ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

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Tome XXVI, nº 3 (2017), p. 645-685.

<http://afst.cedram.org/item?id=AFST_2017_6_26_3_645_0>

Annales de la faculté des sciences de Toulouse Volume XXVI, n° 3, 2017
pp. 645-685

Resurgence and highest level's connection-to-Stokes formulæ for some linear meromorphic differential systems [(*)]

Pascal Remy [(1)]

Abstract. — In this article, we consider a linear meromorphic differential system with several levels. We prove that the Borel transforms of its highest level's reduced formal solutions are summable-resurgent and we give the general form of all their singularities. This one is then precised in restriction to some convenient hypotheses on the geometric configuration of singular points. Next, under the same hypotheses, we state exact formulæ to express some highest level's Stokes multipliers of the initial system in terms of connection constants in the Borel plane, generalizing thus formulæ already displayed by M. Loday-Richaud and the author for systems with a single level. As an illustration, we develop one example.

Résumé. — Dans cet article, nous considérons un système différentiel linéaire méromorphe à multiples niveaux. Nous démontrons que les transformées de Borel de ses solutions formelles réduites de plus haut niveau sont résurgentes-sommables et nous donnons la forme générale de toutes leurs singularités. Celle-ci est ensuite précisée pour certaines configurations géométriques des points singuliers. Pour ces mêmes configurations, nous énonçons également des formules exactes permettant d'exprimer les multiplicateurs de Stokes de plus haut niveau du système initial à l'aide de constantes de connexion dans le plan de Borel, généralisant ainsi les formules déjà données par M. Loday-Richaud et l'auteur pour les systèmes de niveau unique. Ces formules sont illustrées par un exemple.

[(*)] Reçu le 6 septembre 2015, accepté le 29 juin 2016.

Keywords: Linear differential system, multisummability, Stokes phenomenon, Stokes multipliers, resurgence, singularities, connection constants.

Math. classification: 34M03, 34M25, 34M30, 34M35, 34M40.

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Article proposé par Lucia DiVizio.

1. Introduction

All along the article, we are given a positive integer $r \geqslant 1$ and a linear differential system (in short, a differential system or a system) of dimension $n \geqslant 2$ with meromorphic coefficients of order $r+1$ at the origin $0 \in \mathbb{C}$ of the form

$$x^{r+1}\frac{dY}{dx} = A(x)Y \ , \quad A(x) \in M_n(\mathbb{C}\{x\}),\ A(0) \neq 0. \tag{A}$$

Using a finite algebraic extension $x \longmapsto x^\nu$ of the variable x with $\nu \in \mathbb{N}$, $\nu \geqslant 1$, and a meromorphic gauge transformation $Y \longmapsto T(x)Y$ with a suitable polynomial matrix $T(x)$ in x and $1/x$ if needed, we can always assume (see [5]) that system (A) admits for formal fundamental solution at 0 a matrix $\widetilde{Y}(x)$ of the form $\widetilde{Y}(x) = \widetilde{F}(x)x^L e^{Q(1/x)}$ and normalized as follows:

- (N_1)**:** $\widetilde{F}(x) \in M_n(\mathbb{C}[\![x]\!])$ is a formal power series in x satisfying $\widetilde{F}(x) = I_n + O(x^r)$, where I_n denotes the identity matrix of size n,
- (N_2)**:** the matrix $L \in M_n(\mathbb{C})$ of exponents of formal monodromy reads in a Jordan form $L = \bigoplus_{j=1}^{J} (\lambda_j I_{n_j} + J_{n_j})$, where J is an integer $\geqslant 2$, the eigenvalues λ_j satisfy $0 \leqslant \mathrm{Re}(\lambda_j) < 1$ and where
$$J_{n_j} = \begin{cases} 0 & \text{if } n_j = 1 \\ \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} & \text{if } n_j \geqslant 2 \end{cases}$$
is an irreductible Jordan block of size n_j,
- (N_3)**:** $Q\left(\frac{1}{x}\right) = \bigoplus_{j=1}^{J} q_j\left(\frac{1}{x}\right) I_{n_j}$ is a diagonal matrix of polynomials $q_j(1/x)$ in $1/x$ of degree $\leqslant r$ and without constant term which commutes with L.

Recall that normalizations $(N_1) - (N_2)$ guarantee the unicity of $\widetilde{F}(x)$ as formal series solutions of the homological system associated with system (A) (cf. [5]).

The effective calculation of the *Stokes multipliers* of $\widetilde{F}(x)$ (= the non-trivial entries of the *Stokes–Ramis matrices* associated with $\widetilde{Y}(x)$, see Definition 5.1) is crucial in a large number of theoretical and practical problems (calculation of differential Galois groups [24, 25], integrability of some

Hamiltonian systems [26, 27], etc.). Thereby, in the last decades of the twentieth century, several approaches, issuing from the summability and multi-summability theories and essentially based on integral methods such that Cauchy–Heine integral and Laplace transformations, were given by many authors under more or less generic assumptions on system (A) (see for instance [2, 4, 6, 7, 8, 9, 11, 12, 15, 20]).

More recently, in a 2011 article [19], M. Loday-Richaud and the author combined, in the case where system (A) has the *unique* level 1 (see Definition 2.3 for the exact definition of levels), this "summation" approach with the "resurgence" approach due to J. Écalle. Doing that, they derived, from a full description of the resurgent structure of the Borel transform $\widehat{F}$ of $\widetilde{F}$, explicit formulæ relating the Stokes multipliers of $\widetilde{F}$ to *connection constants* given by some analytic continuations of $\widehat{F}$ at its various singular points, providing thus a new efficient tool for the effective calculation of the Stokes multipliers of $\widetilde{F}$.

Afterwards, these so-called *connection-to-Stokes formulæ* were generalized by the author to systems with an arbitrary single level r [32] and to the lowest level of systems with multi-levels [31] by respectively replacing, via the classical method of rank reduction, the initial system by its r-reduced and its lowest level's reduced system. One knows indeed perfectly relate the Stokes–Ramis matrices of the initial system with those of its reduced system [17].

In the present article, we assume that system (A) has multi-levels, say $r_1 < \cdots < r_p$ with $p \geqslant 2$, and we propose to extend these connection-to-Stokes formulæ to the highest level r_p. To do that, the organization of the paper is as follows. In Section 2, we first recall some definitions and basic properties about levels and about the r_p-reduced system $(\boldsymbol{A})$ associated with system (A). In Section 3, we describe the complete resurgent structure of the Borel transforms of the formal solutions of system $(\boldsymbol{A})$. In particular, we show that these functions are summable-resurgent (Theorem 3.3) and we give the general form of all their singularities (Theorem 3.6). These two theorems are then proved in Section 4 by reducing system $(\boldsymbol{A})$ into a convenient scalar linear differential equation with polynomial coefficients and by applying a method similar to the one of [32]. In Section 5, we restrict our study to the case where the Borel transforms we consider have "good" singularities. In this case, we define their connection constants at their various singular points and we relate these ones to the highest level's Stokes multipliers of $\widetilde{F}$ throughout explicit *highest level's connection-to-Stokes formulæ*, generalizing thus formulæ already obtained in [19, 31, 32] . We illustrate this result with a numerical example.

2. Preliminaries

In this section, we recall some definitions we are needed in the sequel and we introduce the highest level's reduced system associated with system (A) which will play a central role all along the article.

2.1. Levels, Stokes values and anti-Stokes directions

Split the matrix $\widetilde{F}(x)$ into J column-blocks fitting to the Jordan block structure of L (for $\ell = 1, \dots, J$, the matrix $\widetilde{F}^{\bullet;\ell}(x)$ has n_ℓ columns):

$$\widetilde{F}(x) = \begin{bmatrix} \widetilde{F}^{\bullet;1}(x) & \widetilde{F}^{\bullet;2}(x) & \cdots & \widetilde{F}^{\bullet;J}(x) \end{bmatrix}.$$

Definition 2.1. — *Let $j, \ell \in \{1, \dots, J\}$ be such that $q_j \not\equiv q_\ell$. We denote*

$$(q_j - q_\ell)\left(\frac{1}{x}\right) = -\frac{\alpha_{j,\ell}}{x^{r_{j,\ell}}} + o\left(\frac{1}{x^{r_{j,\ell}}}\right)$$

with $\alpha_{j,\ell} \neq 0$ and $r_{j,\ell} \in \mathbb{N}^* = \mathbb{N}\backslash\{0\}$. Then,

- *the degree $r_{j,\ell}$ is called* a level of $\widetilde{F}^{\bullet;\ell}(x)$,
- *the coefficient $\alpha_{j,\ell}$ is called* a Stokes value of level $r_{j,\ell}$ of $\widetilde{F}^{\bullet;\ell}(x)$,
- *the directions of maximal decay of $\mathrm{e}^{(q_j - q_\ell)(1/x)}$, i.e. the $r_{j,\ell}$ directions* $\arg(\alpha_{j,\ell})/r_{j,\ell} \mod (2\pi/r_{j,\ell})$ *along which $-\alpha_{j,\ell}/x^{r_{j,\ell}}$ is real negative, are called* anti-Stokes directions of level $r_{j,\ell}$ of $\widetilde{F}^{\bullet;\ell}(x)$.

Note that a Stokes value (resp. an anti-Stokes direction) of $\widetilde{F}^{\bullet;\ell}(x)$ may be with several levels. Note also that the term "*anti-Stokes direction*" is not universal; sometimes, one calls such a direction "*Stokes direction*".

Notation 2.2. — For all $\ell \in \{1, \dots, J\}$, we denote by $\mathcal{R}_\ell := \{\rho_{\ell;1} < \dots < \rho_{\ell;p_\ell}\}$ with $p_\ell \geqslant 1$ the set of all the levels of $\widetilde{F}^{\bullet;\ell}(x)$.

Note that, according to normalization (N_3), all the levels $\rho_{\ell;k}$ of all the $\widetilde{F}^{\bullet;\ell}(x)$ are integer; one refers sometimes this case as the *unramified case*.

Note also that, for all ℓ, we have $\rho_{\ell;p_\ell} \leqslant r$ the rank of system (A). Actually, if there exists ℓ_0 such that $\rho_{\ell_0;p_{\ell_0}} < r$, then $\rho_{\ell;p_\ell} < r$ for all ℓ and polynomials q_j have the same degree r and the same terms of highest degree. One then reduces to the case $\rho_{\ell;p_\ell} = r$ by means of a change of unknown vector of the form $Y = Ze^{q(1/x)}$ with a convenient polynomial $q(1/x) \in x^{-1}\mathbb{C}[x^{-1}]$. Recall that such a change does not affect levels, Stokes values, anti-Stokes directions nor Stokes–Ramis matrices (see Section 5.1 for their exact definition) of system (A).

Definition 2.3. — *One calls*

- level of $\widetilde{F}(x)$ (or of system (A)) *any level of the* $\widetilde{F}^{\bullet;\ell}(x)$*'s,*
- Stokes value of $\widetilde{F}(x)$ (or of system (A)) *any Stokes value of the* $\widetilde{F}^{\bullet;\ell}(x)$*'s,*
- anti-Stokes direction of $\widetilde{F}(x)$ (or of system (A)) *any anti-Stokes direction of the* $\widetilde{F}^{\bullet;\ell}(x)$*'s.*

Notation 2.4. — We denote by $\mathcal{R} := \{r_1 < \cdots < r_p\}$ with $p \geqslant 1$ the set of all the levels of $\widetilde{F}(x)$ (or of system (A)). We have $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_J$ and $r_p = r$ the rank of system (A).

Since the case $p = 1$ was already investigated in great details in [19] (case $r = 1$) and [32] (case $r \geqslant 2$), we suppose from now on $p \geqslant 2$, that is system (A) has at least two levels. Note however that some column-blocks $\widetilde{F}^{\bullet;\ell}(x)$ may have the unique level r, i.e. $p_\ell = 1$ and $\mathcal{R}_\ell = \{r\}$.

2.2. Highest level's reduced system

The *highest level's reduced system* (= r-reduced system) associated with system (A) is the unique system of the variable $t = x^r$ having meromorphic coefficients at $0 \in \mathbb{C}$ and the formal solutions $[\widetilde{Y}(x), x^{-1}\widetilde{Y}(x), \ldots, x^{-(r-1)}\widetilde{Y}(x)]$ for a given choice $x = t^{1/r}$ of a r-th root of t [17]. Such a choice being made, we denote from now on $\mu := e^{-2i\pi/r}$. Then, the r-reduced system of system (A) reads as

$$rt^2 \frac{d\boldsymbol{Y}}{dt} = \boldsymbol{A}(t)\boldsymbol{Y} \tag{$\boldsymbol{A}$}$$

with $\boldsymbol{A}(t) \in M_{rn}(\mathbb{C}\{t\})$ the $rn \times rn$-analytic matrix defined by

$$\boldsymbol{A}(t) = \begin{bmatrix} \boldsymbol{A}^{[0]}(t) & t\boldsymbol{A}^{[r-1]}(t) & \cdots & \cdots & t\boldsymbol{A}^{[1]}(t) \\ \boldsymbol{A}^{[1]}(t) & \boldsymbol{A}^{[0]}(t) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \boldsymbol{A}^{[0]}(t) & t\boldsymbol{A}^{[r-1]}(t) \\ \boldsymbol{A}^{[r-1]}(t) & \cdots & \cdots & \boldsymbol{A}^{[1]}(t) & \boldsymbol{A}^{[0]}(t) \end{bmatrix} - \bigoplus_{u=0}^{r-1} utI_n$$

where the $\boldsymbol{A}^{[u]}(t) \in M_n(\mathbb{C}\{t\})$ are the *r-reduced series of* $A(x)$ uniquely determined by the relation

$$A(x) = \boldsymbol{A}^{[0]}(x^r) + x\boldsymbol{A}^{[1]}(x^r) + \cdots + x^{r-1}\boldsymbol{A}^{[r-1]}(x^r).$$

By construction, system $(\boldsymbol{A})$ has levels $\leqslant 1$. Moreover, it admits as formal fundamental solution at 0 the matrix

$$\widetilde{\boldsymbol{Y}}(t)=\begin{bmatrix}\widetilde{Y}(t^{1/r}) & \widetilde{Y}(\mu t^{1/r}) & \cdots & \widetilde{Y}(\mu^{r-1}t^{1/r})\\ (t^{1/r})^{-1}\widetilde{Y}(t^{1/r}) & (\mu t^{1/r})^{-1}\widetilde{Y}(\mu t^{1/r}) & \cdots & (\mu^{r-1}t^{1/r})^{-1}\widetilde{Y}(\mu^{r-1}t^{1/r})\\ \vdots & \vdots & & \vdots\\ (t^{1/r})^{-(r-1)}\widetilde{Y}(t^{1/r}) & (\mu t^{1/r})^{-(r-1)}\widetilde{Y}(\mu t^{1/r}) & \cdots & (\mu^{r-1}t^{1/r})^{-(r-1)}\widetilde{Y}(\mu^{r-1}t^{1/r})\end{bmatrix}.$$

This one reads more precisely on the form $\widetilde{\boldsymbol{Y}}(t)=\widetilde{\boldsymbol{F}}(t)\widetilde{\boldsymbol{Y}}_0(t)$ with

- $$\widetilde{\boldsymbol{F}}(t)=\begin{bmatrix}\widetilde{\boldsymbol{F}}^{[0]}(t) & t\widetilde{\boldsymbol{F}}^{[r-1]}(t) & \cdots & \cdots & t\widetilde{\boldsymbol{F}}^{[1]}(t)\\ \widetilde{\boldsymbol{F}}^{[1]}(t) & \widetilde{\boldsymbol{F}}^{[0]}(t) & \ddots & & \vdots\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \widetilde{\boldsymbol{F}}^{[0]}(t) & t\widetilde{\boldsymbol{F}}^{[r-1]}(t)\\ \widetilde{\boldsymbol{F}}^{[r-1]}(t) & \cdots & \cdots & \widetilde{\boldsymbol{F}}^{[1]}(t) & \widetilde{\boldsymbol{F}}^{[0]}(t)\end{bmatrix} \text{ and}$$
- $$\widetilde{\boldsymbol{Y}}_0(t)=\begin{bmatrix}(t^{\frac{1}{r}})^{\Lambda_0}e^{Q_0(t)} & (\mu t^{\frac{1}{r}})^{\Lambda_0}e^{Q_1(t)} & \cdots & (\mu^{r-1}t^{\frac{1}{r}})^{\Lambda_0}e^{Q_{r-1}(t)}\\ (t^{\frac{1}{r}})^{\Lambda_1}e^{Q_0(t)} & (\mu t^{\frac{1}{r}})^{\Lambda_1}e^{Q_1(t)} & \cdots & (\mu^{r-1}t^{\frac{1}{r}})^{\Lambda_1}e^{Q_{r-1}(t)}\\ \vdots & \vdots & \ddots & \vdots\\ (t^{\frac{1}{r}})^{\Lambda_{r-1}}e^{Q_0(t)} & (\mu t^{\frac{1}{r}})^{\Lambda_{r-1}}e^{Q_1(t)} & \cdots & (\mu^{r-1}t^{\frac{1}{r}})^{\Lambda_{r-1}}e^{Q_{r-1}(t)}\end{bmatrix}$$

where $Q_k(t)=Q(1/(\mu^k t^{1/r}))$ and $\Lambda_k:=L-kI_n$ for all $k=0,\ldots,r-1$. The formal series $\widetilde{\boldsymbol{F}}^{[u]}(t)\in M_n(\mathbb{C}[[t]])$ are the r-reduced series of $\widetilde{F}(x)$ and are defined in the same way as the $\boldsymbol{A}^{[u]}(t)$'s. In particular, the initial condition $\widetilde{F}(x)=I_n+O(x^r)$ (see normalization (N_1)) implies $\widetilde{\boldsymbol{F}}(t)=I_{rn}+O(t)$.

