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Quantum expanders and growth of group representations

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Abstract. — Let π be a finite dimensional unitary representation of a group G with a generating symmetric n-element set $S \subset G$. Fix $\varepsilon > 0$. Assume that the spectrum of $|S|^{-1} \sum_{s \in S} \pi(s) \otimes \overline{\pi(s)}$ is included in $[-1, 1 - \varepsilon]$ (so there is a spectral gap $\geqslant \varepsilon$). Let $r'_N(\pi)$ be the number of distinct irreducible representations of dimension $\leqslant N$ that appear in π. Then let $R'_{n,\varepsilon}(N) = \sup r'_N(\pi)$ where the supremum runs over all π with n, ε fixed. We prove that there are positive constants δ_ε and c_ε such that, for all sufficiently large integer n (i.e. $n \geqslant n_0$ with n_0 depending on ε) and for all $N \geqslant 1$, we have $\exp \delta_\varepsilon n N^2 \leqslant R'_{n,\varepsilon}(N) \leqslant \exp c_\varepsilon n N^2$. The same bounds hold if, in $r'_N(\pi)$, we count only the number of distinct irreducible representations of dimension exactly $= N$.

Résumé. — Soit π une représentation unitaire de dimension finie d'un groupe G munie d'un ensemble générateur symétrique $S \subset G$ à n-éléments. Fixons $\varepsilon > 0$ et supposons que le spectre de $|S|^{-1} \sum_{s \in S} \pi(s) \otimes \overline{\pi(s)}$ est inclus dans $[-1, 1-\varepsilon]$ (il y a donc un trou spectral $\geqslant \varepsilon$). Soit $r'_N(\pi)$ le nombre de représentations irréductibles distinctes de dimension $\leqslant N$ qui apparaissent dans la décomposition de π. Soit alors $R'_{n,\varepsilon}(N) = \sup r'_N(\pi)$ où le sup court sur toutes les π possibles avec n, ε fixés. Nous démontrons l'existence de constantes positives δ_ε et c_ε telles que, pour tout entier n suffisamment grand (i.e. $n \geqslant n_0$ ou n_0 peut dépendre de ε) et pour tout $N \geqslant 1$, on a $\exp \delta_\varepsilon n N^2 \leqslant R'_{n,\varepsilon}(N) \leqslant \exp c_\varepsilon n N^2$. Les mêmes bornes sont valables si, dans $r'_N(\pi)$, on compte seulement le nombre de représentations irréductibles distinctes de dimension exactement $= N$.

1. Introduction

We wish to formulate and answer a natural extension of a question raised explicitly by Wigderson in several lectures (see e.g. [23, p. 59]) and also

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implicitly in [18]. Although the variant that we answer seems to be much easier, it may shed some light on the original question. Wigderson's question concerns the growth of the number $r_N(G)$ of distinct irreducible representations of dimension $\leqslant N$ that may appear on a finite group G when the order of G is arbitrarily large and all that one knows is that G admits a generating set S of n elements for which the Cayley graph forms an expander with a fixed spectral gap $\varepsilon > 0$. The problem is to find the best bound of the form $r_N(G) \leqslant R(N)$ with $R(N)$ independent of the order of G (but depending on n, ε). We consider a more general framework: the finite group G is replaced by a finite dimensional representation π (playing the role of the regular representation λ_G for finite groups) such that the representation $\pi \otimes \bar{\pi}$ admits a spectral gap, meaning that the trivial representation is isolated with a gap $\geqslant \varepsilon$ from the other irreducible components of $\pi \otimes \bar{\pi}$. When $\pi = \lambda_G$ we recover the previous notion of spectral gap. Let then $r'_N(\pi)$ be the number of distinct irreducible representations of dimension $\leqslant N$ appearing in π (note that $r_N(G) = r'_N(\lambda_G)$), and let $R'(N)$ denote the least upper bound $r'_N(\pi) \leqslant R'(N)$ when the only restriction on π is that n, ε remain fixed (but the dimension of π is arbitrary). We observe that the previously known bound for $R(N)$ namely $R(N) = e^{O(nN^2)}$ is also valid for $R'(N)$ and also that $R(N) \leqslant R'(N)$. Our main result, which follows from the metric entropy estimate for quantum expanders in [20], is that this bound for $R'(N)$ is sharp: there is $\delta > 0$ such that for all n large enough (i.e. $\forall n \geqslant n_0(\varepsilon)$) we have $R'(N) \geqslant e^{\delta n N^2}$ for all N.

