1}^{3}\int_{\mathbb{R}^3} \mathsf{E}_i(x,t)g_i(x)\,\mathrm{d}x = \mathsf{E}_g(t) \qquad \forall\, g \in \mathcal{H}. \tag{3.23}$$

Since the test functions j and g are limited to be divergence free, the operator valued distributions must be interpreted to be divergence free distributions. This, of course, goes with our use of the Coulomb gauge for A and charge zero for E.

Theorem 3.3. — *The operator valued distributions* $\mathsf{E}, \mathsf{B}, \mathsf{A}$ *on* $\mathbb{R}^4$ *satisfy Maxwell's equations* (2.2) *with zero charge and current. Furthermore* $\mathsf{E} = -\dot{\mathsf{A}}$, $\mathsf{B} = \mathrm{d}\mathsf{A}$ *and* $\mathrm{d}^*\mathsf{A} = 0$.

We will need some Gaussian integration by parts formulas to prove this.

Lemma 3.4 (Integration by parts identities). — *Let* $g \in \mathcal{H}$. *Define*

$$u = (\mathrm{d}^*\mathrm{d})^{1/2} g. \tag{3.24}$$

Then

$$\int_{\mathcal{H}} (\partial_g f)(A)\,\mathrm{d}\gamma(A) = \int_{\mathcal{H}} q_u(A) f(A)\,\mathrm{d}\gamma(A), \tag{3.25}$$

$$(\partial_g)^* = -\partial_g + \mathsf{A}_u, \tag{3.26}$$

$$[\mathsf{H}, \mathsf{A}_g] = -2\partial_g + \mathsf{A}_u, \tag{3.27}$$

$$[\mathsf{H}, \partial_g] = -\partial_u \tag{3.28}$$

$$\textit{and} \quad [\mathsf{H}, \partial_g - (\partial_g)^*] = -\mathsf{A}_{\mathrm{d}^*\mathrm{d}g}. \tag{3.29}$$

In (3.25) *the function* f *is an arbitrary polynomial on* $\mathcal{H}$.

Proof. — Using the informal expression (3.4) for the Gaussian measure γ, we can compute

$$\begin{aligned}\int_{\mathcal{H}} (\partial_g f)(A)\,\mathrm{d}\gamma(A) &= \int_{\mathcal{H}} (\partial_g f)(A) e^{-(1/2)\|A\|^2_{H_{1/2}}} \mathcal{D}A \quad Z^{-1} \\ &= \int_{\mathcal{H}} \Big((1/2)\partial_g \|A\|^2_{H_{1/2}}\Big) f(A)\,\mathrm{d}\gamma(A).\end{aligned}$$

But

$$\begin{aligned}(1/2)\partial_g \|A\|^2_{H_{1/2}} &= \langle A, g\rangle_{H_{1/2}} \\ &= \langle A, (\mathrm{d}^*\mathrm{d})^{1/2} g\rangle_{L^2}.\end{aligned}$$

This proves (3.25). Integration by parts and (3.25) show that

$$\begin{aligned}((\partial_g)^*\psi, \phi)_{L^2(\gamma)} &= \int_{\mathcal{H}} \psi(\partial_g \phi)\,\mathrm{d}\gamma(A) \\ &= \int_{\mathcal{H}} \big(-\partial_g\psi + q_u\psi\big)\phi\,\mathrm{d}\gamma,\end{aligned}$$

for polynomials ψ and ϕ. The identity (3.26) now follows because polynomials constitute a core for all the operators in this identity. Further,

$$\begin{aligned}([\mathsf{H}, \mathsf{A}_g]\psi, \phi)_{L^2(\gamma)} &= (\mathsf{H}\mathsf{A}_g\psi, \phi)_{L^2(\gamma)} - (\mathsf{H}\psi, \mathsf{A}_g\phi)_{L^2(\gamma)} \\ &= \int_{\mathcal{H}} \Big(\langle D(q_g\psi), D\phi\rangle_{L^2(\mathbb{R}^3)} - \langle D\psi, D(q_g\phi)\rangle_{L^2(\mathbb{R}^3)}\Big)\,\mathrm{d}\gamma(A).\end{aligned}$$

