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KAWA lecture notes on complex hyperbolic geometry

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Abstract. — These lecture notes are based on a mini-course given at the fifth KAWA Winter School on March 24-29, 2014 at CIRM, Marseille. They provide an introduction to hyperbolicity of complex algebraic varieties namely the geometry of entire curves, and a description of some recent developments.

Résumé. — Ces notes sont issues d'un mini-cours donné à la cinquième École d'Hiver KAWA du 24 au 29 Mars 2014, au CIRM à Marseille. Elles donnent une introduction à l'hyperbolicité des variétés algébriques complexes à savoir la géométrie des courbes entières, ainsi qu'une description de certains développements récents.

1. Introduction

Hyperbolicity of algebraic varieties is the study of the geometry of entire curves. There is a conjectural picture claiming that positivity properties of the canonical line bundle control the distribution of rational and holomorphic curves. After Green, Griffiths and Lang, we expect that entire curves are not Zariski dense in varieties of general type. Recent works confirm this picture in the case of generic projective hypersurfaces of large degree using jet differentials techniques developed since Bloch by Green and Griffiths, Demailly, Siu...

In these lecture notes, we will introduce in Section 2 the problem to characterize which projective varieties are Kobayashi hyperbolic. According to Brody's criterion, a compact complex space X is Kobayashi hyperbolic if and only if there is no non-constant holomorphic curve $\mathbb{C} \to X$ from the complex plane. We explain how to develop a transcendental intersection theory between holomorphic curves and line bundles.

In Section 3, we present the fundamental tool of this intersection theory: the *Tautological inequality* of McQuillan. As applications of this inequality,

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we explain how to recover many classical results: hyperbolicity of compact Riemann surfaces of genus $g \geqslant 2$, of varieties with ample cotangent bundle... Finally, we introduce the Green–Griffiths–Lang conjecture and prove several results confirming it in different cases for surfaces.

Next in Section 4, we discuss jets spaces, one of the main tool to attack the above conjecture. We explain in detail the strategy, and why it may fail in some cases like quotient of bounded symmetric domains.

Finally, in Section 5, we describe alternative strategies for these examples.

2. Entire curves and diverging sequences of discs

Let X be a compact complex manifold equipped with a hermitian metric. If there exists an entire curve i.e. a non-constant holomorphic map $f : \mathbb{C} \to X$ such that $f'(0) \neq 0$, then one can construct a sequence of discs diverging in the following sense. Take $f_n : \Delta \to X$ where Δ is the unit disc and $f_n(z) = f(nz)$, then $\|f_n'(0)\| \to \infty$.

Conversely, from such a sequence of discs, one can construct an entire curve.

THEOREM 2.1 (Brody's lemma). — *Let $f_n : \Delta \to X$, $\|f_n'(0)\| \to \infty$ be a diverging sequence of discs. Then there exists a sequence (r_n) of reparametrizations of $\mathbb{C}$ such that, after passing possibly to a subsequence, $f_n \circ r_n$ converges locally uniformly towards a non-constant entire curve $f : \mathbb{C} \to X$ such that $\|f'\| \leqslant 1$. Such a curve is called a* Brody curve.

Proof. — Without loss of generality, one can suppose that f_n extends to $\overline{\Delta}$. Consider
$$\phi_n := \|f_n'\|(1 - |z|),$$
and a_n the point where $\phi_n(a_n) := M_n$ is maximum.

Then $M_n \geqslant \phi_n(0) = \|f_n'(0)\|$ and therefore this sequence goes to infinity. Now, consider the disc $D(a_n, \epsilon_n)$ where $2\epsilon_n := 1 - |a_n|$. On this disc, $\phi_n(z) \leqslant \phi_n(a_n) = M_n$ therefore $\epsilon_n \|f_n'(z)\| \leqslant 2\epsilon_n \|f_n'(a_n)\|$ and $\|f_n'(z)\| \leqslant 2\|f_n'(a_n)\|$.

One can now define $r_n(z) = a_n + \frac{\epsilon_n}{M_n} z$ and $g_n = f_n \circ r_n$. Then on the disc $D(0, M_n)$, $\|g_n'(z)\| = \frac{\epsilon_n}{M_n}\|f_n'(a_n + \frac{\epsilon_n}{M_n} z)\| \leqslant \frac{2\epsilon_n}{M_n}\|f_n'(a_n)\| = 1$. The sequence (g_n) is equicontinuous and by Ascoli (possibly taking a subsequence) it converges locally uniformly to a holomorphic map $f : \mathbb{C} \to X$. Finally, remark that $\|g_n'(0)\| = \frac{\epsilon_n}{M_n}\|f_n'(a_n)\| = \frac{1}{2}$, and therefore f is not constant. $\square$

An immediate consequence, is the following characterization of the *Kobayashi hyperbolicity* for compact complex manifolds. Recall that the Kobayashi pseudo-metric is defined as follows

$$k_X(p,v) = \inf \left\{ \frac{1}{r} \; \middle| \; \exists \, f : \Delta \to X, f(0) = p, f'(0) = rv \right\}.$$

X is said to be *Kobayashi hyperbolic* if its Kobayashi pseudo-metric is non-degenerate. A consequence of Brody's lemma is the following.

COROLLARY 2.2. — *Let X be a compact complex manifold. X is Kobayashi hyperbolic if and only if X does not contain Brody curves.*

Indeed, "k_X degenerates" means exactly that there is a sequence of discs diverging in the above sense.

A difficult problem is the localization of the entire curve produced by this process. Take a diverging sequence of discs $f_n : \Delta \to X$, $f_n(0) = p$, $\|f_n'(0)\| \to \infty$. Then there is a Brody curve $f : \mathbb{C} \to X$ but $f(\mathbb{C})$ could be far from p as shown in the following example of Winkelmann [28].

Example 2.3. — There is a projective manifold X obtained as a blow-up of an abelian threefold Y such that every Brody curve $g : \mathbb{C} \to X$ lies in the exceptional divisor $E \subset X$. However entire curves can be dense in X.

The notion of diverging sequence of discs leads to the concept of *Ahlfors currents*, but one needs to change the definition of diverging sequences. Recall that a current T on X is a differential form on X whose coefficients are distributions. The pairing $\langle T, \omega \rangle \to \int_X T \wedge \omega$ identifies the vector space of currents of type (p,q) on X with the topological dual of the vector space of smooth $(n-p, n-q)$−forms with compact support on X. Any smooth subvariety Y of X of codimension m defines a current $[Y]$ of type (m,m) by the formula $\langle [Y], \omega \rangle = \int_Y \omega$.

Let X be a complex manifold admitting an entire curve $f : \mathbb{C} \to X$. Let $A(r) = \text{area}(f(D(0,r))$ and $L(r) = \text{length}(f(\partial D(0,r))$. Then we have

LEMMA 2.4 (Ahlfors lemma). — *There exists a sequence $r_n \to \infty$ such that $\frac{L(r_n)}{A(r_n)} \to 0$.*

Proof. — Let $g|\mathrm{d}z|$ be the pull-back of some metric ω on X by f. Then $L(r) = \int_0^{2\pi} g(re^{i\theta}) r \mathrm{d}\theta$ and $A(r) = \int_0^{2\pi} \int_0^r g^2(te^{i\theta}) t \mathrm{d}t \mathrm{d}\theta$. So $A'(r) = \int_0^{2\pi} g^2(te^{i\theta}) r \mathrm{d}\theta$. Now by Cauchy–Schwartz, one has $L(r)^2 \leqslant A'(r) 2\pi r$ and therefore $(\frac{L}{A})^2 \frac{1}{2\pi r} \leqslant \frac{A'}{A^2}$.

