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Harmonic functions on multiplicative graphs and inverse Pitman transform on infinite random paths (*)

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Abstract. — This survey establishes some miscellaneous results on random Littelmann paths and generalized Pitman transform. We describe central probability distributions on Littelmann paths. Next we state a law of large numbers and a central limit theorem for the generalized Pitman transform. We then study harmonic functions on multiplicative graphs defined from the tensor powers of finite-dimensional Lie algebras representations. Finally, we explain there exists an inverse of the generalized Pitman transform defined almost surely on the set of infinite paths remaining in the Weyl chamber and how it can be computed.

Résumé. — Dans cet article de synthèse nous établissons des résultats complémentaires sur les chemins de Littelmann aléatoires et sur la transformée de Pitman généralisée. Nous décrivons les distributions de probabilité centrales sur les chemins de Littelmann. Ensuite nous donnons une loi des grands nombres et un théorème central limite pour la transformée de Pitman généralisée. Nous étudions alors les fonctions harmoniques sur les graphes multiplicatifs définis à partir des puissances tensorielles des représentations irréductibles des algèbres de Lie. Enfin, nous expliquons qu'il existe une transformée inverse de la transformée de Pitman généralisée définie presque sûrement sur les trajectoires infinies qui restent dans la chambre de Weyl et montrons comment elle peut être calculée.

1. Introduction

The goal of this survey is to state some results on the generalized Pitman transform $\mathcal{P}$ introduced by Biane, Bougerol and O'Connell [1] and also on

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harmonic functions introduced in [7] and [9]. Some of them are new, other appear implicitly in the literature but we thought it is useful to properly write them down for a non specialized audience and also for possible future references. The harmonic functions we are interested in appear in the study of the random Littelmann path defined from a simple module V of a Kac–Moody algebra $\mathfrak{g}$ and its conditioning to stay in the dominant Weyl chamber of $\mathfrak{g}$. Roughly speaking, the random path we are interested in is obtained by concatenation of elementary paths randomly chosen among the vertices of the crystal graph B associated to V following a distribution depending on the graph structure of B. It is worth noticing that for $\mathfrak{g} = \mathfrak{sl}_2$, this random path reduces to the random walk on $\mathbb{Z}$ with steps $\{\pm 1\}$ and the transform $\mathcal{P}$ is the usual Pitman transform [15]. Also when V is the defining representation of $\mathfrak{g} = \mathfrak{sl}_{n+1}$, the vertices of B are simply the paths linking 0 to each vector of the standard basis of $\mathbb{R}^{n+1}$ and we notably recover some results by O'Connell exposed in [14]. It appears that many natural random walks can in fact be realized from a suitable choice of the representation V.

We will assume here that $\mathfrak{g}$ is a simple (finite-dimensional) Lie algebra over $\mathbb{C}$ of rank n. The irreducible finite-dimensional representations of $\mathfrak{g}$ are then parametrized by the dominant weights of $\mathfrak{g}$ which are the elements of the set $P_+ = P \cap \mathcal{C}$ where P and $\mathcal{C}$ are the weight lattice and the dominant Weyl chamber of $\mathfrak{g}$, respectively. The random path $\mathcal{W}$ we considered in [9] is defined from the crystal $B(\kappa)$ of the irreducible $\mathfrak{g}$-module $V(\kappa)$ with highest weight $\kappa \in P_+$ (κ is fixed for each $\mathcal{W}$). The crystal $B(\kappa)$ is an oriented graph graded by the weights of $\mathfrak{g}$ whose vertices are Littelmann paths of length 1. The vertices and the arrows of $B(\kappa)$ are obtained by simple combinatorial rules from a path π_κ connecting 0 to κ and remaining in $\mathcal{C}$ (the highest weight path). We endowed $B(\kappa)$ with a probability distribution p compatible with the weight graduation defined from the choice of an n-tuple τ of positive reals (a positive real for each simple root of $\mathfrak{g}$). The probability distribution considered on the successive tensor powers $B(\kappa)^{\otimes \ell}$ is the product distribution $p^{\otimes \ell}$. It has the crucial property to be central: two paths in $B(\kappa)^{\otimes \ell}$ with the same ends have the same probability. We can then define, following the classical construction of a Bernoulli process, a random path $\mathcal{W}$ with underlying probability space $(B(\kappa)^{\otimes \mathbb{Z}_{\geqslant 0}}, p^{\otimes \mathbb{Z}_{\geqslant 0}})$ as the direct limit of the spaces $(B(\kappa)^{\otimes \ell}, p^{\otimes \ell})$. The trajectories of $\mathcal{W}$ are the concatenations of the Littelmann paths appearing in $B(\kappa)$. It makes sense to consider the image of $\mathcal{W}$ by the generalized Pitman transform $\mathcal{P}$. This yields a Markov process $\mathcal{H} = \mathcal{P}(\mathcal{W})$ whose trajectories are the concatenations of the paths appearing in $B(\kappa)$ which remain in the dominant Weyl chamber $\mathcal{C}$. When the drift of $\mathcal{W}$ belongs to the interior of $\mathcal{C}$, we establish in [9] that the law of $\mathcal{H}$ coincides with the law of $\mathcal{W}$ conditioned to stay in $\mathcal{C}$. By setting $W_\ell = \mathcal{W}(\ell)$ for any positive integer ℓ, we obtain in particular a Markov chain $W = (W_\ell)_{\ell \geqslant 1}$ on the dominant weights of $\mathfrak{g}$.

In the spirit of the works of Kerov and Vershik, one can define central probability measures on the space $\Omega_{\mathcal{C}}$ of infinite trajectories associated to $\mathcal{H}$ (i.e. remaining in $\mathcal{P}$). These are the probability measures giving the same probability to any cylinders C_π and $C_{\pi'}$ issued from paths π and π' of length ℓ remaining in $\mathcal{C}$ with the same ends. Alternatively, we can consider the multiplicative graph $\mathcal{G}$ with vertices $(\lambda, \ell) \in P_+ \times \mathbb{Z}_{\geqslant 0}$ and weighted arrows $(\lambda, \ell) \overset{m^\Lambda_{\lambda,\kappa}}{\to} (\Lambda, \ell + 1)$ where $m^\Lambda_{\lambda,\kappa}$ is the multiplicity of the representation $V(\Lambda)$ in the tensor product $V(\lambda) \otimes V(\kappa)$. Each central probability measure on $\Omega_{\mathcal{C}}$ is then characterized by the harmonic function φ on $\mathcal{G}$ associating with each vertex (λ, ℓ), the probability of any cylinder C_π where π is any path of length ℓ remaining in $\mathcal{C}$ and ending at λ. Finally, a third equivalent way to study central probability measures on $\Omega_{\mathcal{C}}$ is to define a Markov chain on $\mathcal{G}$ whose transition matrix is computed from the harmonic function φ. We refer to Section 6.1 for a detailed review.

When $\mathfrak{g} = \mathfrak{sl}_{n+1}$, the elements of P_+ can be regarded as the partitions $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0) \in \mathbb{Z}^n$. Moreover, if we choose $V(\kappa) = V$, the defining representation of $\mathfrak{g} = \mathfrak{sl}_{n+1}$, we have $m^\Lambda_{\lambda,\kappa} \neq 0$ if and only if the Young diagram of Λ is obtained by adding one box to that of λ. The connected component of $\mathcal{G}$ obtained from $(\emptyset, 0)$ thus coincides with the Young lattice $\mathcal{Y}_n$ of partitions with at most n parts (one can obtain the whole Young lattice $\mathcal{Y}$ by working with $\mathfrak{g} = \mathfrak{sl}_\infty$). In that case, Kerov and Vershik (see [6]) completely determined the harmonic function on $\mathcal{Y}$. They showed that these harmonic functions have nice expressions in terms of generalized Schur functions.