Let us now split $\widetilde{\boldsymbol{F}}(t)$ into r column-blocks $\widetilde{\boldsymbol{F}}^{\bullet;v}(t)$ of size $rn\times n$:

$$\widetilde{\boldsymbol{F}}(t)=\begin{bmatrix}\widetilde{\boldsymbol{F}}^{\bullet;1}(t) & \widetilde{\boldsymbol{F}}^{\bullet;2}(t) & \cdots & \widetilde{\boldsymbol{F}}^{\bullet;r}(t)\end{bmatrix},$$

then, each $\widetilde{\boldsymbol{F}}^{\bullet;v}(t)$ into J column-blocks $\widetilde{\boldsymbol{F}}^{\bullet;v,\ell}(t)$ as $\widetilde{F}(x)$ (the matrix $\widetilde{\boldsymbol{F}}^{\bullet;v,\ell}(t)$ has size $rn\times n_\ell$):

$$\widetilde{\boldsymbol{F}}^{\bullet;v}(t)=\begin{bmatrix}\widetilde{\boldsymbol{F}}^{\bullet;v,1}(t) & \widetilde{\boldsymbol{F}}^{\bullet;v,2}(t) & \cdots & \widetilde{\boldsymbol{F}}^{\bullet;v,J}(t)\end{bmatrix}.$$

Let $\theta\in\mathbb{R}/2\pi\mathbb{Z}$ be a non anti-Stokes direction of system (A) and $\boldsymbol{\theta}:=r\theta$. Due to the classical theory of multisummability of linear meromorphic systems (see for instance [2, 3, 11, 16, 17, 18, 23, 25]), the formal series $\widetilde{\boldsymbol{F}}^{\bullet;v,\ell}(t)$ are $\underline{\boldsymbol{\rho_\ell}}$-summable for all $v=1,\ldots,r$ in direction $\boldsymbol{\theta}$ with $\underline{\boldsymbol{\rho_\ell}}:=(\boldsymbol{\rho}_{\ell;1}<\cdots<\boldsymbol{\rho}_{\ell;p_\ell}=1)$ and $\boldsymbol{\rho}_{\ell;k}:=\rho_{\ell;k}/r$. In particular, applying Balser–Tougeron theorem [1] (see also [18, Thm. 7.4.5]), we get the following proposition which provides us a first result about the formal Borel transform[1] $\widehat{\boldsymbol{F}}^{\bullet;v,\ell}(\tau)$ of $\widetilde{\boldsymbol{F}}^{\bullet;v,\ell}(t)$.

Proposition 2.5. — *Let $v \in \{1,\dots,r\}$ and $\ell \in \{1,\dots,J\}$. Then,*

- Case $p_\ell = 1$: $\widehat{\boldsymbol{F}}^{\bullet;v,\ell}(\tau)$ *defines an analytic function on a disc centered at the origin* $0 \in \mathbb{C}$,
- Case $p_\ell \geqslant 2$: $\widehat{\boldsymbol{F}}^{\bullet;v,\ell}(\tau)$ *is* $\underline{\boldsymbol{\rho}'_\ell}$*-summable in direction* $\boldsymbol{\theta}$ *with* $\underline{\boldsymbol{\rho}'_\ell} := (\boldsymbol{\rho}'_{\ell;1} < \dots < \boldsymbol{\rho}'_{\ell;p_\ell-1})$ *and* $\boldsymbol{\rho}'_{\ell;k} := \boldsymbol{\rho}_{\ell;k}/(1-\boldsymbol{\rho}_{\ell;k}) = \rho_{\ell;k}/(r-\rho_{\ell;k})$*; moreover, its sum defines an analytic function on a sector with vertex* 0*, bisected by* $\boldsymbol{\theta}$ *and opening larger than* $\pi/\boldsymbol{\rho}'_{\ell;p_{\ell-1}}$.

We denote below by $\widehat{\boldsymbol{F}}^{\bullet;v,\ell}_{\boldsymbol{\theta}}(\tau)$ *the function thus defined and by* $\Sigma^{v,\ell}_{\boldsymbol{\theta}}$ *its corresponding domain of analyticity.*

In Section 3, we propose to investigate, for any v and ℓ, the *resurgent structure* of $\widehat{\boldsymbol{F}}^{\bullet;v,\ell}_{\boldsymbol{\theta}}$, that is, the analytic continuations of $\widehat{\boldsymbol{F}}^{\bullet;v,\ell}_{\boldsymbol{\theta}}$ outside the domain $\Sigma^{v,\ell}_{\boldsymbol{\theta}}$. In particular, we prove that $\widehat{\boldsymbol{F}}^{\bullet;v,\ell}_{\boldsymbol{\theta}}$ is *summable-resurgent*[2] (Theorem 3.3) and we give the *general form* of all its singularities (Theorem 3.6), generalizing thus the results already obtained by M. Loday-Richaud and the author in [19, 32] for systems with a single level.

3. Resurgence and singularities

In this section, we just state the results allowing to describe the resurgent structure of functions $\widehat{\boldsymbol{F}}^{\bullet;v,\ell}_{\boldsymbol{\theta}}$ above. These ones will be proved later in Section 4. In the sequel, we fix $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ a non anti-Stokes direction of initial system (A) and we set $\boldsymbol{\theta} = r\theta$ as before.

3.1. Summable-resurgence theorem

Recall that a *resurgent* function is an analytic function near the origin which can be analytically continued on all a convenient Riemann surface. More precisely, one has the following.

Definition 3.1 (Resurgent function). — *Let $\Omega \subset \mathbb{C}$ be a finite subset of $\mathbb{C}$ containing 0. A function defined and analytic near 0 is said to be*

[1] Recall that the formal Borel transform of a formal series $\sum_{m\geqslant 0} a_m t^m \in \mathbb{C}[[t]]$ is defined by $a_0\delta + \sum_{m\geqslant 1} a_m \frac{\tau^{m-1}}{(m-1)!}$, where δ denotes the Dirac distribution at 0.

[2] See Definition 3.2.

- resurgent with singular support $\Omega, 0$ *when it can be analytically continued on the whole Riemann surface* $\mathcal{R}_\Omega$ *defined as (the terminal end of) all homotopy classes in* $\mathbb{C}\backslash\Omega$ *of paths issuing from* 0 *and bypassing all points of* Ω *(only homotopically trivial paths are allowed to turn back to* 0*); in particular, such a function is analytic at* 0 *in the first sheet,*
- resurgent with singular support $\Omega, \widetilde{0}$ *when it can be analytically continued on the whole Riemann surface* $\widetilde{\mathcal{R}}_\Omega :=$ *the universal cover of* $\mathbb{C}\backslash\Omega$*.*

We denote by $\mathcal{R}es_{\Omega,0}$ *and* $\mathcal{R}es_{\Omega,\widetilde{0}}$ *the sets of resurgent functions with singular support* $\Omega, 0$ *and of resurgent functions with singular support* $\Omega, \widetilde{0}$*.*

Recall that the difference between $\mathcal{R}_\Omega$ and $\widetilde{\mathcal{R}}_\Omega$ just lies in the fact that $\mathcal{R}_\Omega$ has no branch point at 0 in the first sheet. In particular, we have a natural injection $\mathcal{R}es_{\Omega,0} \hookrightarrow \mathcal{R}es_{\Omega,\widetilde{0}}$. Recall also that the choice of the Riemann surface $\widetilde{\mathcal{R}}_\Omega$ or $\mathcal{R}_\Omega$ only depends on the fact that the function we consider has a singular point or not at the origin $0 \in \mathbb{C}$ (i.e. in the first sheet).

Definition 3.2 (Summable-resurgent function). — *A resurgent function of* $\mathcal{R}es_{\Omega,0}$ *(resp.* $\mathcal{R}es_{\Omega,\widetilde{0}}$*) is said to be* summable-resurgent *if it grows at most exponentially on any bounded sector of infinity of* $\mathcal{R}_\Omega$ *(resp.* $\widetilde{\mathcal{R}}_\Omega$*). We denote by* $\mathcal{R}es^{sum}_{\Omega,0}$ *(resp.* $\mathcal{R}es^{sum}_{\Omega,\widetilde{0}}$*) the set of summable-resurgent functions with singular support* $\Omega, 0$ *(resp.* $\Omega, \widetilde{0}$*). As before, we have a natural injection* $\mathcal{R}es^{sum}_{\Omega,0} \hookrightarrow \mathcal{R}es^{sum}_{\Omega,\widetilde{0}}$*.*

We are now able to state the main result of this section.

Theorem 3.3 (Summable-resurgence theorem). — *Let* $v \in \{1,\dots,r\}$ *and* $\ell \in \{1,\dots,J\}$*. Let* $p_\ell \geqslant 1$ *be the number of levels of* $\widetilde{F}^{\bullet;\ell}(x)$*. Let* $\boldsymbol{\Omega}^*_\ell$ *be the set of Stokes values of level* r *of* $\widetilde{F}^{\bullet;\ell}(x)$ *(see Definition 2.1) and* $\boldsymbol{\Omega}_\ell := \boldsymbol{\Omega}^*_\ell \cup \{0\}$*. Then,*

- Case $p_\ell = 1$: $\widehat{\boldsymbol{F}}^{\bullet;v,\ell}_{\boldsymbol{\theta}}(\tau) \in \mathcal{R}es^{sum}_{\boldsymbol{\Omega}_\ell,0}$,
- Case $p_\ell \geqslant 2$: $\widehat{\boldsymbol{F}}^{\bullet;v,\ell}_{\boldsymbol{\theta}}(\tau) \in \mathcal{R}es^{sum}_{\boldsymbol{\Omega}_\ell,\widetilde{0}}$.

In particular, Theorem 3.3 tells us that the only possible singular points of $\widehat{\boldsymbol{F}}^{\bullet;v,\ell}_{\boldsymbol{\theta}}(\tau)$ are 0 and the Stokes values of level r of $\widetilde{F}^{\bullet;\ell}(x)$. The general form of the singularities of $\widehat{\boldsymbol{F}}^{\bullet;v,\ell}_{\boldsymbol{\theta}}(\tau)$ at these various points is precised just below.

3.2. General form of singularities

Before stating the structure of the singularities of the $\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}}^{\bullet;v,\ell}(\tau)$'s, let us recall some definitions and notations about the singularities. For more details, we refer for instance to [14, 21, 22, 33].

3.2.1. Some spaces of singularities

Denote by $\mathcal{O}$ the space of holomorphic germs at $0 \in \mathbb{C}$ and by $\widetilde{\mathcal{O}}$ the space of holomorphic germs at 0 on the Riemann surface $\widetilde{\mathbb{C}}$ of the logarithm. One calls any element of the quotient space $\mathcal{C} := \widetilde{\mathcal{O}}/\mathcal{O}$ a *singularity at* 0. Recall that $\mathcal{C}$ is also denoted by $\mathtt{SING}_0$ by J. Écalle et al. (cf. [33] for instance). Recall also that the elements of $\mathcal{C}$ are called *micro-functions* by B. Malgrange [21, 22] by analogy with hyper- and micro-functions defined by Sato, Kawai and Kashiwara in higher dimensions.

The elements of $\mathcal{C}$ are usually denoted with a nabla, like $\overset{\triangledown}{h}$, for a singularity of the function h. A representative of $\overset{\triangledown}{h}$ in $\widetilde{\mathcal{O}}$ is often denoted by $\check{h}$ and is called *a major* of h.

It is worth to consider the two natural maps

$$\begin{array}{lll} \mathrm{can} : \widetilde{\mathcal{O}} \longrightarrow \mathcal{C} = \widetilde{\mathcal{O}}/\mathcal{O} & & \text{the canonical map and} \\ \mathrm{var} : \mathcal{C} \longrightarrow \widetilde{\mathcal{O}} & & \text{the variation map,} \end{array}$$

action of a positive turn around 0 defined by $\mathrm{var}\,\overset{\triangledown}{h} = \check{h}(\tau) - \check{h}(\tau e^{-2i\pi})$, where $\check{h}(\tau e^{-2i\pi})$ is the analytic continuation of $\check{h}(\tau)$ along a path turning once clockwise around 0 and close enough to 0 for $\check{h}$ to be defined all along (the result is independent of the choice of the major $\check{h}$). The germ $\mathrm{var}\,\overset{\triangledown}{h}$ is called *the minor* of $\overset{\triangledown}{h}$.

One can not multiply two elements of $\mathcal{C}$, but an element of $\mathcal{C}$ and an element of $\mathcal{O}$: $\alpha\overset{\triangledown}{h} := \mathrm{can}(\alpha\check{h}) = \overset{\triangledown}{(\alpha h)}$ for all $\alpha \in \mathcal{O}$ and $\overset{\triangledown}{h} \in \mathcal{C}$. On the other hand, one can define a convolution product $\circledast$ on $\mathcal{C}$ by setting $\overset{\triangledown}{h}_1 \circledast \overset{\triangledown}{h}_2 := \mathrm{can}(\check{h}_1 *_u \check{h}_2)$, where $\check{h}_1 *_u \check{h}_2$ is the truncated convolution product

$$(\check{h}_1 *_u \check{h}_2)(\tau) := \int_u^{\tau-u} \check{h}_1(\tau-\eta)\check{h}_2(\eta)d\eta \in \widetilde{\mathcal{O}}$$

with u arbitrarily close to 0 satisfying $\tau \in]0,u[$ and $\arg(\tau-u) = \arg(\tau)-\pi$. Note that $\overset{\triangledown}{h}_1 \circledast \overset{\triangledown}{h}_2$ makes sense since it does not depend on u, nor on the choice of the majors $\check{h}_1$ and $\check{h}_2$. The convolution product $\circledast$ is commutative and associative on $\mathcal{C}$ with unit $\delta := \operatorname{can}\left(\frac{1}{2i\pi\tau}\right)$.

The action of $\frac{d}{d\tau}$ on $\mathcal{C}$ is defined by

$$\frac{d}{d\tau}\overset{\triangledown}{h} = \delta' \circledast \overset{\triangledown}{h}$$

and satisfies relation

$$\frac{d}{d\tau}\left(\overset{\triangledown}{h}_1 \circledast \overset{\triangledown}{h}_2\right) = \left(\frac{d}{d\tau}\overset{\triangledown}{h}_1\right) \circledast \overset{\triangledown}{h}_2 = \overset{\triangledown}{h}_1 \circledast \left(\frac{d}{d\tau}\overset{\triangledown}{h}_2\right).$$

Finally, the multiplication by τ is an $\circledast$-derivation, i.e.

$$\tau\left(\overset{\triangledown}{h}_1 \circledast \overset{\triangledown}{h}_2\right) = \left(\tau\overset{\triangledown}{h}_1\right) \circledast \overset{\triangledown}{h}_2 + \overset{\triangledown}{h}_1 \circledast \left(\tau\overset{\triangledown}{h}_2\right).$$

In the sequel of this article, we shall use especially the following classical subspaces of $\mathcal{C}$.

- *The subspace* $\mathcal{C}^{\leqslant 1}$ of singularities for which the variation defines an entire function on all $\widetilde{\mathbb{C}}$ with exponential growth of order $\leqslant 1$ on any bounded sector of infinity[(3)]. Recall that this space is isomorphic to the space of analytic functions with subexponential growth at $0 \in \widetilde{\mathbb{C}}$ [14, pp. 46-48]. In particular, any power t^λ and any exponential $e^{P(t^{1/p})}$ with $p \geqslant 2$ and $P(t)$ polynomial in t of degree $< p$ define singularities in $\mathcal{C}^{\leqslant 1}$.
- *The subspace* $\overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\Omega,0}$ (resp. $\overset{\triangledown}{\mathcal{D}et}{}^{s\text{-}res}_{\Omega,\widetilde{0}}$) of singularities for which there exists a major of the form
$$\sum_{\text{finite}} h_{\alpha,p}(\tau)\tau^\alpha(\ln\tau)^p$$
with $\alpha \in \mathbb{C}$, $p \in \mathbb{N}$ and $h_{\alpha,p}(\tau) \in \mathcal{R}es^{sum}_{\Omega,0}$ (resp. $h_{\alpha,p}(\tau) \in \mathcal{R}es^{sum}_{\Omega,\widetilde{0}}$ holomorphic on a punctured disc at 0 in the first sheet).

Definition 3.4. — *The elements of* $\overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\Omega,0}$ *(resp.* $\overset{\triangledown}{\mathcal{D}et}{}^{s\text{-}res}_{\Omega,\widetilde{0}}$*) are called* summable-resurgent singularities of Nilsson class with singular support $\Omega, 0$ *(resp.* of finite determination with singular support $\Omega, \widetilde{0}$*).*

For any $\omega \in \mathbb{C}^*$, we denote by $\mathcal{C}_{|\omega}$ the space of *singularities at* ω, i.e. the space $\mathcal{C}$ translated from 0 to ω. A function $\check{h}$ is then a major of a singularity at ω if $\check{h}(\omega+\tau)$ is a major of a singularity at 0. In the same way, we define the translated space $\mathcal{C}^{\leqslant 1}{}_{|\omega}$, etc…

[(3)] or, equivalently, for which there exists a major satisfying this same property [14].

3.2.2. Description of singularities

Given $v \in \{1,\dots,r\}$ and $\ell \in \{1,\dots,J\}$, the behavior of $\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}}^{\bullet;v,\ell}$ at a singular point $\omega \in \boldsymbol{\Omega}_\ell^*$ depends, of course, on the "homotopic class" of the path γ of analytic continuation followed from any point $a \neq 0$ of $\Sigma_{\boldsymbol{\theta}}^{v,\ell}$ to a neighborhood of ω. Note in particular that "homotopic class" implies that the behavior of $\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}}^{\bullet;v,\ell}$ does not depend on the choice of a. We denote below by

- $\widehat{\boldsymbol{F}}_{\boldsymbol{\theta};\omega,\gamma}^{\bullet;v,\ell}$ the analytic continuation of $\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}}^{\bullet;v,\ell}$ along the path γ,
- $\overset{\triangledown}{\boldsymbol{F}}{}_{\boldsymbol{\theta};\omega,\gamma}^{\bullet;v,\ell} := \operatorname{can}(\widehat{\boldsymbol{F}}_{\boldsymbol{\theta};\omega,\gamma}^{\bullet;v,\ell})$ the singularity of $\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}}^{\bullet;v,\ell}$ at ω defined by $\widehat{\boldsymbol{F}}_{\boldsymbol{\theta};\omega,\gamma}^{\bullet;v,\ell}$.

Let us now introduce the key notion of front of a singularity [31, 32] .

Definition 3.5. — *Let $\ell \in \{1,\dots,J\}$ and $\omega \in \boldsymbol{\Omega}_\ell^*$ a Stokes value of level r of $\widetilde{F}^{\bullet;\ell}(x)$. We call* front of level r of ω *the set of all the polynomials $(q_j - q_\ell)(1/x)$ with leading terms $-\omega/x^r$. We denote it by $Fr_\ell(\omega)$ and we have*

$$Fr_\ell(\omega) := \left\{-\frac{\omega}{x^r} + q_{\ell,\omega;k}\left(\frac{1}{x}\right) \ ;\ k = 1,\dots,s_{\ell,\omega}\right\},$$

where $s_{\ell,\omega}$ is an integer $\geqslant 1$ and where all the $q_{\ell,\omega;k}(1/x)$ are polynomials in $1/x$ of degree $< r$ and without constant term. Moreover, ω (hence, its corresponding singularity $\overset{\triangledown}{\boldsymbol{F}}{}_{\boldsymbol{\theta};\omega,\gamma}^{\bullet;v,\ell}$ too) is said to be of a good front *when $s_{\ell,\omega} = 1$ and with* a bad front *otherwise.*

In the special case where ω has a good front, we simply denote $q_{\ell,\omega}$ for $q_{\ell,\omega;1}$. Then,

$$Fr_\ell(\omega) = \left\{-\frac{\omega}{x^r} + q_{\ell,\omega}\left(\frac{1}{x}\right)\right\}$$

and we more precisely say that ω (and its corresponding singularity too) has *a good monomial front* when $q_{\ell,\omega} \equiv 0$ and *a good nonmonomial front* otherwise.

