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Equivariant triple intersections [(*)]

Delphine Moussard [(1)]

Abstract. — Given a null-homologous knot K in a rational homology 3-sphere M, and the standard infinite cyclic covering $\widetilde{X}$ of (M,K), we define an invariant of triples of curves in $\widetilde{X}$ by means of equivariant triple intersections of surfaces. We prove that this invariant provides a map ϕ on $\mathfrak{A}^{\otimes 3}$, where $\mathfrak{A}$ is the Alexander module of (M,K), and that the isomorphism class of ϕ is an invariant of the pair (M,K). For a fixed Blanchfield module $(\mathfrak{A},\mathfrak{b})$, we consider pairs (M,K) whose Blanchfield modules are isomorphic to $(\mathfrak{A},\mathfrak{b})$ equipped with a marking, i.e. a fixed isomorphism from $(\mathfrak{A},\mathfrak{b})$ to the Blanchfield module of (M,K). In this setting, we compute the variation of ϕ under null Borromean surgeries and we describe the set of all maps ϕ. Finally, we prove that the map ϕ is a finite type invariant of degree 1 of marked pairs (M,K) with respect to null Lagrangian-preserving surgeries, and we determine the space of all degree 1 invariants with rational values of marked pairs (M,K).

Résumé. — Étant donné un nœud K dans une sphère d'homologie rationnelle M, et le revêtement infini cyclique standard $\widetilde{X}$ de (M,K), on définit un invariant des triplets de courbes dans $\widetilde{X}$, *via* des intersections triples équivariantes de surfaces. On montre que cet invariant fournit une application ϕ sur $\mathfrak{A}^{\otimes 3}$, où $\mathfrak{A}$ est le module d'Alexander de (M,K), et que la classe d'isomorphisme de ϕ est un invariant de la paire (M,K). Pour un module de Blanchfield $(\mathfrak{A},\mathfrak{b})$ fixé, on considère les paires (M,K) dont le module de Blanchfield est isomorphe à $(\mathfrak{A},\mathfrak{b})$, équippées d'un marquage, c'est-à-dire d'un isomorphisme fixé de $(\mathfrak{A},\mathfrak{b})$ vers le module de Blanchfield de (M,K). Dans ce cadre, on calcule la variation de ϕ sous l'effet d'une chirurgie borroméenne nulle, et on décrit l'ensemble de toutes les applications ϕ. Enfin, on montre que l'application ϕ est un invariant de type

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fini de degré 1 des paires marquées (M, K) par rapport aux chirurgies LP nulles, et on détermine l'espace de tous les invariants de degré 1 à valeurs rationnelles des paires marquées (M, K).

1. Introduction

In [5], Garoufalidis and Rozansky introduced a theory of finite type invariants of knots in integral homology spheres with respect to the null-move (the move which defines the Goussarov–Habiro theory of finite type invariants of 3-manifolds), with a nullity condition with respect to the knot. In particular, they proved that the Kricker lift of the Kontsevich integral constructed by Kricker [6] (see also [4]) is a universal finite type invariant of knots in integral homology spheres with trivial Alexander polynomial. In [11], we extended this result to finite type invariants of null-homologous knots in rational homology spheres, with respect to a move called null Lagrangian-preserving surgery (which generalizes the null-move to the setting of rational homology), in the case of a trivial Alexander polynomial. We also studied the case of a non-trivial Alexander polynomial. The study of these theories of finite type invariants gives tools to understand the Kricker lift of the Kontsevich integral and to compare it with other powerful invariants as the one constructed by Lescop [8] by means of equivariant intersections in configuration spaces.

In this paper, we construct and study an invariant of null-homologous knots in rational homology spheres, which appears to have finiteness properties with respect to null Lagrangian-preserving surgeries when a parametrization of the Alexander module (a marking) is fixed. Such a marking is preserved by null Lagrangian-preserving surgeries, hence the theory of finite type invariants can be defined for null-homologous knots in rational homology spheres with a fixed marking and it provides a richer and more faithful theory.

The Kricker invariant organizes the Kontsevich integral into a series of terms ordered by their loop degree given by the first Betti number of the graphs. As proved by Garoufalidis and Rozansky [5, Cor. 1.5], the n-loop part of this invariant is a finite type invariant of degree $2n-2$ with respect to the null-move. The invariant constructed in this paper takes place in some sense beetween the 1-loop part (explicitly given by the Alexander polynomial [6, Thm. 1.0.8]) and the 2-loop part (which coincides with the triple equivariant intersection of Lescop [7] at least for knots in integral homology spheres with trivial Alexander polynomial) of the Kricker invariant, but it exists as a finite type invariant only when a marking of the Alexander module is fixed.

Description of the paper

We consider pairs (M,K), where M is a rational homology 3-sphere and K a null-homologous knot in M. We define an invariant of triples of curves in the associated infinite cyclic covering by means of equivariant triple intersection numbers of surfaces. It provides a map ϕ on $\mathfrak{A}_h = \frac{\mathfrak{A}^{\otimes 3}}{(\otimes_{1\leqslant j\leqslant 3}\beta_j = \otimes_{1\leqslant j\leqslant 3}t\beta_j)}$, where $\mathfrak{A}$ is the Alexander module of (M,K). The isomorphism class of $(\mathfrak{A},\phi)$ is an invariant of the homeomorphism class of (M,K).

Then for a fixed Blanchfield module $(\mathfrak{A},\mathfrak{b})$, i.e. an Alexander module endowed with a Blanchfield form, we consider marked pairs (M,K,ξ), where ξ is an isomorphism from $(\mathfrak{A},\mathfrak{b})$ to the Blanchfield module of (M,K). For such marked pairs, the map ϕ is well-defined, not only up to isomorphism. In this setting, we compute the variation of ϕ under the null-move of Garoufalidis and Rozansky [5], called here null Borromean surgery. As a consequence, we see that the equivariant triple intersection map ϕ is a finite type invariant of degree one of the marked pairs (M,K,ξ) with respect to null Borromean surgeries.

For a fixed Blanchfield module $(\mathfrak{A},\mathfrak{b})$, we identify the rational vector space of all equivariant triple intersection maps ϕ of marked pairs (M,K,ξ) with the space $\mathcal{H} = \frac{\Lambda^3_{\mathbb{Q}}\mathfrak{A}}{(\beta_1\wedge\beta_2\wedge\beta_3 = t\beta_1\wedge t\beta_2\wedge t\beta_3)}$. We study the vector space $\mathcal{H}$ and give bounds for its dimension.

In the last section, we consider null Lagrangian-preserving surgeries, a move which includes the null-move of Garoufalidis and Rozansky and which is transitive on the set of marked pairs (M,K,ξ) for a fixed Blanchfield module. We show that the map ϕ is a finite type invariant of degree one of the marked pairs (M,K,ξ) with respect to null Lagrangian-preserving surgeries. We prove that the map ϕ, together with degree one invariants obtained from the cardinality of $H_1(M;\mathbb{Z})$, provides a universal rational valued degree one invariant of the marked pairs (M,K,ξ) with respect to null Lagrangian-preserving surgeries. We obtain similar results in the case of pairs (M,K,ξ) where M is an integral homology 3-sphere and the marking ξ is defined on the integral Blanchfield module. This part builds upon earlier work by the author in [11, Chap. 6] and [12].

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Conventions and definitions

For $n \in \mathbb{N} \setminus \{0\}$, S^n is the standard n-dimensional sphere.

A $\mathbb{Q}$HS is a *rational homology 3-sphere*, i.e. an oriented compact 3-manifold which has the same homology with rational coefficients as the standard 3-sphere S^3. A *null-homologous knot* in a 3-manifold M is a knot whose class in $H_1(M;\mathbb{Z})$ is trivial. A $\mathbb{Q}$*SK-pair* is a pair (M,K) where M is a $\mathbb{Q}$HS and K is a null-homologous knot in M. Two $\mathbb{Q}$SK-pairs (M,K) and (M',K') are homeomorphic if there is a homeomorphism $h: M \xrightarrow{\cong} M'$ such that $h(K)=K'$.

The *standard genus g handlebody* is the 3-manifold with boundary obtained by adding g 1-handles to a 3-ball.

The boundary of an oriented manifold with boundary is oriented with the “outward normal first” convention. We also use this convention to define the co-orientation of an oriented manifold embedded in another oriented manifold. If U and V are submanifolds of a manifold M, define the orientation of the intersection $U \cap V$ in the following way: an oriented basis of the normal vector space $N_x(U\cap V)$ at a point x can be obtained by taking an oriented basis of $N_x(U)$ followed by an oriented basis of $N_x(V)$. Given an oriented manifold M, we denote by $-M$ the same manifold with opposite orientation.

The homology class of a curve γ in a manifold is denoted by $[\gamma]$.

If $C_1,\dots,C_k$ are transverse integral chains in a manifold M, such that the sum of the codimensions of the C_i equals the dimension of M, $\langle C_1,\dots,C_k\rangle_M$ is the algebraic intersection number of the C_i in M.

For chains C_1 and C_2 in a manifold M, the transversality condition includes $\partial C_1 \cap \partial C_2 = \emptyset$.

Unless otherwise mentioned, all tensor products and exterior products are defined over $\mathbb{Q}$.

2. Statement of the main results

2.1. Equivariant triple intersections

We first recall the definition of the Alexander module. Let (M,K) be a $\mathbb{Q}$SK-pair. Let $T(K)$ be a tubular neighborhood of K. The *exterior* of K is $X = M \setminus Int(T(K))$. Consider the projection $\pi : \pi_1(X) \to \frac{H_1(X;\mathbb{Z})}{torsion} \cong \mathbb{Z}$

and the covering map $p : \widetilde{X} \to X$ associated with its kernel. Then $\widetilde{X}$ is the *infinite cyclic covering* of X. The automorphism group of the covering, $Aut(\widetilde{X})$, is isomorphic to $\mathbb{Z}$. It acts on $H_1(\widetilde{X};\mathbb{Q})$. Denoting the action of a generator τ of $Aut(\widetilde{X})$ as the multiplication by t, we get a structure of $\mathbb{Q}[t^{\pm 1}]$-module on $\mathfrak{A}(M,K) = H_1(\widetilde{X};\mathbb{Q})$.

Definition 2.1. — *The* $\mathbb{Q}[t^{\pm 1}]$*-module* $\mathfrak{A}(M,K)$ *is the* Alexander module *of* (M,K)*.*

The module $\mathfrak{A}(M,K)$ is a finitely generated torsion $\mathbb{Q}[t^{\pm 1}]$-module [13, Prop. 1.2]. Since $\mathbb{Q}[t^{\pm 1}]$ is a principal ideal domain, $\mathfrak{A}(M,K)$ has an annihilator well-defined up to a unit of $\mathbb{Q}[t^{\pm 1}]$. We denote by $\delta_{(M,K)}(t)$ this annihilator normalized so that $\delta_{(M,K)}(t) \in \mathbb{Q}[t]$, $\delta_{(M,K)}(0) \neq 0$ and $\delta_{(M,K)}(1) = 1$. By a slight abuse of notation, for any $\mathbb{Q}$SK-pair, we denote by τ the automorphism of the infinite cyclic covering which induces the multiplication by t in the Alexander module, and for a polynomial $P = \sum_{k\in\mathbb{Z}} a_k t^k \in \mathbb{Q}[t^{\pm 1}]$ and a chain C in the infinite cyclic covering, we denote by $P(\tau)C$ the chain $\sum_{k\in\mathbb{Z}} a_k \tau^k(C)$.

We aim at defining an equivariant triple intersection map on the rational vector space $\mathfrak{A}(M,K)^{\otimes 3}$. We first define equivariant triple intersection numbers.

Definition 2.2. — *Let* C_1*,* C_2*,* C_3 *be integral chains in* $\widetilde{X}$ *such that* $\sum_{1\leqslant j\leqslant 3} \mathrm{codim}(C_j) = 3$*. Assume that* C_1*,* C_2*,* C_3 *are* τ-transverse *in* $\widetilde{X}$*, i.e.* $\tau^{k_1}C_1$*,* $\tau^{k_2}C_2$*, and* $\tau^{k_3}C_3$ *are transverse for all integers* k_1*,* k_2*,* k_3*. The* equivariant triple intersection number *of* C_1*,* C_2 *and* C_3 *is*

$$\langle C_1, C_2, C_3\rangle_e = \sum_{k_2\in\mathbb{Z}}\sum_{k_3\in\mathbb{Z}} \langle C_1, \tau^{-k_2}C_2, \tau^{-k_3}C_3\rangle t_2^{k_2} t_3^{k_3} \quad \in \frac{\mathcal{R}}{(t_1t_2t_3-1)},$$

where $\mathcal{R} = \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]$*. We extend it to rational chains by multilinearity.*

Note that the finiteness of the sum follows from the compactness of the support of integral chains.

Remark. — The quotient by the relation $t_1t_2t_3 = 1$ is not necessary for the definition, but it makes it relevant since it ensures the properties of Lemma 2.3.

We have the following easy *formulae.*

Lemma 2.3. — *The equivariant triple intersection number satisfies:*

- *if* $\mathrm{codim}(C_j) = 1$ *for all* j*, then for any permutation* $\sigma \in \mathcal{S}_3$*, with signature* $\varepsilon(\sigma)$*,* $\langle C_{\sigma(1)}, C_{\sigma(2)}, C_{\sigma(3)}\rangle_e(t_1,t_2,t_3) = \varepsilon(\sigma)\langle C_1,C_2,C_3\rangle_e(t_{\sigma^{-1}(1)}, t_{\sigma^{-1}(2)}, t_{\sigma^{-1}(3)})$*,*

- $\langle P_1(\tau)C_1, P_2(\tau)C_2, P_3(\tau)C_3\rangle_e = P_1(t_1)P_2(t_2)P_3(t_3)\langle C_1, C_2, C_3\rangle_e$, *for all* $P_j \in \mathbb{Q}[t^{\pm 1}]$.

Remark. — The second *formula* gives sense to the term "equivariant" with the equality $\langle \tau C_1, \tau C_2, \tau C_3\rangle_e = t_1t_2t_3\langle C_1, C_2, C_3\rangle_e$, meaning that if the three chains C_i are simultaneously applied the same automorphism of the covering, their equivariant triple intersection number is preserved.

In Section 3, we prove:

Lemma 2.4. — *Let* (M,K) *be a* $\mathbb{Q}$*SK-pair. Let* $\widetilde{X}$ *be the infinite cyclic covering associated with* (M,K)*. Let* β_1*,* β_2*,* β_3 *be elements of* $\mathfrak{A}(M,K)$ *which can be represented by knots in* $\widetilde{X}$*. Let* μ_1*,* μ_2*,* μ_3 *be representatives of the* β_j *whose images in* M *are pairwise disjoint. For* $j = 1,2,3$*, let* $P_j \in \mathbb{Q}[t^{\pm 1}]$ *satisfy* $[P_j(\tau)\mu_j] = 0$ *in* $\mathfrak{A}(M,K)$*. Let* Σ_1*,* Σ_2*,* Σ_3 *be* τ*-transverse rational 2-chains such that* $\partial\Sigma_j = P_j(\tau)\mu_j$*. Then*

$$\langle \Sigma_1, \Sigma_2, \Sigma_3\rangle_e \in \frac{\mathcal{R}}{(t_1t_2t_3 - 1, P_1(t_1), P_2(t_2), P_3(t_3))}$$

does not depend on the choice of the surfaces Σ_j *and of the representatives* μ_j*.*

Let (M,K) be a $\mathbb{Q}$SK-pair. Set

$$\mathfrak{A}_h(M,K) = \frac{\mathfrak{A}(M,K)^{\otimes 3}}{(\beta_1\otimes\beta_2\otimes\beta_3 = t\beta_1\otimes t\beta_2\otimes t\beta_3)}.$$

Set $\mathcal{R}_\delta = \dfrac{\mathcal{R}}{(t_1t_2t_3 - 1, \delta(t_1), \delta(t_2), \delta(t_3))}$, where $\delta(t) = \delta_{(M,K)}(t)$ is the annihilator of $\mathfrak{A}(M,K)$. Define a structure of $\mathcal{R}_\delta$-module on $\mathfrak{A}_h(M,K)$ by

$$t_1^{k_1}t_2^{k_2}t_3^{k_3}.\beta_1\otimes\beta_2\otimes\beta_3 = t^{k_1}\beta_1\otimes t^{k_2}\beta_2\otimes t^{k_3}\beta_3.$$

Lemmas 2.3 and 2.4 imply:

Theorem 2.5. — *Let* $\widetilde{X}$ *be the infinite cyclic covering associated with* (M,K)*. Define a* $\mathbb{Q}$*-linear map* $\phi^{(M,K)} : \mathfrak{A}_h(M,K) \to \mathcal{R}_\delta$ *as follows. If* μ_1*,* μ_2*,* μ_3 *are knots in* $\widetilde{X}$ *whose images in* $M\setminus K$ *are pairwise disjoint, let* Σ_1*,* Σ_2*,* Σ_3 *be* τ*-transverse rational 2-chains such that* $\partial\Sigma_j = \delta(\tau)\mu_j$*, and set*

$$\phi^{(M,K)}([\mu_1]\otimes[\mu_2]\otimes[\mu_3]) = \langle \Sigma_1, \Sigma_2, \Sigma_3\rangle_e.$$

Then the map $\phi^{(M,K)}$ *is well-defined,* $\mathcal{R}_\delta$*-linear, and satisfies*

$$\phi^{(M,K)}(\otimes_{1\leqslant j\leqslant 3}\beta_{\sigma(j)})(t_1,t_2,t_3) = \varepsilon(\sigma)\phi^{(M,K)}(\otimes_{1\leqslant j\leqslant 3}\beta_j)(t_{\sigma^{-1}(1)}, t_{\sigma^{-1}(2)}, t_{\sigma^{-1}(3)}) \quad (\star)$$

for all permutations $\sigma\in\mathcal{S}_3$ *with signature* $\varepsilon(\sigma)$ *and all* $(\beta_1,\beta_2,\beta_3)\in\mathfrak{A}(M,K)^3$*. The isomorphism class of* $(\mathfrak{A}(M,K), \phi^{(M,K)})$ *is an invariant of the homeomorphism class of* (M,K)*.*

Let us precise that the isomorphism class of $(\mathfrak{A}(M,K),\phi^{(M,K)})$ is the set of all pairs $(\mathfrak{A},\phi)$ where $\mathfrak{A}$ is a $\mathbb{Q}[t^{\pm1}]$-module, $\phi : \frac{\mathfrak{A}^{\otimes 3}}{(\otimes_{1\leqslant j\leqslant 3}\beta_j = \otimes_{1\leqslant j\leqslant 3}t\beta_j)} \to \mathcal{R}_\delta$ is a $\mathbb{Q}$-linear map, and there is an isomorphism $\xi : \mathfrak{A} \xrightarrow{\cong} \mathfrak{A}(M,K)$ such that $\phi^{(M,K)}(\otimes_{1\leqslant j\leqslant 3}\xi(\beta_j)) = \phi(\otimes_{1\leqslant j\leqslant 3}\beta_j)$ for all β_1, β_2 and β_3 in $\mathfrak{A}$.

Remark. — So far, we do not need the condition that K is null-homologous. Indeed, we do not even need to work in the exterior of a knot. Given an oriented 3-manifold equipped with an infinite cyclic covering $\widetilde{X}$, one can make the same construction on the torsion submodule of $H_1(\widetilde{X};\mathbb{Q})$, provided that $H_2(\widetilde{X};\mathbb{Q}) = 0$ (necessary in the proof of Lemma 2.4). In this case, the variation under null Borromean surgeries can also be computed as in Section 4.

2.2. Variation under null Borromean surgeries

In order to define marked $\mathbb{Q}$SK-pairs, we recall the definition of the Blanchfield form introduced by Blanchfield in [2].

On an Alexander module $\mathfrak{A}(M,K)$, one can define the *Blanchfield form*, or *equivariant linking pairing*, $\mathfrak{b}^{(M,K)} : \mathfrak{A}(M,K)\times\mathfrak{A}(M,K) \to \frac{\mathbb{Q}(t)}{\mathbb{Q}[t^{\pm1}]}$, as follows. First define the equivariant linking number of two knots.