The term "quantum expander" was coined in [2, 3, 8] to which we refer for background (see also [7, 9]).

2. Main result

Let G be any group with a finite generating set $S \subset G$ with $|S| = n$. For any unitary representation $\pi : \ G \to H_\pi$ we set

$$\lambda(\pi, S) = n^{-1} \sup\{\Re \ \langle \sum_{s \in S} \pi(s)\xi, \xi \rangle \mid \xi \in H_\pi^{\mathrm{inv}\perp}, \|\xi\|_{H_\pi} = 1\}.$$

where $H_\pi^{\mathrm{inv}} \subset H_\pi$ denotes the subspace of all π-invariant vectors. When S is symmetric, $\sum_{s \in S} \pi(s)$ being selfadjoint, the real part sign $\Re$ can be omitted. We then set

$$\varepsilon(\pi, S) = 1 - \lambda(\pi, S).$$

It will be useful to record here the elementary observation that if π is unitarily equivalent to the direct sum $\oplus_{i \in I} \pi_i$ of a family of unitary representations,

then $\lambda(\pi, S) = \sup_{i \in I} \lambda(\pi_i, S)$ and hence

$$\varepsilon(\pi, S) = \inf_{i \in I} \varepsilon(\pi_i, S). \tag{2.1}$$

In particular, if π_1 is contained in π_2, then $\varepsilon(\pi_1, S) \geqslant \varepsilon(\pi_2, S)$.

We denote

$$\varepsilon(G, S) = \inf\{\varepsilon(\pi, S)\}$$

where the infimum runs over all unitary representations $\pi : G \to H_\pi$. Thus the condition

$$\varepsilon(G, S) > 0$$

means that G has Kazhdan's "property (T)", (or in otherwords is a Kazhdan-group), see [1] for more background.

We start by the following result somewhat implicitly due to S. Wassermann [22] and explicitly proved in detail in [6].

PROPOSITION 2.1 ([22, 6]). — *For any $\varepsilon > 0$ there is a constant c_ε such that for any n, any group G and any $S \subset G$ with $|S| = n$ such that $\varepsilon(G, S) \geqslant \varepsilon$, the number $r_N(G)$ of distinct irreducible unitary representations $\sigma : G \to B(H_\sigma)$ with $\dim(H_\sigma) \leqslant N$ is majorized as follows:*

$$r_N(G) \leqslant \exp\left(c_\varepsilon n N^2\right). \tag{2.2}$$

Of course, here distinct means up to unitary equivalence.

Remark 2.2. — Note that it suffices to prove a bound of the same form for the number of distinct irreducible unitary representations $\sigma : G \to B(H_\sigma)$ with $\dim(H_\sigma) = N$. Indeed, if the latter number is denoted by $s_N(G)$, we have $r_N(G) = \sum_{d=1}^{N} s_d(G)$, so that it suffices to have a bound of the form $s_d(G) \leqslant \exp\left(c'_\varepsilon n d^2\right)$ to obtain (2.2). See [14, 15] for some examples of estimates of the growth of $r_N(G)$.

We note that it was originally proved by Wang [21] that for any Kazhdan-group G this number $r_N(G)$ is finite for any N. There is an indication of proof of (2.2) in [22], and detailed proofs appear in [6] (see also [18]). We will prove a simple extension of this bound below.

Recall that a sequence (G_k, S_k) of finite groups equipped with generating sets $S_k \subset G_k$ such that

$$\sup_k |S_k| < \infty, \quad |G_k| \to \infty \quad \text{and} \quad \inf_k \varepsilon(G_k, S_k) > 0$$

is called an expander or an expanding family. This corresponds to the usual notion among *Cayley* graphs to which we restrict the entire discussion. Let $\hat{G}$ denote as usual the (finite) set of all irreducible unitary representations of a finite group G (up to unitary equivalence). We note in passing that it

is well known (and this also can be derived from Proposition 2.1) that any expander satisfies

$$\lim_{k\to\infty} \max\{\dim(H_\sigma) \mid \sigma \in \hat{G}_k\} = \infty. \tag{2.3}$$

We refer the reader to the surveys [10, 17] for more information on expanders.