We are now able to state the main result of this section.

Theorem 3.6 (General form of $\overset{\triangledown}{\boldsymbol{F}}{}_{\boldsymbol{\theta};\omega,\gamma}^{\bullet;v,\ell}$). — *Let $v \in \{1,\dots,r\}$ and $\ell \in \{1,\dots,J\}$. Let $\omega \in \boldsymbol{\Omega}_\ell^*$ be a Stokes value of level r of $\widetilde{F}^{\bullet;\ell}(x)$ and γ a path on $\mathbb{C}\backslash\boldsymbol{\Omega}_\ell$ starting from a point of $\Sigma_{\boldsymbol{\theta}}^{v,\ell}$ and ending in a neighborhood of ω.*

(1) *Suppose that ω has a good front. Let*

$$\mathcal{Q}_{\ell,\omega} = \left\{q_{\ell,\omega}\left(\frac{1}{\mu^{v-1}t^{1/r}}\right) \ ;\ v = 1,\dots,r\right\}$$

with $\mu = e^{-2i\pi/r}$. *Then,*

$$\boxed{\overset{\triangledown}{\boldsymbol{F}}{}^{\bullet;v,\ell}_{\boldsymbol{\theta};\omega,\gamma} \in \sum_{q\in\mathcal{Q}_{\ell,\omega}} \overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\boldsymbol{\Omega}_\ell-\omega,0} \circledast \overset{\triangledown}{e}{}^{q}{}_{|\omega}.}$$

In particular, if ω *has moreover a monomial front, then*

$$\boxed{\overset{\triangledown}{\boldsymbol{F}}{}^{\bullet;v,\ell}_{\boldsymbol{\theta};\omega,\gamma} \in \overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\boldsymbol{\Omega}_\ell-\omega,0|\omega}.}$$

(2) *Suppose that* ω *has a bad front. Let*

$$\mathcal{Q}_{\ell,\omega} = \left\{q_{\ell,\omega;k}\left(\frac{1}{\mu^{v-1}t^{1/r}}\right) \ ; \ k = 1,\dots,s_{\ell,\omega} \text{ and } v = 1,\dots,r\right\}$$

with $\mu = e^{-2i\pi/r}$. *Then*

$$\boxed{\overset{\triangledown}{\boldsymbol{F}}{}^{\bullet;v,\ell}_{\boldsymbol{\theta};\omega,\gamma} \in \sum_{q\in\mathcal{Q}_{\ell,\omega}} \overset{\triangledown}{\mathcal{D}et}{}^{s\text{-}res}_{\boldsymbol{\Omega}_\ell-\omega,\tilde{0}} \circledast \overset{\triangledown}{e}{}^{q}{}_{|\omega}.}$$

Notation $\overset{\triangledown}{e}{}^{q}$ *stands for the singularity of* $\mathcal{C}^{\leqslant 1}$ *defined by* e^q *(see Section 3.2.1).*

A more precise description of singularities with good monomial front will be given later in Section 5. For the moment, let us prove our two main Theorems 3.3 and 3.6.

4. Proofs of Theorems 3.3 and 3.6

Before starting the proofs, let us first begin by some reminders about the Borel transformation which shall play a central role.

4.1. Borel transformation

Definition and properties. Let $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ and let $h(t)$ be a function defined and holomorphic on a domain containing a sector with vertex 0, bisected by α and opening larger than π. Under "good" hypotheses on h at 0 [24], the Borel transform (of level 1) of h in direction α is given by the integral

$$\mathcal{B}_\alpha(h(t))(\tau) = \widehat{h}_\alpha(\tau) := \frac{1}{2i\pi}\int_{\Gamma_\alpha} h(t)e^{-t/\tau}\frac{dt}{t^2},$$

where Γ_α denotes the image by $t \mapsto 1/t$ of a Hankel contour directed by the direction α and oriented positively[(4)] . Using Hankel's formula for the inverse of the Gamma function, we obtain $\mathcal{B}_\alpha(t^\lambda)(\tau) = \tau^{\lambda-1}/\Gamma(\lambda)$ for all $\lambda \in \mathbb{C}\backslash(-\mathbb{N})$ and $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$; when $h(t) = t^{-m}$ with $m \geqslant 1$, we get a natural generalization $\mathcal{B}_\alpha(t^{-m}) = \delta^{(m)}$ the m-th derivative of the Dirac distribution at 0 (hence, the coherence with the definition of the formal Borel transformation, see footnote 1).

The Borel transformation $\mathcal{B}_\alpha$ changes the derivation $t^2\frac{d}{dt}$ into the multiplication by τ and the multiplication by $1/t$ into the derivation $\frac{d}{d\tau}$. In particular, it changes derivation $\frac{d^k}{dt^k}$ into $\frac{d^{k+1}}{d\tau^{k+1}}(\tau^k\frac{d^{k-1}}{d\tau^{k-1}})$ for any $k \geqslant 1$. Moreover, it changes the ordinary product $\cdot$ into the convolution product $*$:

$$h(t)k(t) \longmapsto \widehat{h}_\alpha * \widehat{k}_\alpha(\tau) := \int_0^\tau \widehat{h}_\alpha(\tau-\eta)\widehat{k}_\alpha(\eta)d\eta$$

when $\widehat{h}_\alpha$ and $\widehat{k}_\alpha$ are both integrable at 0 (note that δ is the unit of $*$). Finally, it changes the multiplication by $e^{-\omega/t}$ into the translation by ω.

Some classes of functions. Among all the classes of functions on which one can apply the Borel transformation, we shall actually use in the sequel of the article only those of one of the following two types:

- $h(t) \in \widetilde{\mathcal{O}}$ has a subexponential growth at the origin (we denote below $h(t) \in \widetilde{\mathcal{O}}^{\leqslant \exp}$), that is, for all $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that $|h(t)| \leqslant C_\varepsilon e^{\varepsilon/|t|}$ (or, what amounts the same, $\limsup\limits_{|t|\to 0}(|t|\ln(|h(t)|)) = 0$) uniformly on any bounded sector of the form $\theta_1 < \arg(t) < \theta_2$. In particular, any analytic function $h(t) \in \mathcal{O}$ at 0, any power t^λ of t, any power $(\ln t)^m$ of the logarithm and any exponential $e^{P(t^{-1/p})}$ with a polynomial $P(t)$ in t of degree $< p$ belong to $\widetilde{\mathcal{O}}^{\leqslant \exp}$,
- $h(t)$ is the $\underline{k}$-sum of a $\underline{k}$-summable series $\widetilde{h}(t) \in \mathbb{C}[[t]]$ in direction α with $\underline{k} := (k_1 < \cdots < k_s = 1)$ and $s \geqslant 1$.

For the first one, the existence of $\widehat{h}_\alpha$ is straightaway from the property of h at 0 and one can show that $\widehat{h}_\alpha$ defines in this case an entire function on all $\widetilde{\mathbb{C}}$ (= the Riemann surface of the logarithm) with exponential growth of order $\leqslant 1$ on any bounded sector at infinity. We denote below $\widehat{h}_\alpha(\tau) \in \mathcal{O}^{\leqslant 1}(\widetilde{\mathbb{C}})$. In the special case where $h(t) \in \mathcal{O}$ is analytic at 0, one has more precisely $\widehat{h}_\alpha(\tau) \in \mathcal{O}^{\leqslant 1}(\mathbb{C})$. As for the second one, the existence of $\widehat{h}_\alpha$ is due to the

(4) Observe that we need a contour that ends at 0 since the functions we consider are studied near the origin; if we worked at infinity, we would use a Hankel contour itself.

continuity of h at 0; furthermore, one can check that $\widehat{h}_\alpha$ coincides with the function defined from the Balser–Tougeron theorem by the formal Borel transform of $\widetilde{h}$. In particular, denoting by $\boldsymbol{F}_{\boldsymbol{\theta}}^{\bullet;v,\ell}(t)$ the $\underline{\boldsymbol{\rho}_\ell}$-sum of $\widetilde{\boldsymbol{F}}^{\bullet;v,\ell}(t)$ in direction $\boldsymbol{\theta}$ (see p. 650), we have $\mathcal{B}_{\boldsymbol{\theta}}(\boldsymbol{F}_{\boldsymbol{\theta}}^{\bullet;v,\ell}) = \widehat{\boldsymbol{F}}_{\boldsymbol{\theta}}^{\bullet;v,\ell}$.

Extended Borel transformation. Since the Borel transform $\mathcal{B}_\alpha(h(t))$ of any $h(t) \in \widetilde{\mathcal{O}}^{\leqslant \exp}$ may be integrable, or not, at 0 (see for instance the Borel transform of t^λ just above), the convolution product $*$ may not be defined, as well as the Laplace transform of such functions. To circumvent this problem, the idea consists then in considering $\mathcal{B}_\alpha(h(t))$ not as a function, but as a singularity whose the variation is $\widehat{h}_\alpha(\tau)$. More precisely, one has the following.

Proposition 4.1 (Écalle, [14, pp. 46–48]). — *Let $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ be a direction. Then, the Borel transformation $\mathcal{B}_\alpha$ can be extended to an isomorphism*

$$\mathcal{B}_\alpha^{ext} : \left(\widetilde{\mathcal{O}}^{\leqslant \exp}, +, \cdot, t^2\frac{d}{dt}\right) \longrightarrow \left(\mathcal{C}^{\leqslant 1}, +, \circledast, \tau\cdot\right) \ , \qquad \mathcal{B}_\alpha^{ext}(h) = \overset{\triangledown}{h}_\alpha$$

of $\mathbb{C}$-differential algebras[5] so that $\operatorname{var}(\overset{\triangledown}{h}_\alpha) = \widehat{h}_\alpha$ for all $h \in \widetilde{\mathcal{O}}^{\leqslant \exp}$. Its inverse is the Laplace transformation $\mathcal{L}_\alpha^{ext}$ defined as follows: given $\overset{\triangledown}{h} \in \mathcal{C}^{\leqslant 1}$, $\check{h}$ a major of $\overset{\triangledown}{h}$ and $\widehat{h} = \operatorname{var}(\overset{\triangledown}{h})$,

$$\mathcal{L}_\alpha^{ext}(\overset{\triangledown}{h})(t) := \int_{\gamma_{\alpha,\varepsilon}} \check{h}(\tau)e^{-\tau/t}d\tau + \int_{\varepsilon e^{i\alpha}}^{\infty e^{i\alpha}} \widehat{h}(\tau)e^{-\tau/t}d\tau,$$

where $\gamma_{\alpha,\varepsilon}$ denotes a circle centered at the origin and going from $\varepsilon e^{i(\alpha-2\pi)}$ to $\varepsilon e^{i\alpha}$, $\varepsilon > 0$ small enough.

Note that $\mathcal{L}_\alpha^{ext}(\overset{\triangledown}{h})$ makes sense since it does not depend on the choice of ε nor on the chosen major $\check{h}$; in particular, for a choice $\check{h} \in \mathcal{O}^{\leqslant 1}(\widetilde{\mathbb{C}})$, one has

$$\mathcal{L}_\alpha^{ext}(\overset{\triangledown}{h})(t) := \int_{\gamma_\alpha} \check{h}(\tau)e^{-\tau/t}d\tau,$$

where γ_α denotes a Hankel path directed by direction α and oriented positively. Note also that, if $\widehat{h}$ is integrable at 0, then $\mathcal{L}_\alpha^{ext}(\overset{\triangledown}{h})$ coincides with the "classical" Laplace transform

$$\mathcal{L}_\alpha(\widehat{h})(t) := \int_0^{\infty e^{i\alpha}} \widehat{h}(\tau)e^{-t/\tau}d\tau.$$

[5] The stability of $\mathcal{C}^{\leqslant 1}$ under the convolution product $\circledast$ is due to the fact that, for any singularity $\overset{\triangledown}{h} \in \mathcal{C}^{\leqslant 1}$, one can always choose a major $\check{h}$ in $\mathcal{O}^{\leqslant 1}(\widetilde{\mathbb{C}})$ (see footnote 3).

The following relations are essentially known:

$$(t^{\lambda}(\ln t)^{p})^{\triangledown}_{\alpha} = \operatorname{can}\left(\frac{d^p}{d\lambda^p}\left(\frac{\tau^{\lambda-1}}{(1-e^{-2i\pi\lambda})\Gamma(\lambda)}\right)\right),$$

$$(\overset{\triangledown}{t^{m}})_{\alpha} = \operatorname{can}\left(\frac{\tau^{m-1}\ln\tau}{(m-1)!}\right), \qquad \overset{\triangledown}{1}_{\alpha} = \delta, \qquad (\overset{\triangledown}{t^{-m}})_{\alpha} = \delta^{(m)}$$

for all $\lambda \in \mathbb{C}\backslash\mathbb{Z}$, $p \in \mathbb{N}$, $m \in \mathbb{N}^*$ and $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$. More generally, let $\mathbb{C}[t^{\lambda},(\ln t)^{p}]_{\lambda\in\mathbb{C},p\in\mathbb{N}} \subset \widetilde{\mathcal{O}}^{\leqslant \exp}$ denote the space of $\mathbb{C}$-linear combinations of terms of the form $t^{\lambda}(\ln t)^{p}$ with $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}$. Let $\overset{\triangledown}{\mathbb{C}}_{\alpha}[t^{\lambda},(\ln t)^{p}]_{\lambda\in\mathbb{C},p\in\mathbb{N}}$ be its image by $\mathcal{B}^{ext}_{\alpha}$. Then, for any $\overset{\triangledown}{h} \in \overset{\triangledown}{\mathbb{C}}_{\alpha}[t^{\lambda},(\ln t)^{p}]_{\lambda\in\mathbb{C},p\in\mathbb{N}}$, there exists a major $\check{h}$ in $\mathbb{C}[\tau^{\mu},(\ln\tau)^{q}]_{\mu\in\mathbb{C},q\in\mathbb{N}}$. The following result will be useful later.

Proposition 4.2. — *Let $\Omega \subset \mathbb{C}$ be a finite subset of $\mathbb{C}$ containing* 0. *Let $p \geqslant 2$ and let $q(1/x)$ be a polynomial in $1/x$ of degree $< p$. Then, the four spaces $\overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\Omega,0}$, $\overset{\triangledown}{\mathcal{D}et}{}^{s\text{-}res}_{\Omega,\widetilde{0}}$, $\overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\Omega,0} \circledast \overset{\triangledown}{e}{}^{q(t^{-1/p})}$ and $\overset{\triangledown}{\mathcal{D}et}{}^{s\text{-}res}_{\Omega,\widetilde{0}} \circledast \overset{\triangledown}{e}{}^{q(t^{-1/p})}$ are stable under derivation $\frac{d}{d\tau}$ and under $\circledast$-convolution by an element of $\overset{\triangledown}{\mathbb{C}}_{\alpha}[t^{\lambda},(\ln t)^{p}]_{\lambda\in\mathbb{C},p\in\mathbb{N}}$. They are also stable by multiplication by τ.*

Proof. — We just prove the stability of $\overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\Omega,0} \circledast \overset{\triangledown}{e}{}^{q(t^{-1/p})}$ and $\overset{\triangledown}{\mathcal{D}et}{}^{s\text{-}res}_{\Omega,\widetilde{0}} \circledast \overset{\triangledown}{e}{}^{q(t^{-1/p})}$ by the multiplication by τ. The other stabilities are straightforward and are left to the reader. Let $\overset{\triangledown}{h} \in \overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\Omega,0}$ (resp. $\overset{\triangledown}{\mathcal{D}et}{}^{s\text{-}res}_{\Omega,\widetilde{0}}$). Since the multiplication by τ is a $\circledast$-convolution, we have

$$\tau\left(\overset{\triangledown}{h} \circledast \overset{\triangledown}{e}{}^{q(t^{-1/p})}\right) = \left(\tau\overset{\triangledown}{h}\right) \circledast \overset{\triangledown}{e}{}^{q(t^{-1/p})} + \overset{\triangledown}{h} \circledast \left(\tau\overset{\triangledown}{e}{}^{q(t^{-1/p})}\right),$$

where, due to the properties of the Borel transformation $\mathcal{B}^{ext}_{\alpha}$,

$$\tau\overset{\triangledown}{e}{}^{q(t^{-1/p})} = \mathcal{B}^{ext}_{\alpha}\left(t^2\frac{d}{dt}e^{q(t^{-1/p})}\right) = \overset{\triangledown}{P} \circledast \overset{\triangledown}{e}{}^{q(t^{-1/p})}$$

with

$$P(t) = -\frac{1}{p}t^{(p-1)/p}\frac{dq}{dt}\left(t^{-1/p}\right) \in t\mathbb{C}[t^{-1/p}].$$

Indeed, $e^{q(t^{-1/p})} \in \widetilde{\mathcal{O}}^{\leqslant \exp}$ (note that the choice of the direction α is arbitrary). Consequently,

$$\tau\left(\overset{\triangledown}{h} \circledast \overset{\triangledown}{e}{}^{q(t^{-1/p})}\right) = \left(\tau\overset{\triangledown}{h} + \overset{\triangledown}{h} \circledast \overset{\triangledown}{P}\right) \circledast \overset{\triangledown}{e}{}^{q(t^{-1/p})}$$

with $\tau\overset{\triangledown}{h} + \overset{\triangledown}{h} \circledast \overset{\triangledown}{P} \in \overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\Omega,0}$ (resp. $\overset{\triangledown}{\mathcal{D}et}{}^{s\text{-}res}_{\Omega,\widetilde{0}}$), which completes the proof. □

Let us now consider a $\underline{k}$-summable series $\widetilde{h}(t) \in \mathbb{C}[[t]]$ in direction α with $\underline{k} = (k_1 < \cdots < k_s = 1)$ and $s \geqslant 1$. As before, we denote by $h_\alpha(t)$ its $\underline{k}$-sum in direction α and by $\widehat{h}_\alpha(\tau)$ the Borel transform of h_α in direction α. Assume also that $\widehat{h}_\alpha(\tau) \in \mathcal{R}es^{sum}_{\Omega,0}$ if $s = 1$ and $\widehat{h}_\alpha(\tau) \in \mathcal{R}es^{sum}_{\Omega,\widetilde{0}}$ if $s \geqslant 2$. Then, the extended Borel transformation $\mathcal{B}^{ext}_\alpha$ of Proposition 4.1 above can be applied to h_α and one thereby defines a unique singularity $\overset{\triangledown}{h}_\alpha$ satisfying $\operatorname{var}(\overset{\triangledown}{h}_\alpha) = \widehat{h}_\alpha$ and $\mathcal{L}^{ext}_\alpha(\overset{\triangledown}{h}_\alpha) = h_\alpha$. For example, for $s = 1$ and $\widetilde{h}(t) = a + \widetilde{c}(t) \in \mathbb{C} \oplus t\mathbb{C}[[t]]$, one has

$$\widehat{h}_\alpha(\tau) = a\delta + \widehat{c}_\alpha(\tau) \in \mathbb{C}\delta \oplus \mathcal{R}es^{sum}_{\Omega,0}$$

and

$$\overset{\triangledown}{h}_\alpha = \operatorname{can}\left(\frac{a}{2i\pi\tau} + \frac{\widehat{c}_\alpha(\tau)\ln\tau}{2i\pi}\right).$$

In particular, $\overset{\triangledown}{h}_\alpha \in \overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\Omega,0}$. More generally, one has the following classical result.