But $(Dq_g)(A)\langle v\rangle = \langle v, g\rangle_{L^2(\mathbb{R}^3)}$. That is, $Dq_g = g$, which is constant, i.e., independent of A. Thus the product rule shows that the integrand is

$$\begin{aligned}q_g(A)\langle D\psi, D\phi\rangle_{L^2(\mathbb{R}^3)} + \psi(A)\langle g, D\phi\rangle - q_g(A)\langle D\psi, D\phi\rangle_{L^2(\mathbb{R}^3)} - \langle D\psi, g\rangle\phi(A) \\ = \psi(A)\langle g, D\phi\rangle - \langle D\psi, g\rangle\phi(A) \\ = \psi(A)(\partial_g\phi)(A) - (\partial_g\psi)(A)\phi(A).\end{aligned}$$

Do just one more integration by parts to see that

$$\begin{aligned}&\int_{\mathcal{H}} \Big(\psi(A)(\partial_g\phi)(A) - (\partial_g\psi)(A)\phi(A)\Big)\,\mathrm{d}\gamma(A) \\ &\qquad = \int_{\mathcal{H}} \Big((-(\partial_g\psi)(A) + q_u(A))\phi(A) - (\partial_g\psi)(A)\phi(A)\Big)\,\mathrm{d}\gamma(A),\end{aligned}$$

which is (3.27). Now

$$\begin{aligned}
([\mathsf{H},\partial_g]\psi,\phi)_{L^2(\gamma)} &= \int_{\mathcal{H}}\Big(\langle D\partial_g\psi, D\phi\rangle - \langle D\psi, D\partial_g^*\phi\rangle\Big)\,\mathrm{d}\gamma(A)\\
&= \int_{\mathcal{H}}\Big(\langle D\partial_g\psi, D\phi\rangle + \langle D\psi, D\partial_g\phi\rangle - \langle D\psi, D(q_u\phi)\rangle\Big)\,\mathrm{d}\gamma\\
&= \int_{\mathcal{H}}\Big(\partial_g\langle D\psi, D\phi\rangle - \langle D\psi, D(q_u\phi)\rangle\Big)\,\mathrm{d}\gamma(A)\\
&= \int_{\mathcal{H}}\Big(q_u(A)\langle D\psi, D\phi\rangle - \langle D\psi, D(q_u\phi)\rangle\Big)\,\mathrm{d}\gamma(A)\\
&= \int_{\mathcal{H}}\Big(-\langle D\psi, u\rangle\phi(A)\Big)\,\mathrm{d}\gamma(A),
\end{aligned}$$

which is (3.28). (3.29) now follows from the computation

$$[\mathsf{H},\partial_g-\partial_g^*] = [\mathsf{H},2\partial_g - A_u] = -2\partial_u + \Big(2\partial_u - \mathsf{A}_{(\mathrm{d}^*\mathrm{d})^{1/2}u}\Big). \qquad \square$$

Corollary 3.5.

$$[i\mathsf{H},\mathsf{A}_g] = -\mathsf{E}_g, \qquad g\in\mathcal{H} \tag{3.30}$$

$$[i\mathsf{H},\mathsf{B}_h] = -\mathsf{E}_{d^*h}, \qquad h\in H_{1/2}(\mathbb{R}^3;\Lambda^2) \tag{3.31}$$

$$[i\mathsf{H},\mathsf{E}_g] = \mathsf{B}_{dg}, \qquad g\in\mathcal{H}. \tag{3.32}$$