The question raised by Wigderson in this context can be formulated as follows:

Let

$$R_{n,\varepsilon}(N) = \sup\{r_N(G)\}$$

where the supremum runs over all finite groups G admitting a subset S with $|S| = n$ such that $\varepsilon(G,S) \geqslant \varepsilon$. Actually the question is just as interesting for arbitrary (Kazhdan) groups G, but it is more natural to restrict to finite groups, because there are infinite Kazhdan groups without *any* (nontrivial) finite dimensional representations.

Moreover, since, for a finite group G, all representations are weakly contained in the left regular representation λ_G, we have clearly by (2.1)

$$\varepsilon(G,S) = \varepsilon(\lambda_G, S). \tag{2.4}$$

By (2.2), we have

$$R_{n,\varepsilon}(N) \leqslant \exp\left(c_\varepsilon n N^2\right). \tag{2.5}$$

and a fortiori simply $R_{n,\varepsilon}(N) = \exp O(N^2)$. Wigderson asked whether this upper bound can be improved. More explicitly, what is the precise order of growth of $\log R_{n,\varepsilon}(N)$ when $N \to \infty$. Does it grow like N rather than like N^2? The motivation for this question can be summarized like this: In [18, Th. 1.4] an exponential bound $\exp O(N)$ is proved for a special class of groups G (namely monomial groups), admitting a fixed spectral gap with generating sets of very slowly growing size (but not bounded) and it is asked whether the same exponential bound holds in general for $R_{n,\varepsilon}(N)$. Moreover, in a remark following the proof of [18, Th. 1.4], Meshulam and Wigderson observe that for any prime number $p > 2$, there is a group G_p with a generating set of (unbounded) size $\log p$ admitting a fixed spectral gap and such that $r_p(G) \approx 2^p/p$.

Remark 2.3. — By classical results, originating in the works of Kazhdan and Margulis (see e.g. [16] or [17, Cor. 2.4]), for any fixed $m \geqslant 3$, the family $\{SL_m(\mathbb{Z}_p) \mid p \text{ prime}\}$ is an expander, so that we have (for suitable ℓ, δ)

$$R_{\ell,\delta}(N) \geqslant \sup_p r_N(SL_m(\mathbb{Z}_p)).$$

Similarly, let $\mathcal{G}_k$ denote the symmetric group of all permutations of a k element set. Kassabov [11] proved that the family $\{\mathcal{G}_k \mid k \geqslant 1\}$ forms an

expanding family with respect to subsets $S_k \subset \mathcal{G}_k$ of a fixed size ℓ and a fixed spectral gap $\delta > 0$. Thus we find a lower bound

$$R_{\ell,\delta}(N) \geqslant \sup_k r_N(\mathcal{G}_k).$$

Quite remarkably, it is proved in [13] that the family itself of *all* non-commutative finite simple groups forms an expander (for some suitable n, ε).

Remark 2.4. — However, it seems the resulting lower bounds are still far from being exponential in N. Actually, in many important cases (see e.g. [4]), the proof that certain finite groups G give rise to expanders uses the fact that the smallest dimension of a (non-trivial) irreducible representation on G is $\geqslant c|G|^a$ for some $a > 0$. Then since $|G| = \sum_{\pi \in \hat{G}} \dim(\pi)^2$ the cardinal of $\hat{G}$ is bounded above by $|G|^{1-2a}/c^2$. Therefore, for any $N \geqslant c|G|^a$ we have $r_N(G) \leqslant |G|^{1-2a}/c^2 \leqslant c'N^{(1/a)-2}$, so that the resulting growth implied for $R_{n,\varepsilon}(N)$ is at most polynomial in N. (I am grateful to N. Ozawa for drawing my attention to this point).