Proposition 4.3. — *With conditions as above.*

(1) Case $s = 1$.
  (a) *Let $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}$. Then,*

$$\boxed{\left(h_\alpha(t)t^\lambda(\ln t)^p\right)^{\triangledown}_\alpha \in \overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\Omega,0}.}$$

  (b) *Conversely, let $\overset{\triangledown}{h} = \operatorname{can}(h(\tau)\tau^\lambda(\ln\tau)^p) \in \overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\Omega,0}$ and write $h(\tau)$ in a neighborhood of $0 \in \mathbb{C}$ as*

$$h(\tau) = \sum_{m \geqslant 0} h_m \tau^m.$$

Then, for any direction $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ such that $\Omega \cap]0, \infty e^{i\alpha}[= \varnothing$,

$$\boxed{\mathcal{L}^{ext}_\alpha(\overset{\triangledown}{h}) = \sum_{k=0}^{p}\binom{p}{k} h_{\lambda,p-k;\alpha}(t)t^{\lambda+1}(\ln t)^k,}$$

where, for all $\ell = 0, \ldots, p$, $h_{\lambda,\ell;\alpha}(t)$ is the 1-sum in direction α of the formal series

$$\widetilde{h}_{\lambda,\ell}(t) = 2i\pi \sum_{m \geqslant 0} \frac{d^\ell}{dz^\ell}\left(\frac{e^{-i\pi z}}{\Gamma(1-z)}\right)_{|z=m+1+\lambda} h_m t^m.$$

Moreover, the Borel transform $\widehat{h}_{\lambda,\ell;\alpha}(\tau)$ in direction α of $h_{\lambda,\ell;\alpha}(t)$ belongs to $\mathcal{R}es^{sum}_{\Omega,0}$.

(2) Case $s \geqslant 2$. *Let $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}$. Suppose also that $\widehat{h}_\alpha$ is of finite determination at 0. Then*

$$\boxed{\left(h_\alpha(t)\overset{\triangledown}{t^\lambda}(\ln t)^p\right)_\alpha \in \overset{\triangledown}{\mathcal{D}et}{}^{s\text{-}res}_{\Omega,\widetilde{0}}.}$$

Proof. — The case $s=1$ is proved in [19, 21, 33] and the case $s \geqslant 2$ stems obvious from the properties of variation (see [19, Lem. 3.6] for instance) when $\lambda = p = 0$. In the general case, it is sufficient to remark that

$$\left(h_\alpha(t)\overset{\triangledown}{t^\lambda}(\ln t)^p\right)_\alpha = \overset{\triangledown}{h}_\alpha \circledast \left(\overset{\triangledown}{t^\lambda}(\ln t)^p\right)_\alpha$$

with $\overset{\triangledown}{h}_\alpha \in \overset{\triangledown}{\mathcal{D}et}{}^{s\text{-}res}_{\Omega,\widetilde{0}}$ and $\left(\overset{\triangledown}{t^\lambda}(\ln t)^p\right)_\alpha \in \overset{\triangledown}{\mathbb{C}}_\alpha[t^\mu,(\ln t)^q]_{\mu\in\mathbb{C},q\in\mathbb{N}}$. The result follows then by stability. $\square$

Let us now turn to the proofs of Theorems 3.3 and 3.6.

4.2. Reduction of the proofs

First step. Let us first observe that any of the J column-blocks $\widetilde{F}^{\bullet;\ell}(x)$, $\ell = 1,\dots,J$, of $\widetilde{F}(x)$ associated with the Jordan block-structure of the matrix L of exponents of formal monodromy (see the beginning of Section 2.1) can be positioned at the first place by means of a convenient permutation P on the columns of $\widetilde{Y}(x)$. Furthermore, acting also by P^{-1} on the rows of $\widetilde{Y}(x)$, one can keep the initial normalizations $(N_1)-(N_3)$ of $\widetilde{Y}(x)$; precisely, the new formal fundamental solution $P^{-1}\widetilde{Y}(x)P$ reads $P^{-1}\widetilde{Y}(x)P = P^{-1}\widetilde{F}(x)Px^{P^{-1}LP}e^{P^{-1}Q(1/x)P}$ with $P^{-1}\widetilde{F}(x)P = I_n + O(x^r)$. Consequently, due to the block-structure of matrix $\widetilde{\boldsymbol{F}}(t)$ (see p. 650), it is sufficient to prove Theorems 3.3 and 3.6 in restriction to the first column-block $\widetilde{\boldsymbol{F}}^{\bullet;1,1}(t)$ of $\widetilde{\boldsymbol{F}}(t)$. Recall that $\widetilde{\boldsymbol{F}}^{\bullet;1,1}(t)$ has size $rn \times n_1$.

Moreover, to simplify notations and calculations, we assume also that polynomial q_1 and eigenvalue λ_1 are zero, conditions which can be always fulfilled by means of the transformation $Y \longmapsto x^{-\lambda_1}e^{-q_1(1/x)}Y$ on initial system (A). In particular, writing polynomials $q_j(1/x)$ in the form

$$q_j\left(\frac{1}{x}\right) = -\frac{a_{j,r}}{x^r} - \frac{a_{j,r-1}}{x^{r-1}} - \cdots - \frac{a_{j,1}}{x}$$

with $a_{j,k} \in \mathbb{C}$, this implies that the set $\boldsymbol{\Omega}_1^*$ of Stokes values of level r of $\widetilde{F}^{\bullet;1}(x)$ is the set of all the $a_{j,r} \neq 0$ (see Definition 2.1).

Second step. Let us now apply the cyclic vector lemma due to P. Deligne [13, Lem. II.1.3] and the Birkhoff's algebraisation theorem [10] (see also [34, Thm. 3.3.1]): there exists a meromorphic gauge transformation $\boldsymbol{Y} = \boldsymbol{M}(t)\boldsymbol{Z}$ with $\boldsymbol{M}(t) \in GL_{rn}(\mathbb{C}\{t\}[t^{-1}])$ that changes the r-reduced system $(\boldsymbol{A})$ into a system $({}^{\boldsymbol{M}}\boldsymbol{A})$ which is the companion form of a scalar linear differential equation $Dy(t) = 0$ with polynomial coefficients, of order rn and levels $\leqslant 1$ at the origin (the levels of D are the levels of $(\boldsymbol{A})$). Moreover, multiplying the formal solutions of this equation by a convenient power of t if needed, we can always suppose that $({}^{\boldsymbol{M}}\boldsymbol{A})$ admits for formal fundamental solution at 0 a matrix $\widetilde{\boldsymbol{Z}}(t)$ of the form $\widetilde{\boldsymbol{Z}}(t) = \widetilde{\boldsymbol{G}}(t)\widetilde{\boldsymbol{Y}}_0(t)$, where $\widetilde{\boldsymbol{G}}(t) := \boldsymbol{M}^{-1}(t)\widetilde{\boldsymbol{F}}(t) \in M_{rn}(\mathbb{C}[[t]])$ is a formal power series in t. We refer to p. 650 for the definition of $\widetilde{\boldsymbol{Y}}_0(t)$.

By construction, the two column-blocks $\widetilde{\boldsymbol{F}}^{\bullet;v,\ell}(t)$ and $\widetilde{\boldsymbol{G}}^{\bullet;v,\ell}(t)$ are together $\underline{\boldsymbol{\rho}_\ell}$-summable in direction $\boldsymbol{\theta}$ for all v and ℓ and we have $\boldsymbol{F}_{\boldsymbol{\theta}}^{\bullet;v,\ell}(t) = \boldsymbol{M}(\boldsymbol{t})\boldsymbol{G}_{\boldsymbol{\theta}}^{\bullet;v,\ell}(t)$, where, as before, $\boldsymbol{F}_{\boldsymbol{\theta}}^{\bullet;v,\ell}$ and $\boldsymbol{G}_{\boldsymbol{\theta}}^{\bullet;v,\ell}$ denote their respective $\underline{\boldsymbol{\rho}_\ell}$-sums in direction $\boldsymbol{\theta}$. Thereby, writing $\boldsymbol{M}(t)$ in the form

$$\boldsymbol{M}(t) = \sum_{m=0}^{N} \frac{\boldsymbol{\alpha}_m}{t^m} + \boldsymbol{M}'(t)$$

with $N \in \mathbb{N}$, $\boldsymbol{\alpha}_m \in M_{rn}(\mathbb{C})$ and $\boldsymbol{M}'(t) \in M_{rn}(t\mathbb{C}\{t\})$, we deduce from the properties of the Borel transformation (see Section 4.1 above) that the functions $\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}}^{\bullet;v,\ell}$ and $\widehat{\boldsymbol{G}}_{\boldsymbol{\theta}}^{\bullet;v,\ell}$ satisfy the relation

$$\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}}^{\bullet;v,\ell} = \sum_{m=0}^{N} \boldsymbol{\alpha}_m \frac{d^m}{d\tau^m} \widehat{\boldsymbol{G}}_{\boldsymbol{\theta}}^{\bullet;v,\ell} + \widehat{\boldsymbol{M}}'_{\boldsymbol{\theta}} * \widehat{\boldsymbol{G}}_{\boldsymbol{\theta}}^{\bullet;v,\ell},$$

where $\widehat{\boldsymbol{M}}'_{\boldsymbol{\theta}} \in M_{rn}(\mathcal{O}^{\leqslant 1}(\mathbb{C}))$ and where $\widehat{\boldsymbol{G}}_{\boldsymbol{\theta}}^{\bullet;v,\ell}$ is holomorphic on the same domain $\Sigma_{\boldsymbol{\theta}}^{v,\ell}$ as $\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}}^{\bullet;v,\ell}$. In particular, for $v = \ell = 1$, this relation shows us that it is sufficient to prove Theorems 3.3 and 3.6 with $\widehat{\boldsymbol{G}}_{\boldsymbol{\theta}}^{\bullet;1,1}$ instead of $\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}}^{\bullet;1,1}$.

Third step. This last step is based on the structure of matrix $\widehat{\boldsymbol{G}}_{\boldsymbol{\theta}}^{\bullet;1,1}$. Let us first begin by giving a basis of the space $Sol(D)$ of solutions of $Dy(t) = 0$. It can be built as follows. Let us choose an argument of $\boldsymbol{\theta}$, say its principal determination $\boldsymbol{\theta}^\star \in \,]-2\pi, 0]$ [(6)], and let us denote by $\boldsymbol{Y}_{0;\boldsymbol{\theta}^\star}(t)$ the actual analytic function defined from $\widetilde{\boldsymbol{Y}}_0(t)$ by the choice of $\arg(t)$ close to $\boldsymbol{\theta}^\star$ (denoted below by $\arg(t) \simeq \boldsymbol{\theta}^\star$). Let us also denote by $\boldsymbol{G}_{\boldsymbol{\theta}}(t)$ the sum of $\widetilde{\boldsymbol{G}}(t)$

[(6)] Any choice is convenient. However, to be compatible, on the Riemann sphere, with the usual choice $0 \leqslant \arg(z = 1/x) < 2\pi$ of the principal determination at infinity, we suggest to choose $-2\pi < \arg(x) \leqslant 0$ as principal determination about 0.

in direction $\boldsymbol{\theta}$, i.e. the matrix in which all the column-blocks $\widetilde{\boldsymbol{G}}^{\bullet;v,\ell}(t)$ are replaced by their $\underline{\boldsymbol{\rho}_\ell}$-sum $\boldsymbol{G}_{\boldsymbol{\theta}}^{\bullet;v,\ell}(t)$. Then, the matrix $\boldsymbol{Z}_{\boldsymbol{\theta}}(t) := \boldsymbol{G}_{\boldsymbol{\theta}}(t)\boldsymbol{Y}_{0;\boldsymbol{\theta}^\star}(t)$ is, for $\arg(t) \simeq \boldsymbol{\theta}^\star$, a fundamental solution of the companion form of equation $Dy(t) = 0$. Hence, by considering the rn coefficients of the first line of $\boldsymbol{Z}_{\boldsymbol{\theta}}(t)$, we obtain the following proposition.

Proposition 4.4 (Basis of $Sol(D)$). — *Let $\boldsymbol{g}_{\boldsymbol{\theta}}^{v,\ell,q}(t)$ denote the entry at row 1 and column q of $\boldsymbol{G}_{\boldsymbol{\theta}}^{\bullet;v,\ell}(t)$. Then*

$$\boxed{Sol(D) = \operatorname{vect}(\boldsymbol{z}_{\boldsymbol{\theta}}^{v,\ell,q}(t);\ v = 1,\dots,r,\ \ell = 1,\dots,J,\ q = 1,\dots,n_\ell),}$$

where $\boldsymbol{z}_{\boldsymbol{\theta}}^{v,\ell,q}(t)$ is defined for all v, ℓ and q by

$$\boldsymbol{z}_{\boldsymbol{\theta}}^{v,\ell,q}(t) := \boldsymbol{h}_{\boldsymbol{\theta}}^{v,\ell,q}(t)e^{q_\ell(1/(\mu^{v-1}t^{1/r}))}$$

with

$$\boldsymbol{h}_{\boldsymbol{\theta}}^{v,\ell,q}(t) := \sum_{u=1}^{r}\sum_{p=1}^{q}\mu^{(v-1)(\lambda_\ell-u+1)}\boldsymbol{g}_{\boldsymbol{\theta}}^{u,\ell,p}(t)t^{\frac{\lambda_\ell-u+1}{r}}\frac{\ln^{q-p}(\mu^{v-1}t^{\frac{1}{r}})}{(q-p)!}$$

and $\arg(t) \simeq \boldsymbol{\theta}^\star$. Recall that $\mu = e^{-2i\pi/r}$.

The following description of the first column-block $\boldsymbol{G}_{\boldsymbol{\theta}}^{\bullet;1,1}$ of $\boldsymbol{G}_{\boldsymbol{\theta}}$ is then straightforward by observing that the q-th column of $\boldsymbol{Z}_{\boldsymbol{\theta}}$ reads by construction as

$$\left[\boldsymbol{z}_{\boldsymbol{\theta}}^{1,1,q}, \frac{d}{dt}\boldsymbol{z}_{\boldsymbol{\theta}}^{1,1,q}, \dots, \frac{d^{rn}}{dt^{rn}}\boldsymbol{z}_{\boldsymbol{\theta}}^{1,1,q}\right].$$

Lemma 4.5. — *Let $q \in \{1,\dots,n_1\}$ and $m \in \{0,\dots,rn-1\}$. Then the entry at row $m+1$ and column q of $\boldsymbol{G}_{\boldsymbol{\theta}}^{\bullet;1,1}(t)$ reads as*

$$\frac{d^m\boldsymbol{g}_{\boldsymbol{\theta}}^{1,1,q}}{dt^m} + \sum_{p=1}^{q-1}\sum_{k=q-p}^{m}\binom{m}{k}\frac{(-1)^{k-q+p}(k-q+p)!}{r^{q-p}t^{k-q+p+1}}\frac{d^{m-k}\boldsymbol{g}_{\boldsymbol{\theta}}^{1,1,p}}{dt^{m-k}}$$

with the classical convention $\binom{m}{k} = 0$ if $m < k$. In particular, the condition $\boldsymbol{G}_{\boldsymbol{\theta}}^{\bullet;1,1}(t) \underset{t\to 0}{=} O(1)$ implies $\boldsymbol{g}_{\boldsymbol{\theta}}^{1,1,q}(t) \underset{t\to 0}{=} O(t)$ for all $q \leqslant n_1 - 1$.

In particular, Proposition 4.2, Lemma 4.5 and the properties of the Borel transformation tell us that it is sufficient to prove Theorems 3.3 and 3.6 for the n_1 entries $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,q}$ of the first line of $\widehat{\boldsymbol{G}}_{\boldsymbol{\theta}}^{\bullet;1,1}$, i.e. to prove that the $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,q}$'s satisfy the same statements as $\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}}^{\bullet;1,1}$. This is the subject of the next section.

4.3. Proofs of Theorems 3.3 and 3.6

As we said just above, it is sufficient to prove these two theorems for the functions $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,q}$. Recall that these latter are the Borel transforms in direction $\boldsymbol{\theta}$ of the functions $\boldsymbol{g}_{\boldsymbol{\theta}}^{1,1,q}$ and are defined and holomorphic on the domain $\Sigma_{\boldsymbol{\theta}}^{1,1}$. Before starting the calculations, let us first begin by proving the following preliminary technical lemma.

Lemma 4.6. — *Let $q \in \{1,\dots,n_1\}$. Then, for $\arg(t) \simeq \boldsymbol{\theta}^\star$,*

$$\sum_{p=1}^{q} \boldsymbol{g}_{\boldsymbol{\theta}}^{1,1,p}(t) \frac{\ln^{q-p}(t^{1/r})}{(q-p)!} \in Sol(D).$$

Proof. — We shall prove in fact the following more general statement: for all $u \in \{1,\dots,r\}$ and $q \in \{1,\dots,n_1\}$, we have, for $\arg(t) \simeq \boldsymbol{\theta}^\star$,

$$h_{u,q}(t) := \sum_{p=1}^{q} g_{u,p}(t) \frac{\ln^{q-p}(t^{1/r})}{(q-p)!} \in Sol(D)\ ,\quad g_{u,p}(t) := \boldsymbol{g}_{\boldsymbol{\theta}}^{u,1,p}(t) t^{-\frac{u-1}{r}}.$$

Let us begin with the simplest case $n_1 = q = 1$. According to Proposition 4.4, we have, for all $v = 1,\dots,r$ and $\arg(t) \simeq \boldsymbol{\theta}^\star$, the following equalities

$$\boldsymbol{z}_{\boldsymbol{\theta}}^{v,1,1}(t) = \sum_{u=1}^{r} \overline{\mu}^{(v-1)(u-1)} g_{u,1}(t) = \sum_{u=1}^{r} \overline{\mu}^{(v-1)(u-1)} h_{u,1}(t) \in Sol(D)$$

which can be rewritten as a van der Monde identity. Thereby, all the $h_{u,1}$'s are linear combinations of the $\boldsymbol{z}_{\boldsymbol{\theta}}^{v,1,1}$'s; hence, $h_{u,1}(t) \in Sol(D)$ for all $u = 1,\dots,r$.