Proof. — By (3.9) and (3.26) we have $E_g = i(\partial_g + \partial_g - \mathsf{A}_u) = i(2\partial_g - \mathsf{A}_u)$ where $u = (\mathrm{d}^*\mathrm{d})^{1/2}g$. Hence, by (3.27), we have

$$[i\mathsf{H},\mathsf{A}_g] = i(-2\partial_g + \mathsf{A}_u) = -\mathsf{E}_g, \tag{3.33}$$

which proves (3.30). From (3.27) we also find

$$[i\mathsf{H},\mathsf{B}_h] = [i\mathsf{H},\mathsf{A}_{d^*h}] = -\mathsf{E}_{d^*h}, \tag{3.34}$$

which proves (3.31). By (3.29)

$$[i\mathsf{H},\mathsf{E}_g] = -[\mathsf{H},(\partial_g-\partial_g^*)] = \mathsf{A}_{\mathrm{d}^*\mathrm{d}g} = \mathsf{B}_{dg}, \tag{3.35}$$

which proves (3.32). $\square$

Proof of Theorem 3.3. — We want to show that

$$(\partial/\partial t)\mathsf{E}(x,t) = \mathrm{d}^*B(x,t) \tag{3.36}$$

$$(\partial/\partial t)\mathsf{B}(x,t) = -\mathrm{d}\mathsf{E}(x,t) \tag{3.37}$$

as operator valued distributions on $\mathbb{R}^4$. These distributions are not as wild as this description sounds. One needs only to integrate against test functions in the spatial variables. Multiply (3.36) by a function $g \in C_c^\infty(\mathbb{R}^3; \Lambda^1)$ with $\mathrm{d}^* g = 0$ and integrate to find

$$(\partial/\partial t)\mathsf{E}_g(t) = \mathsf{B}_{dg}(t), \tag{3.38}$$

wherein we have done an integration by parts on the right. This is the distributional interpretation of (3.36) which needs to be proven. From the definition (3.20) and commutation relations (3.32) we see that

$$\begin{aligned}(\partial/\partial t)\mathsf{E}_g(t) &= e^{it\mathsf{H}}[i\mathsf{H}, \mathsf{E}_g]e^{-it\mathsf{H}} \\ &= e^{it\mathsf{H}}\mathsf{B}_{dg}e^{-it\mathsf{H}} \\ &= \mathsf{B}_{dg}(t).\end{aligned}$$

This proves (3.38) and therefore (3.36).

Multiply (3.37) by a function $h \in C_c^\infty(\mathbb{R}^3; \Lambda^2)$ and integrate over $\mathbb{R}^3$ to find

$$(\partial/\partial t)\mathsf{B}_h(t) = -\mathsf{E}_{\mathrm{d}^* h}(t). \tag{3.39}$$

This is the distributional interpretation of (3.37). It follows from the definition (3.19) and the commutation relation (3.31), just as in the preceding argument. This proves (3.37). Similarly the equation $\dot{\mathsf{A}}(x,t) = -\mathsf{E}(x,t)$ just amounts to the equation (3.30), while $\mathsf{B} = \mathrm{d}\mathsf{A}$ is just the definition (3.11) $\square$

Fact. — *The family of operators $\{\mathsf{B}_h, \mathsf{E}_g : h \in H_{1/2}(\mathbb{R}^3; \Lambda^2), g \in \mathcal{H}\}$ is irreducible on $L^2(\mathcal{H}, \gamma)$. The Hilbert space on which we have constructed the operator valued distributions $\mathsf{B}(x,t), \mathsf{E}(x,t)$ satisfying Maxwell's equations is therefore not artificially big. One could, after all, take some classical solution, multiply it by the identity operator on one's favorite Hilbert space and claim that one now has operator valued solutions to Maxwell's equations. Of course the commutation relations would fail and the objective, quantization of Maxwell's equations, would therefore fail.*