Nevertheless, we have:

Remark 2.5 (Communicated by Martin Kassabov). — For suitable n, ε the numbers $R_{n,\varepsilon}(N)$ grow faster than any power of N. In fact, we will prove the

CLAIM. — *There is an expanding family of Cayley graphs (G_k) of groups generated by 3 elements with a positive spectral gap ε and such that for $N_k = 2^{3k}-2$, G_k admits 2^{k^2} distinct irreducible representations of dimension N_k.*

From this claim follows that $R_{3,\varepsilon}(N_k) \geqslant 2^{k^2} \geqslant 2^{(\log(N_k))^2}$, say for all k large enough, and hence

$$R_{n,\varepsilon}(N) \geqslant 2^{(\log(N))^2} \text{ for infinitely many } N\text{'s.}$$

To prove the claim we use the ideas from [12]. Let $\mathcal{R}_k$ denote the (finite) ring $M_k(F_2)$ of $k \times k$ matrices with entries in the field with 2 elements. It is known that the cartesian product $\Pi_k = \mathcal{R}_k^{2^{k^2}}$ of $|\mathcal{R}_k| = 2^{k^2}$ copies of $\mathcal{R}_k$ is generated by 3 elements. Indeed, $\mathcal{R}_k$ itself is generated as a ring by two elements, e.g. $a = e_{12}$ and the shift $b = e_{12} + e_{23} + \cdots + e_{k-1k} + e_{k1}$, then Π_k is generated as a ring by $\{A, B, C\}$ where A (resp. B) is the element with all coordinates equal to a (resp. b), and C is such that its coordinates are in one to one correspondence with the elements of $\mathcal{R}_k$. To check this, let $R \subset \Pi_k$ be the ring generated by $\{A, B, C\}$. Note, by the choice of C, the following easy observation: for any coordinate i, there is $x \in R$ such that $x_i = 0$ but $x_j \neq 0$ for all $j \neq i$. For any subset I of the index set let $p_I : R \to \mathcal{R}_k^I$ be the coordinate projection. One can then prove by induction

on $m = |I|$ that $p_I(R) = \mathcal{R}_k^I$ for all I. Indeed, assume the fact established for $m - 1$. For any I with $|I| = m$ we pick $i \in I$ and we consider the set $\mathcal{I} = \{y \in \mathcal{R}_k^{I \backslash i} \mid (0, y) \in p_I(R)\}$. By the induction hypothesis, $\mathcal{I}$ is an ideal in $\mathcal{R}_k^{I \backslash i}$, but, since $\mathcal{R}_k$ is simple, the above observation implies that $\mathcal{I} = \mathcal{R}_k^{I \backslash i}$, and since a, b generate $\mathcal{R}_k$ we have $p_{\{i\}}(R) = \mathcal{R}_k$, so we obtain $p_I(R) = \mathcal{R}_k^I$.

This implies that the free associative ring $\mathbb{Z}\langle x, y, z \rangle$ (in 3 non-commutative variables) can be mapped onto the product Π_k. Consider now the group $EL_3(\mathbb{Z}\langle x, y, z \rangle)$ generated by the elementary matrices in $GL_3(\mathbb{Z}\langle x, y, z \rangle)$. This is a noncommutative universal lattice in the terminology of [5, 12]. First observe that $EL_3(\mathbb{Z}\langle x, y, z \rangle)$ is generated by 3 elements. Indeed, let α, β generate $SL_3(\mathbb{Z})$. Then α, β, γ will generate $EL_3(\mathbb{Z}\langle x, y, z \rangle)$ where

$$\gamma = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover, by [5, Th. 1.1] $EL_3(\mathbb{Z}\langle x, y, z \rangle)$ has Kazhdan's property T. It follows that the groups

$$G_k = EL_3(\Pi_k)$$

have expanding generating sets with 3 elements. But it turns out that G_k can be identified with the product

$$SL_{3k}(F_2)^{2^{k^2}}.$$

Indeed, firstly one easily checks the natural isomorphism $EL_3(\mathcal{R}_k^{2^{k^2}}) \simeq EL_3(\mathcal{R}_k)^{2^{k^2}}$, secondly it is well known that, since F_2 is a field, $EL_n(F_2) = SL_n(F_2)$ for any n, and hence (taking $n = 3k$) we have a natural isomorphism $EL_3(\mathcal{R}_k) = SL_{3k}(F_2)$; this yields the identification $G_k = SL_{3k}(F_2)^{2^{k^2}}$.