When $n_1 \geqslant 2$, we proceed by induction on q and we suppose that, for a certain $q \in \{1,\dots,n_1-1\}$, $h_{u,p}(t) \in Sol(D)$ for all $u = 1,\dots,r$ and $p = 1,\dots,q$. We must then prove that $h_{u,q+1}(t) \in Sol(D)$ for all $u = 1,\dots,r$. To do that, we apply again Proposition 4.4 which says us that

$$\boldsymbol{z}_{\boldsymbol{\theta}}^{v,1,q+1}(t) = \sum_{u=1}^{r} \sum_{p=1}^{q+1} \overline{\mu}^{(v-1)(u-1)} g_{u,p}(t) \frac{\ln^{q+1-p}(\mu^{v-1} t^{\frac{1}{r}})}{(q+1-p)!} \in Sol(D) \tag{4.1}$$

for all $v = 1,\dots,r$ and $\arg(t) \simeq \boldsymbol{\theta}^\star$. Let us temporarily denote

$$S_u := \sum_{p=1}^{q+1} g_{u,p}(t) \frac{\ln^{q+1-p}(\mu^{v-1} t^{\frac{1}{r}})}{(q+1-p)!} \quad \text{for all } u \in \{1,\dots,r\}$$

and let us apply Newton's formula: for all $p = 1,\dots,q$,

$$\frac{\ln^{q+1-p}(\mu^{v-1} t^{\frac{1}{r}})}{(q+1-p)!} = \frac{\ln^{q+1-p}(t^{\frac{1}{r}})}{(q+1-p)!} + A_{q,p}$$

with

$$A_{q,p} := \sum_{s=1}^{q+1-p} \frac{\ln^s(\mu^{v-1})}{s!} \frac{\ln^{q+1-p-s}(t^{\frac{1}{r}})}{(q+1-p-s)!}.$$

Then,

$$\begin{aligned} S_u &= \sum_{p=1}^{q} g_{u,p} \left(\frac{\ln^{q+1-p}(t^{\frac{1}{r}})}{(q+1-p)!} + A_{q,p} \right) + g_{u,q+1} \\ &= h_{u,q+1} + \sum_{p=1}^{q} \left(g_{u,p} \sum_{s=1}^{q+1-p} \frac{\ln^s(\mu^{v-1})}{s!} \frac{\ln^{q+1-p-s}(t^{\frac{1}{r}})}{(q+1-p-s)!} \right) \\ &= h_{u,q+1} + \sum_{s=1}^{q} \left(\frac{\ln^s(\mu^{v-1})}{s!} \sum_{p=1}^{q+1-s} g_{u,p} \frac{\ln^{q+1-p-s}(t^{\frac{1}{r}})}{(q+1-p-s)!} \right) \\ &= h_{u,q+1} + \sum_{s=1}^{q} \frac{\ln^s(\mu^{v-1})}{s!} h_{u,q+1-s} \end{aligned}$$

and, according to relation (4.1), the following identities

$$\boldsymbol{z}_{\boldsymbol{\theta}}^{v,1,q+1} - \sum_{u=1}^{r} \left(\overline{\mu}^{(v-1)(u-1)} \sum_{s=1}^{q} \frac{\ln^s(\mu^{v-1})}{s!} h_{u,q+1-s} \right) = \sum_{u=1}^{r} \overline{\mu}^{(v-1)(u-1)} h_{u,q+1}$$

hold for all $v = 1, \ldots, r$ and $\arg(t) \simeq \boldsymbol{\theta}^\star$. The fact that $h_{u,q+1}(t) \in Sol(D)$ follows then as in the case $q = 1$ (indeed, the left hand-side of identity above belongs by assumption to $Sol(D)$). This completes the proof. $\square$

Proof of Theorem 3.3. — Let $\widehat{D}\widehat{y}(\tau) = 0$ denote the Borel transformed equation of $Dy(t) = 0$. Recall that, multiplying D by a convenient power of $1/t$ if needed, this equation is again a linear differential equation with polynomial coefficients. Moreover, it has the two following well-known properties.

- (P_1)**:** The singular points of $\widehat{D}$ are the elements of $\boldsymbol{\Omega}_1 = \boldsymbol{\Omega}_1^* \cup \{0\}$.
- (P_2)**:** The levels of $\widehat{D}$ at infinity are $\leqslant 1$.

Recall that property (P_1) can be proved by using the Newton polygons of D and $\widehat{D}$ at 0 (adapt, for instance, the proof of [18, Lem. 5.3.18]). It can also be seen as a consequence of Écalle–Malgrange's theorem (see Proposition 4.7 below). As for property (P_2), it is a classical result and we refer, for instance, to [22, Thm. 1.4] or [18, Prop. 3.3.18].

The proof of Theorem 3.3 proceeds by recursion on the column q. For $q = 1$, Lemma 4.6 above says us that $\boldsymbol{g}_{\boldsymbol{\theta}}^{1,1,1}(t) \in Sol(D)$. Therefore, its Borel transform $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,1}(\tau)$ is a solution defined on $\Sigma_{\boldsymbol{\theta}}^{1,1}$ of $\widehat{D}\widehat{y}(\tau) = 0$ and the result follows from the two properties above. Indeed, property (P_1) and Cauchy–Lipschitz's theorem prove that $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,1}$ can be analytically continued along any path in $\mathbb{C}\backslash\boldsymbol{\Omega}_1$ starting at any point of $\Sigma_{\boldsymbol{\theta}}^{1,1}$; property (P_2) and Ramis index theorems [28] imply the exponential growth of $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,1}$ at infinity.

Let us now suppose that, for a certain $q \in \{1,\ldots,n_1-1\}$, Theorem 3.3 is valid for any $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,p}$ with $p \in \{1,\ldots,q\}$. According to Lemmas 4.5 and 4.6, we have, for $\arg(t) \simeq \boldsymbol{\theta}^\star$,

$$\boldsymbol{g}_{\boldsymbol{\theta}}^{1,1,q+1}(t) + \sum_{p=1}^{q} \frac{\boldsymbol{g}_{\boldsymbol{\theta}}^{1,1,p}(t)}{t} \cdot \frac{t\ln^{q+1-p}(t^{1/r})}{(q+1-p)!} \in Sol(D)$$

with $\boldsymbol{g}_{\boldsymbol{\theta}}^{1,1,p}(t)/t \underset{t\to 0}{=} O(1)$ and $t\ln^{q+1-p}(t^{1/r}) \in \widetilde{\mathcal{O}}^{\leqslant \exp}$. Therefore, applying the Borel transformation in direction $\boldsymbol{\theta}$, the function

$$\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,q+1} + \sum_{p=1}^{q} \frac{1}{(q+1-p)!} \frac{d\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,p}}{d\tau} * \left(t\ln^{\widehat{q+1-p}}(t^{1/r})\right)_{\boldsymbol{\theta}}$$

defines an actual solution of the equation $\widehat{D}\widehat{y}(\tau) = 0$ and the same arguments as the case $q = 1$ show it is summable-resurgent. Note that the convolution products make sense since all the terms are integrable at 0. Indeed, $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,p}$ (hence, its derivative too) admits an asymptotic expansion at 0 in $\mathbb{C}[\![\tau]\!]$ and $\left(t\ln^{\widehat{q+1-p}}(t^{1/r})\right)_{\boldsymbol{\theta}} \in \mathbb{C}[\ln\tau]$ (see Section 4.1). Note also that all these convolution products belong to $\mathcal{R}es^{sum}_{\boldsymbol{\Omega}_1,\widetilde{0}}$. Indeed, we have $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,p} \in \mathcal{R}es^{sum}_{\boldsymbol{\Omega}_1,\widetilde{0}}$ for all $p \leqslant q$ by hypothesis and the space $\mathcal{R}es^{sum}_{\boldsymbol{\Omega}_1,\widetilde{0}}$ is stable under derivation and convolution by an element of $\mathcal{O}^{\leqslant 1}(\widetilde{\mathbb{C}})$. In particular, this proves that $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,q+1}$ satisfies Theorem 3.3, which completes the proof. □

Proof of Theorem 3.6. — The proof of Theorem 3.6 we propose here is based on Écalle–Malgrange's theorem [22, Thm. 2.2] which asserts that the Borel transformation $\mathcal{B}_{\boldsymbol{\theta}}^{ext}$ defines an isomorphism[(7)] (with inverse $\mathcal{L}_{\boldsymbol{\theta}}^{ext}$) from the space $Sol(D)$ of solutions of equation $Dy(t) = 0$ to the space $\overset{\triangledown}{\mathcal{M}}(\widehat{D})$ of micro-solutions of equation $\widehat{D}\widehat{y}(\tau) = 0$ at its various singular points, that is at the points of $\boldsymbol{\Omega}_1$ (see property (P_1) above). Recall that a micro-solution of $\widehat{D}$ at $\omega \in \boldsymbol{\Omega}_1$ is a singularity $\overset{\triangledown}{h} \in \mathcal{C}_\omega$ satisfying $\widehat{D}\overset{\triangledown}{h} = 0$ in $\mathcal{C}_\omega$.

In our case, this theorem reads as follows.

Proposition 4.7 (Écalle–Malgrange). — *Let $v \in \{1,\ldots,r\}$, $\ell \in \{1,\ldots,J\}$ and $q \in \{1,\ldots,n_\ell\}$. Then, the singularity $\overset{\triangledown}{\boldsymbol{z}}_{\boldsymbol{\theta}}^{v,\ell,q} := \mathcal{B}_{\boldsymbol{\theta}}^{ext}(\boldsymbol{z}_{\boldsymbol{\theta}}^{v,\ell,q})$ is*

(7) In [22], B. Malgrange formulates actually this theorem, not in terms of Borel transformation, but in terms of Fourier (= Laplace) transformation.

a micro-solution of $\widehat{D}$ at the point $a_{\ell,r} \in \mathbf{\Omega}_1$. Moreover, denoting by $\overset{\triangledown}{\mathcal{M}}_\omega(\widehat{D})$ the space of micro-solutions of $\widehat{D}$ at $\omega \in \mathbf{\Omega}_1$, we have

$$\boxed{\overset{\triangledown}{\mathcal{M}}_\omega(\widehat{D}) = \operatorname{vect}\left(\overset{\triangledown}{\boldsymbol{z}}{}_{\boldsymbol{\theta}}^{v,\ell,q}\ ,\ v = 1,\ldots,r,\ q = 1,\ldots,n_\ell\right)_{\ell;q_\ell(1/x)\in Fr_1(\omega)}.}$$

The following lemma precises the structure of singularities $\overset{\triangledown}{\boldsymbol{z}}{}_{\boldsymbol{\theta}}^{v,\ell,q}$.

Lemma 4.8. — *Let $v \in \{1,\ldots,r\}$, $\ell \in \{1,\ldots,J\}$ and $q \in \{1,\ldots,n_\ell\}$. Let $\omega = a_{\ell,r}$. According to Definition 3.5, polynomial $q_\ell(1/x)$ reads as*

$$q_\ell\left(\frac{1}{x}\right) = -\frac{\omega}{x^r} + q_{1,\omega;k}\left(\frac{1}{x}\right)$$

with a suitable k in $\{1,\ldots,s_{1,\omega}\}$ (indeed, we have $q_\ell(1/x) \in Fr_1(\omega)$). Then,

- Case ω with a good front: $\overset{\triangledown}{\boldsymbol{z}}{}_{\boldsymbol{\theta}}^{v,\ell,q} \in \overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\mathbf{\Omega}_1-\omega,0} \circledast \overset{\triangledown}{e}{}^{q_{1,\omega;k}(1/(\mu^{v-1}t^{1/r}))}{}_{|\omega}$,
- Case ω with a bad front: $\overset{\triangledown}{\boldsymbol{z}}{}_{\boldsymbol{\theta}}^{v,\ell,q} \in \overset{\triangledown}{\mathcal{D}et}{}^{s\text{-}res}_{\mathbf{\Omega}_1-\omega,\widetilde{0}} \circledast \overset{\triangledown}{e}{}^{q_{1,\omega;k}(1/(\mu^{v-1}t^{1/r}))}{}_{|\omega}$,

where, as before, we set $\mu = e^{-2i\pi/r}$.

Proof of Lemma 4.8. — According to Proposition 4.4, $\boldsymbol{z}_{\boldsymbol{\theta}}^{v,\ell,q}$ reads as

$$\boldsymbol{z}_{\boldsymbol{\theta}}^{v,\ell,q}(t) = \boldsymbol{h}_{\boldsymbol{\theta}}^{v,\ell,q}(t)e^{q_{1,\omega;k}(1/(\mu^{v-1}t^{1/r}))}e^{-\omega/t}$$

with

$$\boldsymbol{h}_{\boldsymbol{\theta}}^{v,\ell,q}(t) := \sum_{u=1}^{r}\sum_{p=1}^{q}\mu^{(v-1)(\lambda_\ell-u+1)}\boldsymbol{g}_{\boldsymbol{\theta}}^{u,\ell,p}(t)t^{\frac{\lambda_\ell-u+1}{r}}\frac{\ln^{q-p}(\mu^{v-1}t^{\frac{1}{r}})}{(q-p)!}.$$

Moreover, due to the definition of the $\boldsymbol{g}_{\boldsymbol{\theta}}^{v,\ell,q}$'s (see Section 4.2), Theorem 3.3 implies that their Borel transforms $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{v,\ell,q}$ are summable-resurgent with singular support $\mathbf{\Omega}_\ell, 0$ if ω has a good front and with singular support $\mathbf{\Omega}_\ell, \widetilde{0}$ otherwise. In particular, Proposition 4.3(1a) tells us that $\overset{\triangledown}{\boldsymbol{h}}{}_{\boldsymbol{\theta}}^{v,\ell,q} \in \overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\mathbf{\Omega}_\ell,0}$ when ω has a good front.

When ω has a bad front, we proceed as follows. Let us first observe that, besides the summable-resurgence, the proof of Theorem 3.3 allows also to prove that the $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,q}$'s are of finite determination at 0 (indeed, D is a linear differential operator with coefficients in $\mathcal{O}$). Thereby, $\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}}^{\bullet;1,1}$ (hence, $\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}}^{\bullet;v,1}$ for all $v = 1,\ldots,r$ too) is of finite determination at 0. Using then the transformation $Y \mapsto x^{-\lambda_\ell}e^{-q_\ell(1/x)}Y$ on initial system (A) and proceeding as above, we easily check that all the $\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}}^{\bullet;v,\ell}$ are of finite determination at 0, so are all the $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{v,\ell,q}$ (recall we have $\boldsymbol{G}_{\boldsymbol{\theta}}^{\bullet;v,\ell} = \boldsymbol{M}^{-1}\boldsymbol{F}_{\boldsymbol{\theta}}^{\bullet;v,\ell}$ with $\boldsymbol{M}^{-1}(t) \in$

$GL_{rn}(\mathbb{C}\{t\}[t^{-1}])$). Consequently, Proposition 4.3(2) applies and implies that $\overset{\triangledown}{\boldsymbol{h}}{}_{\boldsymbol{\theta}}^{v,\ell,q} \in \overset{\triangledown}{\mathcal{D}et}{}^{s\text{-}res}_{\boldsymbol{\Omega}_\ell,\widetilde{0}}$.

To complete the proof, it is sufficient to apply $\mathcal{B}_{\boldsymbol{\theta}}^{ext}$ on $\boldsymbol{z}_{\boldsymbol{\theta}}^{v,\ell,q}$ and to remark, on one hand, that $\boldsymbol{\Omega}_\ell = \boldsymbol{\Omega}_1 - \omega$ and, on the other hand, that $\mathcal{B}_{\boldsymbol{\theta}}^{ext}(e^{-\omega/t})$ is the translation by ω. □

In particular, Proposition 4.7 and Lemma 4.8 allow us to make explicit the general form of all the micro-solutions of $\widehat{D}$. More precisely, due to the definition of the front $Fr_1(\omega)$ of $\omega \in \boldsymbol{\Omega}_1$ (see Definition 3.5), we have the following.

Lemma 4.9. — *Let $\omega \in \boldsymbol{\Omega}_1$ and $\overset{\triangledown}{h} \in \overset{\triangledown}{\mathcal{M}}_\omega(\widehat{D})$ a micro-solution of $\widehat{D}$ at ω.*

(1) *Suppose that ω has a good front. Let*

$$\mathcal{Q}_{1,\omega} = \left\{ q_{1,\omega}\left(\frac{1}{\mu^{v-1}t^{1/r}}\right) \ ; \ v = 1,\dots,r \right\}$$

with $\mu = e^{-2i\pi/r}$. Then

$$\boxed{\overset{\triangledown}{h} \in \sum_{q\in\mathcal{Q}_{1,\omega}} \overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\boldsymbol{\Omega}_1-\omega,0} \circledast \overset{\triangledown}{e}{}^q{}_{|\omega}.}$$

(2) *Suppose that ω has a bad front. Let*

$$\mathcal{Q}_{1,\omega} = \left\{ q_{1,\omega;k}\left(\frac{1}{\mu^{v-1}t^{1/r}}\right) \ ; \ k = 1,\dots,s_{1,\omega} \text{ and } v = 1,\dots,r \right\}$$

with $\mu = e^{-2i\pi/r}$. Then

$$\boxed{\overset{\triangledown}{h} \in \sum_{q\in\mathcal{Q}_{1,\omega}} \overset{\triangledown}{\mathcal{D}et}{}^{s\text{-}res}_{\boldsymbol{\Omega}_1-\omega,\widetilde{0}} \circledast \overset{\triangledown}{e}{}^q{}_{|\omega}.}$$

We are now able to prove Theorem 3.6. Recall that it is sufficient to prove that the $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,q}$ with $q = 1,\dots,n_1$ satisfy the same statement as $\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}}^{\bullet;1,1}$. To do that, we shall proceed, as in the proof of Theorem 3.3, by induction on q. Let $\omega \in \boldsymbol{\Omega}_1$ and let γ be a path in $\mathbb{C}\backslash\boldsymbol{\Omega}_1$ starting at a point of $\Sigma_{\boldsymbol{\theta}}^{1,1}$ and ending in a neighborhood of ω. As before, we denote by $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta};\omega,\gamma}^{1,1,q}$ the analytic continuation of $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,q}$ along γ and by $\overset{\triangledown}{\boldsymbol{g}}{}_{\boldsymbol{\theta};\omega,\gamma}^{1,1,q} := \operatorname{can}(\widehat{\boldsymbol{g}}_{\boldsymbol{\theta};\omega,\gamma}^{1,1,q})$ the singularity at ω defined by $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta};\omega,\gamma}^{1,1,q}$.

For $q=1$, we saw in the proof of Theorem 3.3 that $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,1}$ is a solution of $\widehat{D}\widehat{y}(\tau)=0$ defined and holomorphic on $\Sigma_{\boldsymbol{\theta}}^{1,1}$. Thereby, its analytic continuation $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta};\omega,\gamma}^{1,1,1}$ yields a micro-solution $\overset{\nabla}{\boldsymbol{g}}{}_{\boldsymbol{\theta};\omega,\gamma}^{1,1,1}$ of $\widehat{D}$ at ω and the result follows from Lemma 4.9.