So we have solutions to Maxwell's equations in our structures and therefore wave theory. But we also have now a structure which lends itself to an interpretation in terms of particles: Denote by $\mathcal{P}_n$ the space of polynomials on $\mathcal{H}$ of degree at most n and by $\overline{\mathcal{P}}_n$ the closure of $\mathcal{P}_n$ in $L^2(\mathcal{H}, \gamma)$. Let $\mathcal{F}_n = \overline{\mathcal{P}}_n \ominus \overline{\mathcal{P}}_{n-1}$, $n = 1, 2, \ldots$ and let $\mathcal{F}_0 = \mathcal{P}_0$ (which consists of the constant functions.) Then $L^2(\mathcal{H}, \gamma)$ is the direct sum of these mutually orthogonal subspaces. (As you know well, $\mathcal{F}_n$ is spanned by Hermite polynomials, more precisely, by products of Hermite polynomials of mutually orthogonal coordinates, of total degree n.) Each field operator B_h and E_g carries $\mathcal{F}_n$ into $\mathcal{F}_{n-1} \oplus \mathcal{F}_{n+1}$. Interpretations:

(1) A wave function $\psi \in \mathcal{F}_n$ represents a state of the field containing exactly n photons.

(2) The role of the field operators is to create and annihilate photons.

The two hundred year long dispute as to whether light is a wave phenomenon or a particle phenomenon has now a resolution in this structure: The mathematical structure is big enough to allow both interpretations.

Remark 3.6 (Heuristics on the origin of Gauss measure). — In the interest of greater consistency in the motivation of this example I feel obliged to say a word of "explanation" as to where the Gaussian measure γ comes from as well as the Dirichlet form operator H. I put "explanation" in quotations because the standard reasoning in the physics literature, from its beginnings in 1929 through standard textbooks on quantum field theory today, proceed on this issue by heuristic arguments involving "harmless" subtractions of infinity and meaningless infinite dimensional Lebesque measure in a way that readers of the physics literature can and do simply get used to. Why not us? The argument goes like this. The classical equations of motion of the electromagnetic field (2.6) resemble the Newtonian equations of motion of a system of harmonic oscillators: Both have the form $\ddot{u} = Lu$ for some linear operator L. For N harmonic oscillators $u(t)$ takes its values in $\mathbb{R}^N$ and L, after diagonalization, has the negative square natural frequencies $-\omega_j^2$ on its diagonal. For the electromagnetic field, $u(t)$ lies in the infinite dimensional Hilbert space $\mathcal{K} \equiv \{\omega \in L^2(\mathbb{R}^3; \Lambda^1, \mathrm{d}x) : \mathrm{d}^*\omega = 0\}$ and $L = \Delta$. For N harmonic oscillators one quantizes by taking the quantum Hilbert space to be $L^2(\mathbb{R}^N; \text{Lebesgue measure})$, while its Hamiltonian is a well known sum of N second order differential operators of the form $-\partial^2/\partial x_j^2 + \omega_j^2 x_j^2$. So to quantize the electromagnetic field just replace $\mathbb{R}^N$ by $\mathcal{K}$ and relax about the infinite dimensional Lebesgue measure. For the Hamiltonian just replace the finite sum by the corresponding infinite sum. Surprise: The infinite sum diverges in any reasonable sense. But that doesn't matter because the infinite dimensional Lebesgue measure over $\mathcal{K}$ that we just considered using was meaningless anyway. Resolution:

(1) For N harmonic oscillators subtract off the bottom of the spectrum of the Hamiltonian. (This will be infinite in the electromagnetic case.)

(2) Change the representing measure for the quantum Hilbert space for N harmonic oscillators from Lebesgue measure to the ground state

measure associated to the lowest eigenfunction. Then proceed as before, letting $N \to \infty$. The result is the Gaussian measure and Dirichlet form operator stated in the previous paragraph. A reader interested in seeing this argument carried out in more, but still unavoidably heuristic, detail could look in my notes [6].