Definition 2.6. — *Let (M,K) be a $\mathbb{Q}$SK-pair. Let $\widetilde{X}$ be the associated infinite cyclic covering. Let μ_1 and μ_2 be two knots in $\widetilde{X}$ such that $\mu_1 \cap \tau^k(\mu_2) = \emptyset$ for all $k\in\mathbb{Z}$. Let $P\in\mathbb{Q}[t^{\pm1}]$ satisfy $P(\tau)\mu_1 = \partial S$, where S is an integral 2-chain in $\widetilde{X}$. The* equivariant linking number *of μ_1 and μ_2 is*

$$lk_e(\mu_1,\mu_2) = \frac{1}{P(t)}\sum_{k\in\mathbb{Z}}\langle S,\tau^k(\mu_2)\rangle t^k \quad \in \mathbb{Q}(t).$$

Note that the polynomial P can always be chosen to be a scalar multiple of $\delta_{(M,K)}$. One can easily check that the equivariant linking number is well-defined (independant of the choice of P) and satisfies $lk_e(\mu_1,\mu_2)\in \frac{1}{\delta(t)}\mathbb{Q}[t^{\pm1}]$, $lk_e(\mu_2,\mu_1)(t) = lk_e(\mu_1,\mu_2)(t^{-1})$, and $lk_e(P(\tau)\mu_1,Q(\tau)\mu_2)(t) = P(t)Q(t^{-1})lk_e(\mu_1,\mu_2)(t)$. Now, if β_1 (resp. β_2) is the homology class of μ_1 (resp. μ_2) in $\mathfrak{A}(M,K)$, define $\mathfrak{b}^{(M,K)}(\beta_1,\beta_2)$ by

$$\mathfrak{b}^{(M,K)}(\beta_1,\beta_2) = lk_e(\mu_1,\mu_2) \ mod \ \mathbb{Q}[t^{\pm1}].$$

The Blanchfield form is *hermitian*:

$$\mathfrak{b}^{(M,K)}(\beta_1,\beta_2)(t) = \mathfrak{b}^{(M,K)}(\beta_2,\beta_1)(t^{-1})$$

and

$$\mathfrak{b}^{(M,K)}(P(t)\beta_1, Q(t)\beta_2)(t) = P(t)Q(t^{-1})\, \mathfrak{b}^{(M,K)}(\beta_1,\beta_2)(t)$$

for all $\beta_1, \beta_2 \in \mathfrak{A}(M,K)$ and all $P, Q \in \mathbb{Q}[t^{\pm 1}]$. Moreover, as proved by Blanchfield in [2], it is *non degenerate*: $\mathfrak{b}^{(M,K)}(\beta_1,\beta_2) = 0$ for all $\beta_2 \in \mathfrak{A}(M,K)$ implies $\beta_1 = 0$.

Remark. — The definition and the properties of the triple intersection form are close to those of the Blanchfield form. In particular, the target space of the Blanchfield form can be understood as $\mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}]/(t_1t_2 - 1)$ with one variable corresponding to each argument. The triple intersection form can be viewed as a generalization of the Blanchfield from in the setting of three arguments.

Definition 2.7. — *The* Blanchfield module *of a* $\mathbb{Q}$*SK-pair* (M,K) *is the Alexander module* $\mathfrak{A}(M,K)$ *endowed with the Blanchfield form* $\mathfrak{b}^{(M,K)}$.

Fix an abstract Blanchfield module $(\mathfrak{A},\mathfrak{b})$, i.e. a finitely generated torsion $\mathbb{Q}[t^{\pm 1}]$-module $\mathfrak{A}$ on which the map $x \mapsto (1-t)x$ is an isomorphism and endowed with a non-degenerate hermitian form $\mathfrak{b}$ valued in $\mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}]$ (see [13, Prop. 1.2 and Thm. 1.4]). If ξ is a fixed isomorphism from $(\mathfrak{A},\mathfrak{b})$ to the Blanchfield module of a $\mathbb{Q}$SK-pair (M,K), i.e.

$$\mathfrak{A} \xrightarrow[\cong]{\xi} \mathfrak{A}(M,K) \qquad \text{and} \qquad \mathfrak{b}^{(M,K)}(\xi(x),\xi(y)) = \mathfrak{b}(x,y) \quad \text{for all } x,y \in \mathfrak{A},$$

then (M,K,ξ) is an $(\mathfrak{A},\mathfrak{b})$-*marked* $\mathbb{Q}$*SK-pair*. Let $\mathcal{P}^m(\mathfrak{A},\mathfrak{b})$ be the set of all such $(\mathfrak{A},\mathfrak{b})$-marked $\mathbb{Q}$SK-pairs up to orientation-preserving and marking-preserving homeomorphism. When it does not seem to cause confusion, the image of an element $\beta \in \mathfrak{A}$ by a marking ξ is still denoted by β, and an $(\mathfrak{A},\mathfrak{b})$-marked $\mathbb{Q}$SK-pair is called a marked $\mathbb{Q}$SK-pair. Note that the infinite cyclic covering $\widetilde{X}$ associated with a $\mathbb{Q}$SK-pair (M,K) is well-defined only up to the automorphisms of the covering, which are the τ^k. Hence a marking ξ of (M,K) is defined up to multiplication by a power of t.

For a marked $\mathbb{Q}$SK-pair (M,K,ξ), the equivariant triple intersection map introduced in Theorem 2.5 is well-defined on $\mathfrak{A}_h = \dfrac{\mathfrak{A}\otimes\mathfrak{A}\otimes\mathfrak{A}}{(\otimes_{1\leqslant j\leqslant 3}\beta_j = \otimes_{1\leqslant j\leqslant 3} t\beta_j)}$, not only up to isomorphism, and we denote it by $\phi^{(M,K,\xi)}$. We aim at studying the variation of the map $\phi^{(M,K,\xi)}$ under null Borromean surgeries, that we now define.

The *standard Y-graph* is the graph $\Gamma_0 \subset \mathbb{R}^2$ represented in Figure 2.1. The looped edges of Γ_0 are called *leaves*. The vertex incident to three different edges is the *internal vertex*. With Γ_0 is associated a regular neighborhood $\Sigma(\Gamma_0)$ of Γ_0 in the plane. The surface $\Sigma(\Gamma_0)$ is oriented with the usual convention. This induces an orientation of the leaves and an *orientation of the*

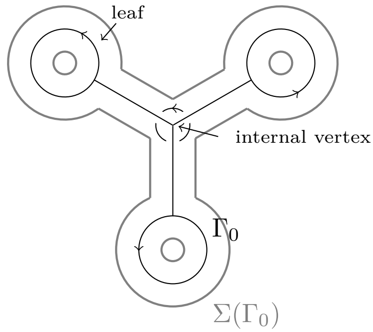


Figure 2.1. The standard Y-graph

internal vertex, i.e. a cyclic order of the three edges which meet at this vertex. Let M be a 3-manifold and let $h : \Sigma(\Gamma_0) \to M$ be an embedding. The image Γ of Γ_0 is a *Y-graph* endowed with its *associated surface* $\Sigma(\Gamma) = h(\Sigma(\Gamma_0))$. The Y-graph Γ is equipped with the framing induced by $\Sigma(\Gamma)$.

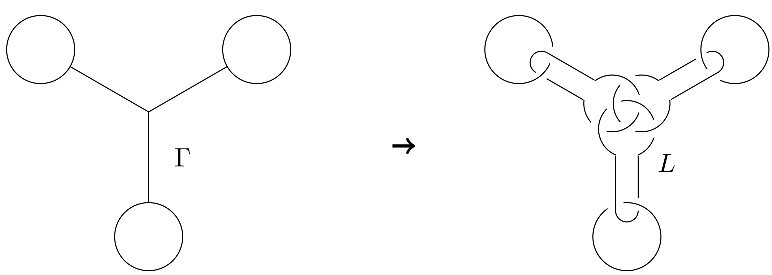


Figure 2.2. Y-graph and associated surgery link

Let Γ be a Y-graph in a 3-manifold M. Let $\Sigma(\Gamma)$ be its associated surface. In $\Sigma(\Gamma) \times [-1, 1]$, associate with Γ the six-component link L represented in Figure 2.2 with the blackboard framing. The *Borromean surgery on* Γ is the usual surgery along the framed link L (the usual surgery replaces an open tubular neighborhood of each component of the link by another solid torus in such a way that the meridian of the reglued torus is identified with the prefered parallel of the initial component). The manifold obtained from M by surgery on Γ is denoted by $M(\Gamma)$.

Let $(M, K, \xi) \in \mathcal{P}^m(\mathfrak{A}, \mathfrak{b})$. Let Γ be a Y-graph in $M \setminus K$. If the map $i_* : H_1(\Gamma; \mathbb{Q}) \to H_1(M \setminus K)$ induced by the inclusion has a trivial image, then Γ is *null in* $M \setminus K$ and the surgery on Γ is a *null Borromean surgery* (null-move in [5]). In this case, the pair $(M,K)(\Gamma)$ obtained from (M,K) by surgery on Γ is again a $\mathbb{Q}$SK-pair. The surgery on Γ induces a canonical isomorphism between the Blanchfield modules of (M,K) and $(M,K)(\Gamma)$ (this is stated in [14, Lem. 2.1] for a more general move called null LP-surgery, see Subsection 2.4). Hence we can define the marked $\mathbb{Q}$SK-pair $(M,K,\xi)(\Gamma)$ obtained from (M,K,ξ) by surgery on Γ.

In Section 4, we prove:

Proposition 2.8. — *Let $(M,K,\xi)\in\mathcal{P}^m(\mathfrak{A},\mathfrak{b})$. Let Γ be a Y-graph, null in $M\setminus K$. Let $\widetilde{\Gamma}$ be a lift of Γ in the infinite cyclic covering $\widetilde{X}$ associated with (M,K). Let γ_1, γ_2, γ_3 be the leaves of $\widetilde{\Gamma}$ in $\mathfrak{A}$ given in an order induced by the orientation of the internal vertex of Γ. For β_1, β_2, β_3 in $\mathfrak{A}$, we have*

$$\phi^{(M,K,\xi)(\Gamma)}(\beta_1\otimes\beta_2\otimes\beta_3)-\phi^{(M,K,\xi)}(\beta_1\otimes\beta_2\otimes\beta_3)$$
$$=\sum_{\sigma\in\mathcal{S}_3}\varepsilon(\sigma)\prod_{j=1}^{3}\delta(t_j)\mathfrak{b}(\beta_j,[\gamma_{\sigma(j)}])(t_j).$$

The following corollary says that the triple intersection map is a degree one invariant of $(\mathfrak{A},\mathfrak{b})$-marked $\mathbb{Q}$SK-pairs with respect to null Borromean surgeries.

Corollary 2.9. — *Let $(M,K,\xi)\in\mathcal{P}^m(\mathfrak{A},\mathfrak{b})$. Let Γ_1 and Γ_2 be disjoint Y-graphs, null in $M\setminus K$. Then the map $\phi^{(M,K,\xi)}-\phi^{(M,K,\xi)(\Gamma_1)}-\phi^{(M,K,\xi)(\Gamma_2)}+\phi^{(M,K,\xi)(\Gamma_1)(\Gamma_2)}$ vanishes on $\mathfrak{A}_h$.*

Proof. — Since the Blanchfield form is preserved by null Borromean surgeries, it follows from Proposition 2.8 that the difference $\phi^{(M,K,\xi)}-\phi^{(M,K,\xi)(\Gamma_1)}$ is not changed when performing the surgery on Γ_2. □

Proposition 2.8 will allow us to give a description of the space of all equivariant triple intersection maps. More precisely, let $\mathbf{\Phi}$ be the rational vector space of all morphisms of $\mathcal{R}_\delta$-modules $\phi:\mathfrak{A}_h\to\mathcal{R}_\delta$ which satisfy the relation $(\star)$ of Theorem 2.5. In Section 6, we prove:

Theorem 2.10. — *Define $\phi^\bullet:\mathcal{P}^m(\mathfrak{A},\mathfrak{b})\to\mathbf{\Phi}$ by $\phi^\bullet(M,K,\xi)=\phi^{(M,K,\xi)}$. Then the rational vector space $\phi^\bullet(\mathcal{P}^m(\mathfrak{A},\mathfrak{b}))$ is isomorphic to*

$$\mathcal{H}=\frac{\Lambda^3\mathfrak{A}}{(\beta_1\wedge\beta_2\wedge\beta_3=t\beta_1\wedge t\beta_2\wedge t\beta_3)}.$$

2.3. Structure of $\mathcal{H}$

Fix an abstract Blanchfield module $(\mathfrak{A}, \mathfrak{b})$. In Section 5, we study the structure of

$$\mathfrak{A}_h = \frac{\mathfrak{A}^{\otimes 3}}{(\beta_1 \otimes \beta_2 \otimes \beta_3 = t\beta_1 \otimes t\beta_2 \otimes t\beta_3)}$$

and

$$\mathcal{H} = \frac{\Lambda^3 \mathfrak{A}}{(\beta_1 \wedge \beta_2 \wedge \beta_3 = t\beta_1 \wedge t\beta_2 \wedge t\beta_3)}.$$

For this study, we consider a decomposition of $\mathfrak{A}$ as a direct sum of cyclic submodules and associated decompositions of $\mathfrak{A}_h$ and $\mathcal{H}$. In order to characterize the equivariant triple intersection maps in Section 6, we choose a decomposition adapted to the Blanchfield form.

By [13, Thm. 1.3], the $\mathbb{Q}[t^{\pm 1}]$-module $\mathfrak{A}$ is a direct sum, orthogonal with respect to $\mathfrak{b}$, of submodules of the following two kinds ($\pi \in \mathbb{Q}[t^{\pm 1}]$ is *symmetric* if $\pi(t^{-1}) = rt^k \pi(t)$ with $r \in \mathbb{Q}^*$ and $k \in \mathbb{Z}$):

- $\frac{\mathbb{Q}[t^{\pm 1}]}{(\pi^n(t))}\eta$ with π prime and symmetric or $\pi(t) = t + 2 + t^{-1}$, $n > 0$, and $\mathfrak{b}(\eta, \eta) = \frac{a}{\pi^n}$ where a is symmetric and prime to π;
- $\frac{\mathbb{Q}[t^{\pm 1}]}{(\pi^n(t))}\eta \oplus \frac{\mathbb{Q}[t^{\pm 1}]}{(\pi^n(t^{-1}))}\eta'$, with either π prime, non symmetric, $\pi(-1) \neq 0$, $n > 0$, or $\pi(t) = 1 + t$, n odd, and in both cases $\mathfrak{b}(\eta, \eta') = \frac{1}{\pi^n}$, $\mathfrak{b}(\eta, \eta) = \mathfrak{b}(\eta', \eta') = 0$.

Note that these submodules are all cyclic except in the second case when $\pi(t) = t + 1$. Define “Blanchfield duals” for the generators:

- in the first case, set $d(\eta) = \eta$,
- in the second case, set $d(\eta) = \eta'$, and $d(\eta') = \eta$.

Index all these generators to obtain a family $(\eta_i)_{1 \leqslant i \leqslant q}$ that generates $\mathfrak{A}$ over $\mathbb{Q}[t^{\pm 1}]$. We finally have a family $(\eta_i)_{1 \leqslant i \leqslant q}$ in $\mathfrak{A}$, an involution d of that family, and polynomials a_i, δ_i in $\mathbb{Q}[t^{\pm 1}]$, that satisfy:

- $\mathfrak{A} = \bigoplus_{i=1}^q \mathfrak{A}_i$, where $\mathfrak{A}_i = \frac{\mathbb{Q}[t^{\pm 1}]}{(\delta_i)}\eta_i$,
- each δ_i is a power of a prime polynomial,
- $\mathfrak{b}(\eta_i, d(\eta_j)) = 0$ if $i \neq j$,
- $\mathfrak{b}(\eta_i, d(\eta_i)) = \frac{a_i}{\delta_i}$, where a_i is prime to δ_i.

For technical simplicity, we denote by m_i the power that appears when we write δ_i as a power of a prime polynomial, and we require that $m_i \geqslant m_{i+1}$ for $1 \leqslant i < q$. Note that m_i is the multiplicity of any complex root of δ_i. Normalize the δ_i so that $\delta_i(t) \in \mathbb{Q}[t]$, $\delta_i(0) \neq 0$ and $\delta_i(1) = 1$.

The well-known result on the structure of finitely generated modules over a principal ideal domain implies that the family of the δ_i's is well-defined up to permutation. Hence if $\mathfrak{A} = \oplus_{1\leqslant i\leqslant q'}\mathfrak{A}'_i$ is another decomposition of $\mathfrak{A}$ satisfying the above conditions, then $q' = q$ and there is a permutation σ of $\{1,\dots,q\}$ such that $\mathfrak{A}'_i$ is isomorphic to $\mathfrak{A}_{\sigma(i)}$. But the decomposition $\mathfrak{A} = \bigoplus_{i=1}^q \mathfrak{A}_i$ is not unique. For instance, if $\mathfrak{A} = \frac{\mathbb{Q}[t^{\pm1}]}{(\delta)}\eta_1 \oplus \frac{\mathbb{Q}[t^{\pm1}]}{(\delta)}\eta_2$ with $\mathfrak{b}(\eta_1,\eta_1) = \mathfrak{b}(\eta_2,\eta_2)$ and $\mathfrak{b}(\eta_1,\eta_2) = 0$, then the decomposition $\mathfrak{A} = \frac{\mathbb{Q}[t^{\pm1}]}{(\delta)}(\eta_1+\eta_2) \oplus \frac{\mathbb{Q}[t^{\pm1}]}{(\delta)}(\eta_1-\eta_2)$ also satisfies the above conditions. When the $\mathfrak{A}_i$'s are fixed, it remains infinitely many possible choices for the generators η_i.

For $\underline{i} = (i_1,i_2,i_3) \in \{1,\dots,q\}^3$, set:

$$\mathfrak{A}(\underline{i}) = \frac{\mathfrak{A}_{i_1}\otimes\mathfrak{A}_{i_2}\otimes\mathfrak{A}_{i_3}}{(\otimes_{1\leqslant j\leqslant 3}\beta_j = \otimes_{1\leqslant j\leqslant 3}t\beta_j)}.$$

We have:

$$\mathfrak{A}_h = \bigoplus_{\underline{i}\in\{1,\dots,q\}^3} \mathfrak{A}(\underline{i}).$$

For $\underline{i} = (i_1,i_2,i_3) \in \{1,\dots,q\}^3$, let $\mathcal{H}(\underline{i})$ be the rational vector subspace of $\mathcal{H}$ generated by the $t^{k_1}\eta_{i_1}\wedge t^{k_2}\eta_{i_2}\wedge t^{k_3}\eta_{i_3}$ for all integers k_1, k_2, k_3. We have

$$\mathcal{H} = \bigoplus_{1\leqslant i_1\leqslant i_2\leqslant i_3\leqslant q} \mathcal{H}(\underline{i}).$$

In Section 5, we prove the following results and we further study the structure of the $\mathfrak{A}(\underline{i})$ and $\mathcal{H}(\underline{i})$ in order to bound their dimensions.

Theorem 2.11. — *Let $\underline{i} = (i_1,i_2,i_3) \in \{1,\dots,q\}^3$. The rational vector space $\mathfrak{A}(\underline{i})$ is non trivial if and only if there are complex roots z_1, z_2, z_3 of δ_{i_1}, δ_{i_2}, δ_{i_3} respectively such that $z_1z_2z_3 = 1$.*

Theorem 2.12. — *Let $\underline{i} = (i_1,i_2,i_3) \in \{1,\dots,q\}^3$. The rational vector space $\mathcal{H}(\underline{i})$ is non trivial if and only if there are complex roots z_1, z_2, z_3 of δ_{i_1}, δ_{i_2}, δ_{i_3} respectively which satisfy:*

- $z_1z_2z_3 = 1$,
- *for $1\leqslant j\leqslant 3$, the multiplicity m_{i_j} is at least the number of indices $l\in\{1,2,3\}$ such that $i_l = i_j$ and $z_l = z_j$.*

Example. — If all the roots of the Alexander polynomial are simple, and if the product of three of them is always different from 1, then $\mathcal{H} = 0$. It is the case, for instance, of the trefoil knot, and of the figure eight knot in S^3. We will study non trivial examples in Section 5.