Integrating one obtains

$$\int_1^{+\infty} \left(\frac{L}{A}\right)^2 \frac{\mathrm{d}r}{2\pi r} \leqslant \frac{1}{A(1)} \leqslant +\infty,$$

which certainly implies that $\liminf_{r \to +\infty} \frac{L}{A} = 0$. $\qquad\square$

Now, suppose that we are given a sequence of discs Δ_n in X diverging in the following sense: $\frac{L(\Delta_n)}{A(\Delta_n)} \to 0$. Then (possibly taking a subsequence) one obtains a closed positive current

$$T = \lim_{n \to +\infty} \frac{[\Delta_n]}{A(\Delta_n)},$$

called an *Ahlfors current*.

These currents play a crucial role in the proof by McQuillan of the Green–Griffiths conjecture for surfaces of general type with positive second Segre number. One of their interest, is that we can use them to present *Nevanlinna theory* as an intersection theory of these currents.

To introduce the classical quantities of this theory, one needs to slightly modify the above definition and adapt the proof of Ahlfors lemma. Let $f : \mathbb{C} \to X$ be an entire curve. Then one defines

$$T_{f,r}(\omega) = \int_0^r A(t) \frac{\mathrm{d}t}{t} := \int_0^r \frac{\mathrm{d}t}{t} \int_{D(0,t)} f^*\omega,$$

$$S_{f,r}(\omega) := \int_0^r L(t) \frac{\mathrm{d}t}{t}.$$

Then one gets the corresponding Ahlfors lemma

LEMMA 2.5. — *There exists a sequence $r_n \to \infty$ such that $\frac{S_{f,r_n}(\omega):}{T_{f,r_n}(\omega)} \to 0$.*

One also obtains in the same way Ahlfors currents. Let $\eta \in A^2(X)$ be a 2-form. Let $\Phi_r(\eta) := \frac{T_{f,r}(\eta)}{T_{f,r}(\omega)}$. This defines a family of positive currents of bounded mass from which one can extract a closed postive current $\Phi := \lim_{r_n} \Phi_{r_n}$. The closedness follows from compactness of X and Stokes: if $\beta \in A^1(X)$ is a one-form then $|T_{f,r}(d\beta)| \leqslant \int_0^r \frac{\mathrm{d}t}{t} \int_{\partial D(t)} |f^*\beta| \leqslant C S_{f,r}\omega$.

A nice result of Duval [14] gives a version of Brody's lemma in this setting.

THEOREM 2.6 (Duval). — *Let X be a compact complex manifold. If there is a diverging sequence of discs in the sense that $\frac{L(\Delta_n)}{A(\Delta_n)} \to 0$. Then there is a non-constant entire curve $f : \mathbb{C} \to X$.*

A consequence is a characterization of Kobayashi hyperbolicity in terms of isoperimetric inequalities.

COROLLARY 2.7. — *Let X be a compact complex manifold. Then holomorphic discs in X satisfy an isoperimetric linear inequality, i.e. there exists a constant C such that* $\mathrm{area}(\Delta) \leqslant C \cdot \mathrm{length}(\partial\Delta)$ *if and only if X is Kobayashi hyperbolic.*

The idea now is to use Ahlfors currents to obtain a transcendental intersection theory between holomorphic curves $\mathbb{C} \to X$ and divisors or line bundles on X.

Let $f : \mathbb{C} \to X$ be an entire curve and $[\Phi] \in H^{1,1}(X,\mathbb{R})$ the class of an Ahlfors current associated to it. The classical first main theorem of Nevanlinna theory can be interpreted as follows.

PROPOSITION 2.8. — *Let $Z \subset X$ be an algebraic hypersurface. If $f(\mathbb{C}) \not\subset Z$ then $[\Phi].[Z] \geqslant 0$.*

Proof. — Let $Z = \{s = 0\}$, and h a hermitian metric on the line bundle $\mathcal{O}(Z)$. By Poincaré–Lelong formula

$$dd^c \ln \|s\|^2 = Z - \Theta_h.$$

Therefore

$$T_{f,r}(\Theta) = -\int_0^r \frac{\mathrm{d}t}{t} \int_{D(t)} dd^c \ln \|s \circ f\|^2 + \sum_{z \in D(r), f(z) \in Z} \mathrm{ord}_z(s \circ f) \ln\left(\frac{r}{|z|}\right).$$

Using Jensen's formula, one obtains

$$T_{f,r}(\Theta) = -\frac{1}{2\pi}\int_0^{2\pi} \ln \|s(f(re^{i\theta}))\| \mathrm{d}\theta + \log \|s(f(0))\| + N_f(Z,r),$$

which is nothing else than the First Main Theorem of Nevanlinna theory

$$T_{f,r}(\Theta) = m_f(Z,r) + N_f(Z,r) + O(1),$$

relating respectively the characteristic function, the proximity function and the counting function. Rescaling, one may suppose that

$$\|s\| \leqslant 1,$$

which implies $[\Phi].[Z] \geqslant 0$. $\qquad\square$

3. The Tautological inequality and some applications

3.1. The inequality

Now we describe, the crucial tool called "Tautological inequality" by McQuillan [21]. The picture to keep in mind which summarizes the setting is

the following.

$$
\begin{array}{ccc}
 & & \mathbb{P}(T_X) \\
 & {\scriptstyle f'} \nearrow & \downarrow {\scriptstyle \pi} \\
\mathbb{C} & \xrightarrow{\;f\;} & X
\end{array}
$$

Let us denote $L = \mathcal{O}_{\mathbb{P}(T_X)}(-1)$, Φ and Φ' the Ahlfors currents normalized so that $\pi_* \Phi' = \Phi$.

THEOREM 3.1 (McQuillan).

$$[\Phi'].L \geqslant 0.$$

This theorem can be seen as a generalization of the following easy remark.

Remark 3.2. — Let $C \subset X$ be a smooth algebraic curve with lifting $C' \subset \mathbb{P}(TX)$. Then $L_{|C'} = \pi^* T_C$ and $L.C' = \chi(C) \geqslant 0$ if and only if C is rational or elliptic i.e. if there exists a non-constant map $f : \mathbb{C} \to C$.

Before giving the proof of the Tautological inequality, we need the following easy lemma whose proof is left to the reader:

LEMMA 3.3.
(1) *Let g be a continuously differentiable, increasing function on $[0, +\infty[$ with $g(r) \to \infty$. Then for $\delta > 0$, we have $g'(r) \leqslant g(r)^{1+\delta}$ for all $r > 0$ outside a set E_δ of finite linear measure.*
(2) *Let g and h be continuous functions on $[0, +\infty[$ with $g > 0$. For $\delta > 0$, suppose $h(r) \leqslant \delta g(r)$ for all $r > 0$ outside a set E_δ of finite linear measure. Then $h(r) = o(g(r))\|$ as $r \to \infty$, where $\|$ means that the estimate holds for $r > 0$ outside some exceptional set of finite linear measure.*

Let us now prove the Tautological inequality.