In [15] Pitman established that the usual (one-dimensional) Pitman transform is almost surely invertible on infinite trajectories (i.e. reversible on a space of trajectories of probability 1). It is then a natural question to ask wether its generalized version $\mathcal{P}$ shares the same invertibility property. Observe that in the case of the defining representation of $\mathfrak{sl}_{n+1}$ (or $\mathfrak{sl}_\infty$), the generalized Pitman transform can be expressed in terms of a Robinson–Schensted (RS) type correspondence. Such an invertibility property was obtained by O'Connell in [14] (for the usual RS correspondence related to ordinary Schur functions) and recently extended by Sniady [16] (for the generalized version of RS correspondence used by Kerov and Vershik and related to the generalized Schur functions). We show that this invertibility property (implicit in fact in [1]) survives beyond type A and for random paths constructed from any irreducible representation.

In what follows, we first prove that the probability distributions p on $B(\kappa)$ we introduced in [7, 8, 9] are precisely all the possible distributions yielding central distributions on $B(\kappa)^{\otimes \ell}$. We also establish a law of large numbers and a central limit theorem for the Markov process $\mathcal{H}$. Here we need our assumption that $\mathfrak{g}$ is finite-dimensional since in this case $\mathcal{P}$ has a particular simple expression as a composition of (ordinary) Pitman transforms. Then we determine the harmonic functions on the multiplicative graph $\mathcal{G}$ for which the associated Markov chain satisfies a law of large numbers. We establish in fact that these Markov chains are exactly the processes H defined in [7] and have simple expressions in terms of the Weyl characters of $\mathfrak{g}$. This can be regarded as an analogue of the result of Kerov and Vershik determining the harmonic functions on the Young lattice. Finally, we prove that the generalized Pitman transform $\mathcal{P}$ is almost surely invertible and explain how its inverse can be computed.

The survey is organized as follows. In Section 2, we recall some background on continuous time Markov processes. Section 3 is a recollection of results on representation theory of Lie algebras and the Littelmann path model. We state in Section 4 the main results of [9] and prove that the probability distributions p introduced in [7] are in fact the only possible yielding central measures on trajectories. The law of large numbers and the central limit theorem for $\mathcal{H}$ are established in Section 5. We study the harmonic functions of the graphs $\mathcal{G}$ in Section 6. In Section 7 we show that the spaces of trajectories for $\mathcal{W}$ and $\mathcal{H}$ both have the structure of dynamical systems coming from the shift operation. We then prove that these dynamical systems are intertwined by $\mathcal{P}$. Finally, we study the inverse of $\mathcal{P}$ in Section 8.

2. Random paths

2.1. Background on Markov chains

Consider a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and a countable set M. A sequence $Y = (Y_\ell)_{\ell \geqslant 0}$ of random variables defined on Ω with values in M is a *Markov chain* when

$$\mathsf{P}(Y_{\ell+1} = \mu_{\ell+1} \mid Y_\ell = \mu_\ell, \ldots, Y_0 = \mu_0) = \mathsf{P}(Y_{\ell+1} = \mu_{\ell+1} \mid Y_\ell = \mu_\ell)$$

for any $\ell \geqslant 0$ and any $\mu_0, \ldots, \mu_\ell, \mu_{\ell+1} \in M$. The Markov chains considered in the sequel will also be assumed time homogeneous, that is $\mathsf{P}(Y_{\ell+1} = \lambda \mid Y_\ell = \mu) = \mathsf{P}(Y_\ell = \lambda \mid Y_{\ell-1} = \mu)$ for any $\ell \geqslant 1$ and $\mu, \lambda \in M$. For all μ, λ in M, the transition probability from μ to λ is then defined by

$$\Pi(\mu, \lambda) = \mathsf{P}(Y_{\ell+1} = \lambda \mid Y_\ell = \mu)$$

and we refer to Π as the transition matrix of the Markov chain Y. The distribution of Y_0 is called the initial distribution of the chain Y.

A *continuous time Markov process* $\mathcal{Y} = (\mathcal{Y}(t))_{t\geqslant 0}$ on $(\Omega, \mathcal{F}, \mathsf{P})$ with values in $\mathbb{R}^n$ is a measurable family of random variables defined on $(\Omega, \mathcal{F}, \mathsf{P})$ such that, for any integer $k \geqslant 1$ and any $0 \leqslant t_1 < \cdots < t_{k+1}$ the conditional distribution[1] of $\mathcal{Y}(t_{k+1})$ given $(\mathcal{Y}(t_1), \cdots, \mathcal{Y}(t_k))$ is equal to the conditional distribution of $\mathcal{Y}(t_{k+1})$ given $\mathcal{Y}(t_k)$; in other words, for almost all $(y_1, \cdots, y_k)$ with respect to the distribution of the random vector $(\mathcal{Y}(t_1), \cdots, \mathcal{Y}(t_k))$ and for all Borelian set $B \subset \mathbb{R}^n$

$$\begin{aligned}\mathsf{P}(\mathcal{Y}(t_{k+1}) \in B \mid \mathcal{Y}(t_1) = y_1, \cdots, \mathcal{Y}(t_k) = y_k)& \\ = \mathsf{P}(\mathcal{Y}(t_{k+1}) \in B \mid \mathcal{Y}(t_k) = y_k)&\,.\end{aligned}$$

We refer to the book [3, Chapter 3], for a description of such processes.

From now on, we consider a $\mathbb{R}^n$-valued Markov process $(\mathcal{Y}(t))_{t\geqslant 0}$ defined on $(\Omega, \mathcal{F}, \mathsf{P})$ and we assume the following conditions:

(1) $M \subset \mathbb{R}^n$

(2) for any integer $\ell \geqslant 0$

$$Y_\ell := \mathcal{Y}(\ell) \in M \qquad \mathsf{P}-\text{almost surely}\,. \tag{2.1}$$

It readily follows that the sequence $Y = (Y_\ell)_{\ell\geqslant 0}$ is a M-valued Markov chain.

(3) for any integer $\ell \geqslant 0$, the conditional distribution of $(\mathcal{Y}(t))_{t\geqslant \ell}$ given Y_ℓ is equal to the one of $(\mathcal{Y}(t))_{t\geqslant 0}$ given Y_0; in other words, for any Borel set $B \subset (\mathbb{R}^n)^{\otimes[0,+\infty[}$ and any $\lambda \in M$, one gets

$$\mathsf{P}((\mathcal{Y}(t))_{t\geqslant \ell} \in B \mid Y_\ell = \lambda) = \mathsf{P}((\mathcal{Y}(t))_{t\geqslant 0} \in B \mid Y_0 = \lambda)\,.$$

In the following, we will assume that the initial distribution of the Markov process $(\mathcal{Y}(t))_{t\geqslant 0}$ has full support, i.e. $\mathsf{P}(\mathcal{Y}(0) = \lambda) > 0$ for any $\lambda \in M$.