4. Logarithmic Sobolev inequality for local states

Notation 4.1. — Suppose that M is the closure of a bounded open set in $\mathbb{R}^3$. Let

$$H_{-1/2}(M) = \{j \in \mathcal{H}^* : \text{support } j \subset M\}. \tag{4.1}$$

Support refers to support as a distribution. Clearly $H_{-1/2}(M)$ is a closed subspace of $\mathcal{H}^*$. Define also

$$\begin{aligned}\mathcal{F}_M = {} & \text{closure in } L^2(\mathcal{H}, \gamma) \text{ of the linear span of the products} \\ & \qquad q_{j_1} \cdots q_{j_n}, \quad j_i \in H_{-1/2}(M),\ i = 1, \ldots, n,\ n = 0, 1, 2, \ldots\end{aligned} \tag{4.2}$$

If the support of j is contained in M, then the function $q_j(A)$ is sensitive only to the values of A in the set M, as we see from the definition (3.5). Thus a measurement of A using the current j will give information about A only in M. The subspace $\mathcal{F}_M$ consists of states of the field dependent only on the behavior of A in M. We refer to $\mathcal{F}_M$ as the *local subspace* of $L^2(\mathcal{H}, \gamma)$ associated to M.

This space can be described also in terms of the actual measure space Ω, γ. The linear functionals q_j on $\mathcal{H}$ extend to measurable functions on Ω. Let Σ_M be the σ field in Ω generated by the coordinate functions $\{q_j ; j \in H_{-1/2}(M)\}$. Then

$$\mathcal{F}_M = L^2(\Omega, \Sigma_M, \gamma). \tag{4.3}$$

Note: There is no space $H_{1/2}(M)$. The definition of such a space would require choosing boundary conditions. It is not needed. The closest space to this is $\mathcal{H} \ominus (H_{-1/2}(M))^0$.

Theorem 4.2. — *Let M be the closure of a bounded open subset of $\mathbb{R}^3$. Let Σ_M denote the σ field defined in Notation 4.1 There is a constant c_M such that*

$$\begin{aligned}&\int_\Omega \psi^2 \log |\psi| \, \mathrm{d}\gamma \leqslant c_M Q(\psi) + \|\psi\|_2^2 \log \ \|\psi\|_2 \\ &\textit{for all } \psi \in \mathcal{F}_M \qquad (\textit{equivalently, for all } \psi \in L^2(\Omega; \Sigma_M, \gamma)).\end{aligned} \tag{4.4}$$

Lemma 4.3 (Poincaré like lemma). — *Suppose that M is the closure of a bounded open set in R^3. Then there is a constant $c_M > 0$, depending only on the volume of M, such that*

$$\|j\|^2_{H_{-1/2}} \leqslant c_M \|j\|^2_{L^2(\mathbb{R}^3)} \quad \textit{for all} \quad j \in H_{-1/2}(M). \tag{4.5}$$

Proof. — Define $\hat{j}(\xi) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} j(x) e^{ix\cdot\xi}\, \mathrm{d}x$. Then

$$\begin{aligned}\|j\|^2_{H_{-1/2}} &= \int_{\mathbb{R}^3} |\xi|^{-1} |\hat{j}(\xi)|^2 \,\mathrm{d}\xi \\ &\leqslant \epsilon^{-1} \int_{|\xi|\geqslant\epsilon} |\hat{j}(\xi)|^2 \,\mathrm{d}\xi + (\sup_{\xi\in\mathbb{R}^3} |\hat{j}(\xi)|^2) \int_{|\xi|<\epsilon} |\xi|^{-1} \,\mathrm{d}\xi \\ &\leqslant \epsilon^{-1} \|j\|^2_{L^2} + 2\pi\epsilon^2 (\sup_{\xi\in\mathbb{R}^3} |\hat{j}(\xi)|^2).\end{aligned}$$

But $|\hat{j}(\xi)| \leqslant (2\pi)^{-3/2} \int_M |j(x)|\,\mathrm{d}x \leqslant (2\pi)^{-3/2} vol(M)^{1/2} \|j\|_{L^2}$. Hence

$$\|j\|^2_{H_{-1/2}} \leqslant \|j\|^2_{L^2} \Big(\epsilon^{-1} + (volM/(2\pi)^2)\epsilon^2\Big).$$

Take $\epsilon = (volM)^{-1/3}$ to find (4.5) with $c_M = (volM)^{1/3} \cdot const$. This proves (4.5). □

Notation 4.4. — The unit Dirichlet form for the measure γ is

$$Q_0(\psi) = \int_{\mathcal{H}} \|(D\psi)(A)\|^2_{H_{-1/2}(\mathbb{R}^3)} \,\mathrm{d}\gamma(A) \tag{4.6}$$

because the gradient of a function on $\mathcal{H}$ satisfies

$$\|\nabla\psi(A)\|_{\mathcal{H}} = \|D\psi(A)\|_{\mathcal{H}^*}. \tag{4.7}$$

Consequently, the standard logarithmic Sobolev inequality holds for the Dirichlet form Q_0.