2.4. Degree one invariants of marked $\mathbb{Q}$SK-pairs

In this subsection, we describe the finiteness and universality properties of the equivariant triple intersection map. Let us define Lagrangian-preserving surgeries.

Definition 2.13. — *For $g \in \mathbb{N}$, a* genus g rational homology handlebody *($\mathbb{Q}$HH) is a 3-manifold which is compact, oriented, and which has the same homology with rational coefficients as the standard genus g handlebody.*

Such a $\mathbb{Q}$HH is connected, and its boundary is necessarily a compact connected oriented surface of genus g.

Definition 2.14. — *The* Lagrangian $\mathcal{L}_A$ *of a $\mathbb{Q}$HH A is the kernel of the map*

$$i_* : H_1(\partial A;\mathbb{Q}) \to H_1(A;\mathbb{Q})$$

induced by the inclusion. Two $\mathbb{Q}$HH's A and B have LP-identified *boundaries if (A,B) is equipped with a homeomorphism $h : \partial A \to \partial B$ such that $h_*(\mathcal{L}_A) = \mathcal{L}_B$.*

The Lagrangian of a $\mathbb{Q}$HH A is indeed a Lagrangian subspace of $H_1(\partial A;\mathbb{Q})$ with respect to the intersection form.

Definition 2.15. — *Let M be a $\mathbb{Q}$HS, let $A \subset M$ be a $\mathbb{Q}$HH, and let B be a $\mathbb{Q}$HH whose boundary is LP-identified with ∂A. Set $M(\frac{B}{A}) = (M \setminus Int(A)) \cup_{\partial A =_h \partial B} B$. We say that the $\mathbb{Q}$HS $M(\frac{B}{A})$ is obtained from M by the* Lagrangian-preserving surgery *or* LP-surgery *$(\frac{B}{A})$.*

Given a $\mathbb{Q}$SK-pair (M,K), a $\mathbb{Q}$HH $A \subset M \setminus K$ is null *in $M \setminus K$ if the map $i_* : H_1(A;\mathbb{Q}) \to H_1(M \setminus K;\mathbb{Q})$ induced by the inclusion has a trivial image. A* null LP-surgery *on (M,K) is an LP-surgery $(\frac{B}{A})$ such that A is null in $M \setminus K$. The $\mathbb{Q}$SK-pair obtained by surgery is denoted by $(M,K)(\frac{B}{A})$.*

Since a null LP-surgery induces a canonical isomorphism beetween the Blanchfield modules of the involved pairs (see Theorem 2.17 below), this move is well-defined on marked $\mathbb{Q}$SK-pairs.

Notation 2.16. — The marked $\mathbb{Q}$SK-pair obtained from a marked $\mathbb{Q}$SK-pair (M,K,ξ) by a null LP-surgery $(\frac{B}{A})$ is denoted by $(M,K,\xi)(\frac{B}{A})$.

A Borromean surgery along a Y-graph Γ in a 3-manifold N can be realized by cutting a regular neighborhood of Γ in N (a standard genus 3 handlebody) and gluing another genus 3 handlebody instead, in a Lagrangian-preserving way (see [10]). Hence Borromean surgeries are a specific kind of LP-surgeries.

Let $\mathcal{F}_0^m$ be the rational vector space generated by all marked $\mathbb{Q}$SK-pairs up to orientation-preserving and marking-preserving homeomorphism. Let $\mathcal{F}_n^m$ denote the subspace of $\mathcal{F}_0^m$ generated by the

$$[(M,K,\xi);(\frac{B_i}{A_i})_{1\leqslant i\leqslant n}]=\sum_{I\subset\{1,\dots,n\}}(-1)^{|I|}(M,K,\xi)((\frac{B_i}{A_i})_{i\in I})$$

for all marked $\mathbb{Q}$SK-pairs (M,K,ξ) and all families of $\mathbb{Q}$HH's $(A_i,B_i)_{1\leqslant i\leqslant n}$, where the A_i are null in $M\setminus K$ and disjoint, and each ∂B_i is LP-identified with the corresponding ∂A_i. Since $\mathcal{F}_{n+1}^m\subset\mathcal{F}_n^m$, this defines a filtration.

Theorem 2.17 ([14, Thm. 1.13]). — *A null LP-surgery induces a canonical isomorphism between the Blanchfield modules of the involved $\mathbb{Q}$SK-pairs. Conversely, any isomorphism between the Blanchfield modules of two $\mathbb{Q}$SK-pairs can be realized by a finite sequence of null LP-surgeries up to multiplication by a power of t.*

Recall that the multiplication by t on the Blanchfield module is induced by the automorphism τ which generates the automorphism group of the infinite cyclic covering associated to the $\mathbb{Q}$SK-pair.

The above result implies in particular that the fitration $(\mathcal{F}_n^m)_{n\in\mathbb{N}}$ splits in the following way. For a given Blanchfield module $(\mathfrak{A},\mathfrak{b})$, let $\mathcal{F}_0^m(\mathfrak{A},\mathfrak{b})$ be the subspace of $\mathcal{F}_0^m$ generated by the $(\mathfrak{A},\mathfrak{b})$-marked $\mathbb{Q}$SK-pairs. Let $(\mathcal{F}_n^m(\mathfrak{A},\mathfrak{b}))_{n\in\mathbb{N}}$ be the filtration defined on $\mathcal{F}_0^m(\mathfrak{A},\mathfrak{b})$ by null LP-surgeries. Then, for $n\in\mathbb{N}$, $\mathcal{F}_n^m$ is the direct sum over all isomorphism classes of Blanchfield modules of the $\mathcal{F}_n^m(\mathfrak{A},\mathfrak{b})$. Set $\mathcal{G}_n^m(\mathfrak{A},\mathfrak{b})=\mathcal{F}_n^m(\mathfrak{A},\mathfrak{b})/\mathcal{F}_{n+1}^m(\mathfrak{A},\mathfrak{b})$ and $\mathcal{G}^m(\mathfrak{A},\mathfrak{b})=\bigoplus_{n\in\mathbb{N}}\mathcal{G}_n^m(\mathfrak{A},\mathfrak{b})$.

An invariant of $(\mathfrak{A},\mathfrak{b})$-marked $\mathbb{Q}$SK-pairs is a map defined on $\mathcal{P}^m(\mathfrak{A},\mathfrak{b})$. Given such an invariant λ valued in an abelian torsion free group Z, one can extend it to a $\mathbb{Q}$-linear map $\tilde{\lambda}:\mathcal{F}_0^m(\mathfrak{A},\mathfrak{b})\to\mathbb{Q}\otimes_{\mathbb{Z}}Z$. The invariant λ is a *finite type invariant of degree at most n of $(\mathfrak{A},\mathfrak{b})$-marked $\mathbb{Q}$SK-pairs with respect to null LP-surgeries* if $\tilde{\lambda}(\mathcal{F}_{n+1}^m(\mathfrak{A},\mathfrak{b}))=0$. The dual of the quotient $\mathcal{G}_n^m(\mathfrak{A},\mathfrak{b})$ is naturally identified with the space of all rational valued finite type invariants of degree n of marked $\mathbb{Q}$SK-pairs with respect to null LP-surgeries, hence a description of $\mathcal{G}_n^m(\mathfrak{A},\mathfrak{b})$ provides a description of this space of invariants. Theorem 2.17 implies $\mathcal{G}_0^m(\mathfrak{A},\mathfrak{b})\cong\mathbb{Q}$.

We studied in [11, Chap. 6] the filtration associated to $\mathbb{Q}$SK-pairs (without marking) and defined a graded space of diagrams which surjects onto the corresponding graded space $\mathcal{G}(\mathfrak{A},\mathfrak{b})$. This work can be adapted to marked $\mathbb{Q}$SK-pairs in order to define a graded space of diagrams and a surjective map from this space to $\mathcal{G}^m(\mathfrak{A},\mathfrak{b})$. We focus here on the degree one case, and we give a complete description of $\mathcal{G}_1^m(\mathfrak{A},\mathfrak{b})$ for an arbitrary isomorphism class $(\mathfrak{A},\mathfrak{b})$ of Blanchfield modules.

In Subsection 6.2, in order to prove Theorem 2.10, we construct an isomorphism $\mathfrak{h} : \phi^\bullet(\mathcal{P}^m(\mathfrak{A},\mathfrak{b})) \xrightarrow{\cong} \mathcal{H}$. Set $\hbar = \mathfrak{h}\circ\phi^\bullet : \mathcal{P}^m(\mathfrak{A},\mathfrak{b}) \to \mathcal{H}$. The following result is a consequence of Theorem 7.10, Corollary 2.9 and Lemma 6.5.

Proposition 2.18. — *The map $\hbar : \mathcal{P}^m(\mathfrak{A},\mathfrak{b}) \to \mathcal{H}$ is a degree at most one invariant of $(\mathfrak{A},\mathfrak{b})$-marked $\mathbb{Q}$SK-pairs with respect to null LP-surgeries.*

For a prime integer p, define a map $\nu_p : \mathcal{F}_0^m \to \mathbb{Q}$ by $\nu_p(M,K,\xi) = v_p(|H_1(M;\mathbb{Z})|)$, where v_p is the p-adic valuation and $|\,.\,|$ denotes the cardinality. By [12, Prop. 0.8], the ν_p are degree 1 invariants of $\mathbb{Q}$HS's, hence they are also degree 1 invariants of $\mathbb{Q}$SK-pairs. The following result is obtained in Section 7 as a consequence of Propositions 7.1 and 7.7.

Theorem 2.19. — *Fix a Blanchfield module $(\mathfrak{A},\mathfrak{b})$. Set*

$$\mathcal{H} = \frac{\Lambda^3\mathfrak{A}}{(\beta_1\wedge\beta_2\wedge\beta_3 = t\beta_1\wedge t\beta_2\wedge t\beta_3)}.$$

Let $(M,K,\xi)\in\mathcal{P}^m(\mathfrak{A},\mathfrak{b})$. For p prime, let B_p be a rational homology ball such that $H_1(B_p;\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$. Then

$$\mathcal{G}_1^m(\mathfrak{A},\mathfrak{b}) \cong \left(\bigoplus_{p\ prime} \mathbb{Q}[(M,K,\xi);\frac{B_p}{B^3}]\right)\oplus\mathcal{H}.$$

Moreover, Propositions 7.1 and 7.7 show that the invariants ν_p together with the map $\hbar$, obtained from the equivariant triple intersection map, form a universal rational valued finite type invariant of degree 1 of $(\mathfrak{A},\mathfrak{b})$-marked $\mathbb{Q}$SK-pairs with respect to null LP-surgeries in the following sense. If $\lambda : \mathcal{P}^m(\mathfrak{A},\mathfrak{b}) \to \mathbb{Q}$ is a degree 1 invariant with respect to null LP-surgeries, then there are maps $f : \mathcal{H}\to\mathbb{Q}$ and $g_p : \mathbb{Q}\to\mathbb{Q}$ for all prime integers p such that $\lambda - (f\circ\hbar + \sum_{p \text{ prime}} g_p\circ\nu_p)$ is a degree 0 invariant, i.e. a constant.

The case of $\mathbb{Z}$SK-pairs

A *$\mathbb{Z}$SK-pair* (M,K) is a $\mathbb{Q}$SK-pair such that M is an *integral homology 3-sphere*, i.e. an oriented compact 3-manifold which has the same homology with integral coefficients as the standard 3-sphere S^3. The *integral Alexander module* of a $\mathbb{Z}$SK-pair (M,K) is the $\mathbb{Z}[t^{\pm1}]$-module $\mathfrak{A}_\mathbb{Z}(M,K) = H_1(\widetilde{X};\mathbb{Z})$, where $\widetilde{X}$ is the infinite cyclic covering associated with (M,K). The *integral Blanchfield module* of (M,K) is the integral Alexander module $\mathfrak{A}_\mathbb{Z}(M,K)$ equipped with the Blanchfield form. Fix an integral Blanchfield module $(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$. If ξ is a fixed isomorphism from $(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$ to the Blanchfield module of a $\mathbb{Z}$SK-pair (M,K), then (M,K,ξ) is an *$(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$-marked $\mathbb{Z}$SK-pair.*

As for $\mathbb{Q}$SK-pairs, this isomorphism ξ is defined up to multiplication by a power of t. Let $\mathcal{P}_{\mathbb{Z}}^m(\mathfrak{A}_{\mathbb{Z}},\mathfrak{b})$ be the set of all such $(\mathfrak{A}_{\mathbb{Z}},\mathfrak{b})$-marked $\mathbb{Z}$SK-pairs up to orientation-preserving and marking-preserving homeomorphism, called marked $\mathbb{Z}$SK-pairs when it does not seem to cause confusion.

Borromean surgeries are well defined on the set of marked $\mathbb{Z}$SK-pairs, since they preserve the integral homology of the manifold. The equivariant triple intersection map is again a degree one invariant of marked $\mathbb{Z}$SK-pairs with respect to null Borromean surgeries. We will see that this invariant contains all the rational valued degree one invariants of marked $\mathbb{Z}$SK-pairs with respect to null Borromean surgeries.

Replacing $\mathbb{Q}$ by $\mathbb{Z}$ in the definitions at the beginning of the subsection, define *integral homology handlebodies* ($\mathbb{Z}$HH), *integral Lagrangians*, *integral LP-surgeries*, and *integral null LP-surgeries*, similarly. Integral LP-surgeries (in particular Borromean surgeries) preserve the homology with integral coefficients of the manifold. Hence they provide a move on the set of integral homology 3-spheres. Integral null LP-surgeries define a move on the set of $\mathbb{Z}$SK-pairs. Moreover, they induce canonical isomorphisms beetween the integral Blanchfield modules of the involved pairs (see Theorem 2.21 below), hence they provide a move on the set of marked $\mathbb{Z}$SK-pairs.

Let $\mathcal{F}_0^{m,\mathbb{Z}}$ be the rational vector space generated by all marked $\mathbb{Z}$SK-pairs up to orientation-preserving homeomorphism. Let $(\mathcal{F}_n^{m,\mathbb{Z}})_{n\in\mathbb{N}}$ be the filtration of $\mathcal{F}_0^{m,\mathbb{Z}}$ defined by integral null LP-surgeries. The following result implies that Borromean surgeries define the same filtration.

Proposition 2.20 ([1, Lem. 4.11]). — *Let A and B be $\mathbb{Z}$HH's whose boundaries are LP-identified. Then A and B can be obtained from one another by a finite sequence of Borromean surgeries in the interior of the $\mathbb{Z}$HH's.*

The following result is the equivalent of Theorem 2.17 in the setting of $\mathbb{Z}$SK-pairs.

Theorem 2.21 ([14, Thm. 1.14]). — *An integral null LP-surgery induces a canonical isomorphism between the integral Blanchfield modules of the involved $\mathbb{Z}$SK-pairs. Conversely, any isomorphism between the integral Blanchfield modules of two $\mathbb{Z}$SK-pairs can be realized by a finite sequence of integral null LP-surgeries, up to multiplication by a power of t.*

This implies that the filtration $(\mathcal{F}_n^{m,\mathbb{Z}})_{n\in\mathbb{N}}$ splits along the isomorphism classes of integral Blanchfield modules. For a given integral Blanchfield module $(\mathfrak{A}_{\mathbb{Z}},\mathfrak{b})$, let $\mathcal{F}_0^{m,\mathbb{Z}}(\mathfrak{A}_{\mathbb{Z}},\mathfrak{b})$ be the subspace of $\mathcal{F}_0^{m,\mathbb{Z}}$ generated by the $(\mathfrak{A}_{\mathbb{Z}},\mathfrak{b})$-marked $\mathbb{Z}$SK-pairs. Let $(\mathcal{F}_n^{m,\mathbb{Z}}(\mathfrak{A}_{\mathbb{Z}},\mathfrak{b}))_{n\in\mathbb{N}}$ be the filtration defined on

$\mathcal{F}_0^{m,\mathbb{Z}}(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$ by integral null LP-surgeries. Then, for $n\in\mathbb{N}$, $\mathcal{F}_n^{m,\mathbb{Z}}$ is the direct sum over all isomorphism classes of integral Blanchfield modules of the $\mathcal{F}_n^{m,\mathbb{Z}}(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$. Set $\mathcal{G}_n^{m,\mathbb{Z}}(\mathfrak{A}_\mathbb{Z},\mathfrak{b})=\mathcal{F}_n^{m,\mathbb{Z}}(\mathfrak{A}_\mathbb{Z},\mathfrak{b})/\mathcal{F}_{n+1}^{m,\mathbb{Z}}(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$. Theorem 2.21 implies $\mathcal{G}_0^{m,\mathbb{Z}}(\mathfrak{A}_\mathbb{Z},\mathfrak{b})\cong\mathbb{Q}$.

An invariant of $(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$-marked $\mathbb{Z}$SK-pairs is a map defined on $\mathcal{P}_\mathbb{Z}^m(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$. Given such an invariant λ valued in an abelian torsion free group Z, one can extend it to a $\mathbb{Q}$-linear map $\tilde{\lambda}:\mathcal{F}_0^{m,\mathbb{Z}}(\mathfrak{A}_\mathbb{Z},\mathfrak{b})\to\mathbb{Q}\otimes_\mathbb{Z} Z$. The invariant λ is a *finite type invariant of degree at most n of $(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$-marked $\mathbb{Z}$SK-pairs with respect to integral null LP-surgeries* if $\tilde{\lambda}(\mathcal{F}_{n+1}^m(\mathfrak{A}_\mathbb{Z},\mathfrak{b}))=0$.

Let $(M,K,\xi)\in\mathcal{P}_\mathbb{Z}^m(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$. The marking ξ induces an $(\mathfrak{A},\mathfrak{b})$-marking $\bar{\xi}$ of (M,K) viewed as a $\mathbb{Q}$SK-pair, where $(\mathfrak{A},\mathfrak{b})=(\mathbb{Q}\otimes\mathfrak{A}_Z,\mathrm{id}_\mathbb{Q}\otimes\mathfrak{b})$. Since $\mathfrak{A}_\mathbb{Z}$ has no $\mathbb{Z}$-torsion by [14, Lem. 5.5], the marking ξ can be recovered from $\bar{\xi}$ and we have a natural injection $\mathcal{P}_\mathbb{Z}^m(\mathfrak{A}_\mathbb{Z},\mathfrak{b})\hookrightarrow\mathcal{P}^m(\mathfrak{A},\mathfrak{b})$. Consider the map $\hbar$ of Proposition 2.18 and its restriction $\hbar:\mathcal{P}_\mathbb{Z}^m(\mathfrak{A}_\mathbb{Z},\mathfrak{b})\to\mathcal{H}$. Corollary 2.9 implies:

Proposition 2.22. — *The map $\hbar:\mathcal{P}_\mathbb{Z}^m(\mathfrak{A}_\mathbb{Z},\mathfrak{b})\to\mathcal{H}$ is a degree at most one invariant of $(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$-marked $\mathbb{Z}$SK-pairs with respect to integral null LP-surgeries.*

In Section 7, we prove:

Theorem 2.23. — *Fix an integral Blanchfield module $(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$. Set $\mathfrak{A}=\mathfrak{A}_\mathbb{Z}\otimes_\mathbb{Z}\mathbb{Q}$. Set $\mathcal{H}=\dfrac{\Lambda^3_\mathbb{Q}\mathfrak{A}}{(\beta_1\wedge\beta_2\wedge\beta_3=t\beta_1\wedge t\beta_2\wedge t\beta_3)}$. Then the map $\hbar:\mathcal{P}_\mathbb{Z}^m(\mathfrak{A}_\mathbb{Z},\mathfrak{b})\to\mathcal{H}$ induces an isomorphism*

$$\mathcal{G}_1^{m,\mathbb{Z}}(\mathfrak{A}_\mathbb{Z},\mathfrak{b})\cong\mathcal{H}.$$

This result shows that the map $\hbar$, obtained from the equivariant triple intersection map, is a universal rational valued finite type invariant of degree 1 of $(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$-marked $\mathbb{Z}$SK-pairs with respect to integral null LP-surgeries in the following sense. If $\lambda:\mathcal{P}_\mathbb{Z}^m(\mathfrak{A}_\mathbb{Z},\mathfrak{b})\to\mathbb{Q}$ is a degree 1 invariant with respect to integral null LP-surgeries, then there is a map $f:\mathcal{H}\to\mathbb{Q}$ such that $\lambda-f\circ\hbar$ is a degree 0 invariant, i.e. a constant.