Proof. — $\frac{\mathrm{d}f}{\mathrm{d}z}$ defines a section $s \in H^0(\mathbb{C}, (f')^* L)$. By Poincaré–Lelong formula:

$$dd^c \ln \|s\|^2 = (s = 0) - (f')^* \Theta.$$

Integrating and using Jensen's formula as well as the concavity of log, one obtains

$$-T_{f',r}(\Theta) \leqslant \int_0^r \frac{\mathrm{d}t}{t} \int_{D(t)} dd^c \ln \left\| \frac{\mathrm{d}f}{\mathrm{d}z} \right\|^2 \leqslant C + \frac{1}{2} \ln \left(\int_0^{2\pi} \left\| \frac{\mathrm{d}f}{\mathrm{d}z}(re^{i\theta}) \right\|^2 \frac{\mathrm{d}\theta}{2\pi} \right).$$

Now, one can write

$$\int_0^{2\pi} \left\| \frac{\mathrm{d}f}{\mathrm{d}z}(re^{i\theta}) \right\|^2 \frac{\mathrm{d}\theta}{2\pi} = \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(r \frac{\mathrm{d}}{\mathrm{d}r} T_{f,r}(\omega) \right).$$

From the above lemma, one has

$$
\begin{aligned}
-T_{f',r}(\Theta) &\leqslant C + \frac{1}{2}\ln\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}rT_{f,r}(\omega)^{1+\delta}\right)\Big\| \\
&\leqslant C + \frac{1}{2}\ln\left(r^{\delta}T_{f,r}(\omega)^{(1+\delta)^2}\right)\Big\| \\
&\leqslant \frac{(1+\delta)^2}{2}\ln T_{f,r}(\omega) + \frac{\delta}{2}\ln(r) + O(1) \leqslant o(T_{f,r}(\omega))\|.
\end{aligned}
$$

One can choose the metric on $\mathbb{P}(T_X)$ of the form $\tilde{\omega} = \pi^*\omega - \epsilon\Theta$, with ϵ small enough. Then the last inequality implies that $[\Phi'].L \geqslant 0$. $\qquad\square$
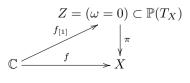
3.2. Applications

To illustrate the importance of the previous inequality, let us show that one can recover many classical results.

3.2.1. Riemann surfaces

Let C be a compact Riemann surface of genus $g \geqslant 2$. We want to show that C is Kobayashi hyperbolic. Suppose that there is an entire curve $f : \mathbb{C} \to C$. Then $\mathbb{P}(T_C) \simeq C$ and $\mathcal{O}(-1) \simeq T_C = -K_C < 0$. Therefore $[\Phi].\mathcal{O}(-1) < 0$ contradicting the tautological inequality $[\Phi].\mathcal{O}(-1) \geqslant 0$.

3.2.2. Symmetric differentials

Let us see why the presence of many symmetric differentials on a projective manifold imposes strong conditions on entire curves. Let X be a projective manifold, $A \to X$ an ample line bundle and $f : \mathbb{C} \to X$ an entire curve. Suppose that we have a section $\omega \in H^0(X, S^m T_X^* \otimes A^{-1})$. Let us prove that $\omega(f, f') \equiv 0$ which means that any entire curve has to satisfy the first-order differential equation defined by ω. Recall the isomorphism $\pi_*\mathcal{O}(m) \simeq S^m T_X^*$. Therefore ω can be seen as a section of $m\mathcal{O}(1) - \pi^*A$ and its zero locus defines a divisor $Z \subset \mathbb{P}(T_X)$.

$$
Z = (\omega = 0) \subset \mathbb{P}(T_X)
$$

We want to prove that $f_{[1]}(\mathbb{C}) \subset Z$. Suppose it is not the case, then by Proposition 2.8, $[\Phi_1].[Z] \geqslant 0$, i.e. $[\Phi_1].(m\mathcal{O}(1) - \pi^*A) \geqslant 0$. But from the tautological inequality 3.1, $[\Phi_1].\mathcal{O}(1) \leqslant 0$ and $-[\Phi].A < 0$, which gives a contradiction. Therefore $f_{[1]}(\mathbb{C}) \subset Z$ i.e. $\omega(f, f') \equiv 0$.

We have just proved that the lifting $f_{[1]}$ of any entire curve has to lie in the base locus of $(m\mathcal{O}(1) - \pi^*A)$ for any ample line bundle A.

PROPOSITION 3.4. — *Let X be a projective manifold, $A \to X$ an ample line bundle and $f : \mathbb{C} \to X$ an entire curve. Suppose there is a section $\omega \in H^0(X, S^m T_X^* \otimes A^{-1})$. Then $\omega(f, f') \equiv 0$*

COROLLARY 3.5. — *Let X be a projective manifold, $A \to X$ an ample line bundle and $f : \mathbb{C} \to X$ an entire curve. Then*

$$f_{[1]}(\mathbb{C}) \subset Bs(\mathcal{O}(m) - \pi^*A).$$

This immediately proves that varieties with ample cotangent bundle are Kobayashi hyperbolic.

COROLLARY 3.6. — *Let X be a projective manifold. If T_X^* is ample then X is hyperbolic.*

Proof. — If T_X^* is ample then for m large enough $\mathcal{O}(m) - \pi^*A$ is ample. Therefore $Bs(\mathcal{O}(m) - \pi^*A) = \emptyset$. □

3.2.3. The Green–Griffiths–Lang conjecture

One of the main open problem concerning the geometry of entire curves is the following

CONJECTURE 3.7 (Green–Griffiths–Lang). — *Let X be a projective variety of general type. Then for every entire curve $f : \mathbb{C} \to X$, the Zariski closure of the image $f(\mathbb{C})$ is a proper subset of X.*

Even for surfaces this conjecture is still open in general. It is known to be true under conditions on Chern classes as in the following result of Lu and Yau [20].

THEOREM 3.8. — *Let X be a smooth algebraic surface of general type such that $c_1^2 > 2c_2$. Then every entire curve $f : \mathbb{C} \to X$ is algebraically degenerate.*

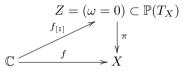*Proof.* — Under the hypothesis $c_1^2 > 2c_2$, one obtains that T_X^* is big. Indeed, by Riemann–Roch

$$\chi(X, S^m T_X^*) = \frac{m^3}{6}(c_1^2 - c_2) + O(m^2).$$

Therefore $h^0(X, S^m T_X^*) + h^2(X, S^m T_X^*) > cm^3$. Now, by Serre duality and the isomorphism $K_X \otimes T_X = T_X^*$, we have $h^2(X, S^m T_X^*) = h^0(X, K_X^{(1-m)} \otimes S^m T_X^*) \leqslant h^0(X, S^m T_X^*)$. The last inequality comes from the fact that X is of general type and in particular, K_X^m is effectif for large m. Finally, we obtain $h^0(X, S^m T_X^*) > \frac{c}{2} m^3$ and T_X^* is big.

So we are in the situation of Proposition 3.4: we have a section $\omega \in H^0(X, S^m T_X^* \otimes A^{-1})$, where A is an ample line bundle. If $f : \mathbb{C} \to X$ is an entire curve, $\omega(f, f') \equiv 0$.

$$Z = (\omega = 0) \subset \mathbb{P}(T_X)$$

We can suppose that Z is an irreducible horizontal surface.

Let us prove that $\mathcal{O}(1)_{|Z}$ is big. In the Picard group, $Z = \mathcal{O}(m) + \pi^* L$. Since it is effective, we have an injection $L^* \hookrightarrow S^m T_X^*$. From the semistability of T_X^*, we have $c_1(L).c_1(X) \leqslant \frac{m}{2} c_1^2(X)$.

Moreover $c_1^2(\mathcal{O}(1)_{|Z}) = c_1^2(\mathcal{O}(1)).(\mathcal{O}(m) + \pi^* L) = m(c_1^2 - c_2) - c_1.L \geqslant \frac{m}{2}(c_1^2 - 2c_2) > 0$.