2.2. Elementary random paths

Consider a $\mathbb{Z}$-lattice $P \subset \mathbb{R}^n$ of rank n. An *elementary Littelmann path* is a piecewise continuous linear map $\pi : [0, 1] \to P_\mathbb{R}$ such that $\pi(0) = 0$ and

[1] Let us recall briefly the definition of the conditional distribution of a random variable given another one. Let X and Y be random variables defined on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with values respectively in $\mathbb{R}^n$ and $\mathbb{R}^m$, $n, m \geqslant 1$. Denote by μ_X the distribution of X, it is a probability measure on $\mathbb{R}^n$. The conditional distribution of Y given X is defined by the following "disintegration" formula: for any Borelian sets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$

$$\mathsf{P}\Big((X \in A) \cap (Y \in B)\Big) = \int_A \mathsf{P}(Y \in B \mid X = x)\,\mathrm{d}\mu_X(x)\,.$$

Notice that the function $x \mapsto \mathsf{P}(Y \in B \mid X = x)$ is a Radon–Nikodym derivative with respect to μ_X and is thus just defined modulo the measure μ_X. The measure $B \mapsto \mathsf{P}(Y \in B \mid X = x)$ is called the *conditional distribution of* Y *given* $X = x$.

$\pi(1) \in P$. Two paths which coincide up to reparametrization are considered as identical.

The set $\mathcal{F}$ of continuous functions from $[0,1]$ to $\mathbb{R}^n$ is equipped with the norm $\|\cdot\|_\infty$ of uniform convergence : for any $\pi \in \mathcal{F}$, one has $\|\pi\|_\infty := \sup_{t\in[0,1]} \|\pi(t)\|$ where $\|\cdot\|$ denotes the Euclidean norm on $P \subset \mathbb{R}^n$. Let B be a *finite set of elementary paths* and fix a probability distribution $p = (p_\pi)_{\pi \in B}$ on B such that $p_\pi > 0$ for any $\pi \in B$. Let X be a random variable with values in B defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and with distribution p (in other words $\mathsf{P}(X = \pi) = p_\pi$ for any $\pi \in B$). The variable X admits a moment of order 1 defined by

$$m := \mathbb{E}(X) = \sum_{\pi \in B} p_\pi \pi \,.$$

The concatenation $\pi_1 * \pi_2$ of two elementary paths π_1 and π_2 is defined by

$$\pi_1 * \pi_2(t) = \begin{cases} \pi_1(2t) & \text{for } t \in [0, \frac{1}{2}]\,, \\ \pi_1(1) + \pi_2(2t-1) & \text{for } t \in [\frac{1}{2}, 1]\,. \end{cases}$$

In the sequel, $\mathcal{C}$ is a closed convex cone in $P \subset \mathbb{R}^n$.

Let B be a set of elementary paths and $(X_\ell)_{\ell \geqslant 1}$ a sequence of i.i.d. random variables with the same law as X where X is the random variable with values in B introduced just above. We define a random process $\mathcal{W}$ as follows: for any $\ell \in \mathbb{Z}_{>0}$ and $t \in [\ell, \ell+1]$,

$$\mathcal{W}(t) := X_1(1) + X_2(1) + \cdots + X_{\ell-1}(1) + X_\ell(t-\ell)\,.$$

The sequence of random variables $W = (W_\ell)_{\ell \in \mathbb{Z}_{\geqslant 0}} := (\mathcal{W}(\ell))_{\ell \geqslant 0}$ is a random walk with set of increments $I := \{\pi(1) \mid \pi \in B\}$.

3. Littelmann paths

3.1. Background on representation theory of Lie algebras

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$ of rank n and $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$ a triangular decomposition. We shall follow the notation and convention of [2]. According to the Cartan–Killing classification, $\mathfrak{g}$ is characterized (up to isomorphism) by its root system R. This root system is determined by the previous triangular decomposition and realized in the Euclidean space $\mathbb{R}^n$. We denote by $\Delta_+ = \{\alpha_i \mid i \in I\}$ the set of simple roots of $\mathfrak{g}$, by R_+ the (finite) set of positive roots. We then have $n = \mathrm{card}(\Delta_+)$ and $R = R_+ \cup R_-$ with $R_- = -R_+$. The root lattice of $\mathfrak{g}$ is the integral lattice $Q = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$. Write $\omega_i, i = 1, \ldots, n$ for the fundamental weights associated with $\mathfrak{g}$. The weight lattice associated with $\mathfrak{g}$ is the integral lattice $P = \bigoplus_{i=1}^{n} \mathbb{Z}\omega_i$. It can be regarded as an integral sublattice of $\mathfrak{h}_{\mathbb{R}}^*$ (the real form of the dual $\mathfrak{h}^*$ of $\mathfrak{h}$). We have $\dim(P) = \dim(Q) = n$ and $Q \subset P$.

The cone of dominant weights for $\mathfrak{g}$ is obtained by considering the positive integral linear combinations of the fundamental weights, that is $P_+ = \bigoplus_{i=1}^{n} \mathbb{Z}_{\geqslant 0}\omega_i$. The corresponding open Weyl chamber is the cone $\mathring{\mathcal{C}} = \bigoplus_{i=1}^{n} \mathbb{R}_{>0}\omega_i$. We also introduce its closure $\mathcal{C} = \bigoplus_{i=1}^{n} \mathbb{R}_{\geqslant 0}\omega_i$. In type A, we shall use the weight lattice of $\mathfrak{gl}_n$ rather than that of $\mathfrak{sl}_n$ for simplicity. We also introduce the Weyl group W of $\mathfrak{g}$ which is the group generated by the orthogonal reflections s_i in the hyperplanes perpendicular to the simple roots $\alpha_i, i = 1, \ldots, n$. Each $w \in \mathsf{W}$ may be decomposed as a product of the $s_i, i = 1, \ldots, n$. All the minimal length decompositions of w have the same length $l(w)$. The group W contains a unique element w_0 of maximal length $l(w_0)$ equal to the number of positive roots of $\mathfrak{g}$, this w_0 is an involution and if $s_{i_1} \cdots s_{i_r}$ is a minimal length decomposition of w_0, we have

$$R_+ = \{\alpha_{i_1}, s_{i_1} \cdots s_{i_a}(\alpha_{i_{a+1}}) \text{ with } a = 1, \ldots, r-1\}. \tag{3.1}$$

Example 3.1. — The root system of $\mathfrak{g} = \mathfrak{sp}_4$ has rank 2. In the standard basis (e_1, e_2) of the Euclidean space $\mathbb{R}^2$, we have $\omega_1 = (1,0)$ and $\omega_2 = (1,1)$. So $P = \mathbb{Z}^2$ and $\mathcal{C} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geqslant x_2 \geqslant 0\}$. The simple roots are $\alpha_1 = e_1 - e_2$ and $\alpha_2 = 2e_2$. We also have $R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$. The Weyl group W is the dihedral group with 8 elements. It acts on $\mathbb{R}^2$ by permuting the coordinates of the vectors and flipping their sign. More precisely, for any $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$, we have $s_1(\beta) = (\beta_2, \beta_1)$ and $s_2(\beta) = (\beta_1, -\beta_2)$. The longest element is $w_0 = -\mathrm{id} = s_1 s_2 s_1 s_2$. One easily verifies we indeed have

$$R_+ = \{\alpha_1, s_1 s_2 s_1(\alpha_2) = \alpha_2, s_1 s_2(\alpha_1) = \alpha_1 + \alpha_2, s_1(\alpha_2) = 2\alpha_1 + \alpha_2\}.$$

We now summarize some properties of the action of W on the weight lattice P. For any weight β, the orbit $\mathsf{W} \cdot \beta$ of β under the action of W intersects P_+ in a unique point. We define a partial order on P by setting $\mu \leqslant \lambda$ if $\lambda - \mu$ belongs to $Q_+ = \bigoplus_{i=1}^{n} \mathbb{Z}_{\geqslant 0}\alpha_i$.