3. Equivariant triple intersections

In this section, we prove Lemma 2.4.

Lemma 3.1. — *Let (M,K) be a $\mathbb{Q}$SK-pair. Let $\widetilde{X}$ be the associated infinite cyclic covering. Then $H_2(\widetilde{X};\mathbb{Q})=0$.*

Proof. — Let Σ be a compact connected oriented surface embedded in M such that $\partial\Sigma = K$. Set $V = M \setminus (\Sigma \times [-1,1])$. Note that V is a rational homology handlebody (see [14, Lem. 3.1]). In particular, $H_2(V;\mathbb{Q}) = 0$. The boundary of V is the union of $\Sigma^+ = \Sigma\times\{1\}$, $\Sigma^- = \Sigma\times\{-1\}$, and $\partial\Sigma\times[-1,1]$. Consider $\mathbb{Z}$ copies V_i of V, and let Σ_i^+, Σ_i^- be the copies of Σ^+ and Σ^- in V_i. The covering $\widetilde{X}$ can be constructed by connecting all the V_i, gluing Σ_i^- and Σ_{i+1}^+ for all $i \in \mathbb{Z}$. Set $\widetilde{V}_e = \cup_{i\in\mathbb{Z}} V_{2i}$ and $\widetilde{V}_o = \cup_{i\in\mathbb{Z}} V_{2i+1}$. Let $\widetilde{\Sigma}$ be the preimage of Σ in $\widetilde{X}$, made of $\mathbb{Z}$ disjoint copies of Σ. We have $\widetilde{\Sigma} = \widetilde{V}_e \cap \widetilde{V}_o$. The Mayer–Vietoris sequence associated with $\widetilde{X} = \widetilde{V}_e \cup \widetilde{V}_o$ yields the exact sequence

$$H_2(\widetilde{V}_e;\mathbb{Q}) \oplus H_2(\widetilde{V}_o;\mathbb{Q}) \longrightarrow H_2(\widetilde{X};\mathbb{Q}) \longrightarrow H_1(\widetilde{\Sigma};\mathbb{Q}) \\ \xrightarrow{\ \iota\ } H_1(\widetilde{V}_e;\mathbb{Q}) \oplus H_1(\widetilde{V}_o;\mathbb{Q}).$$

The module $H_2(\widetilde{V}_e;\mathbb{Q}) \oplus H_2(\widetilde{V}_o;\mathbb{Q})$ is a direct sum of $\mathbb{Z}$ copies of $H_2(V;\mathbb{Q})$, which is trivial. Hence $H_2(\widetilde{V}_e;\mathbb{Q}) \oplus H_2(\widetilde{V}_o;\mathbb{Q}) = 0$. It is well-known that the map ι provides a square, non degenerate presentation of the Alexander module (see [9, Thm. 6.5] for details). In particular, ι is known to be injective. Finally, $H_2(\widetilde{X};\mathbb{Q}) = 0$. □

Lemma 3.2. — *Let N be an oriented 3-manifold. Let C be a rational 3-chain and let Σ_2 and Σ_3 be rational 2-chains, pairwise transverse in N. Then:*

$$\langle \partial C, \Sigma_2, \Sigma_3\rangle = \langle C, \partial\Sigma_2, \Sigma_3\rangle - \langle C, \Sigma_2, \partial\Sigma_3\rangle.$$

Proof. — It suffices to prove the result for pairwise transverse integral chains. Since

$$\partial(C \cap \Sigma_2 \cap \Sigma_3) = \big(\partial C \cap \Sigma_2 \cap \Sigma_3\big) \cup \big(C \cap \partial(\Sigma_2 \cap \Sigma_3)\big),$$

we have $\langle \partial C, \Sigma_2, \Sigma_3\rangle = -\langle C, \partial(\Sigma_2 \cap \Sigma_3)\rangle$. Now,

$$\partial(\Sigma_2 \cap \Sigma_3) = \big(-\partial(\Sigma_2) \cap \Sigma_3\big) \cup \big(\Sigma_2 \cap \partial(\Sigma_3)\big).$$

The announced equality follows. □

Corollary 3.3. — *Let (M,K) be a $\mathbb{Q}$SK-pair. Let $\widetilde{X}$ be the associated infinite cyclic covering. Let C be a rational 3-chain and let Σ_2 and Σ_3 be rational 2-chains, pairwise τ-transverse in $\widetilde{X}$. Then:*

$$\langle \partial C, \Sigma_2, \Sigma_3\rangle_e = \langle C, \partial\Sigma_2, \Sigma_3\rangle_e - \langle C, \Sigma_2, \partial\Sigma_3\rangle_e.$$

Proof. — Apply Lemma 3.2 to C, $\tau^{k_2}\Sigma_2$ and $\tau^{k_3}\Sigma_3$ for all integers k_2, k_3. □

Proof of Lemma 2.4. — Replace Σ_1 by a chain Σ'_1 satisfying the same conditions. Lemma 3.1 shows that there is a rational 3-chain C such that $\partial C = \Sigma'_1 - \Sigma_1$. Compute the difference

$$\langle \Sigma'_1, \Sigma_2, \Sigma_3 \rangle_e - \langle \Sigma_1, \Sigma_2, \Sigma_3 \rangle_e = \langle \partial C, \Sigma_2, \Sigma_3 \rangle_e .$$

By Corollary 3.3,

$$\langle \partial C, \Sigma_2, \Sigma_3 \rangle_e = \langle C, \partial\Sigma_2, \Sigma_3 \rangle_e - \langle C, \Sigma_2, \partial\Sigma_3 \rangle_e .$$

Hence, by Lemma 2.3:

$$\langle \partial C, \Sigma_2, \Sigma_3 \rangle_e = P_2(t_2)\langle C, \mu_2, \Sigma_3 \rangle - P_3(t_3)\langle C, \Sigma_2, \mu_3 \rangle ,$$

and this is trivial in $\dfrac{\mathcal{R}}{(t_1t_2t_3 - 1, P_1(t_1), P_2(t_2), P_3(t_3))}$.

Let μ'_1 be a knot in $\widetilde{X}$, rationally homologous to μ_1, whose image in M is disjoint from the images of μ_2 and μ_3. The difference $\mu'_1 - \mu_1$ is trivial in $H_1(\widetilde{X};\mathbb{Q})$, hence there is a rational 2-chain S such that $\partial S = \mu'_1 - \mu_1$. Choose S τ-transverse to Σ_1, Σ_2, and Σ_3. Set $\hat{S} = P_1(\tau)S$, and $\Sigma'_1 = \hat{S} + \Sigma_1$. We have $\partial\Sigma'_1 = P_1(\tau)\partial S + \partial\Sigma_1 = P_1(\tau)\mu'_1$. Since

$$\langle \hat{S}, \Sigma_2, \Sigma_3 \rangle_e = P_1(t_1)\langle S, \Sigma_2, \Sigma_3 \rangle_e = 0 \quad \text{in} \quad \frac{\mathcal{R}}{(t_1t_2t_3 - 1, P_1(t_1), P_2(t_2), P_3(t_3))},$$

we have $\langle \Sigma'_1, \Sigma_2, \Sigma_3 \rangle_e = \langle \Sigma_1, \Sigma_2, \Sigma_3 \rangle_e$.

Conclude by using the symmetry properties of the equivariant triple intersections. □

4. Variation under null Borromean surgeries

In this section, we prove Proposition 2.8.

It is known that Borromean surgeries preserve the linking number of curves in the complement of the surgery Y-link. The following lemma describes the effect of a Borromean surgery on the triple intersection numbers.

Lemma 4.1. — *Let N be a 3-manifold. Let Γ be a Y-graph in N with leaves ℓ_1, ℓ_2, ℓ_3. Let Σ_1, Σ_2 and Σ_3 be transverse compact surfaces in N. Assume $\Gamma \cap \Sigma_i \cap \Sigma_j = \emptyset$ for $i \neq j$. Then there are surfaces Σ'_1, Σ'_2 and Σ'_3 in $N(\Gamma)$ such that $\partial\Sigma'_i = \partial\Sigma_i$ and*

$$\langle \Sigma'_1, \Sigma'_2, \Sigma'_3 \rangle_{N(\Gamma)} - \langle \Sigma_1, \Sigma_2, \Sigma_3 \rangle_N = \sum_{\sigma \in \mathcal{S}_3} \varepsilon(\sigma) \langle \Sigma_1, \ell_{\sigma(1)} \rangle_N \langle \Sigma_2, \ell_{\sigma(2)} \rangle_N \langle \Sigma_3, \ell_{\sigma(3)} \rangle_N .$$

Proof. — The surgery replaces a tubular neighborhood $T(\Gamma)$ of Γ by another standard handlebody of genus 3 (see Matveev [10]). To each intersection point of Γ with a surface Σ_j corresponds a disk on Σ_j which is removed by the surgery, see Figure 4.1.

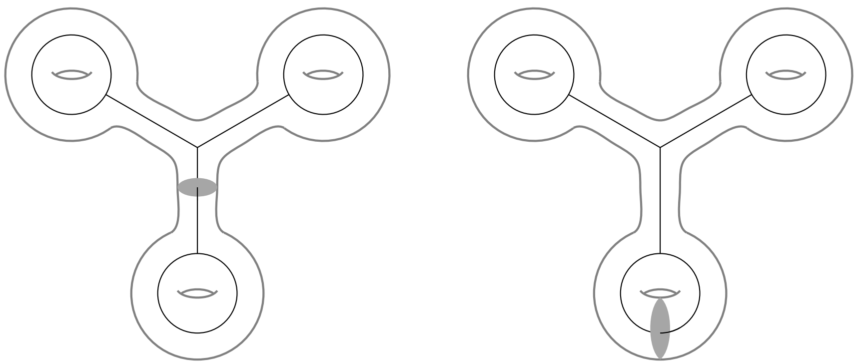


Figure 4.1. Disks of $\Sigma_j \cap T(\Gamma)$ removed by the surgery

In the first case (left part of Figure 4.1) the boundary of the disk is a separating curve of $\partial T(\Gamma)$. The disk can thus be replaced by one of the two subsurfaces of $\partial T(\Gamma)$ defined by this separating curve. Hence we can assume the only intersections of the Σ_j with Γ are on the leaves of Γ. Let D be a disk obtained as the intersection of $T(\Gamma)$ with a surface Σ_j in this latter case (right part of Figure 4.1). We now define a surface F which will replace the disk D after the surgery. One part of F is represented in Figure 4.2: it is made of four pieces in (light and dark) grey.

This grey surface has a boundary inside $T(\Gamma)$ (the boundary of the upper disk in light grey) which is a longitude of a component of the surgery link. Hence this curve bounds a disk after the surgery. Define F in $N(\Gamma)$ as the union of this disk and the grey surface. Replacing each disk of the intersections $T(\Gamma) \cap \Sigma_j$ in this way (with matching orientations), we obtain surfaces Σ'_j in $N(\Gamma)$ such that $\partial \Sigma'_j = \partial \Sigma_j$. It remains to compute the difference of the triple intersection numbers.

Let F_2 denote the surface (after surgery) constructed above, with a part drawn in Figure 4.2, and let F_1 (resp. F_3) be the similar surface corresponding to the left (resp. right) handle. Then the dashed curve represents the intersection $F_1 \cap F_2$, and we have $\langle F_1, F_2, F_3 \rangle = 1$. Note that parallel meridians of the same handle bound parallel surfaces (constructed as F) after the surgery. We obtain the result by counting the intersection points inside the reglued handlebody. $\square$

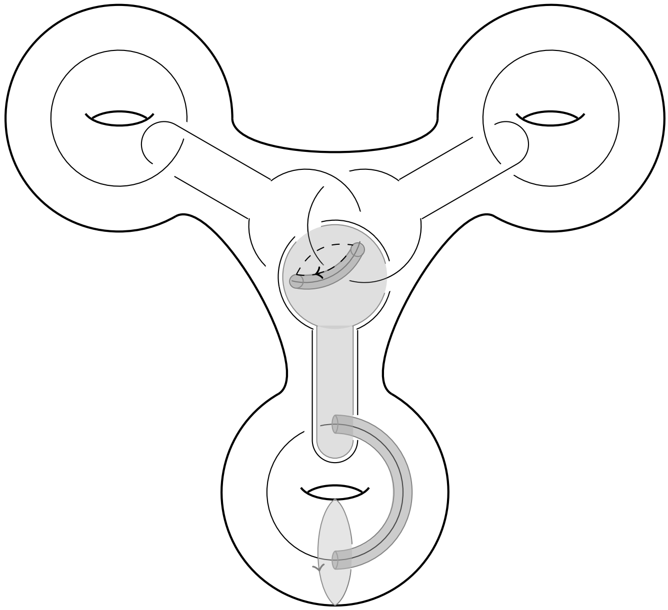


Figure 4.2. Surface in the reglued handlebody

Proof of Proposition 2.8. — Thanks to $\mathbb{Q}$-linearity, it suffices to prove the result for integral homology classes β_j. Consider representatives μ_j of the β_j whose images in $M\setminus K$ are pairwise disjoint and disjoint from Γ. Consider τ-transverse rational 2-chains Σ_j, τ-transverse to $\widetilde{\Gamma}$, such that $\partial\Sigma_j = \delta(\tau)\mu_j$, and $\widetilde{\Gamma}\cap\tau^{k_i}\Sigma_i\cap\tau^{k_j}\Sigma_j=\emptyset$ for $i\neq j$ and $k_i,k_j\in\mathbb{Z}$. The surgery on Γ gives rise to simultaneous surgeries on all the $\tau^k\widetilde{\Gamma}$ in $\widetilde{X}$. Hence, by Lemma 4.1:

$$\begin{aligned}
&\phi^{(M,K,\xi)(\Gamma)}([\mu_1]\otimes[\mu_2]\otimes[\mu_3]) - \phi^{(M,K,\xi)}([\mu_1]\otimes[\mu_2]\otimes[\mu_3])\\
&= \sum_{k_2,k_3\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{\sigma\in\mathcal{S}_3}\varepsilon(\sigma)\langle\Sigma_1,\tau^k\gamma_{\sigma(1)}\rangle\langle\tau^{-k_2}\Sigma_2,\tau^k\gamma_{\sigma(2)}\rangle\\
&\qquad\qquad\langle\tau^{-k_3}\Sigma_3,\tau^k\gamma_{\sigma(3)}\rangle t_2^{k_2}t_3^{k_3}\\
&= \sum_{\sigma\in\mathcal{S}_3}\varepsilon(\sigma)\sum_{k\in\mathbb{Z}}\langle\Sigma_1,\tau^k\gamma_{\sigma(1)}\rangle lk_e(\delta(\tau)\mu_2,\tau^k\gamma_{\sigma(2)})(t_2)\, lk_e(\delta(\tau)\mu_3,\tau^k\gamma_{\sigma(3)})(t_3)\\
&= \sum_{\sigma\in\mathcal{S}_3}\varepsilon(\sigma)\Big(\sum_{k\in\mathbb{Z}}\langle\Sigma_1,\tau^k\gamma_{\sigma(1)}\rangle t_1^k\Big)\delta(t_2)lk_e(\mu_2,\gamma_{\sigma(2)})(t_2)\,\delta(t_3)lk_e(\mu_3,\gamma_{\sigma(3)})(t_3)\\
&= \sum_{\sigma\in\mathcal{S}_3}\varepsilon(\sigma)\prod_{j=1}^{3}\delta(t_j)lk_e(\mu_j,\gamma_{\sigma(j)})(t_j)
\end{aligned}$$

□

5. Structure of $\mathcal{H}$

In this section, we study the structure of $\mathfrak{A}_h$ and $\mathcal{H}$, and we prove Theorems 2.11 and 2.12.

There is a natural surjective map $\mathfrak{A}_h \twoheadrightarrow \mathcal{H}$, which splits into surjective maps $\mathfrak{A}(\underline{i}) \twoheadrightarrow \mathcal{H}(\underline{i})$ for $\underline{i} \in \{1,\dots,q\}^3$. Note that the map $\mathfrak{A}(\underline{i}) \twoheadrightarrow \mathcal{H}(\underline{i})$ is an isomorphism if and only if the i_j are all distinct.

For $1 \leqslant i \leqslant q$, $\mathbb{C}\otimes\mathfrak{A}_i$ can be written:

$$\mathbb{C}\otimes\mathfrak{A}_i = \bigoplus_{\ell=1}^{q_i} \frac{\mathbb{C}[t^{\pm1}]}{((t-z_{i\ell})^{m_i})}\eta_{i\ell},$$

where the $z_{i\ell}$ are complex roots of δ_i, different from 0 and 1. Set:

$$J_{\underline{i}} = \{1,\dots,q_{i_1}\}\times\{1,\dots,q_{i_2}\}\times\{1,\dots,q_{i_3}\}.$$

Let $\underline{\ell} = (\ell_j)_{1\leqslant j\leqslant 3} \in J_{\underline{i}}$. Let $\mathfrak{A}(\underline{i},\underline{\ell})$ be the quotient of $\displaystyle\bigotimes_{1\leqslant j\leqslant 3} \frac{\mathbb{C}[t^{\pm1}]}{((t-z_{i_j\ell_j})^{m_{i_j}})}\eta_{i_j\ell_j}$ by the vector subspace generated by the holonomy relations, namely the relations $\otimes_{1\leqslant j\leqslant 3}\beta_j = \otimes_{1\leqslant j\leqslant 3} t\beta_j$. Then $\mathbb{C}\otimes\mathfrak{A}(\underline{i}) = \bigoplus_{\underline{\ell}\in J_{\underline{i}}} \mathfrak{A}(\underline{i},\underline{\ell})$.

The following lemma implies Theorem 2.11.

Lemma 5.1. — *The complex vector space $\mathfrak{A}(\underline{i},\underline{\ell})$ is non trivial if and only if $\prod_{j=1}^3 z_{i_j\ell_j} = 1$.*

The following sublemma will be useful for rewriting the holonomy relations.

Sublemma 5.2. — *For all $(\beta_j)_{1\leqslant j\leqslant 3}\subset\mathfrak{A}$, for all $(w_j)_{1\leqslant j\leqslant 3}\subset\mathbb{C}$:*

$$\otimes_{1\leqslant j\leqslant 3} t\beta_j = \sum_{I\subset\{1,2,3\}} \otimes_{1\leqslant j\leqslant 3} p_I(w_j)\beta_j,$$

where $p_I(w_j) = \begin{cases} (t-w_j) & \text{if } j\in I \\ w_j & \text{if } j\notin I \end{cases}.$

Proof. — For $1\leqslant j\leqslant 3$, write $t = (t-w_j)+w_j$. □

Proof of Lemma 5.1. — Fix $(\underline{i},\underline{\ell})$, and simplify the notation by setting $z_j = z_{i_j\ell_j}$, $n_j = m_{i_j}$, $\eta_j = \eta_{i_j\ell_j}$ and for $\underline{k} = (k_j)_{1\leqslant j\leqslant 3}\in\mathbb{N}^3$, $[\underline{k}] = \bigotimes_{1\leqslant j\leqslant 3}(t-z_j)^{k_j}\eta_j$. Thanks to Sublemma 5.2, the holonomy relations can be written in terms of these generators, as follows:

$$hol(\underline{k}): \quad [\underline{k}] = \sum_{I\subset\{1,2,3\}} (\prod_{j\notin I} z_j)[\underline{k}+\underline{\delta_I}],$$

where $(\delta_I)_j = \begin{cases} 1 & \text{if } j \in I \\ 0 & \text{if } j \notin I \end{cases}$. We have:

$$\mathfrak{A}(\underline{i}, \underline{\ell}) = \frac{\mathbb{C}\langle [\underline{k}]; 0 \leqslant k_j < n_j \ \forall j \rangle}{\mathbb{C}\langle hol(\underline{k}); 0 \leqslant k_j < n_j \ \forall j \rangle}.$$

First assume $z_1 z_2 z_3 \neq 1$. For $\underline{k} = (k_1, k_2, k_3)$, let $s(\underline{k}) = k_1 + k_2 + k_3$. By decreasing induction on $s(\underline{k})$, we will prove that all the $[\underline{k}]$ vanish in $\mathfrak{A}(\underline{i}, \underline{\ell})$. It is true if $s(\underline{k}) > n_1 + n_2 + n_3 - 3$. Fix $s \geqslant 0$, and assume $[\underline{k}] = 0$ if $s(\underline{k}) > s$. Then, if $s(\underline{k}) = s$, the relation $hol(\underline{k})$ becomes $[\underline{k}] = (z_1 z_2 z_3)[\underline{k}]$, hence $[\underline{k}] = 0$.