Therefore either $\mathcal{O}(1)_{|Z}$ or $\mathcal{O}(-1)_{|Z}$ is big. The last possibility is excluded. Indeed, $(\mathcal{O}(-1)_{|Z}).(\pi^*(K_X)_{|Z}) = c_1 c_1(L) - m c_1^2 \leqslant -\frac{m}{2} c_1^2 < 0$. As we may suppose that X is minimal i.e. K_X nef, its intersection with a big line bundle cannot be negative.

We know that the lifting $f_{[1]} : \mathbb{C} \to Z$ gives an entire curve in Z. Let s be a global section of $(\mathcal{O}(m) - A)_{|Z}$. Then $f_{[1]}(\mathbb{C}) \subset C = (s = 0)$ following the same proof as Proposition 3.4. Indeed, let $[\Phi]$ be the Ahlfors current associated to $f_{[1]}$. Then $[\Phi].[\mathcal{O}(m)_{|Z}] \leqslant 0$ by the Tautological inequality, therefore $[\Phi].(\mathcal{O}(m) - A)_{|Z} < 0$. If $f_{[1]}(\mathbb{C}) \not\subset C$ then $[\Phi].C = [\Phi].(\mathcal{O}(m) - A)_{|Z} \geqslant 0$ which gives a contradiction.

Finally $f(\mathbb{C}) \subset \pi(C)$ and f is algebraically degenerate. $\square$

In fact there is a strong version of the above conjecture.

CONJECTURE 3.9 (Strong Green–Griffiths–Lang conjecture). — *Let X be a projective variety of general type, then there exists a proper Zariski closed subset $Y \subsetneq X$ such that for all non constant holomorphic curves $f : \mathbb{C} \to X$, $f(\mathbb{C}) \subset Y$.*

In the case of surfaces, one may find this locus Y under conditions on the Chern classes following a result of Bogomolov [2].

THEOREM 3.10. — *Let X be a projective surface of general type such that $c_1^2 > c_2$ then there are only finitely many rational or elliptic curves in X.*

Proof. — Let $f : C \to X$ be a rational or elliptic curve. Following the beginning of the proof of Theorem 3.8, we are reduced to the same picture.

$$Z = (\omega = 0) \subset \mathbb{P}(T_X)$$

If $\pi(Z) \neq X$ we are done, so let us suppose that $\pi(Z) = X$. Then Z is equipped with a tautological holomorphic foliation by curves: if $z \in Z$ is a generic point, a neighbourhood U of z induces a foliation on a neighbourhood V of $x = \pi(z)$. Indeed, a point in $U \subset \mathbb{P}(T_X)$ is of the form $(w, [t])$ where w is a point in X and t a tangent vector at this point. This foliation lifts through the isomorphism $U \to V$ induced by π. Leaves are just the derivatives of leaves on V. Tautologically, $f_{[1]} : C \to Z$ is a leaf. So the problem is reduced to the following one: let $(Z, \mathcal{F})$ be a foliated surface of general type, then there are only finitely many algebraic leaves which are rational or elliptic. Therefore one can conclude using a theorem of Jouanolou [17]: either Z has finitely many algebraic leaves or it is a fibration. But in our case, Z is of general type and cannot be ruled or elliptic. □

An immediate corollary is

COROLLARY 3.11. — *The strong Green–Griffiths–Lang conjecture is true for projective surfaces of general type such that $c_1^2 > 2c_2$.*

Proof. — Thanks to Theorem 3.8, any entire curve $f : \mathbb{C} \to X$ is algebraically degenerate so its Zariski closure is a rational or elliptic curve. Therefore from Theorem 3.10, it lies in the finite set of elliptic and rational curves. □

One of the major recent breakthrough in this subject, is the following result of McQuillan [21] which weakens the hypothesis on Chern classes.

THEOREM 3.12. — *The strong Green–Griffiths–Lang conjecture is true for projective surfaces of general type such that $c_1^2 > c_2$.*

As we have explained in the proof of Theorem 3.10, this result can be seen to be a consequence of the following one in the setting of foliated surfaces. Recall that a foliation $\mathcal{F}$ on a smooth variety X is given by a saturated subsheaf $T_{\mathcal{F}} \subset T_X$ stable under Lie bracket.

THEOREM 3.13 (McQuillan [21]). — *Let X be a smooth projective surface of general type with a (possibly singular) foliation $\mathcal{F}$. Let $f : \mathbb{C} \to (X, \mathcal{F})$*

be an entire curve tangent to $\mathcal{F}$. Then f is algebraically degenerate i.e. its image is not Zariski dense.

Let us give the proof only in the case where $\mathcal{F}$ is a *smooth* foliation. The foliation is given by a subbundle $T_{\mathcal{F}} \subset T_X$ which fits into an exact sequence

$$0 \to T_{\mathcal{F}} \to T_X \to \mathcal{N}_{\mathcal{F}} \to 0, \tag{3.1}$$
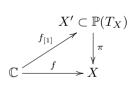
where $\mathcal{N}_{\mathcal{F}}$ is the normal bundle of the foliation. One also introduces the canonical bundle of the foliation $K_{\mathcal{F}} := (\det T_{\mathcal{F}})^*$. For the proof, we need the following proposition about the normal bundle of the foliation.

PROPOSITION 3.14. — *There exists a smooth closed 2-form α on X such that $[\alpha] = c_1(\mathcal{N}_{\mathcal{F}}^*)$ and α vanishes on leaves of the foliation.*

Proof. — Let (U_i) be a covering of X. The foliation is defined by forms $\omega_i \in H^0(U_i, \Omega_X^1)$ satisfying relations $\omega_i = g_{ij}\omega_j$ with $g_{ij} \in H^0(U_i \cap U_j, \mathcal{O}_X^*)$. (g_{ij}) defines a cocycle representing $\mathcal{N}_{\mathcal{F}}^*$. From Frobenius integrability, one has $\mathrm{d}\omega_i = \beta_i \wedge \omega_i$. Therefore $\beta_i \wedge \omega_i = \mathrm{d}\omega_i = \mathrm{d}g_{ij} \wedge \omega_j + g_{ij}\mathrm{d}\omega_j = \left(\frac{\mathrm{d}g_{ij}}{g_{ij}} + \beta_j\right) \wedge \omega_i$. So $\left(\frac{\mathrm{d}g_{ij}}{g_{ij}} + \beta_j - \beta_i\right) \wedge \omega_i = 0$, and $\frac{\mathrm{d}g_{ij}}{g_{ij}} + \beta_j - \beta_i \in H^0(U_i \cap U_j, \mathcal{N}_{\mathcal{F}}^*)$. Let η_i be a $\mathcal{C}^\infty$ section of $\mathcal{N}_{\mathcal{F}}^*$ over U_i such that $\frac{\mathrm{d}g_{ij}}{g_{ij}} + \beta_j - \beta_i = \eta_i - \eta_j$ over $U_i \cap U_j$. Then $\alpha = \mathrm{d}(\eta_i + \beta_i)$ defines a smooth closed form over X representing $c_1(\mathcal{N}_{\mathcal{F}}^*)$. η_i vanishes on leaves of the foliation. $d\beta_i$ is a holomorphic 2-form. Finally, α vanishes on leaves of the foliation. $\square$

Let us now give the proof of Theorem 3.13 for smooth foliations.