To conclude, we will use the fact that $SL_{3k}(F_2)$ admits a nontrivial irreducible representation π with dimension $N_k = 2^{3k} - 2$. (Just consider its action by permutation on the projective space, which has $2^{3k} - 1$ elements; the action is transitive and doubly transitive, therefore the associated Koopman representation π is irreducible and of dimension $2^{3k} - 2$). This immediately produces 2^{k^2} distinct irreducible representations of dimension N_k on $SL_{3k}(F_2)^{2^{k^2}}$. Indeed, it is an elementary fact that if $\Gamma = \Gamma_1 \times \cdots \times \Gamma_m$ is a product group, and if $\pi_1, \ldots, \pi_m$ are arbitrary nontrivial irreducible representations on the factor groups $\Gamma_1, \ldots, \Gamma_m$, then the representations $\widetilde{\pi}_j$ defined on Γ by $\widetilde{\pi}_j(g) = \pi_j(g_j)$ are distinct (meaning not unitarily equivalent), irreducible on Γ and $\dim(\widetilde{\pi}_j) = \dim(\pi_j)$ for any j. So taking all Γ_j's equal to $SL_{3k}(F_2)$, with $\pi_j = \pi$ and $m = 2^{k^2}$, we obtain the announced claim.

In any case, the problem of finding the correct behaviour of $\log R_{n,\varepsilon}(N)$ (or of $R_{n,\varepsilon}(N)$ itself) when $N \to \infty$ appears to be still wide open.

In this paper we consider a modified version of this question involving "quantum expanders" and show that for this (much easier) modified version, N^2 is the correct order of growth.

The term "quantum expander" was introduced in [8] and [2, 3], independently, to designate a sort of non-commutative, or matricial, analogue of expanders, as follows.

Fix an integer n. Consider an n tuple of $N \times N$ unitary matrices, say $u = (u_j) \in U(N)^n$. We view each of them u_j as a linear operator on the N-dimensional Hilbert space H. Then $u_j \otimes \overline{u_j}$ is naturally viewed as a linear operator on the (Hilbert space sense) tensor product $H \otimes \bar{H}$. Using the (canonical) identification $H^* \simeq \bar{H}$, the tensor product $H \otimes \bar{H}$ can be isometrically identified with the space of linear operators from H to H equipped with the Hilbert–Schmidt norm denoted by $\| \ \|_2$ (sometimes called the Frobenius norm in the present finite dimensional context). Then, the identity operator $Id_H : \ H \to H$ defines a distinguished element of $H \otimes \bar{H}$ that we denote by I.

We set

$$\lambda(u) = n^{-1} \sup \left\{ \Re \left\langle \left(\sum_1^n u_j \otimes \bar{u}_j \right) \xi, \xi \right\rangle \ \middle| \ \xi \in H \otimes \bar{H}, \ \xi \perp I, \ \|\xi\|_{H \otimes \bar{H}} = 1 \right\},$$

and

$$\varepsilon(u) = 1 - \lambda(u).$$

In other words, with the preceding identifications, the condition $\varepsilon(u) \geqslant \varepsilon$ means that for any $x \in M_N$ with $\mathrm{tr}(x) = 0$ we have

$$\Re \sum \mathrm{tr}(u_j x u_j^* x^*) \leqslant (1 - \varepsilon)\|x\|_2,$$

where $\|x\|_2 = (\mathrm{tr}(x^* x))^{1/2}$. When $T = \sum_1^n u_j \otimes \bar{u}_j$ is self adjoint (in particular when the set $\{u_1, \cdots, u_n\}$ is selfadjoint) the real part $\Re$ can be omitted in the two preceding lines.