Now assume $z_1 z_2 z_3 = 1$. In this case, the holonomy relations get simplified:

$$hol(\underline{k}) : \qquad \sum_{\emptyset \neq I \subset \{1,2,3\}} (\prod_{j \notin I} z_j)[\underline{k} + \underline{\delta_I}] = 0.$$

The generator $[0, 0, 0]$ does not appear in any of these relations. Hence $\mathfrak{A}(\underline{i}, \underline{\ell}) \neq 0$. $\square$

Examples.

(1) Let $\mathfrak{A} = \dfrac{\mathbb{Q}[t^{\pm 1}]}{(t^4 + 1)} \eta_1 \oplus \dfrac{\mathbb{Q}[t^{\pm 1}]}{(t^2 + 1)} \eta_2$. Let $\zeta = e^{i\frac{\pi}{4}}$. Then:

$$\mathbb{C} \otimes \mathfrak{A} = \frac{\mathbb{C}[t^{\pm 1}]}{(t - \zeta)} \eta_{11} \oplus \frac{\mathbb{C}[t^{\pm 1}]}{(t - \zeta^3)} \eta_{12} \oplus \frac{\mathbb{C}[t^{\pm 1}]}{(t + \zeta)} \eta_{13} \oplus \frac{\mathbb{C}[t^{\pm 1}]}{(t + \zeta^3)} \eta_{14} \oplus \frac{\mathbb{C}[t^{\pm 1}]}{(t - i)} \eta_{21} \oplus \frac{\mathbb{C}[t^{\pm 1}]}{(t + i)} \eta_{22}$$

The space $\mathfrak{A}(\underline{i}, \underline{\ell})$ is non trivial if and only if the set $\{(i_1, l_1), (i_2, l_2), (i_3, l_3)\}$ is, up to permutation, one of the following ones:

$$\{(1,1),(1,3),(2,1)\},\ \{(1,2),(1,2),(2,1)\},\ \{(1,4),(1,4),(2,1)\},$$
$$\{(1,1),(1,1),(2,2)\},\ \{(1,2),(1,4),(2,2)\},\ \{(1,3),(1,3),(2,2)\}.$$

There are 24 different non trivial $\mathfrak{A}(\underline{i}, \underline{\ell})$, and each has complex dimension 1, hence $\dim_{\mathbb{Q}}(\mathfrak{A}_h) = 24$.

(2) Let $\mathfrak{A} = \dfrac{\mathbb{Q}[t^{\pm 1}]}{((t + 1 + t^{-1})^m)}$, $m > 0$. In this case, $q = 1$ and $\mathfrak{A}_h = \mathfrak{A}(1, 1, 1)$. Over the complex numbers, we have

$$\mathbb{C} \otimes \mathfrak{A} = \frac{\mathbb{C}[t^{\pm 1}]}{((t - j)^m)} \eta_{11} \oplus \frac{\mathbb{C}[t^{\pm 1}]}{((t - j^2)^m)} \eta_{12}$$

and

$$\mathbb{C} \otimes \mathfrak{A}_h = \mathfrak{A}((1,1,1),(1,1,1)) \oplus \mathfrak{A}((1,1,1),(2,2,2)),$$

where both the two components of this direct sum are non trivial. In particular, $\mathfrak{A}_h$ has dimension at least 2.

Proof of Theorem 2.12. — Fix $\underline{i} \in \{1,\dots,q\}^3$ such that $i_1 \leqslant i_2 \leqslant i_3$ and $\underline{\ell} \in J_{\underline{i}}$. Set $z_j = z_{i_j \ell_j}$, $n_j = m_{i_j}$, and $\eta_j = \eta_{i_j \ell_j}$. For $\underline{k} = (k_j)_{1\leqslant j\leqslant 3} \in \mathbb{N}^3$, set $[\underline{k}]_{\mathcal{H}} = (t-z_1)^{k_1}\eta_1 \wedge (t-z_2)^{k_2}\eta_2 \wedge (t-z_3)^{k_3}\eta_3$. Let $\mathcal{H}(\underline{i},\underline{\ell})$ denote the complex vector subspace of $\mathbb{C}\otimes_{\mathbb{Q}}\mathcal{H}$ generated by the $[\underline{k}]_{\mathcal{H}}$. Note that:

$$\mathbb{C}\otimes_{\mathbb{Q}}\mathcal{H}(\underline{i}) = \bigoplus_{\underline{\ell}\in J^o_{\underline{i}}} \mathcal{H}(\underline{i},\underline{\ell}),$$

where $J^o_{\underline{i}}$ is the set of all $\underline{\ell}$ in $J_{\underline{i}}$ such that, for $j=1,2$, if $i_j = i_{j+1}$, then $\ell_j \leqslant \ell_{j+1}$. Assume $\underline{\ell}\in J^o_{\underline{i}}$. We shall prove that $\mathcal{H}(\underline{i},\underline{\ell}) \neq 0$ if and only if $z_1z_2z_3 = 1$ and for $1\leqslant j\leqslant 3$, n_j is at least the number of occurrences of (i_j,ℓ_j) in $((i_1,\ell_1),(i_2,\ell_2),(i_3,\ell_3))$. If $z_1z_2z_3\neq 1$, $\mathfrak{A}(\underline{i},\underline{\ell})=0$ implies $\mathcal{H}(\underline{i},\underline{\ell}) = 0$. For the end of the proof, assume $z_1z_2z_3=1$. In this case, note that the holonomy relation $hol(\underline{k})$ relates generators $[\underline{k}']_{\mathcal{H}}$ such that $s(\underline{k}') > s(\underline{k})$.

If the (i_j,l_j) are all distinct, then $\mathcal{H}(\underline{i},\underline{\ell}) \cong \mathfrak{A}(\underline{i},\underline{\ell}) \neq 0$.

Assume $(i_1,\ell_1) = (i_2,\ell_2) \neq (i_3,\ell_3)$. If $n_1=n_2=1$, the anti-symmetry implies $\mathcal{H}(\underline{i},\underline{\ell})=0$. Otherwise $n_1=n_2\geqslant 2$. In this case, the space $\mathcal{H}(\underline{i},\underline{\ell})$ is defined by the generators $[\underline{k}]_{\mathcal{H}}$ with $k_1<k_2$ and the holonomy relations $hol(\underline{k})$ with $k_1<k_2$, rewritten in terms of these generators. Indeed, a relation $hol(k_1,k_1,k_3)$ is trivial, and a relation $hol(k_2,k_1,k_3)$ is equivalent to $hol(k_1,k_2,k_3)$. The generator $[0,1,0]_{\mathcal{H}}$ is non trivial since it does not appear in any relation $hol(\underline{k})$ with $k_1<k_2$. The proof is the same whenever there are exactly two different (i_j,ℓ_j).

Assume $(i_1,\ell_1)=(i_2,\ell_2)=(i_3,\ell_3)$. If $n_1=n_2=n_3\leqslant 2$, then $\mathcal{H}(\underline{i},\underline{\ell})=0$. Otherwise $n_1=n_2=n_3\geqslant 3$. In this case, the space $\mathcal{H}(\underline{i},\underline{\ell})$ is defined by the generators $[\underline{k}]_{\mathcal{H}}$ with $k_1<k_2<k_3$ and the holonomy relations $hol(\underline{k})$ with $k_1<k_2<k_3$, rewritten in terms of these generators. The generator $[0,1,2]_{\mathcal{H}}$ does not appears in any of these relations. Hence $\mathcal{H}(\underline{i},\underline{\ell})\neq 0$. □

Examples.

(1) For $\mathfrak{A} = \dfrac{\mathbb{Q}[t^{\pm1}]}{(t^4+1)}\eta_1 \oplus \dfrac{\mathbb{Q}[t^{\pm1}]}{(t^2+1)}\eta_2$, we have:

$$\mathbb{C}\otimes\mathcal{H} = \mathcal{H}((1,1,2),(1,3,1)) \oplus \mathcal{H}((1,1,2),(2,4,2)),$$

and $\dim(\mathcal{H}) = 2$.

(2) For $\mathfrak{A} = \dfrac{\mathbb{Q}[t^{\pm1}]}{((t+1+t^{-1})^m)}$, $\mathcal{H}$ is trivial if $m\leqslant 2$. If $m\geqslant 3$,

$$\mathbb{C}\otimes\mathcal{H} = \mathcal{H}((1,1,1),(1,1,1)) \oplus \mathcal{H}((1,1,1),(2,2,2)),$$

with both components non trivial. Hence $\mathcal{H}$ has dimension at least 2.

In the remaining of the section, we further study the structure of $\mathfrak{A}(\underline{i},\underline{\ell})$, and we provide bounds for the dimension of $\mathcal{H}$.

Lemma 5.3. — *Fix $(\underline{i},\underline{\ell})$, and simplify the notation by setting $z_j = z_{i_j\ell_j}$, $n_j = m_{i_j}$, $\eta_j = \eta_{i_j\ell_j}$ and for $\underline{k} = (k_j)_{1\leqslant j\leqslant 3} \in \mathbb{N}^3$, $[\underline{k}] = \bigotimes_{1\leqslant j\leqslant 3}(t-z_j)^{k_j}\eta_j$. Assume $z_1z_2z_3 = 1$. Assume $n_1 \geqslant n_2 \geqslant n_3$. Then the vector space $\mathfrak{A}(\underline{i},\underline{\ell})$ is generated by the family $([0,k_2,k_3])_{0\leqslant k_j<n_j}$. If $n_2+n_3 \leqslant n_1+1$, this family is a basis of $\mathfrak{A}(\underline{i},\underline{\ell})$, and hence $\dim_{\mathbb{C}}\mathfrak{A}(\underline{i},\underline{\ell}) = n_2n_3$. If $n_2+n_3 > n_1+1$, then $n_2n_3 - \frac{1}{2}(n_2+n_3-n_1)(n_2+n_3-n_1-1) \leqslant \dim_{\mathbb{C}}\mathfrak{A}(\underline{i},\underline{\ell}) \leqslant n_2n_3$.*

Note that if the (i_j,l_j) are all distinct, then $\mathcal{H}(\underline{i},\underline{\ell}) \cong \mathfrak{A}(\underline{i},\underline{\ell})$, and the above statements hold for $\mathcal{H}(\underline{i},\underline{\ell})$.

Sublemma 5.4. — *If $s \geqslant n_2+n_3-1$, the following equivalence holds:*

$$(\mathit{hol}(\underline{k}) \text{ for all } \underline{k} \text{ such that } s(\underline{k}) \geqslant s-1) \Leftrightarrow ([\underline{k}] = 0 \text{ for all } \underline{k} \text{ such that } s(\underline{k}) \geqslant s).$$

Proof. — We proceed by decreasing induction on s. For $s > n_1+n_2+n_3-3$, the result is trivial. Fix s such that $n_2+n_3-1 \leqslant s \leqslant n_1+n_2+n_3-3$. Let $\underline{k} = (k_1,k_2,k_3)$ satisfy $s(\underline{k}) = s$. If $k_1 = 0$, the condition on s implies $[\underline{k}] = 0$. Assume $k_1 > 0$. Consider the relation:

$$\mathit{hol}(k_1-1,k_2,k_3): \quad z_2z_3[k_1,k_2,k_3] + z_1z_3[k_1-1,k_2+1,k_3] + z_1z_2[k_1-1,k_2,k_3+1] = 0.$$

By increasing induction on k_1, we can replace this relation by $[k_1,k_2,k_3] = 0$. This uses all the relations $\mathit{hol}(\underline{k})$ for $s(\underline{k}) = s-1$, except the relations $\mathit{hol}(n_1-1,k_2,k_3)$, but those get trivial. □

Proof of Lemma 5.3. — Let $V(s)$ be the complex vector subspace of $\mathfrak{A}(\underline{i},\underline{\ell})$ generated by the $[0,h_2,h_3]$ such that $h_2+h_3 \geqslant s$. Again by decreasing induction on s, we prove that for $s \leqslant n_2+n_3-2$, $[\underline{k}] \in V(s)$ if $s(\underline{k}) = s$. Fix s such that $0 < s \leqslant n_2+n_3-2$. Consider $\underline{k} = (k_1,k_2,k_3)$ such that $s(\underline{k}) = s$ and $k_1 > 0$. By the induction hypothesis, the relation $\mathit{hol}(k_1-1,k_2,k_3)$ implies:

$$z_2z_3[k_1,k_2,k_3] + z_1z_3[k_1-1,k_2+1,k_3] + z_1z_2[k_1-1,k_2,k_3+1] \in V(s+1).$$

Conclude by increasing induction on k_1.

We have seen that the relation $\mathit{hol}(k_1-1,k_2,k_3)$ expresses $[k_1,k_2,k_3]$ in terms of the $[0,h_2,h_3]$ with $h_2+h_3 \geqslant s(\underline{k})$. These generators $[0,h_2,h_3]$ may be related by the relations $\mathit{hol}(n_1-1,k_2,k_3)$. If $n_2+n_3 \leqslant n_1+1$, there is no relation $\mathit{hol}(n_1-1,k_2,k_3)$ such that $n_1-1+k_2+k_3 < n_2+n_3-2$. If $n_2+n_3 > n_1+1$, an easy computation shows that there are $\frac{1}{2}(n_2+n_3-n_1)(n_2+n_3-n_1-1)$ pairs (k_2,k_3) of integers such that $0 \leqslant k_i < n_i$ and

$n_1-1+k_2+k_3<n_2+n_3-2$ (note that this last condition implies $k_i<n_i$ for $i=2,3$). □

Let Ξ be the set of all $(\underline{i},\underline{\ell})$ such that $1\leqslant i_1\leqslant i_2\leqslant i_3\leqslant q$, $\underline{\ell}\in J^o_{\underline{i}}$, $z_{i_1\ell_1}z_{i_2\ell_2}z_{i_3\ell_3}=1$, and for $j=1,2,3$, the multiplicity m_{i_j} is a least the number of occurences of (i_j,ℓ_j) in $((i_1,\ell_1),(i_2,\ell_2),(i_3,\ell_3))$. By Theorem 2.12:

$$\mathbb{C}\otimes\mathcal{H}=\sum_{(\underline{i},\underline{\ell})\in\Xi}\mathcal{H}(\underline{i},\underline{\ell}).$$

Recall that if $i\leqslant i'$, $m_i\geqslant m_{i'}$.

Theorem 5.5. — *For $(\underline{i},\underline{\ell})=((i_1,i_2,i_3),(\ell_1,\ell_2,\ell_3))\in\Xi$, set $b(\underline{i},\underline{\ell})=m_{i_2}m_{i_3}-\frac{1}{2}(m_{i_2}+m_{i_3}-m_{i_1})(m_{i_2}+m_{i_3}-m_{i_1}-1)$ if the (i_j,ℓ_j) are all distinct and $m_{i_2}+m_{i_3}\leqslant m_{i_1}+1$, $b(\underline{i},\underline{\ell})=m_{i_2}m_{i_3}$ if the (i_j,ℓ_j) are all distinct and $m_{i_2}+m_{i_3}>m_{i_1}+1$, $b(\underline{i},\underline{\ell})=1$ otherwise. Set:*

$$B(\underline{i},\underline{\ell})=\begin{cases}m_{i_2}m_{i_3} & \text{if the } (i_j,\ell_j) \text{ are all distinct,}\\ m_{i_3}(m_{i_1}-1) & \text{if } (i_1,\ell_1)=(i_2,\ell_2)\neq(i_3,\ell_3),\\ \frac{1}{2}m_{i_2}(m_{i_2}-1) & \text{if } (i_1,\ell_1)\neq(i_2,\ell_2)=(i_3,\ell_3),\\ \frac{1}{2}(m_{i_1}-1)(m_{i_1}-2) & \text{if } (i_1,\ell_1)=(i_2,\ell_2)=(i_3,\ell_3).\end{cases}$$

Then:

$$\sum_{(\underline{i},\underline{\ell})\in\Xi}b(\underline{i},\underline{\ell})\leqslant\dim_{\mathbb{Q}}(\mathcal{H})\leqslant\sum_{(\underline{i},\underline{\ell})\in\Xi}B(\underline{i},\underline{\ell}).$$

Proof. — We want to bound the dimension of $\mathcal{H}(\underline{i},\underline{\ell})$. If the (i_j,ℓ_j) are all distinct, this is done in Lemma 5.3. In the other cases, the non-triviality is given by Theorem 2.12, and it remains to compute the upper bound.

First assume that $(i_1,l_1)=(i_2,l_2)\neq(i_3,l_3)$. In this case, Lemma 5.3 and the anti-symmetry imply that $\mathcal{H}(\underline{i},\underline{\ell})$ is generated by the $[0,k_2,k_3]_{\mathcal{H}}$ such that $0<k_2<m_{i_2}$ and $0\leqslant k_3<m_{i_3}$. Hence $\dim(\mathcal{H}(\underline{i},\underline{\ell}))\leqslant m_{i_3}(m_{i_2}-1)$.

Now assume that $(i_1,l_1)\neq(i_2,l_2)=(i_3,l_3)$. Then $\mathcal{H}(\underline{i},\underline{\ell})$ is generated by the $[0,k_2,k_3]_{\mathcal{H}}$ such that $0\leqslant k_2<k_3<m_{i_2}$. Hence $\dim(\mathcal{H}(\underline{i},\underline{\ell}))\leqslant\frac{1}{2}m_{i_2}(m_{i_2}-1)$.

Finally assume that $(i_1,l_1)=(i_2,l_2)=(i_3,l_3)$. Then $\mathcal{H}(\underline{i},\underline{\ell})$ is generated by the $[0,k_2,k_3]_{\mathcal{H}}$ such that $0<k_2<k_3<m_{i_1}$. Hence $\dim(\mathcal{H}(\underline{i},\underline{\ell}))\leqslant\frac{1}{2}(m_{i_1}-1)(m_{i_1}-2)$. □

Examples.

(1) Let $\mathfrak{A}=\dfrac{\mathbb{Q}[t^{\pm1}]}{((t^2+1)^3)}\eta_1\oplus\dfrac{\mathbb{Q}[t^{\pm1}]}{((t+1)^2)}\eta_2$. Then:

$$\mathbb{C}\otimes\mathfrak{A}=\frac{\mathbb{C}[t^{\pm1}]}{((t-i)^3)}\eta_{11}\oplus\frac{\mathbb{C}[t^{\pm1}]}{((t+i)^3)}\eta_{12}\oplus\frac{\mathbb{C}[t^{\pm1}]}{((t+1)^2)}\eta_{21}.$$

The space $\mathfrak{A}(\underline{i},\underline{\ell})$ is non trivial for $\underline{i}=(1,1,2)$ and $\underline{\ell}=(1,1,1)$ or $(2,2,1)$. The treatement of both cases is the same. Lemma 5.3 gives $5\leqslant \dim(\mathfrak{A}(\underline{i},\underline{\ell}))\leqslant 6$. Moreover, the proof provides the following presentation:

$$\mathfrak{A}(\underline{i},\underline{\ell})=\frac{\mathbb{C}\langle[0,0,0],[0,0,1],[0,1,0],[0,1,1],[0,2,0],[0,2,1]\rangle}{\mathbb{C}\langle hol(2,0,0)\rangle}.$$

Writing down all the relations $hol(\underline{k})$ for $0\leqslant k_j<n_j$ and $s(\underline{k})=2$, we see that $hol(2,0,0)$ implies $[0,2,1]=0$. Finally $\dim_{\mathbb{C}}(\mathfrak{A}(\underline{i},\underline{\ell}))=5$, and $\dim_{\mathbb{Q}}(\mathfrak{A}_h)=10$.