Proof. — Let us suppose that f is Zariski dense and we will derive a contradiction. The foliation $\mathcal{F}$ defines a surface $X' \subset \mathbb{P}(T_X)$ such that $\mathcal{O}(-1)_{|X'} = \pi^* T_{\mathcal{F}|X'}$.

$$
\begin{array}{ccc}
 & & X' \subset \mathbb{P}(T_X) \\
 & \stackrel{f_{[1]}}{\nearrow} & \downarrow \pi \\
\mathbb{C} & \stackrel{f}{\longrightarrow} & X
\end{array}
$$

Now we can apply the Tautological inequality 3.1 upstairs i.e. for $f_{[1]} : \mathbb{C} \to X'$.

$$0 \leqslant [\Phi'].\mathcal{O}(-1)_{|X'} = [\Phi].T_{\mathcal{F}} = -[\Phi].K_{\mathcal{F}}.$$

From the exact sequence (3.1), one obtains the relation

$$K_X = \mathcal{N}_{\mathcal{F}}^* \otimes K_{\mathcal{F}}.$$

From Proposition 3.14, one has $[\Phi].\mathcal{N}_{\mathcal{F}}^* = 0$. Therefore,

$$[\Phi].K_X = [\Phi].K_{\mathcal{F}} + [\Phi].\mathcal{N}_{\mathcal{F}}^* \leqslant 0.$$

But K_X is big, so it is the sum of an ample and an effective line bundle, $K_X = A + E$. This gives a contradiction:

$$[\Phi].K_X = [\Phi].A + [\Phi].E \geqslant 0$$

by Proposition 2.8. In the general (singular) case, the scheme of the proof is the same: one can generalize both inequalities $[\Phi].K_{\mathcal{F}} \leqslant 0$ and $[\Phi].\mathcal{N}_{\mathcal{F}}^* \leqslant 0$. Nevertheless, the presence of singularities make the proof much more difficult. First, one has to "resolve" the singularities in the sense of Seidenberg, and then make a local analysis at singular points. We refer to [21] and [6] for details. □

4. Jets spaces

4.1. Definitions

Following the setting presented by Demailly in [8], we introduce the important tool of jets spaces. We put ourselves in the category of *directed* manifolds i.e. pairs (X, V) of complex manifolds and $V \subset T_X$ holomorphic subbundles.

Let $\tilde{X} = \mathbb{P}(V)$ the projectivized bundle of lines of V. Then we define $\pi : \tilde{X} \to X$ and $\tilde{V} := \pi_*^{-1}\mathcal{O}_{\tilde{X}}(-1)$. In other words $\tilde{V} \subset T_{\tilde{X}}$ is the holomorphic subbundle with fibers

$$\tilde{V}_{(x,[v])} = \left\{ \xi \in T_{\tilde{X}\,(x,[v])} \;\middle|\; \pi_*\xi \in \mathbb{C}v \right\}.$$

It fits into the exact sequence

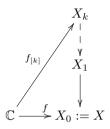$$0 \to T_{\tilde{X}/X} \to \tilde{V} \to \mathcal{O}_{\tilde{X}}(-1) \to 0,$$

and rank $\tilde{V} = $ rank $V = r$.

Given a holomorphic map $f : \Delta_R \to (X, V)$, i.e. a holomorphic map from the disc of radius R to X and tangent to V, there is a natural lifting $\tilde{f} : \Delta_R \to (\tilde{X}, \tilde{V})$ given by $(f(t), [f'(t)])$.

Inductively, we define the projectivized k-jet bundle: $X_0 = X$, $V_0 = V$, $X_k = \widetilde{X_{k-1}}$ and $V_k = \widetilde{V_{k-1}}$. There are natural "vertical divisors" $D_k := \mathbb{P}(T_{X_{k-1}/X_{k-2}})$.

4.2. The Green–Griffiths locus

The picture to keep in mind for the study of entire curves in complex manifolds is the following

$$
\begin{array}{c}
X_k \\
\end{array}
$$

The idea is that the study of entire curves $f : \mathbb{C} \to X$ will go through the study of lifted curves $f_{[k]} : \mathbb{C} \to X_k$ in jets spaces.

Let us see some applications of the Tautological inequality 3.1 in this setting. The first obvious one is the following.

PROPOSITION 4.1. — *Let X be a smooth projective variety and $f : \mathbb{C} \to X$ an entire curve. Take the k-jet lifting $f_{[k]} : \mathbb{C} \to X_k$ and $[\Phi_k]$ its Ahlfors current. Then*

$$[\Phi_k].\mathcal{O}_{X_k}(1) \leqslant 0.$$

COROLLARY 4.2. — *Let $V \subset X_k$ be the Zariski closure of $f_{[k]}(\mathbb{C})$. Then $\mathcal{O}_{X_k}(1)_{|V}$ is not big.*

Proof. — From the previous Proposition 4.1, we have $[\Phi_k].\mathcal{O}_{X_k}(1) \leqslant 0$. Now if $\mathcal{O}_{X_k}(1)_{|V}$ is big, then $[\Phi_k].\mathcal{O}_{X_k}(1) > 0$, which gives a contradiction. □

This naturally leads to the following definition of the *Green–Griffiths locus*. Let A be an ample line bundle on X and $\pi_k : X_k \to X$. Let $B_{k,l} \subset X_k$ be the base locus of $\mathcal{O}_{X_k}(l) \otimes \pi_k^* A^{-1}$. We set $B_k := \bigcap_{l \in \mathbb{N}} B_{k,l}$. The we define the Green–Griffiths locus as

$$GG := \bigcap_{k \in \mathbb{N}} \pi_k(B_k).$$

Remark 4.3.

(1) In fact, one should remove all vertical divisors D_k in the definition of GG, but we omit this point for simplicity.
(2) The definition of the Green–Griffiths locus GG does not depend on the chosen ample line bundle A.

As a consequence of what we have already seen, this locus contains all entire curves.

PROPOSITION 4.4. — *Let X be a smooth projective variety and $f : \mathbb{C} \to X$ an entire curve. Then $f_{[k]}(\mathbb{C}) \subset B_k$ for all k. In particular, $f(\mathbb{C}) \subset GG$.*

Proof. — If there exists k such that $f_{[k]}(\mathbb{C}) \not\subset B_k$, then there exists a divisor D, zero locus of a section of $\mathcal{O}_{X_k}(l) \otimes \pi_k^* A^{-1}$ such that $f_{[k]}(\mathbb{C}) \not\subset D$.

From the Tautological inequality 4.1, $[\Phi_k].\mathcal{O}_{X_k}(l) \leqslant 0$. On the other hand, $[\Phi_k].\mathcal{O}_{X_k}(l) = [\Phi_k].(D + \pi_k^* A) > 0$. □

These results lead to a natural strategy to attack the proof of the Green–Griffiths–Lang conjecture: prove that the Green–Griffiths locus is a proper subvariety.

This was successfully proved in the case of hypersurfaces of high degree in the projective space by Diverio, Merker and Rousseau [12].

THEOREM 4.5. — *Let $X \subset \mathbb{P}^{n+1}$ be a generic hypersurface. If $\deg X \geqslant 2^{n^5}$ then $GG \neq X$. In particular, the strong Green–Griffiths–Lang conjecture is true for X.*

Remark 4.6. — After these notes were written, some important improvements have been obtained on the hyperbolicity of generic complete intersections of large degree in the projective space. Brotbek, Darondeau [4] and independently Xie [29] have proved the ampleness of the cotangent bundle of generic complete intersections of high enough codimension and high enough degree. For hypersurfaces, Brotbek [3] has obtained the hyperbolicity of generic hypersurfaces of high enough degree which has been made effective by Deng [11].