In group theoretic language, if $\pi : \ \mathbf{F}_n \to U(N)$ is the group representation on the free group $\mathbf{F}_n$, equipped with a set of n free generators $S = \{g_1, \cdots, g_n\}$, such that $\pi(g_j) = u_j$ $(1 \leqslant j \leqslant n)$, then we have

$$\varepsilon(u) = \varepsilon(\pi \otimes \overline{\pi}, S).$$

DEFINITION 2.6. — *A sequence $\{u(k) \mid k \in \mathbb{N}\}$ with each $u(k) \in U(N_k)^n$ such that $N_k \to \infty$ (with n remaining fixed) and $\inf_k \{\varepsilon(u(k))\} > 0$ is called a quantum expander. We say that n is its degree and $\inf_k \{\varepsilon(u(k))\} > 0$ its spectral gap.*

Remark 2.7. — The existence of quantum expanders can be deduced as follows from that of expanders. Recalling (2.4), assume given a finite group G and $S \subset G$ as before such that $\varepsilon(G, S) = \varepsilon(\lambda_G, S) \geqslant \varepsilon > 0$. Recall that each $\sigma \in \hat{G}$ is contained in λ_G. Let $\pi \in \hat{G}$. Since any representation on G without invariant vectors, being a direct sum of non trivial irreps, is weakly contained in λ_G, the representation $\rho = \pi \otimes \overline{\pi}$ restricted to $H_\rho^{\mathrm{inv}\perp}$ is weakly contained in the non trivial part of λ_G. In particular, we have by (2.1)

$$\lambda(\rho, S) \leqslant \lambda(\lambda_G, S).$$

Therefore, we have

$$\varepsilon(\pi \otimes \overline{\pi}, S) \geqslant \varepsilon(\lambda_G, S) \geqslant \varepsilon.$$

Thus if we are given an expander (G_k, S_k) as above, say with $S_k = \{s_1(k), \ldots, s_n(k)\}$, we can choose by (2.3) $\sigma_k \in \hat{G}_k$ such that $\dim(H_{\sigma_k}) \to \infty$, and if we set $u_j(k) = \sigma_k(s_j(k))$ $(1 \leqslant j \leqslant n)$, then $u(k) = \{u_1(k), \ldots, u_n(k)\}$ forms a quantum expander.

The next statement is a simple generalization of Proposition 2.1

PROPOSITION 2.8. — *For any $0 < \varepsilon < 1$ there is a constant $c'_\varepsilon > 0$ for which the following holds. Let G be any group and let $\pi : G \to B(H)$ be any unitary representation on a finite dimensional Hilbert space H. Let us assume that there is an n-element subset $S \subset G$ and $\varepsilon > 0$ such that*

$$\varepsilon(\pi \otimes \overline{\pi}, S) \geqslant \varepsilon.$$

In other words, π satisfies the following spectral gap condition:

$$\lambda(\pi \otimes \overline{\pi}, S) \leqslant 1 - \varepsilon \tag{2.6}$$

Let $\pi = \oplus_{t \in T} \pi_t$ be the decomposition into distinct *irreducibles (where each π_t has multiplicity $d_t \geqslant 1$), then*

$$|\{t \in T \mid \dim(\pi_t) \leqslant N\}| \leqslant \exp c'_\varepsilon n N^2. \tag{2.7}$$

Proof. — Let $\sigma = \oplus_{t \in T} \pi_t$ be the direct sum where each component is included with multiplicity equal to 1. We may clearly view σ as a subpresentation of π, acting on a subspace $K \subset H$ so that the orthogonal projection $Q : H \to K$ is intertwining, i.e. satisfies $Q\pi = \sigma Q$. Then we also have $(Q \otimes \bar{Q})(\pi \otimes \bar{\pi}) = (\sigma \otimes \bar{\sigma})(Q \otimes \bar{Q})$, from which it is easy to derive that if we denote $V_\pi = H_{\pi \otimes \bar{\pi}}^{\mathrm{inv}}$, we have $(Q \otimes \bar{Q})V_\pi = V_\sigma$ and $(Q \otimes \bar{Q})V_\pi^\perp = V_\sigma^\perp$. This implies

$$\lambda(\sigma \otimes \overline{\sigma}, S) \leqslant \lambda(\pi \otimes \overline{\pi}, S) \leqslant 1 - \varepsilon.$$

Thus, replacing π by σ, we may as well assume that the multiplicities d_t are all equal to 1.