Let $U(\mathfrak{g})$ be the enveloping algebra associated to $\mathfrak{g}$. Each finite dimensional $\mathfrak{g}$ (or $U(\mathfrak{g})$)-module M admits a decomposition in weight spaces $M = \bigoplus_{\mu \in P} M_\mu$ where

$$M_\mu := \{v \in M \mid h(v) = \mu(h)v \text{ for any } h \in \mathfrak{h} \text{ and some } \mu(h) \in \mathbb{C}\}.$$

This means that the action of any $h \in \mathfrak{h}$ on the weight space M_μ is diagonal with eigenvalue $\mu(h)$. In particular, $(M \oplus M')_\mu = M_\mu \oplus M'_\mu$. The Weyl group W acts on the weights of M and for any $\sigma \in \mathsf{W}$, we have $\dim M_\mu = \dim M_{\sigma \cdot \mu}$. For any $\gamma \in P$, let e^γ be the generator of the group algebra $\mathbb{C}[P]$ associated with γ. By definition, we have $e^{\gamma}e^{\gamma'} = e^{\gamma+\gamma'}$ for any $\gamma, \gamma' \in P$ and the group W acts on $\mathbb{C}[P]$ as follows: $w(e^{\gamma}) = e^{w(\gamma)}$ for any $w \in \mathsf{W}$ and any $\gamma \in P$.

The character of M is the Laurent polynomial $\mathrm{char}(M) := \sum_{\mu\in P} \dim(M_\mu) e^{\mu}$ in $\mathbb{C}[P]$ where $\dim(M_\mu)$ is the dimension of the weight space M_μ.

The irreducible finite dimensional representations of $\mathfrak{g}$ are labelled by the dominant weights. For each dominant weight $\lambda \in P_+$, let $V(\lambda)$ be the irreducible representation of $\mathfrak{g}$ associated with λ. The category $\mathcal{C}$ of finite dimensional representations of $\mathfrak{g}$ over $\mathbb{C}$ is semisimple: each module decomposes into irreducible components. The category $\mathcal{C}$ is equivalent to the (semisimple) category of finite dimensional $U(\mathfrak{g})$-modules (over $\mathbb{C}$). Roughly speaking, this means that the representation theory of $\mathfrak{g}$ is essentially identical to the representation theory of the *associative* algebra $U(\mathfrak{g})$. Any finite dimensional $U(\mathfrak{g})$-module M decomposes as a direct sum of irreducibles $M = \bigoplus_{\lambda\in P_+} V(\lambda)^{\oplus m_{M,\lambda}}$ where $m_{M,\lambda}$ is the multiplicity of $V(\lambda)$ in M. Here we slightly abuse the notation and also denote by $V(\lambda)$ the irreducible finite dimensional $U(\mathfrak{g})$-module associated with λ.

When $M = V(\lambda)$ is irreducible, we set $s_\lambda := \mathrm{char}(M) = \sum_{\mu\in P} K_{\lambda,\mu} e^{\mu}$ with $\dim(M_\mu) = K_{\lambda,\mu}$. Then $K_{\lambda,\mu} \neq 0$ only if $\mu \leqslant \lambda$. Recall also that the characters can be computed from the Weyl character formula but we do not need this approach in the sequel.

Given κ, μ in P_+ and a nonnegative integer ℓ, we define the tensor product multiplicities $f^{\ell}_{\lambda/\mu,\kappa}$ by

$$V(\mu) \otimes V(\kappa)^{\otimes \ell} \simeq \bigoplus_{\lambda\in P_+} V(\lambda)^{\oplus f^{\ell}_{\lambda/\mu,\kappa}}. \tag{3.2}$$

For $\mu = 0$, we set $f^{\ell}_{\lambda,\kappa} = f^{\ell}_{\lambda/0,\kappa}$. When there is no risk of confusion, we write simply $f^{\ell}_{\lambda/\mu}$ (resp. f^{ℓ}_{λ}) instead of $f^{\ell}_{\lambda/\mu,\kappa}$ (resp. $f^{\ell}_{\lambda,\kappa}$). We also define the multiplicities $m^{\lambda}_{\mu,\kappa}$ by

$$V(\mu) \otimes V(\kappa) \simeq \bigoplus_{\mu\rightsquigarrow\lambda} V(\lambda)^{\oplus m^{\lambda}_{\mu,\kappa}} \tag{3.3}$$

where the notation $\mu \rightsquigarrow \lambda$ means that $\lambda \in P_+$ and $V(\lambda)$ appears as an irreducible component of $V(\mu)\otimes V(\kappa)$. We have in particular $m^{\lambda}_{\mu,\kappa} = f^{1}_{\lambda/\mu,\kappa}$.

3.2. Littelmann path model

We now give a brief overview of the Littelmann path model. We refer to [4, 11, 12, 13] for examples and a detailed exposition. Consider a Lie algebra $\mathfrak{g}$ and its root system realized in the Euclidean space $P_{\mathbb{R}} = \mathbb{R}^n$. We fix a scalar product $\langle \cdot , \cdot \rangle$ on $P_{\mathbb{R}}$ invariant under W. For any root α, we set $\alpha^{\vee} = \frac{2\alpha}{\langle \alpha,\alpha \rangle}$. We define the notion of elementary continuous piecewise linear paths in $P_{\mathbb{R}}$ as we did in §2.2. Let $\mathcal{L}$ be the set of elementary paths η having *only rational turning points* (i.e. whose inflexion points have rational coordinates) and ending in P i.e. such that $\eta(1) \in P$. We then define the weight of the path η by $\mathrm{wt}(\eta) = \eta(1)$. Given any path $\eta \in \mathcal{L}$, we define its reverse path $r(\eta) \in \mathcal{L}$ by

$$r(\eta)(t) = \eta(1-t) - \eta(1)\,.$$

Observe the map r is an involution on $\mathcal{L}$. Littelmann associated to each simple root $\alpha_i, i = 1, \dots, n$, some root operators $\tilde{e}_i$ and $\tilde{f}_i$ acting on $\mathcal{L} \cup \{\mathbf{0}\}$. We do not need their complete definition in the sequel and refer to the above mentioned papers for a complete review. Recall nevertheless that roots operators $\tilde{e}_i$ and $\tilde{f}_i$ essentially act on a path η by applying the symmetry s_α on parts of η and we have

$$\tilde{f}_i(\eta) = r\tilde{e}_i r(\eta)\,. \tag{3.4}$$

These operators therefore preserve the length of the paths since the elements of W are isometries. Also if $\tilde{f}_i(\eta) = \eta' \neq \mathbf{0}$, we have

$$\tilde{e}_i(\eta') = \eta \text{ and } \mathrm{wt}(\tilde{f}_i(\eta)) = \mathrm{wt}(\eta) - \alpha_i\,. \tag{3.5}$$

By drawing an arrow $\eta \overset{i}{\to} \eta'$ between the two paths η, η' of $\mathcal{L}$ as soon as $\tilde{f}_i(\eta) = \eta'$ (or equivalently $\eta = \tilde{e}_i(\eta')$), we obtain a Kashiwara crystal graph with set of vertices $\mathcal{L}$. By abuse of notation, we yet denote it by $\mathcal{L}$ which so becomes a colored oriented graph. For any $\eta \in \mathcal{L}$, we denote by $B(\eta)$ the connected component of η i.e. the subgraph of $\mathcal{L}$ generated by η by applying operators $\tilde{e}_i$ and $\tilde{f}_i$, $i = 1, \dots, n$. For any path $\eta \in \mathcal{L}$ and $i = 1, \dots, n$, set $\varepsilon_i(\eta) = \max\{k \in \mathbb{Z}_{\geqslant 0} \mid \tilde{e}_i^k(\eta) = \mathbf{0}\}$ and $\varphi_i(\eta) = \max\{k \in \mathbb{Z}_{\geqslant 0} \mid \tilde{f}_i^k(\eta) = \mathbf{0}\}$.