Coming back to the general Green–Griffiths–Lang conjecture, Green and Griffiths proved [16] that for projective surfaces of general type there exists k sufficiently large such that $B_k \neq X_k$.

The generalization to higher dimensions is a recent result of Demailly [9].

THEOREM 4.7. — *If X is a projective variety of general type, then $\mathcal{O}_{X_k}(1)$ is big on X_k for some k sufficiently large. In particular $B_k \neq X_k$. In other words, all non-constant holomorphic curves $f : \mathbb{C} \to X$ into X satisfy a non-trivial differential equation $P(f, f', \ldots, f^{(k)}) \equiv 0$.*

To obtain the Green–Griffiths–Lang conjecture from this statement would require to eliminate the derivatives $f', \ldots, f^{(k)}$ in the equation and show that these curves satisfy non-trivial *algebraic* equations.

This has motivated the following question of Lang [19].

QUESTION 4.8. — *Is $GG \neq X$ for varieties of general type?*

It turns out that foliations are very useful to answer this question as recently shown in [13] and illustrated by the following result.

THEOREM 4.9. — *Let $(X, \mathcal{F})$ be a projective manifold foliated by curves. If $\mathcal{K}_{\mathcal{F}}$ is not big then $GG = X$.*

Proof. — Let $A \to X$ be an ample line bundle and suppose for simplicity that $\mathcal{F}$ is a smooth foliation corresponding to an exact sequence

$$0 \to T_{\mathcal{F}} \to T_X \to \mathcal{N}_{\mathcal{F}} \to 0.$$

We consider the directed manifolds $(X, T_{\mathcal{F}}) \subset (X, T_X)$. The inductive procedure described above starts with $Z_0 = X = X_0$, $Z_1 = \mathbb{P}(T_{\mathcal{F}}) \subset X_1 = \mathbb{P}(T_X)$ and gives k-jets spaces $Z_k \subset X_k$. At each step Z_k is obtained projectivizing a rank 1 vector bundle, so all Z_k are in fact isomorphic to X: $Z_k \simeq X$.

Now, we have the following isomorphisms

$$\mathcal{O}_{X_k}(-1)_{|Z_k} \simeq \mathcal{O}_{Z_k}(-1) \simeq \pi_{0,k}^* T_{\mathcal{F}}.$$

Therefore

$$\mathcal{O}_{X_k}(m) \otimes \pi_{0,k}^* A^{-1}\Big|_{|Z_k} \simeq \pi_{0,k}^* \mathcal{K}_{\mathcal{F}}^m \otimes A^{-1}.$$

The last bundle has no non-zero holomorphic sections by hypothesis. So, $Z_k \subset B_k$ for all k which proves that $GG = X$.

In the general case where $\mathcal{F}$ is not necessarily smooth, one argues as above on the dense open subset U where $\mathcal{F}|_U$ is smooth, which shows that $U \subset GG$. Since GG is a closed set, one obtains again that $GG = X$. □

4.3. Examples

Let us see now several examples where we can prove that the Green–Griffiths locus covers the whole variety.

4.3.1. Products

Let $X = C_1 \times C_2$ be a product of two compact Riemann surfaces of genus $g(C_i) \geqslant 2$. Then X is hyperbolic, of general type. Nevertheless, taking one of the tautological foliation by curves coming from one factor and applying Theorem 4.9, one sees that $GG = X$. This means that hyperbolicity of this product *cannot* be proved by the previously described strategy using jets spaces.

4.3.2. Product-quotients

Let $X = C_1 \times C_2/G$ be a quotient by a finite group. The minimal resolution $S \to X$ is a called a product-quotient surface following [1]. One easily gets that $GG = X$ but if the action is not free, Green–Griffiths–Lang conjecture does not seem to be known in this case.

4.3.3. Hilbert modular surfaces

Let $X = \Delta \times \Delta/\Gamma$ be a smooth compact irreducible surface uniformized by the bi-disc, $\Gamma \subset SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$. Then $GG = X$. Indeed, X has two natural foliations, $\mathcal{F}$, $\mathcal{G}$ such that $\text{kod } K_{\mathcal{F}} = \text{kod } K_{\mathcal{G}} = -\infty$. This can be seen as a consequence of the following lemma.

LEMMA 4.10. — *Let* $\eta = f(z,w)(\mathrm{d}z)^m$ *be a symmetric differential of degree* m *on* $\Delta_z \times \Delta_w$. *Suppose that* η *is* Γ-*invariant. Then* η *vanishes identically.*

Proof. — $(z,w) \to |f(z,w)|(1 - |z|^2)^m$ is a Γ-invariant smooth function thus descends on the compact quotient $X = \Delta \times \Delta/\Gamma$. It attains its maximum at $p \in X$. Let $\{(z_j, w_j)\} \subset \Delta \times \Delta$ be the preimage of p. For each j, the map $z_j \times \Delta_w : w \to f(z_j, w)$ attains a maximum at the interior point w_j. It is constant. Therefore $\frac{\partial f}{\partial w}(z_j, w) = 0$. Γ has dense image in $SL_2(\mathbb{R})$ so $\{z_j\}$ is dense in Δ and $\frac{\partial f}{\partial w} \equiv 0$. η only depens on z: it is a symmetric differential of degree m on Δ_z, invariant by a dense subgroup of $SL_2(\mathbb{R})$. It has to vanish identically: $\eta \equiv 0$. □

Remark 4.11. — If S is a smooth projective surface of general type and $\mathcal{F}$ a (possibly singular, with at most reduced singularities) holomorphic foliation by curves whose canonical bundle $K_{\mathcal{F}}$ is not big then the birational classification of foliations obtained by Brunella and McQuillan (see [5], [22] and [7]) tells us that $\mathcal{F}$ is necessarily of the following two types:

- A Hilbert modular foliation, and thus S is a Hilbert modular surface, if $\kappa(K_{\mathcal{F}}) = -\infty$.
- An isotrivial fibration of genus $\geqslant 2$, if $\kappa(K_{\mathcal{F}}) = 1$.

Remark 4.12. — One may remark that such surfaces $X = \Delta \times \Delta/\Gamma$ satisfy $c_1^2 = 2c_2$ and appear as a limit case of Theorem 3.8 where the technique of jets spaces was enough to conclude under the hypothesis $c_1^2 > 2c_2$.

Remark 4.13. — Generalizations to non-compact quotients Δ^n/Γ of finite volume are discussed in [13].

4.3.4. Siegel modular varieties

This example will differ from the preceding ones since we will not have natural foliations to use. The idea will be to use the existence of totally geodesic polydiscs.

THEOREM 4.14. — *Let $X = \mathbb{H}_g/\Gamma$, where $\mathbb{H}_g$ is the Siegel upper-half space of rank g and $\Gamma \subset \mathrm{Sp}(2g, \mathbb{R})$ commensurable with $\mathrm{Sp}(2g, \mathbb{Z})$, $n \geqslant 2$, then the Green–Griffiths locus $GG(\overline{X}) = \overline{X}$.*

Proof. — There is a totally geodesic polydisk $\Delta^g \hookrightarrow \mathbb{H}_g$,

$$z = (z_1, \ldots, z_g) \to z^* = \mathrm{diag}(z_1, \ldots, z_g)$$

of dimension g consisting of diagonal matrices

$$\{Z = (z_{ij})/z_{ij} = 0 \text{ for } i \neq j\} \subset \mathbb{H}_g.$$

This corresponds to an embedding $SL(2, \mathbb{R})^g \hookrightarrow \mathrm{Sp}(2g, \mathbb{R})$:

$$M = (M_1, \ldots, M_g) \to M^* = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix},$$

where $M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, and $a^* = \mathrm{diag}(a_1, \ldots, a_g)$ is the corresponding diagonal matrix.