Let us now suppose that, for a certain $q\in\{1,\dots,n_1-1\}$, Theorem 3.6 is valid for any $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,p}$ with $p\in\{1,\dots,q\}$. As in the case $q=1$, we first derive from the proof of Theorem 3.3 that the function

$$\widehat{\boldsymbol{g}}_{\boldsymbol{\theta};\omega,\gamma}^{1,1,q+1}+\sum_{p=1}^{q}\frac{1}{(q+1-p)!}\frac{d\widehat{\boldsymbol{g}}_{\boldsymbol{\theta};\omega,\gamma}^{1,1,p}}{d\tau}*\big(t\widehat{\ln^{q+1-p}}(t^{1/r})\big)_{\boldsymbol{\theta}}$$

yields a micro-solution of $\widehat{D}$ at ω. Moreover, due to our hypothesis and Lemma 4.9, this micro-solution has the same general structure as all the singularities $\overset{\nabla}{\boldsymbol{g}}{}_{\boldsymbol{\theta};\omega,\gamma}^{1,1,p}$ for $p=1,\dots,q$ (which is, of course, of one of the two forms given in Lemma 4.9). Applying then Proposition 4.2 (indeed, $\big(t\widehat{\ln^{q+1-p}}(t^{1/r})\big)_{\boldsymbol{\theta}}\in\mathbb{C}[\tau^{\nu},(\ln\tau)^{s}]_{\nu\in\mathbb{C},s\in\mathbb{N}}$), we easily check that this common structure is transmitted to the singularity $\overset{\nabla}{\boldsymbol{g}}{}_{\boldsymbol{\theta};\omega,\gamma}^{1,1,q+1}$. Hence, our result, which completes the proof of Theorem 3.6. □

5. Application to the effective calculation of some highest level's Stokes multipliers

In this section, we propose to apply the results of Section 3 to the effective calculation of highest level's Stokes multipliers of initial system (A) (see Definition 5.3 below for their exact definition). More precisely, we propose to make explicit *highest level's connection-to-Stokes formulæ* for some geometric configurations of highest level's Stokes values of $\widetilde{F}(x)$, generalizing thus formulæ already displayed by M. Loday-Richaud and the author in [19, 31, 32] for systems with a single level or for the lowest level of systems with multi-levels.

As we said in Section 4 above, we can restrict ourselves to the highest level's Stokes multipliers associated with the first column-block $\widetilde{F}^{\bullet;1}(x)$ of $\widetilde{F}(x)$, which we denote from now on by $\widetilde{f}(x)$. Furthermore, for notational convenience, we assume again $q_1\equiv 0$ and $\lambda_1=0$, conditions which can be always fulfilled by means of the transformation $Y\longmapsto x^{-\lambda_1}e^{-q_1(1/x)}Y$ on initial system (A).

Finally, we write as before the polynomials $q_j(1/x)$ in the form

$$q_j\left(\frac{1}{x}\right)=-\frac{a_{j,r}}{x^r}-\frac{a_{j,r-1}}{x^{r-1}}-\cdots-\frac{a_{j,1}}{x}\quad\text{with } a_{j,k}\in\mathbb{C}\text{ for all } k.$$

Before starting the calculations, let us first begin by some reminders about the *Stokes phenomenon* and the *Stokes–Ramis matrices*.

5.1. Stokes phenomenon and Stokes–Ramis matrices

Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be a direction and $\theta^\star$ its principal determination in $]-2\pi, 0]$ (cf. footnote 6). When θ is not an anti-Stokes direction of $\widetilde{F}(x)$ (see Definition 2.3), the theory of multisummability [2, 3, 11, 16, 17, 18, 23, 25] tells us that $\widetilde{F}(x)$ is $\underline{r}$-summable in direction θ with $\underline{r} = (r_1 < \cdots < r_p = r)$ the p-tuple of all the levels of $\widetilde{F}(x)$ (see Notation 2.4). Then, denoting its sum by $F_\theta(x)$, one can define the sum $Y_\theta(x)$ of the formal fundamental solution $\widetilde{Y}(x)$ in direction θ by setting $Y_\theta(x) := F_\theta(x)Y_{0;\theta^\star}(x)$, where $Y_{0;\theta^\star}(x)$ is the actual analytic function $Y_{0;\theta^\star}(x) = x^L e^{Q(1/x)}$ defined by the choice $\arg(x) \simeq \theta^\star$.

Stokes phenomenon. Let us now suppose that θ is an anti-Stokes direction of $\widetilde{F}(x)$. For $\eta > 0$ small enough, $\widetilde{F}(x)$ is $\underline{r}$-summable in every direction of $]\theta-\eta, \theta+\eta[\setminus\{\theta\}$. One can then define the two *lateral sums* $F_{\theta^-}(x)$ and $F_{\theta^+}(x)$ of $\widetilde{F}(x)$ in direction θ as the respective analytic continuations of the sums $F_{\theta'}(x)$ and $F_{\theta''}(x)$ with any $\theta' \in]\theta-\eta, \theta[$ and $\theta'' \in]\theta, \theta+\eta[$. In particular, they are defined on a common sector with vertex 0, bisected by θ and opening π/r [25].

The *Stokes phenomenon* of system (A) stems from the fact that the sums F_{θ^-} and F_{θ^+} are not analytic continuations from each other in general. This defect of analyticity is quantified by the collection of *Stokes–Ramis automorphisms* $St_{\theta^\star} : Y_{\theta^+} \longmapsto Y_{\theta^-}$ for all the anti-Stokes directions $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ of $\widetilde{F}(x)$, where $Y_{\theta^\pm}$ denote the lateral sums of $\widetilde{Y}(x)$ at θ defined by $Y_{\theta^\pm}(x) = F_{\theta^\pm}(x)Y_{0;\theta^\star}(x)$ for $\arg(x) \simeq \theta^\star$.

Stokes–Ramis matrices. The *Stokes–Ramis matrices*[8] are then defined as matrix representations of the $St_{\theta^\star}$'s in $GL_n(\mathbb{C})$. More precisely, one has the following.

Definition 5.1 (Stokes–Ramis matrix). — *One calls* Stokes–Ramis matrix associated with $\widetilde{Y}(x)$ in direction θ *the matrix of $St_{\theta^\star}$ in the basis Y_{θ^+}.*

[8] In the literature, a Stokes matrix has a more general meaning where one allows to compare any two asymptotic solutions whose domains of definition overlap. According to the custom initiated by J.-P. Ramis [30] in the spirit of Stokes' work, we exclude this case here. We consider only matrices providing the transition between the sums on each side of a same anti-Stokes direction.

We still denote it by $St_{\theta^\star}$; it is uniquely determined by the relation

$$\boxed{Y_{\theta^-}(x) = Y_{\theta^+}(x)St_{\theta^\star} \quad \text{for } \arg(x) \simeq \theta^\star.} \tag{5.1}$$

Let us now split $St_{\theta^\star} = [St_{\theta^\star}^{j;\ell}]$ into blocks fitting to the Jordan block-structure of matrix L of exponents of formal monodromy (for $j, \ell = 1, \ldots, J$, the matrix $St_{\theta^\star}^{j;\ell}$ has size $n_j \times n_\ell$). Then $St_{\theta^\star}^{j;j} = I_{n_j}$ and $St_{\theta^\star}^{j;\ell} = 0$ if θ is not a direction of maximal decay of polynomial $q_j - q_\ell$, i.e. is not an anti-Stokes direction of $\widetilde{F}^{\bullet;\ell}(x)$. Otherwise, the entries of $St_{\theta^\star}^{j;\ell}$ are called *Stokes multipliers of $\widetilde{F}^{\bullet;\ell}(x)$ in direction θ.*

Factorization of Stokes–Ramis matrices. The factorization of matrices $St_{\theta^\star}$ by levels was first proved by J.-P. Ramis in [29, 30] by using the factorization theorem of $\widetilde{F}(x)$; a quite different proof based on Stokes cocycles and mainly algebraic was given later by M. Loday-Richaud in [16].

Theorem 5.2 (Factorization of $St_{\theta^\star}$, [16, 29, 30]). — *With notations as above, the Stokes–Ramis matrix $St_{\theta^\star}$ can be written as*

$$St_{\theta^\star} = St_{r_1;\theta^\star} \ldots St_{r_p;\theta^\star} \quad , \ St_{r_k;\theta^\star} = [St_{r_k;\theta^\star}^{j;\ell}] \in GL_n(\mathbb{C})$$

where, for all $k = 1, \ldots, p$, $St_{r_k;\theta^\star}^{j;j} = I_{n_j}$ and $St_{r_k;\theta^\star}^{j;\ell} = 0$ if θ is not a direction of maximal decay of $q_j - q_\ell$ or $r_{j,\ell} \neq r_k$ (recall that $r_{j,\ell}$ denotes the degree of polynomial $q_j - q_\ell \not\equiv 0$, see Definition 2.1).

Definition 5.3 (Stokes multipliers of level r_k). — *Let $k \in \{1, \ldots, p\}$.*

(1) *The matrix $St_{r_k;\theta^\star}$ is called* Stokes–Ramis matrix of level r_k associated with $\widetilde{Y}(x)$ in direction θ.
(2) *When θ is a direction of maximal decay of $q_j - q_\ell$ and $r_{j,\ell} = r_k$, the entries of $St_{r_k;\theta^\star}^{j;\ell}$ are called* Stokes multipliers of level r_k of $\widetilde{F}^{\bullet;\ell}(x)$ in direction θ.

Recall that a relation similar to (5.1) can be written for each Stokes–Ramis matrix $St_{r_k;\theta^\star}$ by replacing the lateral sums Y_{θ^+} and Y_{θ^-} by suitable "generalized" sums of $\widetilde{Y}(x)$ at θ [25, Thm. 9, p. 366].

As we said at the beginning of Section 5, we are interested here below just in the highest level's Stokes multipliers (= the Stokes multipliers of level $r_p = r$) of $\widetilde{f}(x)$, that is, in the Stokes multipliers located at the first column-block $St_{r;\theta^\star}^{\bullet;1}$ of $St_{r;\theta^\star}$, which we denote below by $st_{r;\theta^\star}$.

5.2. Highest level's Stokes multipliers and rank reduction

According to the normalization $q_1 \equiv 0$ and Definition 2.1, the highest level's anti-Stokes directions of $\widetilde{f}(x)$ are all the directions of maximal decay of exponentials $e^{q_j(1/x)}$ with polynomials q_j of degree r, i.e. all the collections of the r directions $\theta_0, \theta_1, \ldots, \theta_{r-1} \in \mathbb{R}/2\pi\mathbb{Z}$ regularly distribued around the origin $x = 0$ which are given by the r-th roots of the highest level's Stokes values $a_{j,r} \neq 0$ of $\widetilde{f}(x)$.

Let us now choose such a collection (θ_k) and suppose, to fix ideas, that their principal determinations $\theta_k^\star \in]-2\pi, 0]$ satisfy $-2\pi < \theta_{r-1}^\star < \cdots < \theta_1^\star < \theta_0^\star \leqslant 0$.

As before, we denote by $\mathbf{\Omega}_1^*$ the set of all the highest level's Sokes values of $\widetilde{f}(x)$. We also denote $\boldsymbol{\theta} := r\theta_0$ and $\mathbf{\Omega}_{1;\boldsymbol{\theta}}^*$ the set of the Stokes values of $\mathbf{\Omega}_1^*$ with argument $\boldsymbol{\theta}$.

By construction, $\boldsymbol{\theta}$ is a highest level's anti-Stokes direction (= anti-Stokes direction of level 1) of $\widetilde{\boldsymbol{F}}^{\bullet;1,1}(t) := \widetilde{\boldsymbol{f}}(t)$. Then, applying [17, Prop. 4.2] and the generalized multisummability theorem due to J. Martinet and J.-P. Ramis [25, Thm. 9, p. 366], one can relate the highest level's Stokes–Ramis matrices $(St_{r;\theta_k^\star})_{k=0,\ldots,r-1}$ to the highest level's Stokes–Ramis matrix associated with $\widetilde{\boldsymbol{Y}}(t)$ in direction $\boldsymbol{\theta}$. More precisely, using the Balser–Tougeron theorem [1] (see also [18, Thm. 7.4.5]), we have the following.

Proposition 5.4. — *Let $\eta > 0$ be small enough so that*

- $\widetilde{\boldsymbol{F}}(t)$ *is summable in every direction of* $[\boldsymbol{\theta} - \eta, \boldsymbol{\theta} + \eta]\backslash\{\boldsymbol{\theta}\}$,
- $\Sigma_{\boldsymbol{\theta}-\eta}^{v,\ell} \cap [0, \infty e^{i(\boldsymbol{\theta}+\eta)}[\neq \varnothing$ *for all* $v \in \{1, \ldots, r\}$ *and* $\ell \in \{1, \ldots, J\}$.

Then, for $\arg(t) \simeq \boldsymbol{\theta}^\star$,

$$\mathcal{L}_{\boldsymbol{\theta}^-}(\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}^-})(t)\boldsymbol{Y}_{0;\boldsymbol{\theta}^\star}(t) = \mathcal{L}_{\boldsymbol{\theta}^+}(\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}^-})(t)\boldsymbol{Y}_{0;\boldsymbol{\theta}^\star}(t)\left(\bigoplus_{k=0}^{r-1} St_{r;\theta_k^\star}\right), \tag{5.2}$$

where $\boldsymbol{\theta}^\pm := \boldsymbol{\theta} \pm \eta$, $\mathcal{L}_{\boldsymbol{\theta}^\pm}$ denotes the Laplace transformation in direction $\boldsymbol{\theta}^\pm$, $\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}^-} := \left[\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}^-}^{\bullet;v,\ell}\right]$ and where $\boldsymbol{Y}_{0;\boldsymbol{\theta}^\star}(t)$ is the actual analytic function defined from $\widetilde{\boldsymbol{Y}}_0(t)$ (see page 650) by the choice of $\arg(t)$.

Let us now write the Stokes–Ramis matrices $St_{r;\theta_k^\star}$ in the form $St_{r;\theta_k^\star} = I_n + C_{r;\theta_k^\star}$ (hence, $C_{r;\theta_k^\star}^{j;j} = 0$ and $C_{r;\theta_k^\star}^{j;\ell} = St_{r;\theta_k^\star}^{j;\ell}$ if $j \neq \ell$). Then identity (5.2) has the following "additive" form:

$$(\mathcal{L}_{\boldsymbol{\theta}^-} - \mathcal{L}_{\boldsymbol{\theta}^+})(\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}^-})(t) = \mathcal{L}_{\boldsymbol{\theta}^+}(\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}^-})(t)\boldsymbol{Y}_{0;\boldsymbol{\theta}^\star}(t)\left(\bigoplus_{k=0}^{r-1} C_{r;\theta_k^\star}\right)\boldsymbol{Y}_{0;\boldsymbol{\theta}^\star}^{-1}(t), \tag{5.3}$$

where $\boldsymbol{Y}^{-1}_{0;\boldsymbol{\theta}^\star}(t)$ is the $rn \times rn$-matrix defined by

$$\frac{1}{r}\begin{bmatrix} (t^{-\frac{1}{r}})^{\Lambda_0}e^{-Q_0(t)} & (t^{-\frac{1}{r}})^{\Lambda_1}e^{-Q_0(t)} & \cdots & (t^{-\frac{1}{r}})^{\Lambda_{r-1}}e^{-Q_0(t)} \\ (\overline{\mu}t^{-\frac{1}{r}})^{\Lambda_0}e^{-Q_1(t)} & (\overline{\mu}t^{-\frac{1}{r}})^{\Lambda_1}e^{-Q_1(t)} & \cdots & (\overline{\mu}t^{-\frac{1}{r}})^{\Lambda_{r-1}}e^{-Q_1(t)} \\ \vdots & \vdots & \ddots & \vdots \\ (\overline{\mu}^{r-1}t^{-\frac{1}{r}})^{\Lambda_0}e^{-Q_{r-1}(t)} & (\overline{\mu}^{r-1}t^{-\frac{1}{r}})^{\Lambda_1}e^{-Q_{r-1}(t)} & \cdots & (\overline{\mu}^{r-1}t^{-\frac{1}{r}})^{\Lambda_{r-1}}e^{-Q_{r-1}(t)} \end{bmatrix}.$$

Recall that $\mu = e^{-2i\pi/r}$, $Q_k(t) = Q(1/(\mu^k t^{1/r}))$ and $\Lambda_k := L - kI_n$.

Notation 5.5. — In the sequel, we shall use the following notations.

- Given a matrix M of size $n \times m$ with $m \geqslant 1$, we split M into J row-blocks $M^{j;\bullet}$, $j = 1, \dots, J$, of size $n_j \times m$ according to the Jordan block-structure of matrix L of exponents of formal monodromy.
- Given a matrix M of size $rn \times m$ with $m \geqslant 1$, we first split M into r row-blocks $M^{u;\bullet}$, $u = 1, \dots, r$, of size $n \times m$; then, each $M^{u;\bullet}$ into J row-blocks $M^{u,j;\bullet}$ of size $n_j \times m$ as above.

The following Proposition 5.6 stems from the restriction of identity (5.3) to the first column-block and allows to relate the highest level's Stokes multipliers $st_{r;\theta_k^\star}$ of $\widetilde{f}(x)$ to the summable-resurgent function $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}(\tau)$.

Proposition 5.6. — *Let $\eta > 0$ be as in Proposition 5.4 and $\boldsymbol{\theta}^\pm = \boldsymbol{\theta} \pm \eta$. Then, for* $\arg(t) \simeq \boldsymbol{\theta}^\star$,

$$(\mathcal{L}_{\boldsymbol{\theta}^-} - \mathcal{L}_{\boldsymbol{\theta}^+})(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-})(t) = \mathcal{L}_{\boldsymbol{\theta}^+}(\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}^-})(t)\boldsymbol{M}_{\boldsymbol{\theta}^\star}(t), \tag{5.4}$$

where $\boldsymbol{M}_{\boldsymbol{\theta}^\star}(t)$ is the $rn \times n_1$-matrix defined, for all u and j, by

$$\boldsymbol{M}^{u,j;\bullet}_{\boldsymbol{\theta}^\star}(t) = \begin{cases} \dfrac{1}{r}\displaystyle\sum_{k=0}^{r-1}(\mu^k t^{\frac{1}{r}})^{\Lambda_{j,u-1}} st^{j;\bullet}_{r;\theta_k^\star}(\mu^k t^{\frac{1}{r}})^{-J_{n_1}} e^{q_j(1/(\mu^k t^{1/r}))} & \text{if } a_{j,r} \in \boldsymbol{\Omega}^*_{1;\boldsymbol{\theta}} \\ 0 & \text{otherwise} \end{cases}$$

with $\Lambda_{j,u-1} := (\lambda_j - u + 1)I_{n_j} + J_{n_j}$. Recall that $\mu = e^{-2i\pi/r}$.