By Theorem 5.5, $1\leqslant \dim(\mathcal{H}(\underline{i},\underline{\ell}))\leqslant 4$. Since $\mathcal{H}(\underline{i},\underline{\ell})$ it is a quotient of $\mathfrak{A}(\underline{i},\underline{\ell})$, $[\underline{k}]_{\mathcal{H}}=0$ if $s(\underline{k})\geqslant 3$. Using the anti-symmetry, we obtain:

$$\mathcal{H}(\underline{i},\underline{\ell})=\frac{\mathbb{C}\langle[0,1,0],[0,1,1],[0,2,0]\rangle}{\mathbb{C}\langle hol(0,1,0)\rangle}.$$

The relation $hol(0,1,0)$ implies $[0,1,1]_{\mathcal{H}}=\pm i\,[0,2,0]_{\mathcal{H}}$, hence $\dim_{\mathbb{C}}(\mathcal{H}(\underline{i},\underline{\ell}))=2$, and $\dim_{\mathbb{Q}}(\mathcal{H})=4$.

(2) For $\mathfrak{A}=\dfrac{\mathbb{Q}[t^{\pm 1}]}{((t+1+t^{-1})^m)}$, we consider $\mathfrak{A}(\underline{i},\underline{\ell})$ for $\underline{i}=(1,1,1)$ and $\underline{\ell}=(1,1,1)$ or $(2,2,2)$. By Lemma 5.3:

$$\frac{1}{2}m(m+1)\leqslant \dim(\mathfrak{A}(\underline{i},\underline{\ell}))\leqslant m^2.$$

For $m>1$, this does not give the exact dimension. For low values of m, it can be computed by hand following the method of Lemma 5.3. The space $\mathfrak{A}(\underline{i},\underline{\ell})$ is generated by the $[0,k_2,k_3]$ up to the relations $hol(m-1,k_2,k_3)$ for $k_2+k_3\leqslant m-2$. We obtain:

$$\dim(\mathfrak{A}(\underline{i},\underline{\ell}))=\begin{cases}3 & \text{if } m=2\\ 7 & \text{if } m=3\\ 12 & \text{if } m=4\end{cases}.$$

Now consider $\mathcal{H}(\underline{i},\underline{\ell})$ for $m\geqslant 3$. It is non trivial and of dimension at most $\frac{1}{2}(m-1)(m-2)$. Once again, the dimension can be computed by hand for low values of m. The same argument as in Sublemma 5.4 shows that $[\underline{k}]_{\mathcal{H}}=0$ if $s(\underline{k})\geqslant 2m-2$. Hence $\mathcal{H}(\underline{i},\underline{\ell})$ is generated by the $[\underline{k}]_{\mathcal{H}}$ with $0\leqslant k_1<k_2<k_3<m$ and $s(\underline{k})\leqslant 2m-3$, up to the relations $hol(k_1,k_2,k_3)$ with $0\leqslant k_1<k_2<k_3<m$ and $s(\underline{k})\leqslant 2m-4$.

m	3	4	5	6	7
$\dim(\mathcal{H}(\underline{i},\underline{\ell}))$	1	1	2	3	4

6. Decomposition and characterization of ϕ

6.1. Realization of rational homology classes by knots

The goal of this section is to prove Theorem 2.10 which identifies the set of equivariant triple intersection maps with the space $\mathcal{H}$. We first prove the following proposition which reduces Theorem 2.10 to an algebraic problem.

Fix a Blanchfield module $(\mathfrak{A}, \mathfrak{b})$.

Definition 6.1. — *Let $(M, K, \xi) \in \mathcal{P}^m(\mathfrak{A}, \mathfrak{b})$. Let $\widetilde{X}$ be the infinite cyclic covering associated with (M, K). A homology class $\eta \in \mathfrak{A}$ is* realizable for *(M, K, ξ) if there is a knot J in $\widetilde{X}$ such that $[J] = \eta$.*

Proposition 6.2. — *Let $(M, K, \xi) \in \mathcal{P}^m(\mathfrak{A}, \mathfrak{b})$. For all $\eta \in \mathfrak{A}$, there is a marked $\mathbb{Q}$SK-pair $(M', K', \xi') \in \mathcal{P}^m(\mathfrak{A}, \mathfrak{b})$ such that $\phi^{(M',K',\xi')} = \phi^{(M,K,\xi)}$ and η is realizable for (M', K', ξ').*

Remark. — For any marked $\mathbb{Q}$SK-pair (M, K, ξ), there are infinitely many elements of the Alexander module that are not realizable in (M, K, ξ). Indeed, the integral Alexander module $H_1(\widetilde{X}, \mathbb{Z})$ is finitely generated over $\mathbb{Z}[t^{\pm 1}]$, and we have $\mathfrak{A}(M, K) = \mathbb{Q} \otimes H_1(\widetilde{X}, \mathbb{Z})$.

In order to prove the above proposition, we introduce a specific kind of LP-surgeries. Recall LP-surgeries were defined in Subsection 2.4.

Definition 6.3. — *For $d \in \mathbb{N} \setminus \{0\}$, a* d-torus *is a rational homology torus T_d such that there are simple closed curves α, β in ∂T_d, and γ in T_d which satisfy:*

- $\langle \alpha, \beta \rangle_{\partial T_d} = 1$,
- $H_1(\partial T_d; \mathbb{Z}) = \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\beta]$,
- $H_1(T_d; \mathbb{Z}) = \frac{\mathbb{Z}}{d\mathbb{Z}}[\alpha] \oplus \mathbb{Z}[\gamma]$,
- $[\beta] = d[\gamma]$.

A meridian of T_d *is a simple closed curve on ∂T_d homologous to α.*
A (null) d-surgery *is a (null) LP-surgery $(\frac{T_d}{T})$ where T is a standard solid torus and T_d is a d-torus.*

For any $d \in \mathbb{N} \setminus \{0\}$, there exists a d-torus (in [12, Lem. 2.5], such a d-torus is constructed by a relevant gluing of a 2-handle on a standard genus 2 handlebody).

Lemma 6.4. — *Let T_d be a d-torus. Let m_1, m_2, m_3 be disjoint meridians of T_d. There are rational 2-chains S_1, S_2, S_3 in T_d such that $\partial S_j = dm_j$*

for $j = 1, 2, 3$. For any such chains, pairwise transverse, the triple intersection number $\langle S_1, S_2, S_3\rangle$ is trivial.

Proof. — The existence of the S_j is clear since $d[m_j] = 0$ in $H_1(T_d; \mathbb{Z})$. Let us check that $\langle S_1, S_2, S_3\rangle$ does not depend on the choice of the S_j. Replace S_1 by a chain S_1' satisfying the same conditions. Since $H_2(T_d; \mathbb{Q}) = 0$, there is a rational 3-chain C such that $\partial C = S_1' - S_1$. We have:

$$\langle S_1', S_2, S_3\rangle - \langle S_1, S_2, S_3\rangle = \langle \partial C, S_2, S_3\rangle.$$

By Lemma 3.2:

$$\langle \partial C, S_2, S_3\rangle = \langle C, \partial S_2, S_3\rangle - \langle C, S_2, \partial S_3\rangle.$$

Since $m_2 \cap S_3 = \emptyset$ and $S_2 \cap m_3 = \emptyset$, we obtain $\langle S_1', S_2, S_3\rangle = \langle S_1, S_2, S_3\rangle$.

Let $N = [0, 1] \times S^1 \times S^1$ be a collar neighborhood of ∂T_d in T_d, parametrized so that:

- $\{1\} \times S^1 \times S^1 = \partial T_d$,
- for $j = 1, 2, 3$, $m_j = \{1\} \times S^1 \times \{z_j\}$ with $z_j \in S^1$.

Consider the 2-chains S_j in the homeomorphic copy $\overline{T_d \setminus N}$ of T_d, so that:

- for $j = 1, 2, 3$, $\partial S_j = dm_j^0$, where $m_j^0 = \{0\} \times S^1 \times \{z_j\}$.

For $j = 1, 2, 3$, let A_j be the annulus in N whose slice is represented in Figure 6.1. Set $S_j' = S_j + dA_j$.

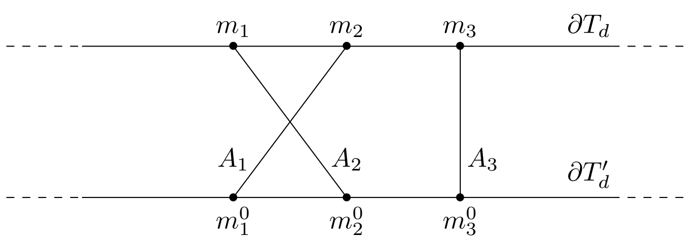


Figure 6.1. Slices of the annuli A_j in N.

Since $\partial S_1' = dm_2$, $\partial S_2' = dm_1$, and $\partial S_3' = dm_3$, the independance with respect to the surfaces implies $\langle S_2', S_1', S_3'\rangle = \langle S_1, S_2, S_3\rangle$. But by construction, $\langle S_1', S_2', S_3'\rangle = \langle S_1, S_2, S_3\rangle$. Finally, $\langle S_1, S_2, S_3\rangle = 0$. ☐

Lemma 6.5. — *Null d-surgeries on marked $\mathbb{Q}SK$-pairs preserve the equivariant triple intersection map.*

Proof. — Let $(M,K,\xi)\in\mathcal{P}^m(\mathfrak{A},\mathfrak{b})$. Let $(\frac{T_d}{T})$ be a null d-surgery defined on (M,K,ξ). Let $\widetilde{T}$ be a lift of T in the infinite cyclic covering $\widetilde{X}$ associated with (M,K). The infinite cyclic covering $\widetilde{X}'$ associated with $(M,K,\xi)(\frac{T_d}{T})$ is obtained from $\widetilde{X}$ by the surgeries $(\frac{T_d^{(k)}}{\tau^k(\widetilde{T})})$ for all $k\in\mathbb{Z}$, where the $T_d^{(k)}$ are copies of T_d. Note that $\mathfrak{A}$ is generated over $\mathbb{Q}$ by the homology classes which are realizable by simple closed curves in $\widetilde{X}\setminus\sqcup_{k\in\mathbb{Z}}\tau^k(\widetilde{T})$. Hence it suffices to prove that the triple equivariant intersection is preserved for the homology classes of disjoint knots μ_1, μ_2, μ_3 in $\widetilde{X}\setminus\sqcup_{k\in\mathbb{Z}}\tau^k(\widetilde{T})$. Let Σ_1, Σ_2, Σ_3, be τ-transverse rational 2-chains, τ-transverse to $\widetilde{T}$, such that $\partial\Sigma_i=\delta(\tau)\mu_i$. Assume no τ-translate of $\widetilde{T}$ meets any of the pairwise intersections of the τ-translates of the Σ_i. The 2-chains $\Sigma_i'=\Sigma_i\cap(\widetilde{X}\setminus\sqcup_{k\in\mathbb{Z}}\tau^k(\widetilde{T}))$ are preserved by the surgery. The boundary of Σ_i' in $\widetilde{X}'$ is the sum of $\delta(\tau)\mu_i$ and of a $\mathbb{Q}$-linear combination of meridians of the $T_d^{(k)}$. Use Lemma 6.4 to add to the Σ_i' rational 2-chains in the $T_d^{(k)}$ so that their boundaries reduce to $\delta(\tau)\mu_i$, without adding triple intersection points. □

Proof of Proposition 6.2. — Let $\eta\in\mathfrak{A}$. Let d be a positive integer such that $d\eta$ is realizable for (M,K,ξ). Let $\widetilde{J}$ be a knot in the infinite cyclic covering $\widetilde{X}$ associated with (M,K), whose image J in $M\setminus K$ is also a knot, and such that $[\widetilde{J}]=d\eta$. Let $T(J)$ be a tubular neighborhood of J which lifts to a tubular neighborhood $T(\widetilde{J})$ of $\widetilde{J}$. Let T_d be a d-torus. Fix an LP-identification of ∂T_d and $\partial T(J)$. Set $(M',K',\xi')=(M,K,\xi)(\frac{T_d}{T(J)})$. The covering $\widetilde{X}'$ can be obtained from $\widetilde{X}$ by simultaneous surgeries $\left(\frac{T_d^{(k)}}{\tau^k(T(\widetilde{J}))}\right)$ for all $k\in\mathbb{Z}$, where the $T_d^{(k)}$ are copies of T_d. Let $\gamma\subset T_d$ be a knot such that $d[\gamma]=[\ell(J)]$, where $\ell(J)$ is a parallel of J in $\partial T(J)$ (which is preserved by the surgery). Note that all the parallels of J have the same rational homology class in (M',K') as well as in (M,K). Let $\widetilde{\Gamma}$ be the lift of γ in $T_d^{(0)}$, so that $d[\widetilde{\Gamma}]=[\ell(\widetilde{J})]$, where $\ell(\widetilde{J})$ is the lift of $\ell(J)$ in $\partial T_d^{(0)}$. We have $[\widetilde{\Gamma}]=\eta$. Conclude with Lemma 6.5. □

6.2. Study of the map ϕ

In this subsection, we decompose the equivariant triple intersection map and we study the target spaces in order to prove Theorem 2.10. Fix a Blanchfield module $(\mathfrak{A},\mathfrak{b})$. Let δ be the normalized annihilator of $\mathfrak{A}$. Define a decomposition of $\mathfrak{A}$ and associated notation as in Subsection 2.3.

For $\underline{i} \in \{1, \dots, q\}^3$, set:

$$\mathcal{R}(\underline{i}) = \frac{\mathcal{R}}{(t_1t_2t_3 - 1, \delta_{i_1}(t_1), \delta_{i_2}(t_2), \delta_{i_3}(t_3))}.$$

Define a structure of $\mathcal{R}(\underline{i})$-module on $\mathfrak{A}(\underline{i})$ by:

$$t_1^{k_1}t_2^{k_2}t_3^{k_3}. \otimes_{1\leqslant j\leqslant 3} \beta_j = \otimes_{1\leqslant j\leqslant 3} t^{k_j}\beta_j.$$

Then $\mathfrak{A}(\underline{i})$ is a free cyclic $\mathcal{R}(\underline{i})$-module generated by $\eta_{\underline{i}} := \eta_1 \otimes \eta_2 \otimes \eta_3$.

Lemmas 2.3 and 2.4 imply:

Proposition 6.6. — *Let $(M,K,\xi) \in \mathcal{P}^m(\mathfrak{A},\mathfrak{b})$. Let $\widetilde{X}$ be the infinite cyclic covering associated with (M,K). Let $\underline{i} = (i_1,i_2,i_3) \in \{1,\dots,q\}^3$. Define a $\mathbb{Q}$-linear map $\phi_{\underline{i}}^{(M,K,\xi)} : \mathfrak{A}(\underline{i}) \to \mathcal{R}(\underline{i})$ as follows. If μ_1, μ_2, μ_3 are knots in $\widetilde{X}$ whose images in $M \setminus K$ are pairwise disjoint and such that $[\mu_j] \in \mathfrak{A}_{i_j}$ for $j = 1,2,3$, let Σ_1, Σ_2, Σ_3 be τ-transverse rational 2-chains such that $\partial\Sigma_j = \delta_{i_j}(\tau)\mu_j$, and set*

$$\phi_{\underline{i}}^{(M,K,\xi)}([\mu_1]\otimes[\mu_2]\otimes[\mu_3]) = \langle \Sigma_1,\Sigma_2,\Sigma_3\rangle_e.$$

Then the map $\phi_{\underline{i}}^{(M,K,\xi)}$ is well-defined and $\mathcal{R}(\underline{i})$-linear.

When it does not seem to cause confusion, the map $\phi_{\underline{i}}^{(M,K,\xi)}$ (resp. $\phi^{(M,K,\xi)}$) is denoted by $\phi_{\underline{i}}$ (resp. ϕ). Note that the maps $\phi_{\underline{i}}$ depend on the decomposition of $\mathfrak{A}$ and on the normalization of the δ_i's.

It is easy to see that the maps $\phi_{\underline{i}}$ and ϕ are related by:

$$\phi(\beta_1\otimes\beta_2\otimes\beta_3) = \frac{\delta(t_1)\delta(t_2)\delta(t_3)}{\delta_{i_1}(t_1)\delta_{i_2}(t_2)\delta_{i_3}(t_3)}\phi_{\underline{i}}(\beta_1\otimes\beta_2\otimes\beta_3)$$

for $\beta_1\otimes\beta_2\otimes\beta_3 \in \mathfrak{A}(\underline{i})$. This implies in particular that $\phi(\mathfrak{A}(\underline{i}))$ is contained in the ideal of $\mathcal{R}_\delta$ generated by $\frac{\delta(t_1)\delta(t_2)\delta(t_3)}{\delta_{i_1}(t_1)\delta_{i_2}(t_2)\delta_{i_3}(t_3)}$. Let $\hat{\mathbf{\Phi}}$ be the set of all $\phi \in \mathbf{\Phi}$ which satisfy this condition. For any $\phi \in \hat{\mathbf{\Phi}}$, the above relation defines associated maps $\phi_{\underline{i}} : \mathfrak{A}(\underline{i}) \to \mathcal{R}(\underline{i})$.

Note that the linearity implies that the map ϕ is encoded in the datum of the family of the $\phi(\eta_{\underline{i}})$, or equivalently of the $\phi_{\underline{i}}(\eta_{\underline{i}})$. For $\underline{i}$ fixed, the map $\phi_{\underline{i}}$ is encoded in $\phi_{\underline{i}}(\eta_{\underline{i}})$.

For $\underline{i}$ such that the i_j are all distinct, we will see below that any element of $\mathcal{R}(\underline{i})$ is a $\phi_{\underline{i}}^{(M,K,\xi)}(\eta_{\underline{i}})$ for some marked $\mathbb{Q}$SK-pair (M,K,ξ). In general, the image may be restricted in the following sense. There is a surjective map $p_{\underline{i}} : \mathcal{R}(\underline{i}) \twoheadrightarrow \mathcal{H}(\underline{i})$ given by $p_{\underline{i}}(t_1^{k_1}t_2^{k_2}t_3^{k_3}) = t^{k_1}\eta_{i_1} \wedge t^{k_2}\eta_{i_2} \wedge t^{k_3}\eta_{i_3}$. It corresponds to the natural projection $\mathfrak{A}(\underline{i}) \twoheadrightarrow \mathcal{H}(\underline{i})$ via the isomorphism $\mathcal{R}(\underline{i}) \cong \mathfrak{A}(\underline{i})$ given by $t_1^{k_1}t_2^{k_2}t_3^{k_3} \mapsto t^{k_1}\eta_{i_1} \otimes t^{k_2}\eta_{i_2} \otimes t^{k_3}\eta_{i_3}$. Note that $\ker(p_{\underline{i}})$ is not an ideal of $\mathcal{R}(\underline{i})$, and that we cannot define an $\mathcal{R}(\underline{i})$-module structure on $\mathcal{H}(\underline{i})$ as we did on $\mathfrak{A}(\underline{i})$. The following lemma implies that we do not lose information when composing the map $\phi_{\underline{i}}$ by the surjection $p_{\underline{i}}$.

Lemma 6.7. — *Let $\underline{i} = (i_1, i_2, i_3) \in \{1, \dots, q\}^3$. There is a rational vector subspace $\mathcal{R}(\underline{i})^a$ of $\mathcal{R}(\underline{i})$, which contains $\phi_{\underline{i}}(\eta_{\underline{i}})$ for all $\phi \in \hat{\mathbf{\Phi}}$, such that $p_{\underline{i}}$ induces an isomorphism $\mathcal{R}(\underline{i})^a \cong \mathcal{H}(\underline{i})$.*

Proof. — If the i_j are all distinct, the map $p_{\underline{i}}$ is an isomorphism, and $\mathcal{R}(\underline{i})^a = \mathcal{R}(\underline{i})$. Assume the i_j are not all distinct.