More generally, taking $A \in GL(g, \mathbb{R})$ one can consider the map $\Delta^g \hookrightarrow \mathbb{H}_g$, given by

$$z = (z_1, \ldots, z_g) \to A^t z^* A.$$

In order to take quotients, one defines

$$\Gamma_A := \left\{ M \in Sl(2, \mathbb{R})^n / \begin{pmatrix} A^t & 0 \\ 0 & A^{-1} \end{pmatrix} M^* \begin{pmatrix} A^t & 0 \\ 0 & A^{-1} \end{pmatrix}^{-1} \in \Gamma \right\}.$$

Indeed we have a modular embedding

$$\phi_A : \Delta^g/\Gamma_A \to X.$$

Considering a totally real number field $K/\mathbb{Q}$ of degree g with the embedding $K \hookrightarrow \mathbb{R}^n$, $\omega \to (\omega^{(1)}, \ldots, \omega^{(g)})$, the matrices $A = (\omega_i^{(j)})$ where $\omega_1, \ldots, \omega_n$ is a basis of K have the property that Γ_A is commensurable with the Hilbert modular group of K [15]. These matrices A are obviously dense in $GL(g, \mathbb{R})$.

As explained in the preceding examples 4.3.3 and 4.13, we have that $\phi_A(\Delta^g/\Gamma_A) \subset GG(\overline{X})$. By density, we finally get

$$GG(\overline{X}) = \overline{X}. \qquad \square$$

4.4. All or nothing principle

All preceding examples are related to quotients of bounded symmetric domains. In fact, the following Theorem [13] tells us that in this case we have a dichotomy for the Green–Griffiths locus.

THEOREM 4.15. — *Let X be a projective manifold uniformized by a bounded symmetric domain Ω. Then, either $\Omega \simeq \mathbb{B}^n$, T_X^* is ample and thus $GG(X) = \emptyset$, or $GG(X) = X$.*

Let us give the ideas of the proof of this theorem and refer to [13] for details.

As we have seen in the proof of Theorem 3.13 directions given by leaves of the foliation were used to define a subvariety of the projectivized tangent bundle. In the case where we have no foliations to start with, Mok has introduced in [23] a nice tool: the *characteristic bundle* $\mathcal{S} \subset \mathbb{P}(T_X)$. It is defined by taking the directions given by *characteristic vectors* i.e. minimizing the holomorphic sectional curvature.

$\mathcal{S}$ is equipped with a smooth tautological foliation $\mathcal{F}$ whose leaves are lifting of *minimal* discs i.e. tangent to *characteristic* vectors. We have $\mathcal{K}_{\mathcal{F}} = \mathcal{O}(1)_{|S}$.

The crucial property of this foliation reminiscent of the Hilbert modular case is the following one.

PROPOSITION 4.16.
$$\mathrm{kod}(\mathcal{S}, \mathcal{K}_{\mathcal{F}}) = -\infty.$$

Proof. — We proceed by contradiction and suppose that there exists an integer $m > 0$ and a non zero section $\sigma \in H^0\big(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(m)\big)$. By a slight abuse of notation, we still call h the restriction of the natural hermitian metric h on $\mathcal{O}_{P(T_X)}(-1)$ to $\mathcal{O}_{\mathcal{S}}(-1)$. Then, $g := (h^m + \sigma \otimes \overline{\sigma})^{1/m}$ defines a hermitian metric on $\mathcal{O}_{\mathcal{S}}(-1)$ which still has semi–negative curvature, since σ is holomorphic.

In this situation, $g = C \cdot h$ for some positive constant C as explained in [13] following ideas of [23]. It implies that $\|\sigma\|_{h^{-m}}$ is constant. If it were a non zero constant this would imply that σ never vanishes and thus $\mathcal{O}_{\mathcal{S}}(m)$ is holomorphically trivial. But this is impossible since the following inequality holds:
$$\int_{\mathcal{S}} c_1\big(\mathcal{O}_{P(T_X)}(1)\big) \wedge \nu^{\dim \mathcal{S}-1} > 0,$$
where ν is the Kähler form on $P(T_X)$ given by $c_1\big(\mathcal{O}_{P(T_X)}(1), h^{-1}\big) + \pi^*\omega$.

But then $\|\sigma\|_{h^{-m}} \equiv 0$ and σ is identically zero, too. $\qquad\square$

Then Theorem 4.15 is an easy consequence of Theorem 4.9.

4.5. Strong general type

In order to overcome the difficulty described in the preceding section when using the jets spaces techniques, Demailly has very recently introduced in [10] a condition called "*strong general type*" property related to a certain *jet-semistability* property of the tangent bundle.

Let us describe briefly these ideas using the same notations as above for jets spaces and refer to [10] for details. If (X, V) is a directed manifold, we say that it is of general type if K_V is a big line bundle. If $Z \subsetneq X_k$ is an irreducible subvariety, there is a natural structure $W \subset T_Z$ induced by V_k. The linear subspace $W \subset T_Z \subset T_{X_k}|_Z$ is defined to be the closure $W := \overline{T_{Z'} \cap V_k}$, taken on a suitable Zariski open set $Z' \subset Z_{reg}$ where the intersection $T_{Z'} \cap V_k$ has constant rank and is a subbundle of $T_{Z'}$.

We say that (Z, W) is of general type modulo $X_k \to X$ if either $W = 0$ or $\text{rank } W \geqslant 1$ and there exists $p \in \mathbb{Q}^+$ such that $K_W \otimes \mathcal{O}_{X_k}(p)_{|Z}$ is big over Z.

DEFINITION 4.17. — *Let (X, V) be a directed pair where X is projective. We say that (X, V) is strongly of general type if it is of general type and for every irreducible $Z \subsetneq X_k$, $Z \not\subset D_k := \mathbb{P}(T_{X_{k-1}/X_{k-2}})$ that projects onto X, the induced directed structure $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$.*

All examples of Section 4.3 provide examples which are of general type without being strongly of general type. The nice fact about this condition is that it enables using jets spaces to obtain the Green–Griffiths–Lang conjecture as explained in [10].

THEOREM 4.18. — *Let (X, V) be a directed pair that is strongly of general type. Then the Green–Griffiths–Lang conjecture holds true for (X, V) namely there exists a proper algebraic variety $Y \subsetneq X$ such that every non constant holomorphic curve $f : \mathbb{C} \to X$ tangent to V satisfies $f(\mathbb{C}) \subset Y$.*

5. Quotients of bounded symmetric domains

5.1. Hyperbolicity modulo the boundary

Let us illustrate this setting with the classical case of Riemann surfaces.

By the uniformization theorem, compact Riemann surfaces of genus $\geqslant 2$ are quotient of the disc Δ/Γ and therefore hyperbolic.

Now, consider $X = \mathbb{C} \setminus \{a, b\}$. It is again uniformized by the disc but if we take its compactification $\overline{X} = \mathbb{P}^1$, it is obviously non-hyperbolic. So the *compactification* process can change hyperbolic properties.

Finally consider $X = \mathbb{H}/SL_2(\mathbb{Z})$ the moduli space of elliptic curves. It is well-known that $X \simeq \mathbb{C}$. The phenomenon which causes this loss of hyperbolicity is the *torsion*.