The set $\mathcal{L}_{\min \mathbb{Z}}$ of *integral paths is the set of* paths η such that $m_\eta(i) = \min_{t \in [0,1]}\{\langle \eta(t), \alpha_i^{\vee} \rangle\}$ belongs to $\mathbb{Z}$ for any $i = 1, \dots, n$. We also recall that we have

$$\mathcal{C} = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle x, \alpha_i^{\vee} \rangle \geqslant 0\} \text{ and } \mathring{\mathcal{C}} = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle x, \alpha_i^{\vee} \rangle > 0\}\,.$$

Any path η such that $\mathrm{Im}\,\eta \subset \mathcal{C}$ satisfies $m_\eta(i) = 0$ so belongs to $\mathcal{L}_{\min \mathbb{Z}}$. One gets the

Proposition 3.2. — *Let η and π two paths in $\mathcal{L}_{\min \mathbb{Z}}$. Then:*

(1) *the concatenation $\pi * \eta$ belongs to $\mathcal{L}_{\min \mathbb{Z}}$,*

(2) *for any $i = 1, \dots, n$ we have*

$$\tilde{e}_i(\eta * \pi) = \begin{cases} \eta * \tilde{e}_i(\pi) & \text{if } \varepsilon_i(\pi) > \varphi_i(\eta) \\ \tilde{e}_i(\eta) * \pi & \text{otherwise,} \end{cases}$$
$$\text{and} \quad \tilde{f}_i(\eta * \pi) = \begin{cases} \tilde{f}_i(\eta) * \pi & \text{if } \varphi_i(\eta) > \varepsilon_i(\pi) \\ \eta * \tilde{f}_i(\pi) & \text{otherwise.} \end{cases} \tag{3.6}$$

In particular, $\tilde{e}_i(\eta*\pi) = \mathbf{0}$ if and only if $\tilde{e}_i(\eta) = \mathbf{0}$ and $\varepsilon_i(\pi) \leqslant \varphi_i(\eta)$ for any $i = 1, \dots, n$.

(3) *$\tilde{e}_i(\eta) = \mathbf{0}$ for any $i = 1, \dots, n$ if and only if $\operatorname{Im} \eta$ is contained in $\mathcal{C}$.*

The following theorem summarizes crucial results by Littelmann (see [11, 12, 13]).

Theorem 3.3. — *Consider λ, μ and κ dominant weights and choose arbitrarily elementary paths η_λ, η_μ and η_κ in $\mathcal{L}$ such that $\operatorname{Im} \eta_\lambda \subset \mathcal{C}$, $\operatorname{Im} \eta_\mu \subset \mathcal{C}$ and $\operatorname{Im} \eta_\kappa \subset \mathcal{C}$ and joining respectively 0 to λ, 0 to μ and 0 to κ.*

(1) *We have $B(\eta_\lambda) := \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} \eta_\lambda \mid k \in \mathbb{Z}_{\geqslant 0} \text{ and } 1 \leqslant i_1, \cdots, i_k \leqslant n\} \setminus \{\mathbf{0}\}$. In particular $\mathrm{wt}(\eta) - \mathrm{wt}(\eta_\lambda) \in Q_+$ for any $\eta \in B(\eta_\lambda)$.*

(2) *All the paths in $B(\eta_\lambda)$ have the same length than η_λ.*

(3) *The paths on $B(\eta_\lambda)$ belong to $\mathcal{L}_{\min \mathbb{Z}}$.*

(4) *If η'_λ is another elementary path from 0 to λ such that $\operatorname{Im} \eta'_\lambda$ is contained in $\mathcal{C}$, then $B(\eta_\lambda)$ and $B(\eta'_\lambda)$ are isomorphic as oriented graphs i.e. there exists a bijection $\theta : B(\eta_\lambda) \to B(\eta'_\lambda)$ which commutes with the action of the operators $\tilde{e}_i$ and $\tilde{f}_i$, $i = 1, \dots, n$.*

(5) *We have*

$$s_\lambda = \sum_{\eta \in B(\eta_\lambda)} e^{\eta(1)}. \tag{3.7}$$

(6) *For any $b \in B(\eta_\lambda)$ we have $\mathrm{wt}(b) = \sum_{i=1}^{n} (\varphi_i(b) - \varepsilon_i(b))\omega_i$.*

(7) *For any $i = 1, \dots, n$ and any $b \in B(\eta_\lambda)$, let $s_i(b)$ be the unique path in $B(\eta_\lambda)$ such that*

$$\varphi_i(s_i(b)) = \varepsilon_i(b) \text{ and } \varepsilon_i(s_i(b)) = \varphi_i(b)$$

(in other words, s_i acts on each i-chain $\mathcal{C}_i$ as the symmetry with respect to the center of $\mathcal{C}_i$). The actions of the s_i's extend to an action[(2)] *of W on $\mathcal{L}$ which stabilizes $B(\eta_\lambda)$. In particular, for any $w \in \mathsf{W}$ and any $b \in B(\eta_\lambda)$, we have $w(b) \in B(\eta_\lambda)$ and $\mathrm{wt}(w(b)) = w(\mathrm{wt}(b))$.*

(2) This action, defined from the crystal structure on paths (see [4, §11]), should not be confused with the pointwise action of the Weyl group on the paths which does not stabilize the crystal $B(\eta_\lambda)$.

(8) *Given any integer* $\ell \geqslant 0$, *set*

$$B(\eta_\mu) * B(\eta_\kappa)^{*\ell} = \left\{ \pi = \eta * \eta_1 * \cdots * \eta_\ell \in \mathcal{L} \;\middle|\; \begin{array}{r} \eta \in B(\eta_\mu) \text{ and } \eta_k \in B(\eta_\kappa) \\ \text{for any } k = 1, \dots, \ell \end{array} \right\}. \quad (3.8)$$

The graph $B(\eta_\mu) * B(\eta_\kappa)^{*\ell}$ *is contained in* $\mathcal{L}_{\min \mathbb{Z}}$.

(9) *The multiplicity* $m_{\mu,\kappa}^\lambda$ *defined in* (3.3) *is equal to the number of paths of the form* $\mu * \eta$ *with* $\eta \in B(\eta_\kappa)$ *contained in* $\mathcal{C}$.

(10) *The multiplicity* $f_{\lambda/\mu}^\ell$ *defined in* (3.2) *is equal to cardinality of the set*

$$H_{\lambda/\mu}^\ell := \left\{ \pi \in B(\eta_\mu) * B(\eta_\kappa)^{*\ell} \;\middle|\; \begin{array}{r} \tilde{e}_i(\pi) = 0 \quad \text{for any } i = 1, \dots, n \\ \text{and } \pi(1) = \lambda \end{array} \right\}.$$

Each path $\pi = \eta * \eta_1 * \cdots * \eta_\ell \in H_{\lambda/\mu}^\ell$ *satisfies* $\operatorname{Im} \pi \subset \mathcal{C}$ *and* $\eta = \eta_\mu$.

Remarks 3.4.

(1) Combining (3.5) with Assertions (1) and (5) of Theorem 3.3, one may check that the function $e^{-\lambda} s_\lambda$ is in fact a polynomial in the variables $T_i = e^{-\alpha_i}$, namely

$$s_\lambda = e^\lambda S_\lambda(T_1, \dots, T_n) \quad (3.9)$$

where $S_\lambda \in \mathbb{C}[X_1, \dots, X_n]$.