Lemma 4.5. — *Suppose that M is the closure of a bounded open set in $\mathbb{R}^3$. Then*

$$Q_0(\psi) \leqslant c_M Q(\psi) \quad \textit{for all } \psi \in \mathcal{F}_M. \tag{4.8}$$

Proof. — Since polynomials form a core for both Q_0 and Q, it suffices to prove (4.8) when ψ is a polynomial in the q_js. We can do the computation in this nice smooth category for each $A \in \mathcal{H}$. In view of the definitions (4.6) and (3.16) it suffices to show that

$$\|(D\psi)(A)\|^2_{\mathcal{H}^*} \leqslant c_M \|(D\psi)(A)\|^2_{L^2(\mathbb{R}^3)} \tag{4.9}$$

for all $A \in \mathcal{H}$ when $\psi(A)$ is a polynomial in the linear functionals $q_j(A)$ with each $j \in H_{-1/2}(M)$. But

$$(Dq_j)(A)\langle g\rangle = (\partial_g q_j)(A) = \langle j, g\rangle_{L^2} \quad \forall\, g \in \mathcal{H}. \tag{4.10}$$

Therefore, by the product rule,

$$(\partial_g \psi)(A) = \langle j_A, g\rangle_{L^2}, \tag{4.11}$$

where, for each $A \in \mathcal{H}$, j_A is a finite linear combination of elements $j_i \in H_{-1/2}(M)$. Hence $j_A \in H_{-1/2}(M)$ for each $A \in H_{1/2}(\mathbb{R}^3)$. It now follows from (4.5) that

$$\|j_A\|_{H_{-1/2}} \leqslant c_M \|j_A\|_{L^2}. \tag{4.12}$$

This proves (4.9). □

Proof of Theorem 4.2. — Since Q_0 is the unit Dirichlet form for the measure γ the standard logarithmic Sobolev inequality assures that

$$\int_\Omega \psi^2 \log|\psi| \,\mathrm{d}\gamma \leqslant Q_0(\psi) + \|\psi\|_2^2 \log \ \|\psi\|_2 \quad \forall\, \psi \in L^2(\gamma). \tag{4.13}$$

But the inequality (4.8) shows that, for ψ which are Σ_M measurable, Q_0 is dominated by $c_M Q$. (4.4) now follows. □

5. The road to Γ_2

In truth, the mathematical content of this note has been merely to establish the relation between the unit quadratic form Q_0, for which we know the validity of a logarithmic Sobolev inequality, and the form Q, which is handed to us by the physics of the electromagnetic field. The proof of the relation (4.8) is just the few lines in the proof of Lemma 4.3. But these aspects of the free electromagnetic field form a useful template for what to look for when constructing the corresponding structures for the Yang–Mills theory. There are big changes in the structures that will be needed. Some of these have already been established, [3, 4, 5, 7, 8]. Here I want to sketch the changes that I anticipate (read “hope”) will be useful for implementing this program for construction of the quantized Yang–Mills field.