As we have recalled in Section 2, every complex variety Y can be equipped with the Kobayashi pseudo-metric K_Y. Its integrated form d_Y is the *Kobayashi pseudo-distance*. It is the largest pseudo-distance such that all holomorphic map $f : (\Delta, d_P) \to (Y, d_Y)$ from the Poincaré disc is distance decreasing: $f^* d_Y \leqslant d_P$.

DEFINITION 5.1. — *Let $W \subset Y$. Y is Kobayashi hyperbolic modulo W if $d_Y(p, q) = 0$ for $p \neq q$ implies that $p, q \in W$.*

Remark 5.2. — One has the obvious implication that if Y is hyperbolic modulo W then any entire curve $f : \mathbb{C} \to Y$ has its image $f(\mathbb{C}) \subset W$: it is *Brody hyperbolic* modulo W. But if $W \neq \emptyset$ the converse analogous to Corollary 2.2 is not known.

The problem we are interested in is the following. Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain, $\Gamma \subset \mathrm{Aut}(\Omega)$ an arithmetic group. $X = \Omega/\Gamma$ a quotient of finite volume. What can be said about the hyperbolicity of $\overline{X}$?

The following result is recently obtained in [26].

THEOREM 5.3. — *There exists a subgroup $\Gamma_1 \subset \Gamma$ of finite index such that $\overline{X_1} := \overline{\Omega/\Gamma_1}$ is Kobayashi hyperbolic modulo $D_1 := \overline{X_1} \setminus X_1$.*

Example 5.4. — Let $X = \mathbb{H}/SL_2(\mathbb{Z})$ and $\Gamma_n := \ker(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/n\mathbb{Z}))$. Then $\overline{X_n} := \overline{\mathbb{H}/\Gamma_n}$ has genus $\geqslant 2$ if $n \geqslant 7$ [27].

Remark 5.5. — The preceding theorem is a generalization of a result of Nadel [25] who gives Brody hyperbolicity in the same setting.

Let us give the ideas of the proof of Theorem 5.3 refering to [26] for details. In order to control the degeneracy of the Kobayashi pseudo-distance, we will construct pseudo-distances on $\overline{X} = \overline{\Omega/\Gamma}$ satisfying the distance-decreasing property.

There is a natural metric to start with: the Bergman metric g on Ω normalized so that $\mathrm{Ricci}\, g = -g$. Its holomorphic sectional curvature is known to be negative $\leqslant -\gamma$ $(\gamma \in \mathbb{Q}^+)$.

From [24], we know that g induces a *good* singular metric $h := (\det g)^{-1}$ on $K_{\overline{X}} + D$. Two facts will be particularly useful for us.

(1) g has Poincaré growth near $D = \overline{X} \setminus X$. In local coordinates, if $D = \{z_1 \ldots z_k = 0\}$, $g = O(g_P)$ where g_P is the Poincaré metric

$$g_P = \sum_{i=1}^{k} \frac{\mathrm{d}z_i \otimes \mathrm{d}\overline{z_i}}{|z_i|^2 \log^2 |z_i|^2} + \sum_{i=k+1}^{n} \mathrm{d}z_i \otimes \mathrm{d}\overline{z_i}.$$

(2) if s is a section of $K_{\overline{X}} + D$, then $\|s\|_h = O(\log^{2N} |z_1 \ldots z_k|)$ (N is an integer > 0).

The idea now is to construct a function $\psi : \overline{X} \to \mathbb{R}^+$ such that $\psi.g$ will extend and define a pseudo-metric on $\overline{X}$.

Such a function will be obtained thanks to the following proposition.

PROPOSITION 5.6. — *Let $s \in H^0(\overline{X}, l(K_{\overline{X}} + D))$ be such that*

- $s(x) \neq 0$.
- *s vanishes on D with multiplicity $m > \frac{l}{\gamma}$.*

Take $\psi := \|s\|_h^{2 \frac{(\gamma - \epsilon)}{l}}$, $\gamma - \epsilon > \frac{l}{m}$. Then there exists $\beta > 0$ such that $\tilde{g} := \beta.\psi.g$ is a distance decreasing pseudo-metric on $\overline{X}$. In particular, the Kobayashi pseudo-distance is bounded below:

$$d_{\widetilde{g}} \leqslant d_{\overline{X}}.$$

Proof. — The two facts described above easily show that $\psi.g$ extends as a pseudo-metric on $\overline{X}$. Now, we have to achieve the distance decreasing property. Let $f : \Delta \to \overline{X}$ be a holomorphic map.

$$i\partial\overline{\partial} \log f^* \widetilde{g} \geqslant (\gamma - \epsilon) f^* \operatorname{Ricci} g + \gamma f^* g$$

$$\geqslant -(\gamma - \epsilon) f^* g + \gamma f^* g = \epsilon f^* g \geqslant \left(\frac{\epsilon}{\sup \psi}\right).f^* \psi g.$$

Therefore one can choose $\beta := \frac{\epsilon}{\sup \psi}$. $\qquad \square$

As we have just seen, one is reduced to find sections of $l(K_{\overline{X}} + D)$ with the desired properties. This is achieved thanks to the following result of Kollár [18].

THEOREM 5.7. — *Let Z be a smooth projective variety of dimension n, L nef and big over Z. Let $x \in Z$ be a point such that any irreducible subvariety containing x has positive degree with respect to L. Then for any $m > \binom{n+1}{2}$, there exists $s \in H^0(Z, K_Z + mL)$ such that $s(x) \neq 0$.*

One can apply this theorem with $Z = \overline{X}$, $L = K_{\overline{X}} + D$ and $x \in X$. Then there exists $s_0 \in H^0(\overline{X}, K_{\overline{X}} + (\binom{n+1}{2} + 1)(K_{\overline{X}} + D)) = H^0(\overline{X}, (\binom{n+1}{2} + 2) \times (K_{\overline{X}} + D) - D)$.

So s_0 is a section of $(\binom{n+1}{2} + 2)(K_{\overline{X}} + D)$ vanishing on D such that $s(x) \neq 0$. Now, according to Mumford [24], for any positive integer m there exists $\Gamma_1 \subset \Gamma$ such that the induced map $\pi : \overline{X_1} \to \overline{X}$ is ramified at order $\geqslant m$ along D_1. To finish the proof one can take $m > \frac{(\binom{n+1}{2} + 2)}{\gamma}$, and $s := \pi^* s_0 \in H^0(\overline{X_1}, (\binom{n+1}{2} + 2)(K_{\overline{X_1}} + D) - mD)$.

5.2. Siegel modular varieties

As an application of the preceding method, let us take $\Omega = \mathbb{H}_g$ the Siegel upper-half space, $\Gamma = \mathrm{Sp}(2g, \mathbb{Z})$, and $\Gamma(n) := \ker(\mathrm{Sp}(2g, \mathbb{Z}) \to \mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z}))$. One usually denotes $\mathcal{A}_g(n) := \mathbb{H}_g/\Gamma(n)$ and we can take $\gamma = \frac{2}{g(g+1)}$.

In this case, one obtains (see [26] for details):

THEOREM 5.8. — $\mathcal{A}_g(n)$ *is Kobayashi hyperbolic modulo D if $n > 6g$.*

Remark 5.9. — In the case $g = 1$, one recovers $n \geqslant 7$ as in Example 5.4.

Remark 5.10. — This improves the bound obtained by Nadel [25] for Brody hyperbolicity $n \geqslant \max(\frac{g(g+1)}{2}, 28)$.

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