Set:

$$\mathcal{S} = \{\sigma \in \mathcal{S}_3 \text{ such that } i_{\sigma(j)} = i_j \text{ for } j = 1, 2, 3\} \subset \mathcal{S}_3,$$

$$\mathcal{R}^a = \{P \in \mathcal{R} \,|\, P(t_{\sigma(1)}, t_{\sigma(2)}, t_{\sigma(3)}) = \varepsilon(\sigma) P(t_1, t_2, t_3) \;\forall\, \sigma \in \mathcal{S}\},$$

and let $\mathcal{R}^s$ be the rational vector subspace of $\mathcal{R}$ generated by the polynomials $P \in \mathcal{R}$ such that $P(t_{\tau(1)}, t_{\tau(2)}, t_{\tau(3)}) = P(t_1, t_2, t_3)$ for some transposition $\tau \in \mathcal{S}$.

Sublemma 6.8. — $\mathcal{R} = \mathcal{R}^s \oplus \mathcal{R}^a$.

Proof of Sublemma 6.8. — Let $P \in \mathcal{R}$. Set:

$$P^a(t_1, t_2, t_3) = \frac{1}{|\mathcal{S}|} \sum_{\sigma \in \mathcal{S}} \varepsilon(\sigma) P(t_{\sigma(1)}, t_{\sigma(2)}, t_{\sigma(3)}),$$

where $|\,.\,|$ stands for the cardinality. We have $P^a \in \mathcal{R}^a$.

We shall check that $\mathcal{R}^s \cap \mathcal{R}^a = 0$ and that for $P \in \mathcal{R}$, $P - P^a$ is in $\mathcal{R}^s$. It is clear if $\mathcal{S} \neq \mathcal{S}_3$. Assume $\mathcal{S} = \mathcal{S}_3$.

Let $P \in \mathcal{R}^s \cap \mathcal{R}^a$. Since $P \in \mathcal{R}^a$, $P = P^a$, and since $P \in \mathcal{R}^s$, $P = P_{12} + P_{13} + P_{23}$, where each P_{ij} is invariant under the transposition (ij). We have $P^a = P_{12}^a + P_{13}^a + P_{23}^a$, and each term in this sum is trivial. Hence $P = 0$.

For $P(t_1, t_2, t_3) = t_1^{k_1} t_2^{k_2} t_3^{k_3}$, with $(k_1, k_2, k_3) \in \mathbb{Z}^3$, we have:

$$\begin{aligned} P(t_1,t_2,t_3) &- P^a(t_1,t_2,t_3) \\ &= \frac{1}{6}(t_1^{k_1} t_2^{k_3} t_3^{k_2} + t_1^{k_3} t_2^{k_1} t_3^{k_2}) + \frac{1}{3}(t_1^{k_1} t_2^{k_2} t_3^{k_3} + t_1^{k_2} t_2^{k_1} t_3^{k_3}) \\ &\quad - \frac{1}{6}(t_1^{k_2} t_2^{k_3} t_3^{k_1} + t_1^{k_2} t_2^{k_1} t_3^{k_3}) - \frac{1}{3}(t_1^{k_3} t_2^{k_1} t_3^{k_2} + t_1^{k_3} t_2^{k_2} t_3^{k_1}) \\ &\quad + \frac{1}{2}(t_1^{k_1} t_2^{k_2} t_3^{k_3} + t_1^{k_3} t_2^{k_2} t_3^{k_1}). \end{aligned}$$

In this expression, each parenthesized term is invariant under some transposition. Finally $\mathcal{R} = \mathcal{R}^s \oplus \mathcal{R}^a$. □

Let $\mathcal{I}$ be the ideal $(t_1t_2t_3 - 1, \delta_{i_1}(t_1), \delta_{i_2}(t_2), \delta_{i_3}(t_3)) \subset \mathcal{R}$. We have:

$$\mathcal{R}(\underline{i}) = \frac{\mathcal{R}}{\mathcal{I}}.$$

Set $\mathcal{I}^s = \mathcal{I} \cap \mathcal{R}^s$ and $\mathcal{I}^a = \mathcal{I} \cap \mathcal{R}^a$.

Sublemma 6.9. — $\mathcal{I} = \mathcal{I}^s \oplus \mathcal{I}^a$.

Proof of Sublemma 6.9. — It is clear that $\mathcal{I}^s \cap \mathcal{I}^a = 0$. Let $P \in \mathcal{I}$. Writing P as a combination of the generators of $\mathcal{I}$, we see that $P(t_{\sigma(1)}, t_{\sigma(2)}, t_{\sigma(3)}) \in \mathcal{I}$ for all $\sigma \in \mathcal{S}$. Hence $P^a \in \mathcal{I}^a$, and it follows that $P - P^a \in \mathcal{I}^s$. □

We finally have the decomposition

$$\mathcal{R}(\underline{i}) = \mathcal{R}(\underline{i})^s \oplus \mathcal{R}(\underline{i})^a,$$

where $\mathcal{R}(\underline{i})^s = \dfrac{\mathcal{R}^s}{\mathcal{I}^s}$ and $\mathcal{R}(\underline{i})^a = \dfrac{\mathcal{R}^a}{\mathcal{I}^a}$. Since $\mathcal{R}(\underline{i})^a \cong \dfrac{\mathcal{R}(\underline{i})}{\mathcal{R}(\underline{i})^s}$ and $\mathcal{R}(\underline{i})^s \subset \ker(p_{\underline{i}})$, we have the following commutative diagram of rational vector spaces, where the isomorphism $\mathcal{R}(\underline{i}) \xrightarrow{\cong} \mathfrak{A}(\underline{i})$ is given by $t_1^{k_1}t_2^{k_2}t_3^{k_3} \mapsto t_1^{k_1}t_2^{k_2}t_3^{k_3}.\eta_{\underline{i}}$. This isomorphism identifies $\mathcal{R}(\underline{i})^s$ with the subspace of $\mathfrak{A}(\underline{i})$ generated by the anti-symmetry relations. Hence $p_{\underline{i}|\mathcal{R}(\underline{i})^a}$ also is an isomorphism.

$$\begin{array}{ccc} \mathcal{R}(\underline{i}) & \twoheadrightarrow & \mathcal{R}(\underline{i})^a \\ \cong\downarrow & \searrow^{p_{\underline{i}}} & \downarrow\cong \\ \mathfrak{A}(\underline{i}) & \twoheadrightarrow & \mathcal{H}(\underline{i}) \end{array}$$

Relation $(\star)$ implies $\phi_{\underline{i}}(\eta_{\underline{i}}) \in \mathcal{R}(\underline{i})^a$. □

The map ϕ is completely determined by the $\phi_{\underline{i}}(\eta_{\underline{i}})$ for $\underline{i} = (i_1, i_2, i_3)$ such that $i_1 \leqslant i_2 \leqslant i_3$. Since $\mathcal{H}$ is the direct sum of the $\mathcal{H}(\underline{i})$ for these $\underline{i}$, the above lemma implies that the datum of ϕ is finally encoded in the element $\mathfrak{h}(\phi) := \sum_{\underline{i}\in\Xi} p_{\underline{i}} \circ \phi_{\underline{i}}(\eta_{\underline{i}})$. This holds for any $\phi \in \hat{\mathbf{\Phi}}$, hence we obtain an injective map $\mathfrak{h} : \hat{\mathbf{\Phi}} \hookrightarrow \mathcal{H}$. Note that this map depends on the choice of a decomposition $\mathfrak{A} = \bigoplus_{1\leqslant i\leqslant q} \mathfrak{A}_i$. To obtain Theorem 2.10, it remains to prove the following lemma.

Lemma 6.10. — *The map $\hbar = \mathfrak{h} \circ \phi^\bullet : \mathcal{P}^m(\mathfrak{A}, \mathfrak{b}) \to \mathcal{H}$ is surjective.*

Proof. — We prove that, for $\underline{i} = (i_1, i_2, i_3) \in \{1, \dots, q\}^3$, any element of $\mathcal{R}(\underline{i})^a$ is equal to $\phi_{\underline{i}}^{(M,K,\xi)}(\eta_{\underline{i}})$ for some $(M, K, \xi) \in \mathcal{P}^m(\mathfrak{A}, \mathfrak{b})$.

Let $(M, K, \xi) \in \mathcal{P}^m(\mathfrak{A}, \mathfrak{b})$. We shall prove that, for any $r \in \mathbb{Q}$ and $(k_1, k_2, k_3) \in \mathbb{Z}^3$, there is a Y-graph Γ, null in $M \setminus K$, such that

$$\phi_{\underline{i}}^{(M,K,\xi)(\Gamma)}(\eta_{\underline{i}}) - \phi_{\underline{i}}^{(M,K,\xi)}(\eta_{\underline{i}}) = r \sum_{\sigma\in\mathcal{S}} \varepsilon(\sigma) \prod_{1\leqslant j\leqslant 3} t_j^{k_{\sigma(j)}}, \tag{6.1}$$

where $\mathcal{S} = \{\sigma \in \mathcal{S}_3$ such that $i_{\sigma(j)} = i_j$ for $j = 1,2,3\} \subset \mathcal{S}_3$. Since these differences generate $\mathcal{R}(\underline{i})^a$ as an additive group, this will prove that we can obtain any element of $\mathcal{R}(\underline{i})^a$.

Fix $r \in \mathbb{Q}$ and $(k_1,k_2,k_3) \in \mathbb{Z}^3$. Let Γ be a Y-graph, null in $M \setminus K$. Let $\widetilde{\Gamma}$ be a lift of Γ in the infinite cyclic covering $\widetilde{X}$ associated with (M,K). Let γ_1, γ_2, γ_3 be the homology classes in $\mathfrak{A}$ of the leaves of $\widetilde{\Gamma}$, given in an order induced by the orientation of the internal vertex of Γ. By Proposition 2.8

$$\phi_{\underline{i}}^{(M,K,\xi)(\Gamma)}(\eta_{\underline{i}}) - \phi_{\underline{i}}^{(M,K,\xi)}(\eta_{\underline{i}}) = \sum_{\sigma\in\mathcal{S}_3} \varepsilon(\sigma) \prod_{j=1}^{3} \delta_{i_j}(t_j)\mathfrak{b}(\eta_{i_j},[\gamma_{\sigma(j)}])(t_j).$$

Set $\beta_j = t^{-k_j} a_{i_j}^{-1}(t^{-1}) d(\eta_{i_j})$ for $j = 1,2,3$, where the inverse of a_{i_j} is defined modulo δ_{i_j}. We want to choose Γ such that $[\gamma_1] = r\beta_1$ and $[\gamma_j] = \beta_j$ for $j = 2,3$. These homology classes may not be realizable for (M,K,ξ). Use Proposition 6.2 to replace (M,K,ξ) with another marked $\mathbb{Q}$SK-pair (still denoted by (M,K,ξ)), so that the map ϕ remains unchanged and the required homology classes are realizable. Then we can define Γ as desired and we obtain Equality (6.1).

This completes the proof since Proposition 2.8 implies that $\phi_{\underline{i}'}(\eta_{\underline{i}'})$ is modified by the surgery on Γ if and only if $\underline{i}'$ is a permutation of $\underline{i}$. □

7. Degree one invariants of marked $\mathbb{Q}$SK-pairs

7.1. The Borromean subquotient

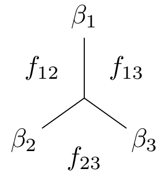
Fix a Blanchfield module $(\mathfrak{A},\mathfrak{b})$. Let $\mathcal{F}_1^{m,b}(\mathfrak{A},\mathfrak{b})$ be the rational vector subspace of $\mathcal{F}_1^{m}(\mathfrak{A},\mathfrak{b})$ generated by the brackets $[(M,K,\xi);\frac{B}{A}]$ where $(\frac{B}{A})$ is a Borromean surgery. Let $\mathcal{G}_1^{m,b}(\mathfrak{A},\mathfrak{b})$ be the image of $\mathcal{F}_1^{m,b}(\mathfrak{A},\mathfrak{b})$ in the quotient $\mathcal{G}_1^{m}(\mathfrak{A},\mathfrak{b})$. In this subsection, we study $\mathcal{G}_1^{m,b}(\mathfrak{A},\mathfrak{b})$ and we prove:

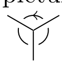
Proposition 7.1. — *Set* $\mathcal{H} = \dfrac{\Lambda^3\mathfrak{A}}{(\beta_1\wedge\beta_2\wedge\beta_3 = t\beta_1\wedge t\beta_2\wedge t\beta_3)}$. *The map* $\hbar : \mathcal{P}^m(\mathfrak{A},\mathfrak{b}) \to \mathcal{H}$ *of Proposition 2.18 induces an isomorphism* $\mathcal{G}_1^{m,b}(\mathfrak{A},\mathfrak{b}) \cong \mathcal{H}$.

The main point of the proof is the construction of a well-defined map $\varphi : \mathcal{H} \to \mathcal{G}_1^{m,b}(\mathfrak{A},\mathfrak{b})$.

Let $(M,K,\xi) \in \mathcal{P}^m(\mathfrak{A},\mathfrak{b})$. For a Y-graph Γ null in $M \setminus K$, the bracket in $\mathcal{F}_1^{m,b}(\mathfrak{A},\mathfrak{b})$ associated with the surgery along Γ is denoted by $[(M,K,\xi);\Gamma]$.

Define a *Y-diagram* as a unitrivalent graph with one oriented trivalent vertex and three univalent vertices, equipped with the following labellings:



where $\beta_i \in \mathfrak{A}$, and the $f_{ij} \in \mathbb{Q}(t)$ satisfy $f_{ij} \mod \mathbb{Q}[t^{\pm 1}] = \mathfrak{b}(\beta_i, \beta_j)$. In the pictures, the orientation of the trivalent vertex is given by the cyclic order  .

We wish to realize Y-diagrams by Y-graphs in $M \setminus K$. Let Γ be a Y-graph null in $M \setminus K$. Fix a lift $p \in \widetilde{X}$ of the internal vertex of Γ. If ℓ is a leaf of Γ, let $\hat{\ell}$ be the extension of ℓ in Γ (see Figure 7.1), and let $\tilde{\ell}$ be the lift of

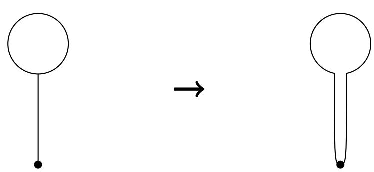


Figure 7.1. Extension of a leaf in a Y-graph

$\hat{\ell}$ with basepoint p. The null Y-graph Γ is a *realization of D in (M, K, ξ)* if there is a numbering ℓ_1, ℓ_2, ℓ_3 of its leaves, coherent with the cyclic order of its internal vertex, such that the following conditions are satisfied:

- for all i, $[\tilde{\ell}_i] = \beta_i$,
- for all $i < j$, $lk_e(\tilde{\ell}_i, \tilde{\ell}_j) = f_{ij}$.

If such a realization exists, the Y-diagram D is *realizable in* (M, K, ξ). Note that the Y-diagram D is realizable if and only if each β_i is realizable for (M, K, ξ) (see Definition 6.1).

Let $(M, K, \xi) \in \mathcal{P}^m(\mathfrak{A}, \mathfrak{b})$. Let $\mathcal{F}_2^{m,b}(M, K, \xi)$ be the subset of $\mathcal{F}_2^m(\mathfrak{A}, \mathfrak{b})$ generated by the $[(M, K, \xi); \Gamma_1, \Gamma_2]$ for all Y-graphs Γ_1 and Γ_2 null in $M \setminus K$.

Lemma 7.2 ([11, Chap. 6, Lem. 2.11]). — *Let $(M, K, \xi) \in \mathcal{P}^m(\mathfrak{A}, \mathfrak{b})$. Let D be a Y-diagram. Let Γ be a realization of D in (M, K, ξ). Then the class of $[(M, K, \xi); \Gamma]$ modulo $\mathcal{F}_2^{m,b}(M, K, \xi)$ does not depend on the realization of D.*

This result allows us to define the bracket $[(M,K,\xi);D]$ for a realizable Y-diagram D as the class of $[(M,K,\xi);\Gamma]$ modulo $\mathcal{F}_2^{m,b}(M,K,\xi)$ for any realization Γ of D.

Lemma 7.3. — *Let $(M,K,\xi)\in\mathcal{P}^m(\mathfrak{A},\mathfrak{b})$. Let $k,k'\in\mathbb{Z}$. Assume that the Y-diagrams D and D' represented in Figure 7.2 are realizable in (M,K,ξ). Then D_h, D_a and D_s are realizable in (M,K,ξ) and the following relations hold:*

$$[(M,K,\xi);D]=[(M,K,\xi);D_h] \tag{Hol}$$

$$[(M,K,\xi);D]+[(M,K,\xi);D_a]=0 \tag{AS}$$

$$[(M,K,\xi);D_s]=k\,[(M,K,\xi);D]+k'\,[(M,K,\xi);D'] \tag{LV}$$

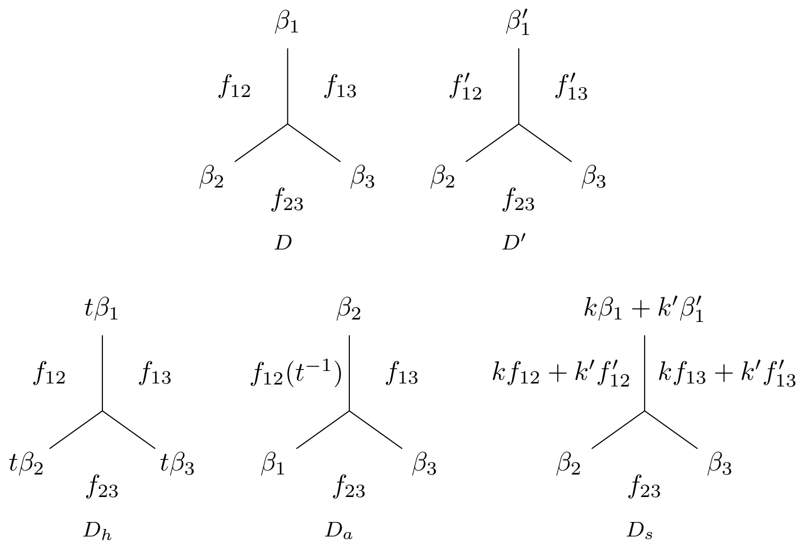
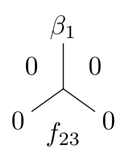


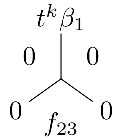
Figure 7.2. Y-diagrams

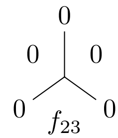
Proof. — Relation (Hol) is obtained by letting the internal vertex of a realization Γ of D turn once around the knot K. Relation (AS) follows from [3, Cor. 4.6]. Relation (LV) follows from [11, Chap. 6, Lem. 2.10]. $\square$

Lemma 7.4. — *If $D=$* 

is a Y-diagram realizable in (M,K,ξ), then

$$[(M,K,\xi);D]=0.$$

Proof. — For $k \in \mathbb{Z}$, set $D_k =$ 

. Set $D_{triv} =$ 

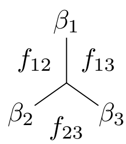
. Let $\delta(t) = \sum_{k\in\mathbb{Z}} a_k t^k$ be the annihilator of $\mathfrak{A}$ normalized with integral coefficients. By Relation (LV),

$$\sum_{k\in\mathbb{Z}} a_k[(M,K,\xi);D_k] = [(M,K,\xi);D_{triv}] = 0.$$

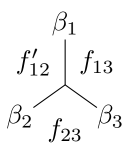
Moreover, Relation (Hol) implies $[(M,K,\xi);D] = [(M,K,\xi);D_k]$ for all $k \in \mathbb{Z}$. Finally,

$$\delta(1)[(M,K,\xi);D] = \sum_{k\in\mathbb{Z}} a_k[(M,K,\xi);D_k] = 0.$$

This completes the proof since $\delta(1) \neq 0$. □

Lemma 7.5. — *Let* $D =$ 

be a Y-diagram realizable in (M,K,ξ)*. Then the bracket* $[(M,K,\xi);D]$ *does not depend on the equivariant linking numbers* f_{ij}*.*

Proof. — Set $D' =$ 

and $P(t) = f'_{12} - f_{12} \in \mathbb{Q}[t^{\pm1}]$. Due to Relation (LV), we can assume $P(t) \in \mathbb{Z}[t^{\pm1}]$.