(2) Using Assertion (1) of Theorem 3.3, we obtain $m_{\mu,\kappa}^\lambda \neq 0$ only if $\mu + \kappa - \lambda \in Q_+$. Similarly, when $f_{\lambda/\mu}^{\kappa,\ell} \neq 0$ one necessarily has $\mu + \ell\kappa - \lambda \in Q_+$.

4. Random paths from Littelmann paths

In this section, we recall some results of [9]. We also introduce the notion of central probability distribution on elementary Littelmann paths and show these distributions coincide with those used in the seminal works [1, 14] and also in our previous papers [7, 8, 9].

4.1. Central probability measure on trajectories

Consider $\kappa \in P_+$ and a path $\pi_\kappa \in \mathcal{L}$ from 0 to κ such that $\operatorname{Im} \pi_\kappa$ is contained in $\mathcal{C}$. Let $B(\pi_\kappa)$ be the connected component of $\mathcal{L}$ containing π_κ.

Assume that $\{\pi_1,\dots,\pi_\ell\}$ is a family of elementary paths in $B(\pi_\kappa)$; the path $\pi_1\otimes\cdots\otimes\pi_\ell$ of length ℓ is defined by: for all $k\in\{1,\dots,\ell-1\}$ and $t\in[k,k+1]$

$$\pi_1\otimes\cdots\otimes\pi_\ell(t)=\pi_1(1)+\cdots+\pi_k(1)+\pi_{k+1}(t-k). \tag{4.1}$$

Let $B(\pi_\kappa)^{\otimes\ell}$ be the set of paths of the form $b=\pi_1\otimes\cdots\otimes\pi_\ell$ where $\pi_1,\dots,\pi_\ell$ are elementary paths in $B(\pi_\kappa)$; there exists a bijection Δ between $B(\pi_\kappa)^{\otimes\ell}$ and the set $B^{*\ell}(\pi_\kappa)$ of paths in $\mathcal{L}$ obtained by concatenations of ℓ paths of $B(\pi_\kappa)$:

$$\Delta:\begin{cases} B(\pi_\kappa)^{\otimes\ell} & \longrightarrow & B(\pi_\kappa)^{*\ell} \\ \pi_1\otimes\cdots\otimes\pi_\ell & \longmapsto & \pi_1*\cdots*\pi_\ell \end{cases}. \tag{4.2}$$

In fact $\pi_1\otimes\cdots\otimes\pi_\ell$ and $\pi_1*\cdots*\pi_\ell$ coincide up to a reparametrization and we define the weight of $b=\pi_1\otimes\cdots\otimes\pi_\ell$ setting

$$\mathrm{wt}(b):=\mathrm{wt}(\pi_1)+\cdots+\mathrm{wt}(\pi_\ell)=\pi_1(1)+\cdots+\pi_\ell(1)\,.$$

The duality map (which is an involution) r on $\eta\in B(\pi_\kappa)^{\otimes\ell}$ is such that

$$r(\eta)(t)=\eta(\ell-t)-\eta(0)$$

for any $t\in[0,\ell]$.

Consider p a probability distribution on $B(\pi_\kappa)$ such that $p_\pi>0$ for any $\pi\in B(\pi_\kappa)$. For any integer $\ell\geqslant 1$, we endow $B(\pi_\kappa)^{\otimes\ell}$ with the product density $p^{\otimes\ell}$. That is we set $p^{\otimes\ell}_\pi=p_{\pi_1}\times\cdots\times p_{\pi_\ell}$ for any $\pi=\pi_1\otimes\cdots\otimes\pi_\ell\in B(\pi_\kappa)^{\otimes\ell}$. Here, we follow the classical construction of a Bernoulli process. Write $\Pi_\ell : B(\pi_\kappa)^{\otimes\ell}\to B(\pi_\kappa)^{\otimes\ell-1}$ the projection defined by $\Pi_\ell(\pi_1\otimes\cdots\otimes\pi_{\ell-1}\otimes\pi_\ell)=\pi_1\otimes\cdots\otimes\pi_{\ell-1}$; the sequence $(B(\pi_\kappa)^{\otimes\ell},\Pi_\ell,p^{\otimes\ell})_{\ell\geqslant 1}$ is a projective system of probability spaces. We denote by $\Omega=(B(\pi_\kappa)^{\otimes\mathbb{Z}_{\geqslant 0}},p^{\otimes\mathbb{Z}_{\geqslant 0}})$ its projective limit. The elements of $B(\pi_\kappa)^{\otimes\mathbb{Z}_{\geqslant 0}}$ are infinite sequences $\omega=(\pi_\ell)_{\ell\geqslant 1}$ we call trajectories. By a slight abuse of notation, we will write $\Pi_\ell(\omega)=\pi_1\otimes\cdots\otimes\pi_\ell$. We also write $\mathsf{P}=p^{\otimes\mathbb{Z}_{\geqslant 0}}$ for short. For any $b\in B(\pi_\kappa)^{\otimes\ell}$, we denote by $U_b=\{\omega\in\Omega\mid\Pi_\ell(\omega)=b\}$ the cylinder defined by π in Ω.

Definition 4.1. — *The probability distribution* $\mathsf{P}=p^{\otimes\mathbb{Z}_{\geqslant 0}}$ *is* central *on* Ω *when for any* $\ell\geqslant 1$ *and any vertices* b *and* b' *in* $B(\pi_\kappa)^{\otimes\ell}$ *such that* $\mathrm{wt}(b)=\mathrm{wt}(b')$ *we have* $\mathsf{P}(U_b)=\mathsf{P}(U_{b'})$.

Remark 4.2. — The probability distribution P is *central* when for any integer $\ell\geqslant 1$ and any vertices b,b' in $B(\pi_\kappa)^{\otimes\ell}$ such that $\mathrm{wt}(b)=\mathrm{wt}(b')$, we have $p^{\otimes\ell}_b=p^{\otimes\ell}_{b'}$. We indeed have $U_b=b\otimes\Omega$ and $U_{b'}=b\otimes\Omega$. Hence $\mathsf{P}(U_b)=p^{\otimes\ell}_b$ and $\mathsf{P}(U_{b'})=p^{\otimes\ell}_{b'}$.

The following proposition shows that P can only be central when the probability distribution p on $B(\pi_\kappa)$ is compatible with the graduation of $B(\pi_\kappa)$ by the set of simple roots. This justifies the restriction we did in [7, 9] on the probability distributions we have considered on $B(\pi_\kappa)$. This restriction will also be relevant in the remaining of this paper.