Note that the left hand-side of (5.4) can be read as the Laplace integral

$$(\mathcal{L}_{\boldsymbol{\theta}^-} - \mathcal{L}_{\boldsymbol{\theta}^+})(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-})(t) = \int_{\gamma'_{\boldsymbol{\theta}}} \widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}(\tau)e^{-\tau/t}d\tau, \tag{5.5}$$

where $\gamma'_{\boldsymbol{\theta}}$ is a Hankel type path going along the straight line $[0, \infty e^{i\boldsymbol{\theta}}[$ from infinity to 0 and back to infinity passing positively all singular points of $\boldsymbol{\Omega}^*_{1;\boldsymbol{\theta}}$ on both ways. Thereby, using the summable-resurgence of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}(\tau)$ (see Theorem 3.3), we shall now be able to relate the Stokes multipliers $st^{j;\bullet}_{r;\theta_k^\star}$ of $\widetilde{f}(x)$ to the singularities of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}(\tau)$ at its various singular points $\omega \in \boldsymbol{\Omega}^*_{1;\boldsymbol{\theta}}$.

Let us first begin by introducing the notion of *principal singularity* of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}(\tau)$.

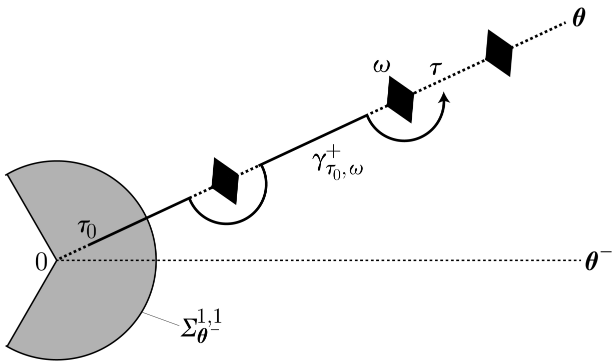

Figure 5.1. A path $\gamma^+_{\tau_0,\omega}$ when $\Sigma^{1,1}_{\boldsymbol{\theta}^-}$ is a sector with opening $< 2\pi$

5.3. Principal singularity

Let $\omega \in \boldsymbol{\Omega}^*_{1;\boldsymbol{\theta}}$. As we said at the beginning of Section 3.2.2, the singularity $\overset{\triangledown}{\boldsymbol{f}}_{\boldsymbol{\theta}^-;\omega,\gamma}$ of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}(\tau)$ at ω depends on the chosen path γ for the analytic continuation of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}(\tau)$ and meanwhile, on the chosen determination of the argument around ω.

Here below, we consider a path $\gamma^+_{\tau_0,\omega}$ defined as follows:

- τ_0 is a point of $\Sigma^{1,1}_{\boldsymbol{\theta}^-} \cap [0, \infty e^{i\boldsymbol{\theta}}[$ [(9)], which is also assumed in the first sheet of $\widetilde{\mathcal{R}}_{\boldsymbol{\Omega}_1}$ when $\Sigma^{1,1}_{\boldsymbol{\theta}^-}$ is a sector with opening $\geqslant 2\pi$,
- $\gamma^+_{\tau_0,\omega}$ is a path starting at τ_0, going along the straight line $[0, \omega]$ to a point τ close to ω and avoiding all points of $\boldsymbol{\Omega}^*_{1;\boldsymbol{\theta}} \cap [0, \omega]$ to the right as shown on Figure 5.1 below,
- we choose as before the principal determination of the variable τ around ω in $]-2\pi, 0]$.

The analytic continuation $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-;\omega,+} := \widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-;\omega,\gamma^+_{\tau_0,\omega}}$ is called *right analytic continuation of* $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}$ *at* ω. Note that it does not depend on the choice of τ_0. The principal singularity of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}$ at ω is then defined as follows.

Definition 5.7 (Principal singularity). — *We call* principal singularity of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}$ at ω *the singularity* $\overset{\triangledown}{\boldsymbol{f}}_{\boldsymbol{\theta}^-;\omega,+}$ *defined by the right analytic continuation* $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-;\omega,+}$ *of* $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}$ *at* ω*. A major* $\check{\boldsymbol{f}}_{\boldsymbol{\theta}^-;\omega,+}$ *of* $\overset{\triangledown}{\boldsymbol{f}}_{\boldsymbol{\theta}^-;\omega,+}$ *is called* principal major*.*

(9) The existence of such a point τ_0 stems from the choice of η which implies $\sum^{1,1}_{\boldsymbol{\theta}^-} \cap [0, \infty e^{i(\boldsymbol{\theta}+\eta)}[\neq \varnothing$ and, consequently, $\sum^{1,1}_{\boldsymbol{\theta}^-} \cap [0, \infty e^{i\boldsymbol{\theta}}[\neq \varnothing$ (see Proposition 5.4).

5.4. Highest level's Stokes multipliers vs principal singularities in a case of a GG-Configuration

The relations between the highest level's Stokes multipliers of $\widetilde{f}(x)$ and the principal singularities of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}(\tau)$ at its various singular points of $\boldsymbol{\Omega}^*_{1;\boldsymbol{\theta}}$ (and, consequently, the highest level's connection-to-Stokes formulæ in view in this section; see Section 5.5, Theorem 5.15) strongly depend on the nature and on the geometric configuration of the elements of $\boldsymbol{\Omega}^*_{1;\boldsymbol{\theta}}$. Henceforth, in the rest of the article, we restrict ourselves to the following *Good Geometric Configuration* (in short, GG-Configuration).

Definition 5.8 (GG-Configuration). — *The set* $\boldsymbol{\Omega}^*_{1;\boldsymbol{\theta}}$ *is said to have a* GG-Configuration *when* all *its elements have a* good *front.*

Note in particular that this condition implies the following property.

($\mathcal{P}$)**:** *For all* $\omega \in \boldsymbol{\Omega}^*_{1;\boldsymbol{\theta}}$, *the front* $Fr_1(\omega)$ *is a singleton*

$$\left\{-\frac{\omega}{x^r} + q_{1,\omega}\left(\frac{1}{x}\right)\right\}$$

with a suitable polynomial $q_{1,\omega}(1/x)$ *in* $1/x$ *of degree* $< r$.

Let us now turn to identity (5.4) of Proposition 5.6.

Without changing the value of the integral (5.5) (use the summable-resurgence of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}$; see Theorem 3.3), the path $\gamma'_{\boldsymbol{\theta}}$ can be deformed into a union $\gamma'_{\boldsymbol{\theta}} = \bigcup_{\omega\in\boldsymbol{\Omega}^*_{1;\boldsymbol{\theta}}} \gamma'_{\boldsymbol{\theta}}(\omega)$ of Hankel type paths $\gamma'_{\boldsymbol{\theta}}(\omega)$ with asymptotic direction $\boldsymbol{\theta}$ around each Stokes value $\omega \in \boldsymbol{\Omega}^*_{1;\boldsymbol{\theta}}$. Hence, by means of a translation from ω to 0 and by replacing $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}$ by one of its principal majors $\check{\boldsymbol{f}}_{\boldsymbol{\theta}^-;\omega,+}$ at each $\omega \in \boldsymbol{\Omega}^*_{1;\boldsymbol{\theta}}$, the following identity holds

$$(\mathcal{L}_{\boldsymbol{\theta}^-} - \mathcal{L}_{\boldsymbol{\theta}^+})(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-})(t) = \sum_{\omega\in\boldsymbol{\Omega}^*_{1;\boldsymbol{\theta}}} e^{-\omega/t}\mathcal{L}^{ext}_{\boldsymbol{\theta}^+}\left(\check{\boldsymbol{f}}_{\boldsymbol{\theta}^-;\omega,+}(\omega+\tau)\right)(t), \tag{5.6}$$

where $\mathcal{L}^{ext}_{\boldsymbol{\theta}^+}$ denotes the Laplace transformation in direction $\boldsymbol{\theta}^+$ defined in Proposition 4.1 and where Theorem 3.6 implies

$$\operatorname{can}\left(\check{\boldsymbol{f}}_{\boldsymbol{\theta}^-;\omega,+}(\omega+\tau)\right) \in \sum_{v=1}^{r} \overset{\nabla}{\mathcal{N}}il^{s\text{-}res}_{\boldsymbol{\Omega}_1-\omega,0} \circledast \overset{\nabla}{e}{}^{q_{1,\omega}(1/(\mu^{v-1}t^{1/r}))}.$$

On the other hand, the right hand side of identity (5.4) can be written in form similar to (5.6):

$$\mathcal{L}_{\boldsymbol{\theta}^+}(\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}^-})(t)\boldsymbol{M}_{\boldsymbol{\theta}^\star}(t) = \sum_{\omega\in\boldsymbol{\Omega}^*_{1;\boldsymbol{\theta}}} e^{-\omega/t}\boldsymbol{M}_{\boldsymbol{\theta}^\star;\omega}(t)$$

with
$$\boldsymbol{M}_{\boldsymbol{\theta}^\star;\omega}(t) := \sum_{v=1}^{r} \sum_{\ell; q_\ell(1/x)\in Fr_1(\omega)} \mathcal{L}_{\boldsymbol{\theta}^+}(\widehat{\boldsymbol{F}}^{\bullet;v,\ell}_{\boldsymbol{\theta}^-})(t) S_{\omega,v,\ell}$$
and
$$S_{\omega,v,\ell} := \frac{1}{r}\sum_{k=0}^{r-1} (\mu^k t^{\frac{1}{r}})^{\Lambda_{\ell,v-1}} st^{\ell;\bullet}_{r;\theta^\star_k} (\mu^k t^{\frac{1}{r}})^{-J_{n_1}} e^{q_{1,\omega}(1/(\mu^k t^{1/r}))}.$$
Moreover, for any $\omega \in \boldsymbol{\Omega}^*_{1;\boldsymbol{\theta}}$ and any ℓ such that $q_\ell(1/x) \in Fr_1(\omega)$, property $(\mathcal{P})$ implies that $\widetilde{F}^{\bullet;\ell}(x)$ has the unique level r. Thereby, due to Theorem 3.3 and Balser–Tougeron theorem, $\widetilde{\boldsymbol{F}}^{\bullet;v,\ell}(t)$ is 1-summable in direction $\boldsymbol{\theta}^+$ for all $v = 1,\dots,r$ and its 1-sum coincides with $\mathcal{L}_{\boldsymbol{\theta}^+}(\widehat{\boldsymbol{F}}^{\bullet;v,\ell}_{\boldsymbol{\theta}^-})(t)$.

Hence, by applying a method similar to the one of [19, Prop. 4.1] (see also [32, § 4.3]), we obtain the following result.

Proposition 5.9. — *Let $\eta > 0$ be as in Proposition 5.4 and $\boldsymbol{\theta}^\pm = \boldsymbol{\theta}\pm\eta$. Let $\omega \in \boldsymbol{\Omega}^*_{1;\boldsymbol{\theta}}$. Then, the identity*
$$\mathcal{L}^{ext}_{\boldsymbol{\theta}^+}\left(\check{\boldsymbol{f}}_{\boldsymbol{\theta}^-;\omega,+}(\omega+\tau)\right)(t) = \boldsymbol{M}_{\boldsymbol{\theta}^\star;\omega}(t) \tag{5.7}$$
holds for $\arg(t) \simeq \boldsymbol{\theta}^\star$.

Remark 5.10. — When $\boldsymbol{\Omega}^*_{1;\boldsymbol{\theta}}$ has not a GG-Configuration, that is $\boldsymbol{\Omega}^*_{1;\boldsymbol{\theta}}$ has (at least) one Stokes value with a bad front, it seems that Proposition 5.9 above is still valid. Nevertheless, we will not treat this case in this article because calculations become much more complicated. Indeed, these Stokes values having no longer a good front, the corresponding singularities are no longer in the Nilsson class and the corresponding column-blocks $\widetilde{\boldsymbol{F}}^{\bullet;v,\ell}(t)$ are no longer 1-summable, but multisummable. This will be studied in great details in a further article.

We are now able to state the highest level's connection-to-Stokes formulæ considered this section.

5.5. Highest level's connection-to-Stokes formulæ in a case of a GG-Configuration

In this section, we fix $\eta > 0$ as in Proposition 5.4 and we denote $\boldsymbol{\theta}^\pm = \boldsymbol{\theta}\pm\eta$ as before.

Let us now choose a Stokes value $\omega \in \boldsymbol{\Omega}^*_{1;\boldsymbol{\theta}}$. According to our assumption of GG-Configuration, ω has a good front. Moreover, applying if needed the following technical lemma due to M. Loday-Richaud, we can also suppose that ω has actually a *good monomial* front.

Lemma 5.11 (M. Loday-Richaud, [15]). — *Let $q_\omega(1/x)$ the unique element of the front $Fr_1(\omega)$ of ω. Then,*

(1) *there exists a change of the variable x of the form*
$$x = \frac{y}{1+\alpha_1 y + \cdots + \alpha_{r-1}y^{r-1}} \quad , \ \alpha_1,\dots,\alpha_{r-1}\in\mathbb{C} \tag{5.8}$$
such that the polar part of $q_\omega(1/x(y))$ reads as $-\omega/y^r$,

(2) *the Stokes–Ramis matrices (hence, the highest level's Stokes–Ramis matrices) of system (A) are preserved by the change of variable* (5.8).

Note that, although Lemma 5.11 is proved in [15] in the case of systems of dimension 2 (hence, with a single level), it can be extended to any system of dimension $\geqslant 3$. Indeed, the change of variable (5.8) being tangent to identity, it "preserves" levels, Stokes values and summation operators.

Let us now introduce the *connection constants* of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}(\tau)$ at ω.

Connection constants. As we saw in Theorem 3.6, the principal singularity of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}(\tau)$ at ω belongs to the Nilsson class $\overset{\nabla}{\mathcal{N}il}{}^{s\text{-}res}_{\boldsymbol{\Omega}_1-\omega,0|\omega}$. The following proposition gives us a much more precise description.

Proposition 5.12 (Principal singularity with a good monomial front). *The principal singularity of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}(\tau)$ at ω admits a major $\check{\boldsymbol{f}}_{\boldsymbol{\theta}^-;\omega,+}$ of the form*

$$\boxed{\check{\boldsymbol{f}}^{u,j;\bullet}_{\boldsymbol{\theta}^-;\omega,+}(\omega+\tau) = \tau^{\frac{\lambda_j-u+1}{r}-1}\tau^{\frac{J_{n_j}}{r}}\boldsymbol{K}^{u,j;\bullet}_{\omega^\star,+}\tau^{-\frac{J_{n_1}}{r}} + rem^{u,j;\bullet}_{\omega^\star,+}(\tau)}$$

for all $u=1,\dots,r$ and $j=1,\dots,J$ with a remainder
$$rem^{u,j;\bullet}_{\omega^\star,+}(\tau) := \sum_{\ell;q_\ell(1/x)\in Fr_1(\omega)}\ \sum_{v=1}^{r}\tau^{\frac{\lambda_\ell-v+1}{r}}\boldsymbol{R}^{u,j;\bullet}_{\lambda_\ell,v;\omega^\star,+}(\ln\tau)$$

where

- $\boldsymbol{K}^{u,j;\bullet}_{\omega^\star,+}$ *denotes a constant $n_j\times n_1$-matrix such that $\boldsymbol{K}^{u,j;\bullet}_{\omega^\star,+}=0$ as soon as $q_j(1/x)\notin Fr_1(\omega)$,*
- $\boldsymbol{R}^{u,j;\bullet}_{\lambda_\ell,v;\omega^\star,+}(X)$ *denotes a $n_j\times n_1$-polynomial matrix with coefficients in $\mathcal{R}es^{sum}_{\boldsymbol{\Omega}_1-\omega,0}$ whose the columns are of log-degree*

$$N[\ell] = \begin{cases}\begin{bmatrix}(n_\ell-1) & (n_\ell-1)+1 & \cdots & (n_\ell-1)+(n_1-1)\end{bmatrix} & \text{if } \lambda_\ell\neq 0\\ \begin{bmatrix}n_\ell & n_\ell+1 & \cdots & n_\ell+(n_1-1)\end{bmatrix} & \text{if } \lambda_\ell=0.\end{cases}$$

Proof. — It is sufficient to apply the Borel transformation $\mathcal{B}^{ext}_{\boldsymbol{\theta}^+}$ to identity (5.7) and to observe that normalizations $(N_1)-(N_2)$ imply, on one hand,

$\widetilde{\boldsymbol{F}}^{\bullet;v,\ell}(t) = I_{rn}^{\bullet;v,\ell} + O(t)$ and, on the other hand, that the eigenvalues λ_j of matrix L of exponents of formal monodromy do not differ from an integer. Proposition 5.12 follows then from Proposition 4.3(1). Calculations are left to the reader. □

Remark 5.13. — Like Proposition 5.9 (see Remark 5.10), it seems that Proposition 5.12 above is also still valid when $\boldsymbol{\Omega}_{1;\boldsymbol{\theta}}^{*}$ has Stokes values with a bad front. As before, we refer to a further article for more details.

The connection constants of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}(\tau)$ at ω are then defined as follows.

Definition 5.14 (Connection constants). — *We call* connection constants of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}$ at ω *all the nontrivial entries of matrices* $\boldsymbol{K}_{\omega^\star,+}^{u,j;\bullet}$.

Note that, in practice, the matrix $\boldsymbol{K}_{\omega^\star,+}^{u,j;\bullet}$ can be determined as the coefficient of the monomial $\tau^{(\lambda_j-u+1)/r-1}$ in the major $\check{\boldsymbol{f}}_{\boldsymbol{\theta}^-;\omega,+}^{u,j;\bullet}(\omega+\tau)$.

Highest level's connection-to-Stokes formulæ. We are now able to state the main result of this section.

Theorem 5.15 (Highest level's connection-to-Stokes formulæ). — *Let $j \in \{1,\dots,J\}$ be such that $q_j(1/x) \in Fr_1(\omega)$. Then the data of the highest level's Stokes multipliers $(st_{r;\theta_k^\star}^{j;\bullet})_{k=0,\dots,r-1}$ of $\widetilde{f}(x)$ and the data of the connection constants $(\boldsymbol{K}_{\omega^\star,+}^{u,j;\bullet})_{u=1,\dots,r}$ of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}(\tau)$ at ω are equivalent and are related, for all $k=0,\dots,r-1$, by the relations*

$$\boxed{st_{r;\theta_k^\star}^{j;\bullet} = \sum_{u=1}^{r} \mu^{k((u-1)I_{n_j}-L_j)} \boldsymbol{I}_{\omega^\star}^{u,j;\bullet} \mu^{kJ_{n_1}}} \tag{5.9}$$

where $\mu = e^{-2i\pi/r}$, $L_j = \lambda_j I_{n_j} + J_{n_j}$ is the j-th Jordan block of matrix L and where $\boldsymbol{I}_{\omega^\star}^{u,j;\bullet}$ is the integral

$$\boxed{\boldsymbol{I}_{\omega^\star}^{u,j;\bullet} := \int_{\gamma_0} \tau^{\frac{\lambda_j-u+1}{r}-1} \tau^{\frac{J_{n_j}}{r}} \boldsymbol{K}_{\omega^\star,+}^{u,j;\bullet} \tau^{-\frac{J_{n_1}}{r}} e^{-\tau} d\tau} \tag{5.10}$$

with γ_0 a Hankel type path around the nonnegative real axis $\mathbb{R}^+$ with argument from -2π to 0.