Change 1. — In Maxwell’s theory of electromagnetism the basic fields were the electric and magnetic fields. The auxiliary field A exists, as we saw, because one of Maxwell’s equations in (2.1) asserts that the 2-form F on $\mathbb{R}^4$ is closed. But in the Yang–Mills extension of this theory the 1-form A on $\mathbb{R}^4$ is a basic object. It has the geometric interpretation of a connection form over space-time, $\mathbb{R}^4$. Moreover it takes its values in the Lie algebra $\mathfrak{k}$ of a compact Lie group K: $A = \sum_{i=1}^4 A_i \,\mathrm{d}x^i$ with each function $A_i : \mathbb{R}^4 \to \mathfrak{k}$. The correct group K has to be determined by experiment. The group $K = SU(2) \times SU(3) \times U(1)$ is currently most in fashion. The 2-form F described in (2.5) is replaced now by the curvature of the connection form A. It is given by

$$F = DA + A \wedge A. \tag{5.1}$$

Maxwell's theory corresponds to the choice $K = U(1)$, whose Lie algebra is $i\mathbb{R}$. The quadratic term is zero in Maxwell's theory, giving (2.5). But in the general Yang–Mills case, terms in the wedge product, such as $A_i A_j \, \mathrm{d}x^i \wedge \mathrm{d}x^j + A_j A_i \, \mathrm{d}x^j \wedge \mathrm{d}x^i = [A_i, A_j] \, \mathrm{d}x^i \wedge \mathrm{d}x^j$, are no longer zero. We therefore no longer have the nice simple linear relationship (2.5). The resulting theory is immediately non-linear. The wave equation is replaced by a non-linear wave equation, whose solution for appropriate initial values in $H_{1/2}$ is still not established, in spite of much work, begun in 1979 by I. Segal, [9]. This is part of the theory of non-linear hyperbolic equations and does not relate significantly to our present interest in infinite dimensional measures and logarithmic Sobolev inequalities.

Change 2. — The heuristic arguments leading to the presence of a Gaussian measure with covariance given by the space $H_{-1/2}(\mathbb{R}^3)$, as we used above, are gone: The Yang–Mills theory is in no reasonable sense an assembly of harmonic oscillators. Instead the Hilbert space $H_{1/2}(\mathbb{R}^3)$, whose extension we used to support the measure γ, has to be replaced by some infinite dimensional differentiable Riemannian manifold which substitutes for $H_{1/2}(\mathbb{R}^3)$. Let's refer to this desired manifold as $\mathcal{Y}_{1/2}$. This non-linear manifold should also have some kind of big extension that supports the presumed ground state measure, analogous to γ. Just what $H_{1/2}$ should mean for this non-linear manifold and what "the" large extension should be is up to the beholder to decide. It must be consistent with what we know about the need for a large support space for measures resembling γ. Underlying all such constructions is the need to maintain Lorentz invariance.

Change 3. — One of the big changes from the body of this note, that one must expect, is illuminating to discuss. The linear functionals q_j on $H_{1/2}(\mathbb{R}^3; \Lambda^1)$, that were central to the whole discussion above, are no longer conceptually meaningful when the gauge group K is not commutative. The relevant substitute for the linear functionals q_j is more or less agreed upon in the physics literature. But solving the technical problems associated with this substitute is a long way off because the substitute is a much more singular function of the connection form A than the functions q_j are. The differential geometric significance of these functions (holonomy) is very compelling, however, and the conventional wisdom is that these singular functions are here to stay. These are parallel transport operators around closed curves in $\mathbb{R}^3$ with respect to the connection form A. Here is how closed curves relate to the divergence free currents that we have been using in this paper.

A generalized current on $\mathbb{R}^3$ is customarily defined as a linear functional on C_c^∞ 1-forms ω on $\mathbb{R}^3$. If, for example $j : \mathbb{R}^3 \to \mathbb{R}^3$, is a locally integrable vector field, then the functional

$$\omega \mapsto \int_{\mathbb{R}^3} \omega(x)\langle j(x)\rangle \,\mathrm{d}^3x \equiv \langle \omega, j\rangle \tag{5.2}$$

is a nice linear functional on C_c^∞ 1-forms. These do not include all the linear functionals arising from $j \in H_{-1/2}$ because $H_{-1/2}$ includes some distributions which are not functions. But the linear functional (5.2) is well defined for $j \in H_{-1/2}$ also. Now suppose that $C : [0,1] \to \mathbb{R}^3$ is a piecewise smooth curve. It defines a linear functional on 1-forms by the definition

$$\omega \to \int_0^1 \omega(C(t))\langle \dot{C}(t)\rangle \,\mathrm{d}t. \tag{5.3}$$