Let $H = \oplus_{t \in T} H_t$ denote the decomposition corresponding to $\pi = \oplus_{t \in T} \pi_t$. We have $\pi \otimes \bar{\pi} = \oplus_{t,r \in T} \pi_t \otimes \overline{\pi_r}$, with associated decomposition $H \otimes \bar{H} =$

$\oplus_{t,r \in T} H_t \otimes \overline{H_r}$. From this follows that the subspace $V_\pi \subset H \otimes \bar{H}$ of $\pi \otimes \bar{\pi}$-invariant vectors is equal to $\oplus_{t,r \in T} V_{t,r}$ where $V_{t,r} \subset H_t \otimes \overline{H_r}$ is the subspace of invariant vectors of $\pi_t \otimes \overline{\pi_r}$. Since for any $t \neq r \in T$, $\pi_t \not\simeq \pi_r$, by Schur's lemma $V_{t,r} = \{0\}$, and hence $V_\pi \subset \oplus_{t \in T} V_{t,t}$. In particular, this shows that

$$\forall t \neq r \in T \quad H_t \otimes \overline{H_r} \subset V_\pi^\perp.$$

Let $T' = \{t \in T \mid \dim(\pi_t) = N\}$. It suffices to show an estimate of the form

$$|T'| \leqslant \exp c_\varepsilon n N^2. \tag{2.8}$$

Let $\mathcal{H}$ be the Hilbert space obtained by equipping M_N^n with the norm

$$\|x\|_{\mathcal{H}}^2 = N^{-1} n^{-1} \sum_1^n \operatorname{tr}(x_j^* x_j).$$

Let $S = \{s_1, \cdots, s_n\}$. For any $t \in T'$ we define $x(t) \in M_N^n$ by

$$x(t)_j = \pi_t(s_j) \quad 1 \leqslant j \leqslant n.$$

Note that, by our normalization, $\|x(t)\|_{\mathcal{H}} = 1$ for any $t \in T'$. Moreover, since for any $t \neq r \in T$ $\pi_t \not\simeq \pi_r$, by Schur's lemma the representation $\pi_t \otimes \overline{\pi_r}$ has no invariant vector, and hence lies inside $(\pi \otimes \bar{\pi})_{|V_\pi^\perp}$. Therefore, by (2.1)

$$\lambda(\pi_t \otimes \overline{\pi_r}, S) \leqslant \lambda(\pi \otimes \bar{\pi}, S),$$

and hence for any unit vector $\xi \in H_{\pi_t} \otimes \overline{H_{\pi_r}}$ we have

$$n^{-1} \Re(\sum_{s \in S} (\pi_t \otimes \overline{\pi_r}) \xi, \xi)) \leqslant 1 - \varepsilon.$$

In particular, if $t \neq r \in T'$, we may realize π_t, π_r as representations on the same N-dimensional space, so that taking $\xi = N^{-1/2} I$ we find

$$\Re \langle x(t), x(r) \rangle_{\mathcal{H}} = (nN)^{-1} \Re \left(\sum_{s \in S} \operatorname{tr}(\pi_t(s)^* \pi_r(s)) \right) \leqslant 1 - \varepsilon,$$

which implies

$$\|x(t) - x(r)\|_{\mathcal{H}} \geqslant \sqrt{2\varepsilon}.$$

Thus we have $|T'|$ points in the unit sphere of $\mathcal{H}$ that are $\sqrt{2\varepsilon}$-separated. Since $\dim(\mathcal{H}) = nN^2$, (2.8) follows immediately by a well known elementary volume argument (see e.g. [19, p. 57]). □

Remark 2.9. — To derive Proposition 2.1 from the preceding statement, consider, in the situation of Proposition 2.1, a finite set $\{\sigma_t \mid t \in T\}$ of distinct finite dimensional irreducible representations of G, let π be their direct sum and let $\rho = \pi \otimes \bar{\pi}$. By the assumption in Proposition 2.1, we know $\varepsilon(\rho, S) \geqslant \varepsilon$, and hence (2.7) implies $|T| \leqslant \exp c_\varepsilon' n N^2$. Applying this to $\pi = \lambda_G$, this shows that Proposition 2.8 contains Proposition 2.1.