Theorem 5.15 is derived from Propositions 5.9 and 5.12 and from Proposition 4.3(1b). The proof is similar to the ones detailed in [19, § 4.3] and [32, § 4.3] and is left to the reader.

Observe that relation (5.9) is similar to the one obtained in [32] for systems with a unique level. In particular, an expanded form providing each entry of formula (5.9) can be found in [32, Cor. 4.6]. This can be useful for effective numerical calculations. Here below, we recall this expanded form in the special case where the matrix L of exponents of formal monodromy is diagonal: $L = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

In this case, the matrices $st^{j;\bullet}_{r;\theta^\star_k}$ and $\boldsymbol{K}^{u,j;\bullet}_{\omega^\star,+}$ are reduced to just one entry which we respectively denote $st^{j}_{r;\theta^\star_k}$ and $\boldsymbol{K}^{u,j}_{\omega^\star,+}$. Then, identity (5.10) becomes

$$\int_{\gamma_0} \tau^{\frac{\lambda_j-u+1}{r}-1} \boldsymbol{K}^{u,j}_{\omega^\star,+} e^{-\tau} d\tau = 2i\pi \frac{e^{-i\pi\frac{\lambda_j-u+1}{r}}}{\Gamma\left(1-\frac{\lambda_j-u+1}{r}\right)} \boldsymbol{K}^{u,j}_{\omega^\star,+}$$

and the highest level's connection-to-Stokes formulæ (5.9) become

$$\boxed{st^{j}_{r;\theta^\star_k} = 2i\pi \sum_{u=1}^{r} \mu^{k(u-1-\lambda_j)} \frac{e^{-i\pi\frac{\lambda_j-u+1}{r}}}{\Gamma\left(1-\frac{\lambda_j-u+1}{r}\right)} \boldsymbol{K}^{u,j}_{\omega^\star,+}} \tag{5.11}$$

for all $k = 0, \ldots, r-1$.

Effective calculation. Theorem 5.15 above tells us in particular that the effective calculation of the highest level's Stokes multipliers of $\widetilde{f}(x)$ at any Stokes value $\omega \in \boldsymbol{\Omega}^*_{1;\boldsymbol{\theta}}$ can be reduced, after applying Lemma 5.11 if needed, to the effective calculation of the connection constants at ω. We develop in this sense a numerical example in Section 5.6 below.

Before starting the calculations, let us first recall that, according to initial normalizations $(N_1)-(N_2)$ on initial system (A) (page 646), the matrix $\widetilde{\boldsymbol{f}}(t)$ is uniquely determined by the first n_1 columns

$$rt^2 \frac{d\boldsymbol{f}}{dt} = \boldsymbol{A}(t)\boldsymbol{f} - t\boldsymbol{f} J_{n_1} \tag{$\boldsymbol{A_H}$}$$

of the homological system of the r-reduced system $(\boldsymbol{A})$ jointly with the initial condition $\widetilde{\boldsymbol{f}}(0) = I_{rn,n_1}$ = the first n_1 columns of the identity matrix of size rn (see [5]). Thereby, the sum $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}$ itself is completely determined by the convolution system $(\boldsymbol{A}^*_{\boldsymbol{H}})$ deduced from $(\boldsymbol{A_H})$ by a Borel transformation. Note however that, in the special case where the matrix $A(x)$ of initial system (A) has rational coefficients, convolution system $(\boldsymbol{A}^*_{\boldsymbol{H}})$ can actually be replaced by a convenient linear differential system.

5.6. Example

In this section, we consider the system

$$x^3\frac{dY}{dx}=\begin{bmatrix}0&0&0&0\\x^3&ix&0&0\\x^2&0&2+\frac{x^2}{2}&0\\x^2&-x^2&x^2&4\end{bmatrix}Y \tag{5.12}$$

of dimension $n=4$ and rank $r=2$ together with its formal fundamental solution $\widetilde{Y}(x)=\widetilde{F}(x)x^Le^{Q(1/x)}$, where

- $Q\left(\frac{1}{x}\right)=\operatorname{diag}\left(0,-\frac{i}{x},-\frac{1}{x^2},-\frac{2}{x^2}\right)$, $\qquad L=\operatorname{diag}\left(0,0,\frac{1}{2},0\right)$,
- $\widetilde{F}(x)=\begin{bmatrix}1&0&0&0\\\widetilde{f}^2(x)&1&0&0\\\widetilde{f}^3(x)&0&1&0\\\widetilde{f}^4(x)&*&*&1\end{bmatrix}$ satisfies $\widetilde{F}(x)=I_4+O(x^2)$. More precisely,

$$\begin{cases}\widetilde{f}^2(x)=ix^2+2x^3+O(x^4) & \in x^2\mathbb{C}[\![x]\!]\\ \widetilde{f}^3(x)=-\frac{1}{2}x^2+O(x^4) & \in x^2\mathbb{C}[\![x^2]\!]\\ \widetilde{f}^4(x)=-\frac{1}{4}x^2+O(x^4) & \in x^2\mathbb{C}[\![x]\!]\,.\end{cases} \tag{5.13}$$

As before, we denote by $\widetilde{f}(x)$ the first column of $\widetilde{F}(x)$. According to calculations above, system (5.12) has levels $(1,2)$ and the set of highest level's Stokes values of $\widetilde{f}(x)$ is $\boldsymbol{\Omega}_1^*=\{1,2\}$. In particular, the highest level's anti-Stokes directions of $\widetilde{f}(x)$ are given by the unique collection $(\theta_0=0,\theta_1=-\pi)$ generated by $\tau=1$ and $\tau=2$ and the corresponding highest level's Stokes–Ramis matrices $St_{2;\theta_k^\star}$ read as

$$St_{2;\theta_k^\star}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\st^3_{2;\theta_k^\star}&0&1&0\\st^4_{2;\theta_k^\star}&*&*&1\end{bmatrix},\qquad k=0,1.$$

Furthermore, using notations as above, we have $\boldsymbol{\theta}=2\theta_0=0$ and $\boldsymbol{\Omega}^*_{1;0}=\boldsymbol{\Omega}^*_1=\{1,2\}$. Thereby, $\boldsymbol{\Omega}^*_{1;0}$ has a GG-Configuration and $\tau=1$ and $\tau=2$ have both a good monomial front. Consequently, Theorem 5.15 applies and tells us that the two highest level's Stokes multipliers $st^3_{2;0}$ and $st^3_{2;-\pi}$ (resp. $st^4_{2;0}$ and $st^4_{2;-\pi}$) are expressed in terms of the connection constants $\boldsymbol{K}^{1,3}_{1,+}$ and $\boldsymbol{K}^{2,3}_{1,+}$ (resp. $\boldsymbol{K}^{1,4}_{2,+}$ and $\boldsymbol{K}^{2,4}_{2,+}$) of $\widehat{\boldsymbol{f}}_{0^-}(\tau)$ at $\tau=1$ (resp. $\tau=2$). More precisely, since the matrix L is diagonal, identity (5.11) applies and implies relations

$$\boxed{\begin{aligned} st^3_{2;0} &= \frac{(1+i)\pi\sqrt{2}}{\Gamma\left(\frac{3}{4}\right)}\boldsymbol{K}^{1,3}_{1,+} - (4-4i)\Gamma\left(\frac{3}{4}\right)\boldsymbol{K}^{2,3}_{1,+},\\ st^3_{2;-\pi} &= \frac{(-1+i)\pi\sqrt{2}}{\Gamma\left(\frac{3}{4}\right)}\boldsymbol{K}^{1,3}_{1,+} + (4+4i)\Gamma\left(\frac{3}{4}\right)\boldsymbol{K}^{2,3}_{1,+},\\ st^4_{2;0} &= 2i\pi\boldsymbol{K}^{1,4}_{2,+} - 4\sqrt{\pi}\boldsymbol{K}^{2,4}_{2,+}, \qquad st^4_{2;-\pi} = 2i\pi\boldsymbol{K}^{1,4}_{2,+} + 4\sqrt{\pi}\boldsymbol{K}^{2,4}_{2,+}\end{aligned}} \tag{5.14}$$

(recall indeed that $\mu = e^{-i\pi}$ since $r = 2$). It remains to calculate the connection constants. To do that, we proceed as follows.

Let us first observe that, according to the definition of $\widetilde{\boldsymbol{F}}(t)$ (see page 650) and relations (5.13), $\widetilde{\boldsymbol{f}}(t)$ is of the form

$$\widetilde{\boldsymbol{f}}(t) = \begin{bmatrix}\widetilde{\boldsymbol{f}}^1(t)\\ \widetilde{\boldsymbol{f}}^2(t)\end{bmatrix} \text{ with } \widetilde{\boldsymbol{f}}^1(t) = \begin{bmatrix}1\\ \widetilde{\boldsymbol{f}}^{1,2}(t)\\ \widetilde{\boldsymbol{f}}^{1,3}(t)\\ \widetilde{\boldsymbol{f}}^{1,4}(t)\end{bmatrix} \text{ and } \widetilde{\boldsymbol{f}}^2(t) = \begin{bmatrix}0\\ \widetilde{\boldsymbol{f}}^{2,2}(t)\\ \widetilde{\boldsymbol{f}}^{2,3}(t)\\ \widetilde{\boldsymbol{f}}^{2,4}(t)\end{bmatrix},$$

the formal series $\widetilde{\boldsymbol{f}}^{u,j}(t) \in t\mathbb{C}[[t]]$ satisfying

$$\begin{cases} \widetilde{\boldsymbol{f}}^{1,2}(t) = it + O(t^2), & \widetilde{\boldsymbol{f}}^{1,3}(t) = -\frac{1}{2}t + O(t^2), \quad \widetilde{\boldsymbol{f}}^{1,4}(t) = -\frac{1}{4}t + O(t^2),\\ \widetilde{\boldsymbol{f}}^{2,2}(t) = 2t + O(t^2), & \widetilde{\boldsymbol{f}}^{2,3}(t) = 0, \qquad\qquad\qquad \widetilde{\boldsymbol{f}}^{2,4}(t) = O(t^2).\end{cases}$$

Let us now apply relation $(\boldsymbol{A_H})$. Then $\widetilde{\boldsymbol{f}}(t)$ is uniquely determined by the system

$$2t^2\frac{d\boldsymbol{f}}{dt} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & t^2 & it & 0 & 0\\ t & 0 & 2+\frac{t}{2} & 0 & 0 & 0 & 0 & 0\\ t & -t & t & 4 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -t & 0 & 0 & 0\\ t & i & 0 & 0 & 0 & -t & 0 & 0\\ 0 & 0 & 0 & 0 & t & 0 & 2-\frac{t}{2} & 0\\ 0 & 0 & 0 & 0 & t & -t & t & 4-t \end{bmatrix}\boldsymbol{f}$$

together with the initial condition $\widetilde{\boldsymbol{f}}(0) = I_{8,1}$ and, consequently, the $\widetilde{\boldsymbol{f}}^{u,j}(t)$'s are uniquely determined as formal series solutions of the following system

$$\begin{cases} 2t^2\dfrac{d\widetilde{\boldsymbol{f}}^{1,2}}{dt} = it\widetilde{\boldsymbol{f}}^{2,2}\,, \qquad 2t^2\dfrac{d\widetilde{\boldsymbol{f}}^{2,2}}{dt} = t + i\widetilde{\boldsymbol{f}}^{1,2} - t\widetilde{\boldsymbol{f}}^{2,2} \\ 2t^2\dfrac{d\widetilde{\boldsymbol{f}}^{1,3}}{dt} = t + \left(2+\dfrac{t}{2}\right)\widetilde{\boldsymbol{f}}^{1,3}\,, \qquad \widetilde{\boldsymbol{f}}^{2,3} = 0 \\ 2t^2\dfrac{d\widetilde{\boldsymbol{f}}^{1,4}}{dt} = t - t\widetilde{\boldsymbol{f}}^{1,2} + t\widetilde{\boldsymbol{f}}^{1,3} + 4\widetilde{\boldsymbol{f}}^{1,4}\,, \qquad 2t^2\dfrac{d\widetilde{\boldsymbol{f}}^{2,4}}{dt} = -t\widetilde{\boldsymbol{f}}^{2,2} + (4-t)\widetilde{\boldsymbol{f}}^{2,4} \end{cases}$$

satisfying $\widetilde{\boldsymbol{f}}^{u,j}(t) = O(t)$. This brings thereby us, after a Borel transformation, to the following properties.

- The formal Borel transforms $\widehat{\boldsymbol{f}}^{1,2}$ and $\widehat{\boldsymbol{f}}^{2,2}$ satisfy relations

$$\begin{cases} \widehat{\boldsymbol{f}}^{2,2} = -2i\dfrac{d}{d\tau}\left(\tau\widehat{\boldsymbol{f}}^{1,2}\right) \\ 4\tau^2\dfrac{d^2\widehat{\boldsymbol{f}}^{1,2}}{d\tau^2} + (14\tau+1)\dfrac{d\widehat{\boldsymbol{f}}^{1,2}}{d\tau} + 6\widehat{\boldsymbol{f}}^{1,2} = 0 \end{cases}$$

  Therefore, according to the Newton polygon at 0 of $(*)$, $\widehat{\boldsymbol{f}}^{1,2}$ (hence, $\widehat{\boldsymbol{f}}^{2,2}$) is 1-summable in any direction $\theta \neq 0$. In particular, functions $\widehat{\boldsymbol{f}}^{u,2}_{0^-}$'s are given, for instance, by the 1-sums of $\widehat{\boldsymbol{f}}^{u,2}$'s in direction 0. Moreover, since 0 is the only singular point of $(*)$, these functions can be analytically continued on the whole Riemann surface $\widetilde{\mathbb{C}}$ of the logarithm.
- $\widehat{\boldsymbol{f}}^{2,3} = \widehat{\boldsymbol{f}}^{2,3}_{0^-} = 0$ and $\widehat{\boldsymbol{f}}^{1,3}$ defines an analytic function at 0 which is the unique solution of the differential equation

$$2(\tau-1)\frac{d\widehat{\boldsymbol{f}}^{1,3}}{d\tau} + \frac{3}{2}\widehat{\boldsymbol{f}}^{1,3} = 0\,, \quad \widehat{\boldsymbol{f}}^{1,3}(0) = -\frac{1}{2}.$$

  In particular, we have $\widehat{\boldsymbol{f}}^{1,3} = \widehat{\boldsymbol{f}}^{1,3}_{0^-}$ and, for all $|\tau| < 1$,

$$\widehat{\boldsymbol{f}}^{1,3}_{0^-}(\tau) = -\frac{1}{2}(1-\tau)^{-3/4}.$$

- According to calculations above, the functions $\widehat{\boldsymbol{f}}^{1,4}_{0^-}$ and $\widehat{\boldsymbol{f}}^{2,4}_{0^-}$ are uniquely determined by the differential equations

$$\begin{cases} 2(\tau-2)\dfrac{d\widehat{\boldsymbol{f}}^{1,4}_{0^-}}{d\tau} + 2\widehat{\boldsymbol{f}}^{1,4}_{0^-} = -\widehat{\boldsymbol{f}}^{1,2}_{0^-} + \widehat{\boldsymbol{f}}^{1,3}_{0^-}\,, \quad \widehat{\boldsymbol{f}}^{1,4}_{0^-}(0) = -\dfrac{1}{4} \\ 2(\tau-2)\dfrac{d\widehat{\boldsymbol{f}}^{2,4}_{0^-}}{d\tau} + 3\widehat{\boldsymbol{f}}^{2,4}_{0^-} = -\widehat{\boldsymbol{f}}^{2,2}_{0^-}\,, \quad \widehat{\boldsymbol{f}}^{2,4}_{0^-}(0) = 0 \end{cases}$$

(recall indeed that the $\widehat{\boldsymbol{f}}_{0^-}^{u,4}$'s are continuous at 0 with $\widehat{\boldsymbol{f}}_{0^-}^{u,4}(0) = \widehat{\boldsymbol{f}}^{u,4}(0)$). Then, since the homogeneous equations are analytic at 0 and since the functions on the right hand side are integrable at 0, Lagrange method (= variation of constant) tells us that, for all $|\tau| < 2$,

$$\begin{cases} \widehat{\boldsymbol{f}}_{0^-}^{1,4}(\tau) = \dfrac{1}{\tau-2}\left(-\dfrac{1}{2} + (1-\tau)^{1/4} - \dfrac{1}{2}\displaystyle\int_0^\tau \widehat{\boldsymbol{f}}_0^{1,2}(\eta)d\eta\right), \\ \widehat{\boldsymbol{f}}_{0^-}^{2,4}(\tau) = \dfrac{1}{2}(2-\tau)^{-3/2}\displaystyle\int_0^\tau \widehat{\boldsymbol{f}}_0^{2,2}(\eta)(2-\eta)^{1/2}d\eta, \end{cases}$$

where the last integral can be written in the form

$$\int_0^\tau \widehat{\boldsymbol{f}}_0^{2,2}(\eta)(2-\eta)^{1/2}d\eta = \beta + (2-\tau)^{3/2}g(\tau)$$

with $\beta \in \mathbb{C}$ and $g(\tau)$ analytic at $\tau = 2$.

Hence, applying Definition 5.14, the connection constants $\boldsymbol{K}_{1,+}^{u,3}$ and $\boldsymbol{K}_{2,+}^{u,4}$ are given by

$$\boxed{\boldsymbol{K}_{1,+}^{1,3} = \frac{1+i}{2\sqrt{2}} \qquad \boldsymbol{K}_{1,+}^{2,3} = 0 \qquad \boldsymbol{K}_{2,+}^{1,4} = \alpha \qquad \boldsymbol{K}_{2,+}^{2,4} = \frac{i\beta}{2}}$$

with

$$\boxed{\alpha = -\frac{1}{2} + \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} - \frac{1}{2}\int_0^2 \widehat{\boldsymbol{f}}_0^{1,2}(\eta)d\eta}$$

and, consequently, identities (5.14) imply

$$\boxed{\begin{array}{ll} st_{2;0}^3 = \dfrac{i\pi}{\Gamma\left(\dfrac{3}{4}\right)} & st_{2;-\pi}^3 = -\dfrac{\pi}{\Gamma\left(\dfrac{3}{4}\right)} \\ st_{2;0}^4 = 2i\sqrt{\pi}(\alpha\sqrt{\pi} - \beta) & st_{2;-\pi}^4 = 2i\sqrt{\pi}(\alpha\sqrt{\pi} + \beta) \end{array}}$$

Note that, although system (5.12) may seem a little bit involved, it is actually simple enough to allow *exact* calculations. This "simplicity" is due to the fact that its matrix is triangular. Of course, such a case is anecdotal and, in a more general situation, i.e. for systems for which the matrices are not triangular, such exact calculations are not possible anymore. Nevertheless, it is worth to be treated since it allows to easily illustrate formulæ (5.9).

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