Let D_0, D_1, D_2 be the Y-diagrams represented in Figure 7.3.

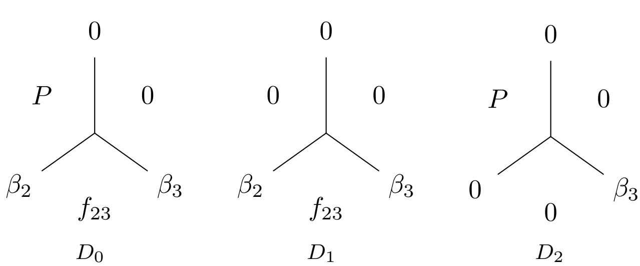


Figure 7.3. Y-diagrams

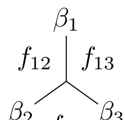
Relation (LV) implies

$$[(M,K,\xi);D'] = [(M,K,\xi);D] + [(M,K,\xi);D_0]$$

and

$$[(M,K,\xi);D_0] = [(M,K,\xi);D_1] + [(M,K,\xi);D_2].$$

Now, by Lemma 7.4, $[(M,K,\xi);D_2] = 0$, and by (LV), $[(M,K,\xi);D_1] = 0$. Hence $[(M,K,\xi);D'] = [(M,K,\xi);D]$ as desired. □

Finally, for a Y-diagram $D =$ 

, the bracket $[(M,K,\xi);D]$ depends on the β_i only. Hence the relation (AS) implies that this bracket depends on $\beta_1 \wedge \beta_2 \wedge \beta_3 \in \mathcal{H}$ only.

Now, for $\beta_1 \wedge \beta_2 \wedge \beta_3 \in \mathcal{H}$ such that each β_i is realizable for (M,K,ξ), one can define $[(M,K,\xi);\beta_1 \wedge \beta_2 \wedge \beta_3] \in \mathcal{G}_1^m(\mathfrak{A},\mathfrak{b})$ as the class of $[(M,K,\xi);D]$ in $\mathcal{G}_1^m(\mathfrak{A},\mathfrak{b})$ for any Y-diagram D whose univalent vertices are colored by β_i for $i = 1,2,3$, with the right cyclic order.

If $\beta_1 \wedge \beta_2 \wedge \beta_3$ is any tensor in $\mathcal{H}$, there are non trivial integers n_1, n_2, n_3 such that $n_i\beta_i$ is realizable for (M,K,ξ) for $i = 1,2,3$. Set

$$[(M,K,\xi);\beta_1 \wedge \beta_2 \wedge \beta_3] = \frac{1}{n_1 n_2 n_3}[(M,K,\xi);n_1\beta_1 \wedge n_2\beta_2 \wedge n_3\beta_3].$$

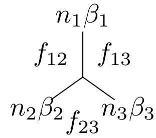
By (LV), this definition does not depend on the triple of integers (n_1,n_2,n_3).

We finally obtain a well-defined $\mathbb{Q}$-linear map

$$\begin{array}{cccc} \varphi: & \mathcal{H} & \to & \mathcal{G}_1^m(\mathfrak{A},\mathfrak{b}) \\ & \beta_1 \wedge \beta_2 \wedge \beta_3 & \mapsto & [(M,K,\xi);\beta_1 \wedge \beta_2 \wedge \beta_3] \end{array}.$$

The next lemma shows that this map is canonical.

Lemma 7.6. — *Let (M,K,ξ) and (M',K',ξ') be marked $\mathbb{Q}$SK-pairs in $\mathcal{P}^m(\mathfrak{A},\mathfrak{b})$. Let $\beta_1 \wedge \beta_2 \wedge \beta_3 \in \mathcal{H}$. Then $[(M',K',\xi');\beta_1 \wedge \beta_2 \wedge \beta_3] = [(M,K,\xi);\beta_1 \wedge \beta_2 \wedge \beta_3]$.*

Proof. — Set $\zeta = \xi' \circ \xi^{-1}$. By Theorem 2.17, (M',K',ξ') can be obtained from (M,K,ξ) by a finite sequence of null LP-surgeries which induces the isomorphism ζ (up to multiplication by a power of t). First assume that the sequence contains a single surgery $(\frac{A'}{A})$. Let $\widetilde{X}$ be the infinite cyclic covering associated with (M,K). Let n_1, n_2, n_3 be non trivial integers such that each $n_i\beta_i$ is realizable by a simple closed curve in $\widetilde{X}$ which does not meet the preimage of A. Let $\Gamma \subset (M \setminus K) \setminus A$ be a Y-graph null in $M \setminus K$ which realizes the Y-diagram $D =$ 

for any coherent values of the f_{ij}. Then

$$[(M,K,\xi);\Gamma,\frac{A'}{A}] = [(M,K,\xi);\Gamma] - [(M',K',\xi');\Gamma].$$

In (M',K',ξ'), Γ still realizes D.

The case of several surgeries easily follows. □

Proof of Proposition 7.1. — It is easy to see that $\varphi(\mathcal{H}) = \mathcal{G}_1^{m,b}(\mathfrak{A},\mathfrak{b})$. So we have a surjective $\mathbb{Q}$-linear map $\mathcal{H} \twoheadrightarrow \mathcal{G}_1^{m,b}(\mathfrak{A},\mathfrak{b})$. Now, the map $\hbar : \mathcal{P}^m(\mathfrak{A},\mathfrak{b}) \to \mathcal{H}$ defines a $\mathbb{Q}$-linear map $\tilde{\hbar} : \mathcal{F}_0^m(\mathfrak{A},\mathfrak{b}) \to \mathcal{H}$. The restriction of $\tilde{\hbar}$ to $\mathcal{F}_1^{m,b}(\mathfrak{A},\mathfrak{b})$ is surjective. The proof of this claim is exactly the proof of Lemma 6.10 without the two first lines. By Proposition 2.18, $\tilde{\hbar}$ induces a surjective map $\mathcal{G}_1^{m,b}(\mathfrak{A},\mathfrak{b}) \twoheadrightarrow \mathcal{H}$ still denoted by $\tilde{\hbar}$. Since $\mathcal{H}$ has a finite dimension, $\varphi : \mathcal{H} \to \mathcal{G}_1^{m,b}(\mathfrak{A},\mathfrak{b})$ and $\tilde{\hbar} : \mathcal{G}_1^{m,b}(\mathfrak{A},\mathfrak{b}) \to \mathcal{H}$ are isomorphisms. □

7.2. Degree one invariants of marked $\mathbb{Z}$SK-pairs

In this subsection, we prove Theorem 2.23 following the proof of Proposition 7.1.

Fix an integral Blanchfield module $(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$. Let $(M,K,\xi) \in \mathcal{P}_\mathbb{Z}^m(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$. Since the space $\mathcal{F}_2^{m,b}(M,K,\xi)$ is a subspace of $\mathcal{F}_2^{m,\mathbb{Z}}(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$, one can define $[(M,K,\xi);\beta_1\wedge\beta_2\wedge\beta_3] \in \mathcal{G}_1^{m,\mathbb{Z}}(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$ for $\beta_1\wedge\beta_2\wedge\beta_3 \in \mathcal{H}$ as in the previous subsection. Once again, this does not depend on the chosen marked $\mathbb{Z}$SK-pair. The only difference in the proof of Lemma 7.6 is that we apply Theorem 2.21 and we use integral null LP-surgeries. Hence we have a well-defined, canonical and surjective map $\varphi^\mathbb{Z} : \mathcal{H} \twoheadrightarrow \mathcal{G}_1^{m,\mathbb{Z}}(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$ defined by $\varphi^\mathbb{Z}(\beta_1\wedge\beta_2\wedge\beta_3) = [(M,K,\xi);\beta_1\wedge\beta_2\wedge\beta_3]$ for any $(M,K,\xi) \in \mathcal{P}_\mathbb{Z}^m(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$.

Proof of Theorem 2.23. — We have a surjective map between finite dimensional vector spaces $\varphi^\mathbb{Z} : \mathcal{H} \twoheadrightarrow \mathcal{G}_1^{m,\mathbb{Z}}(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$. As in the proof of Proposition 7.1, we want to prove that the map $\hbar$ of Proposition 2.18 provides a surjective map from $\mathcal{G}_1^{m,\mathbb{Z}}(\mathfrak{A}_\mathbb{Z},\mathfrak{b})$ onto $\mathcal{H}$. We must be more careful in this case since the proof of the surjectivity of $\hbar$ in Lemma 6.10 makes use of d-surgeries in the application of Proposition 6.2. These d-surgeries do not preserve the homology with integral coefficients of the manifold M, hence they do not define a move on the set of marked $\mathbb{Z}$SK-pairs. So $\hbar(\mathcal{P}^m_{\mathbb{Z}}(\mathfrak{A}_{\mathbb{Z}},\mathfrak{b}))$ may not be the whole $\mathcal{H}$, but, rereading the proof of Lemma 6.10, one easily sees that $\hbar(\mathcal{P}^m_{\mathbb{Z}}(\mathfrak{A}_{\mathbb{Z}},\mathfrak{b}))\subset\mathcal{H}$ generates $\mathcal{H}$ as a $\mathbb{Q}$-vector space. Hence $\hbar$ induces a surjective $\mathbb{Q}$-linear map $\mathcal{F}^{m,\mathbb{Z}}_0(\mathfrak{A}_{\mathbb{Z}},\mathfrak{b})\twoheadrightarrow\mathcal{H}$, which provides a surjective $\mathbb{Q}$-linear map $\tilde{\hbar}:\mathcal{G}^{m,\mathbb{Z}}_1(\mathfrak{A}_{\mathbb{Z}},\mathfrak{b})\twoheadrightarrow\mathcal{H}$. Finally, $\varphi^{\mathbb{Z}}$ and $\tilde{\hbar}$ are isomorphisms. □

7.3. Description of $\mathcal{G}^m_1(\mathfrak{A},\mathfrak{b})$

In this subsection, we prove the following result which, together with Proposition 7.1, implies Theorem 2.19. Fix a Blanchfield module $(\mathfrak{A},\mathfrak{b})$.

Proposition 7.7. — *Let $(M,K,\xi)\in\mathcal{P}^m(\mathfrak{A},\mathfrak{b})$. For p prime, let B_p be a rational homology ball such that $H_1(B_p;\mathbb{Z})\cong\mathbb{Z}/p\mathbb{Z}$. Then*

$$\mathcal{G}^m_1(\mathfrak{A},\mathfrak{b})\cong\left(\bigoplus_{p\ prime}\mathbb{Q}\,[(M,K,\xi);\frac{B_p}{B^3}]\right)\bigoplus\mathcal{G}^{m,b}_1(\mathfrak{A},\mathfrak{b}).$$

The invariants ν_p defined in Subsection 2.4 satisfy $\nu_p([(M,K,\xi);\frac{B_q}{B^3}])=\delta_{pq}$, where δ_{pq} is the Kronecker symbol. Hence $\bigoplus_{p\text{ prime}}\mathbb{Q}\,[(M,K,\xi);\frac{B_p}{B^3}]$ is indeed a direct sum. Note that $[(M,K,\xi);\frac{B_p}{B^3}]\in\mathcal{G}^m_1(\mathfrak{A},\mathfrak{b})$ does not depend on the marked $\mathbb{Q}$SK-pair (M,K,ξ), due to Theorem 2.17 and

$$[(M,K,\xi);\frac{B_p}{B^3}]-[(M,K,\xi)(\frac{A'}{A});\frac{B_p}{B^3}]=[(M,K,\xi);\frac{B_p}{B^3},\frac{A'}{A}].$$

Set $\mathcal{K}=\bigoplus_{p\text{ prime}}\mathbb{Q}[(M,K,\xi);\frac{B_p}{B^3}])\subset\mathcal{G}^m_1(\mathfrak{A},\mathfrak{b})$.

Lemma 7.8. — $\mathcal{K}\cap\mathcal{G}^{m,b}_1(\mathfrak{A},\mathfrak{b})=0$.

Proof. — Since Borromean surgeries preserve the homology, the invariants ν_p are trivial on $\mathcal{G}^{m,b}_1(\mathfrak{A},\mathfrak{b})$. Let $G\in\mathcal{K}\cap\mathcal{G}^{m,b}_1(\mathfrak{A},\mathfrak{b})$. On the one hand, G is a linear combination of the $[(M,K,\xi);\frac{B_p}{B^3}]$, and on the other hand, $\nu_p(G)=0$ for any prime integer p. Hence $G=0$. □

It remains to prove that $\mathcal{K}\oplus\mathcal{G}^{m,b}_1(\mathfrak{A},\mathfrak{b})$ is the whole $\mathcal{G}^m_1(\mathfrak{A},\mathfrak{b})$. We first reduce the set of generators of $\mathcal{G}^m_1(\mathfrak{A},\mathfrak{b})$ using results from [12]. Recall that d-surgeries were defined in Subsection 6.1.

Definition 7.9. — *An* elementary surgery *is an LP-surgery among the following ones:*

(1) *connected sum (genus 0),*
(2) *d-surgery (genus 1),*
(3) *Borromean surgery (genus 3).*

Theorem 7.10 ([12, Thm. 1.15]). — *If A and B are two $\mathbb{Q}$HH's with LP-identified boundaries, then B can be obtained from A by a finite sequence of elementary surgeries and their inverses in the interior of the $\mathbb{Q}$HH's.*

Corollary 7.11. — *The space $\mathcal{F}_1^m(\mathfrak{A},\mathfrak{b})$ is generated by the $[(M,K,\xi);\frac{E'}{E}]$ where $(M,K,\xi)\in\mathcal{P}^m(\mathfrak{A},\mathfrak{b})$ and $(\frac{E'}{E})$ is an elementary null LP-surgery.*

Proof. — Let $[(M,K,\xi);\frac{A'}{A}]\in\mathcal{F}_1^m(\mathfrak{A},\mathfrak{b})$. By Theorem 7.10, A and A' can be obtained from one another by a finite sequence of elementary surgeries and their inverses. Write $A'=A(\frac{E_1'}{E_1})\dots(\frac{E_k'}{E_k})$. For $0\leqslant j\leqslant k$, set $A_j=A(\frac{E_1'}{E_1})\dots(\frac{E_j'}{E_j})$. Then

$$[(M,K,\xi);\frac{A'}{A})]=\sum_{j=1}^k[(M,K,\xi)(\frac{A_{j-1}}{A_0});\frac{E_j'}{E_j}].$$

Conclude with $[(M,K,\xi);\dfrac{E'}{E}]=-[(M,K,\xi)(\dfrac{E'}{E});\dfrac{E}{E'}]$. □

Let $\mathcal{F}_0^{\mathbb{Q}\mathrm{HS}}$ be the rational vector space generated by all $\mathbb{Q}$HS's up to orientation-preserving homeomorphism. Let $(\mathcal{F}_n^{\mathbb{Q}\mathrm{HS}})_{n\in\mathbb{N}}$ be the filtration of $\mathcal{F}_0^{\mathbb{Q}\mathrm{HS}}$ defined by LP-surgeries. Let $\mathcal{G}_n^{\mathbb{Q}\mathrm{HS}}=\mathcal{F}_n^{\mathbb{Q}\mathrm{HS}}/\mathcal{F}_{n+1}^{\mathbb{Q}\mathrm{HS}}$ be the associated quotients.

Lemma 7.12 ([12, Prop. 1.8]). — *For each prime integer p, let B_p be a rational homology ball such that $H_1(B_p;\mathbb{Z})\cong\frac{\mathbb{Z}}{p\mathbb{Z}}$. Then $\mathcal{G}_1^{\mathbb{Q}HS}=\bigoplus_{p\ \mathrm{prime}}\mathbb{Q}[S^3;\dfrac{B_p}{B^3}]$.*

Lemma 7.13. — *For each prime integer p, let B_p be a rational homology ball such that $H_1(B_p;\mathbb{Z})\cong\frac{\mathbb{Z}}{p\mathbb{Z}}$. Let $(M,K,\xi)\in\mathcal{P}^m(\mathfrak{A},\mathfrak{b})$. Let B be a rational homology ball. Then $[(M,K,\xi);\dfrac{B}{B^3}]$ is a rational linear combination of the $[(M,K,\xi);\dfrac{B_p}{B^3}]$ and elements of $\mathcal{F}_2^m(\mathfrak{A},\mathfrak{b})$.*

Proof. — By Lemma 7.12, there is a relation

$$[S^3;\frac{B}{B^3}]=\sum_{p\ \mathrm{prime}}a_p[S^3;\frac{B_p}{B^3}]+\sum_{j\in J}b_j[N_j;\frac{C_j'}{C_j},\frac{D_j'}{D_j}],$$

where J is a finite set, the a_p and b_j are rational numbers, the a_p are all trivial except a finite number, and for $j\in J$, $[N_j;\frac{C_j'}{C_j},\frac{D_j'}{D_j}]\in\mathcal{F}_2^{\mathbb{Q}\mathrm{HS}}$. Make the connected sum of each $\mathbb{Q}$HS in the relation with M. We obtain

$$[(M,K,\xi);\frac{B}{B^3}]=\sum_{p\ \mathrm{prime}}a_p[(M,K,\xi);\frac{B_p}{B^3}]+\sum_{j\in J}b_j[(M\sharp N_j,K,\xi);\frac{C_j'}{C_j},\frac{D_j'}{D_j}].\ \square$$

To complete the proof of Proposition 7.7, we need the following result about degree 1 invariants of *framed rational homology tori*, i.e. rational homology tori with a prefered parallel. Finite type invariants of framed rational homology tori are defined as for $\mathbb{Q}$SK-pairs (see [12, §5.1] for details).

Lemma 7.14 ([12, Cor. 5.10]). — *For any prime integer p, let M_p be a $\mathbb{Q}HS$ such that $H_1(M_p;\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$. Let T_0 be a framed standard torus. If μ is a degree 1 invariant of framed rational homology tori such that $\mu(T_0) = 0$ and $\mu(T_0 \sharp M_p) = 0$ for any prime integer p, then $\mu = 0$.*

Proof of Proposition 7.7. — Let $\lambda \in (\mathcal{F}_1^m(\mathfrak{A},\mathfrak{b}))^*$ be such that $\lambda(\mathcal{F}_2^m(\mathfrak{A},\mathfrak{b})) = 0$. Assume that $\lambda(\mathcal{K} \oplus \mathcal{G}_1^{m,b}(\mathfrak{A},\mathfrak{b})) = 0$. Let us prove that $\lambda = 0$. Due to Corollary 7.11, it suffices to prove that λ vanishes on the brackets defined by elementary surgeries. It is clear for elementary surgeries of genus 3, and for elementary surgeries of genus 0 it follows from Lemma 7.13.

Consider a bracket $[(M,K,\xi);\frac{T_d}{T_0})]$, where $(M,K,\xi) \in \mathcal{P}^m(\mathfrak{A},\mathfrak{b})$, T_0 is a standard torus null in $M \setminus K$, and T_d is a d-torus for some positive integer d. Fix a parallel of T_0. If T is a framed rational homology torus, set $\bar{\lambda}(T) = \lambda\left([(M,K,\xi);\frac{T}{T_0}]\right)$, where the LP-identification $\partial T \cong \partial T_0$ identifies the prefered parallels. Then $\bar{\lambda}$ is a degree 1 invariant of framed rational homology tori:

$$\bar{\lambda}\left([T;\frac{A'_1}{A_1},\frac{A'_2}{A_2}]\right) = -\lambda\left([(M,K,\xi)(\frac{T}{T_0});\frac{A'_1}{A_1},\frac{A'_2}{A_2}]\right) = 0.$$

Moreover, we have $\bar{\lambda}(T_0) = 0$ and $\bar{\lambda}(T_0(\frac{B_p}{B^3})) = 0$. Thus, by Lemma 7.14, $\bar{\lambda} = 0$. □

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