For any finite dimensional unitary representation $\pi : G \to B(H)$ on an arbitrary group, let us denote by $r'_N(\pi)$ the number of distinct irreducible representations appearing in the decomposition of π of dimension at most N. Let then

$$R'_{n,\varepsilon}(N) = \sup r'_N(\pi)$$

where the sup runs over all π's and G's admitting an n-element generating set $S \subset G$ such that

$$\varepsilon(\pi \otimes \bar{\pi}, S) \geqslant \varepsilon.$$

Note that $r'_N(\lambda_G) = r_N(G)$ and hence

$$R_{n,\varepsilon}(N) \leqslant R'_{n,\varepsilon}(N).$$

With this notation (2.7) means that

$$R'_{n,\varepsilon}(N) \leqslant \exp c'_\varepsilon nN^2.$$

While it seems very difficult to give a good lower bound for $R_{n,\varepsilon}(N)$, we can answer the analogous question for $R'_{n,\varepsilon}(N)$: Indeed, the main result of [20] (see [20, Th. 1.3]), which follows, implies the desired lower bound when reformulated in terms of representations.

THEOREM 2.10 ([20]). — *For each $0 < \varepsilon < 1$, there is a constant $\beta_\varepsilon > 0$ such that and for all sufficiently large integer n (i.e. $n \geqslant n_0$ with n_0 depending on ε) and for all $N \geqslant 1$, there is a subset $\mathcal{T} \subset U(N)^n$ with*

$$|\mathcal{T}| \geqslant \exp \beta_\varepsilon nN^2$$

such that

$$\forall u \neq v \in \mathcal{T} \quad \|\sum_1^n u_j \otimes \overline{v_j}\| \leqslant n(1-\varepsilon) \quad \text{(we call these "ε-separated")},$$

and $\varepsilon(u) \geqslant \varepsilon$ for all $u \in \mathcal{T}$ (we call these "ε-quantum expanders"). More precisely, for all $u \in \mathcal{T}$ we have

$$\|(\sum u_j \otimes \overline{u_j})_{|I^\perp}\| \leqslant n(1-\varepsilon).$$

THEOREM 2.11. — *The estimate in Proposition 2.8 is best possible in the sense that for any $0 < \varepsilon < 1$ there is a constant $\beta_\varepsilon > 0$ such that for any n large enough (i.e. $n \geqslant n_0(\varepsilon)$), for any $N \geqslant 1$ there is a group G and a finite dimensional representation π on G satisfying (2.6) and admitting a decomposition $\pi = \oplus_{t \in T} \pi_t$, with distinct irreducibles π_t each with multiplicity 1 (or any specified value $\geqslant 1$) and acting on an N-dimensional space, with*

$$|T| \geqslant \exp \beta_\varepsilon nN^2.$$

Proof. — Fix $N > 1$. Let $T \subset U(N)^n$ be the subset appearing in Theorem 2.10, i.e. T is such that $|T| \geqslant \exp \beta_\varepsilon nN^2$ and $\forall t \neq r \in T$ we have

$$\|\sum t_j \otimes \bar{r}_j\| \leqslant n(1-\varepsilon), \tag{2.9}$$

and also
$$\|(\sum t_j \otimes \bar{t}_j)_{|I^\perp}\| \leqslant n(1-\varepsilon). \qquad (2.10)$$

Let $s_j = \oplus_{t \in T} t_j \in U(m)$ with $m = |T|N$, and let $G \subset U(m)$ be the subgroup generated by $S = \{s_1, \cdots, s_n\}$. Note that $\pi(G) \subset \oplus_{t \in T} M_N$. Let $\pi : G \to U(m)$ be the inclusion map viewed as a representation on G. Let $P_t : \oplus_{t \in T} M_N \to M_N$ be the $*$-homomorphism corresponding to the projection onto the coordinate of index t. For any $t \in T$, let $\pi_t : G \to U(N)$ be the representation defined by $\pi_t = P_t(\pi)$. Then, by definition, we have $\pi = \oplus_{t \in T} \pi_t$. By the spectral gap condition (2.10) the commutant of $\pi_t(S)$ (which is but the commutant of $\{t_1, \cdots, t_n\}$) is reduced to the scalars, so π_t is irreducible, and by (2.9) for any $t \neq r \in T$ the representations π_t and π_r are not unitarily equivalent. $\qquad \square$

Remark 2.12. — In particular, this means that $\forall n \geqslant n_0(\varepsilon)$ and $\forall N$
$$R'_{n,\varepsilon}(N) \geqslant \exp \beta_\varepsilon n N^2.$$

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