This is clearly parametrization independent. In this way a curve defines a generalized current. The notion of divergence free current can be readily formulated for generalized currents thus: For $\lambda \in C_c^\infty(\mathbb{R}^3)$, the identity $\langle \mathrm{d}\lambda, j\rangle = \langle \lambda, \mathrm{d}^* j\rangle$ shows that $\mathrm{d}^* j = 0$ if and only if $\langle \omega, j\rangle = 0$ for all smooth exact 1-forms ω with compact support. This justifies the *definition* that a generalized current is divergence free if it is zero on all smooth exact 1-forms ω with compact support. In the case of the current (5.3), if we put $\omega = \mathrm{d}\lambda$, then we find $\int_0^1 (\mathrm{d}\lambda)(C(t))\langle \dot{C}(t)\rangle \,\mathrm{d}t = \int_0^1 (\mathrm{d}/\mathrm{d}t)\lambda(C(t))\,\mathrm{d}t = \lambda(C(1)) - \lambda(C(0))$, which is zero for all such λ if and only if $C(1) = C(0)$. Thus the generalized current induced by a curve has divergence zero if and only if the curve is closed. To understand how singular such a current is, write $j(x) = \int_0^1 \delta(x - C(t))\dot{C}(t)\,\mathrm{d}t$, which is the customary representation of this current in the physics literature. Such a distribution is highly singular, since it's supported on a curve. It is not in $H_{-1/2}(\mathbb{R}^3)$. It is not among the currents that we have been dealing with in this paper. Yet it is the one that must be used when dealing with the non-commutative theory. Moreover, the simple bilinear pairing (3.5) between a current j and a connection form A over $\mathbb{R}^3$ must be replaced by the following holonomy function.

Consider the solution to the parallel transport equation $\dot{g}(t)g(t)^{-1} = A(C(t))\langle \dot{C}(t)\rangle$, $g(0) = e_K$, where $g : [0,1] \to K$. Define $W_C(A) = \chi(g(1))$ for some character χ on K. This loop dependent function of A, the so-called Wilson loop function, is widely regarded in the physics literature as a natural replacement for the functions $q_j(A)$ of (3.5). In this sense our divergence free currents $j \in H_{-1/2}$ will be replaced by closed curves. These holonomy functions (after suitable regularization, as addressed in [3, 4, 5, 7, 8]) descend to functions on the configuration space $\mathcal{Y}_{1/2}$ and to the enlargement of this space on which one should hope to produce the non-Gaussian analog of γ. Not only must the differential geometry of the manifold $\mathcal{Y}_{1/2}$ be understood, but also the measure that replaces γ. One can hope that the measure itself can be produced by the Feynman–Kac formula for an infinite dimensional state space by a much explored machinery that goes under the name of Euclidean quantum field theory. Many efforts in this direction in the 1970's and 1980's failed, largely (in my opinion) because of the unavailability of regularized versions of the Wilson loop functions $A \mapsto W_C(A)$.

It is the author's hope that once the Yang–Mills analog, η, of the Gaussian measure is constructed, further analysis of these singular functions will be facilitated by use of non-Gaussian logarithmic Sobolev inequalities for the Yang–Mills Hamiltonian acting on $L^2(\mathcal{Y}_{1/2}, \eta)$. Of course establishing non-Gaussian logarithmic Sobolev inequalities is a different ball game from establishing Gaussian ones. The only techniques on the horizon (my horizon) that seem feasible for attacking the problem of logarithmic Sobolev inequalities in this highly non-Gaussian context, once the differential geometry (infinite dimensional) and integration theory (infinite dimensional) are established, are Dominique's Γ_2 methods. A simple calculation shows that Dominique's 65th birthday will occur in less than five years. This author hopes that by then the necessary infinite dimensional differential geometry and integration theory will be in place and ready.

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