Proposition 4.3. — *The following assertions are equivalent*

(1) *The probability distribution* P *is central.*

(2) *There exists an* n*-tuple* $\tau = (\tau_1, \ldots, \tau_n) \in]0, +\infty[^n$ *such that for each arrow* $\pi \xrightarrow{i} \pi'$ *in* $B(\pi_\kappa)$*, we have the relation* $p_{\pi'} = p_\pi \times \tau_i$*.*

Proof. — Assume probability distribution P is central. For any path $\pi \in B(\pi_\kappa)$, we define the depth $d(\pi)$ as the number of simple roots appearing in the decomposition of $\kappa - \mathrm{wt}(\pi)$ on the basis of simple roots (see Assertion (1) of Theorem 3.3). This is also the length of any path joining π_κ to π in the crystal graph $B(\pi_\kappa)$. We have to prove that $\frac{p_{\pi'}}{p_\pi}$ is a constant depending only on i as soon as we have an arrow $\pi \xrightarrow{i} \pi'$ in $B(\pi_\kappa)$. For any $k \geqslant 1$, we set $B(\pi_\kappa)_k = \{\pi \in B(\pi_\kappa) \mid d(\pi) \leqslant k\}$. We will proceed by induction and prove that $\frac{p_{\pi'}}{p_\pi}$ is a constant depending only on i as soon as there is an arrow $\pi \xrightarrow{i} \pi'$ in $B(\pi_\kappa)_k$. This is clearly true in $B(\pi_\kappa)_1$ since there is at most one arrow i starting from π_κ. Assume, the property is true in $B(\pi_\kappa)_k$ with $k \geqslant 1$. Consider π' in $B(\pi_\kappa)_{k+1}$ and an arrow $\pi \xrightarrow{i} \pi'$ in $B(\pi_\kappa)_{k+1}$. We must have $\pi \in B(\pi_\kappa)_k$. If $B(\pi_\kappa)_k$ does not contain any arrow $\xrightarrow{i}$, there is nothing to verify. So assume there is at least an arrow $\pi_1 \xrightarrow{i} \pi_2$ in $B(\pi_\kappa)_k$. In $B(\pi_\kappa)^{\otimes 2}$, we have $\mathrm{wt}(\pi_1 \otimes \pi') = \mathrm{wt}(\pi_1) + \mathrm{wt}(\pi) - \alpha_i$ since $\mathrm{wt}(\pi') = \mathrm{wt}(\pi) - \alpha_i$. Similarly, we have $\mathrm{wt}(\pi_2 \otimes \pi) = \mathrm{wt}(\pi_1) - \alpha_i + \mathrm{wt}(\pi)$ since $\mathrm{wt}(\pi_2) = \mathrm{wt}(\pi_1) - \alpha_i$. Thus $\mathrm{wt}(\pi_1 \otimes \pi') = \mathrm{wt}(\pi_2 \otimes \pi)$. Since P is central, we deduce from the above remark the equality $p^{\otimes 2}(\pi_1 \otimes \pi') = p^{\otimes 2}(\pi_2 \otimes \pi)$. This yields $p_{\pi_1} p_{\pi'} = p_{\pi_2} p_\pi$. Hence $\frac{p_{\pi'}}{p_\pi} = \frac{p_{\pi_2}}{p_{\pi_1}}$. So by our induction hypothesis, $\frac{p_{\pi'}}{p_\pi}$ is equal to a constant which only depends on i.

Conversely, assume there exists an n-tuple $\tau = (\tau_1, \ldots, \tau_n) \in]0, +\infty[^n$ such that for each arrow $\pi \xrightarrow{i} \pi'$ in $B(\pi_\kappa)$, we have the relation $p_{\pi'} = p_\pi \times \tau_i$. Consider vertices b, b' in $B(\pi_\kappa)^{\otimes \ell}$ such that $\mathrm{wt}(b) = \mathrm{wt}(b')$. Since b and b' have the same weight, we derive from (3.5) that the paths from π_κ to b and the paths from π_κ to b' contain the same number (says a_i) of arrows $\xrightarrow{i}$ for any $i = 1, \ldots, n$. We therefore have $p_b = p_{b'} = p_{\pi_\kappa} \tau_1^{a_1} \cdots \tau_n^{a_n}$ and the probability distribution P is central. □

4.2. Central probability distributions on elementary paths

In the remaining of the paper, we fix the n-tuple $\tau = (\tau_1, \ldots, \tau_n) \in]0, +\infty[^n$ and assume that P is a central distribution on Ω defined from τ (in the sense of Definition 4.1. For any $u = u_1\alpha_1 + \cdots + u_n\alpha_n \in Q$, we set $\tau^u = \tau_1^{u_1} \cdots \tau_n^{u_n}$. Since the root and weight lattices have both rank n, any weight $\beta \in P$ also decomposes in the form $\beta = \beta_1\alpha_1 + \cdots + \beta_n\alpha_n$

with possibly non integral coordinates β_i. The transition matrix between the bases $\{\omega_i, i = 1, \dots, n\}$ and $\{\alpha_i, i = 1, \dots, n\}$ (regarded as bases of $P_{\mathbb{R}}$) being the Cartan matrix of $\mathfrak{g}$ whose entries are integers, the coordinates β_i are rational. We will also set $\tau^\beta = \tau_1^{\beta_1} \cdots \tau_n^{\beta_n}$.

Let $\pi \in B(\pi_\kappa)$: by Assertion (1) of Theorem 3.3, one obtains

$$\pi(1) = \mathrm{wt}(\pi) = \kappa - \sum_{i=1}^{n} u_i(\pi)\alpha_i$$

where $u_i(\pi) \in \mathbb{Z}_{\geqslant 0}$ for any $i = 1, \dots, n$. We define $S_\kappa(\tau) := S_\kappa(\tau_1, \dots, \tau_n) = \sum_{\pi \in B(\pi_\kappa)} \tau^{\kappa - \mathrm{wt}(\pi)}$.

Definition 4.4. — *We define the probability distribution* $p = (p_\pi)_{\pi \in B(\pi_\kappa)}$ *on* $B(\pi_\kappa)$ *associated with* τ *by setting* $p_\pi = \frac{\tau^{\kappa - \mathrm{wt}(\pi)}}{S_\kappa(\tau)}$.

Remark 4.5. — By Assertion (3) of Theorem 3.3, for another elementary path π'_κ from 0 to κ such that $\operatorname{Im} \pi'_\kappa$ is contained in $\mathcal{C}$, there exists an isomorphism Θ between the crystals $B(\pi_\kappa)$ and $B(\pi'_\kappa)$. For p' the central probability distribution defined from τ on $B(\pi'_\kappa)$, one gets $p_\pi = p'_{\Theta(\pi)}$ for any $\pi \in B(\pi_\kappa)$. Therefore, the probability distributions we use on the graph $B(\pi_\kappa)$ are invariant by crystal isomorphisms and also the probabilistic results we will establish in the paper.

The following proposition gathers results of [7, Lemma 7.2.1] and [9, Proposition 5.4]. Recall that $m = \sum_{\pi \in B(\pi_\kappa)} p_\pi \pi$. We set $\overline{m} = m(1)$.

Proposition 4.6.

(1) *We have* $\overline{m} \in \mathring{\mathcal{C}}$ *if and only if* $\tau_i \in]0, 1[$ *for any* $i = 1, \dots, n$.
(2) *Denote by* L *the common length of the paths in* $B(\pi_\kappa)$. *Then, the length of* m *is less or equal to* L.

Set $\mathcal{M}_\kappa = \{\overline{m} \mid \tau = (\tau_1, \dots, \tau_n) \in]0, +\infty[\}$ be the set of all vectors $\overline{m}$ obtained from the central distributions defined on $B(\pi_\kappa)$. Observe that $\mathcal{M}_\kappa$ only depends on κ and not of the choice of the highest path π_κ. This is the set of possible means obtained from central probability distributions defined on $B(\pi_\kappa)$. We will also need the set

$$\mathcal{D}_\kappa = \mathcal{M}_\kappa \cap \mathring{\mathcal{C}} = \{\overline{m} \in \mathcal{M}_\kappa \mid \tau_i \in]0, 1[, i = 1, \dots, n\} \tag{4.3}$$

of drifts in $\mathring{\mathcal{C}}$.

Example 4.7. — We resume Example 3.1 and consider the Lie algebra $\mathfrak{g} = \mathfrak{sp}_4$ of type C_2 for which $P = \mathbb{Z}^2$ and $\mathcal{C} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geqslant x_2 \geqslant 0\}$.

For $\kappa = \omega_1$ and π_κ the line between 0 and ε_1, we have $B(\pi_\kappa) = \{\pi_1, \pi_2, \pi_{\overline{2}}, \pi_{\overline{1}}\}$ where each π_a is the line between 0 and e_a (with the convention $e_{\overline{2}} = -e_2$ and $e_{\overline{1}} = -e_1$). The underlying crystal graph is

$$\pi_1 \xrightarrow{1} \pi_2 \xrightarrow{2} \pi_{\overline{2}} \xrightarrow{1} \pi_{\overline{1}} \,.$$

For $(\tau_1, \tau_2) \in]0, +\infty[^2$, we obtain the probability distribution on $B(\pi_\kappa)$

$$p_{\pi_1} = \frac{1}{1+\tau_1+\tau_1\tau_2+\tau_1^2\tau_2}\,, \qquad p_{\pi_2} = \frac{\tau_1}{1+\tau_1+\tau_1\tau_2+\tau_1^2\tau_2}\,,$$

$$p_{\pi_{\overline{2}}} = \frac{\tau_1\tau_2}{1+\tau_1+\tau_1\tau_2+\tau_1^2\tau_2} \qquad \text{and} \qquad p_{\pi_{\overline{2}}} = \frac{\tau_1^2\tau_2}{1+\tau_1+\tau_1\tau_2+\tau_1^2\tau_2}\,.$$

So we have

$$\overline{m} = \frac{1}{1+\tau_1+\tau_1\tau_2+\tau_1^2\tau_2}((1-\tau_1^2\tau_2)\varepsilon_1 + (\tau_1 - \tau_1\tau_2)\varepsilon_2)\,.$$

When the pair (τ_1, τ_2) runs over $]0, 1[^2$, one verifies by a direct computation that $\mathcal{D}_\kappa$ coincides with the interior of the triangle with vertices $0, e_1, e_2$.

Remark 4.8. — In the previous example, it is easy to show by a direct calculation that the closure $\overline{\mathcal{M}_\kappa}$ of $\mathcal{M}_\kappa$ is the convex hull of the weight $\{\pm e_1, \pm e_2\}$ of the representation $V(\omega_1)$ considered (i.e. the interior of the square with vertices $\{\pm e_1, \pm e_2\}$). In general, one can show that $\overline{\mathcal{M}_\kappa}$ is contained in the convex hull of the weights of $V(\kappa)$. The problem of determining, for any dominant weight κ, wether or not both sets coincide seems to us interesting and not immediate.

4.3. Random paths of arbitrary length

With the previous convention, the product probability measure $p^{\otimes \ell}$ on $B(\pi_\kappa)^{\otimes \ell}$ satisfies

$$p^{\otimes \ell}(\pi_1 \otimes \cdots \otimes \pi_\ell) = p(\pi_1)\cdots p(\pi_\ell) = \frac{\tau^{\ell\kappa - (\pi_1(1)+\cdots+\pi_\ell(1))}}{S_\kappa(\tau)^\ell} = \frac{\tau^{\ell\kappa - \mathrm{wt}(b)}}{S_\kappa(\tau)^\ell}\,. \tag{4.4}$$

Let $(X_\ell)_{\ell \geqslant 1}$ be a sequence of i.i.d. random variables with values in $B(\pi_\kappa)$ and law $p = (p_\pi)_{\pi \in B(\pi_\kappa)}$; for any $\ell \geqslant 1$ we thus obtain

$$\mathsf{P}(X_\ell = \pi) = p_\pi \text{ for any } \pi \in B(\pi_\kappa)\,. \tag{4.5}$$

Consider $\mu \in P$. The random path $\mathcal{W}$ starting at μ is defined from the probability space Ω with values in $P_\mathbb{R}$ by

$$\mathcal{W}(t) := \Pi_\ell(\mathcal{W})(t) = \mu + (X_1 \otimes \cdots \otimes X_{\ell-1} \otimes X_\ell)(t) \text{ for } t \in [\ell-1, \ell]\,.$$

For any integer $\ell \geqslant 1$, we set $W_\ell = \mathcal{W}(\ell)$. The sequence $W = (W_\ell)_{\ell \geqslant 1}$ defines a random walk starting at $W_0 = \mu$ whose increments are the weights of the representation $V(\kappa)$. The following proposition was established in [9, Proposition 4.6] (we recall that the numbers $K_{\lambda,\mu}$ are defined in §3.1).

Proposition 4.9.

(1) *For any* $\beta, \eta \in P$, *one gets*

$$\mathsf{P}(W_{\ell+1} = \beta \mid W_\ell = \eta) = K_{\kappa,\beta-\eta}\frac{\tau^{\kappa+\eta-\beta}}{S_\kappa(\tau)}.$$

(2) *Consider* $\lambda, \mu \in P^+$ *we have*

$$\mathsf{P}(W_\ell = \lambda, W_0 = \mu, \mathcal{W}(t) \in \mathcal{C} \ \text{ for any } t \in [0,\ell]) = f^\ell_{\lambda/\mu}\frac{\tau^{\ell\kappa+\mu-\lambda}}{S_\kappa(\tau)^\ell}.$$

In particular

$$\mathsf{P}(W_{\ell+1} = \lambda, W_\ell = \mu, \mathcal{W}(t) \in \mathcal{C} \ \text{ for any } t \in [\ell,\ell+1]) = m^\lambda_{\mu,\kappa}\frac{\tau^{\kappa+\mu-\lambda}}{S_\kappa(\tau)}.$$

4.4. The generalized Pitman transform

By Assertion (8) of Theorem 3.3, we know that $B(\pi_\kappa)^{\otimes\ell}$ is contained in $\mathcal{L}_{\min\mathbb{Z}}$. Therefore, if we consider a path $\eta \in B(\pi_\kappa)^{\otimes\ell}$, its connected component $B(\eta)$ is contained in $\mathcal{L}_{\min\mathbb{Z}}$. Now, if $\eta^h \in B(b)$ is such that $\tilde{e}_i(\eta^h) = 0$ for any $i = 1,\dots,n$, we should have $\operatorname{Im}\eta^h \subset \mathcal{C}$ by Assertion (3) of Proposition 3.2. Assertion (3) of Theorem 3.3 thus implies that η^h is the unique highest weight path in $B(\eta) = B(\eta^h)$. Similarly, there is a unique lowest path η_l in $B(\eta)$ such that $\tilde{f}_i(\eta_l) = 0$ for any $i = 1,\dots,n$. This permits to define the *generalized Pitman* transform on $B(\pi_\kappa)^{\otimes\ell}$ as the map $\mathcal{P}$ which associates with any $\eta \in B(\pi_\kappa)^{\otimes\ell}$ the unique path $\mathcal{P}(\eta) \in B(\eta)$ such that $\tilde{e}_i(\mathcal{P}(\eta)) = 0$ for any $i = 1,\dots,n$. By definition, we have $\operatorname{Im}\mathcal{P}\ (\eta) \subset \mathcal{C}$ and $\mathcal{P}(\eta)(\ell) \in P_+$. One can also define a dual Pitman transform $\mathcal{E}$ which associates with any $\eta \in B(\pi_\kappa)^{\otimes\ell}$ the unique path $\mathcal{E}\ (\eta) \in B(\eta)$ such that $\tilde{f}_i(\mathcal{E}(\eta)) = 0$ for any $i = 1,\dots,n$. By (3.4), we have in fact

$$\mathcal{E} = r\mathcal{P}r.$$

As observed in [1] the path transformation $\mathcal{P}$ can be made more explicit (recall we have assumed that $\mathfrak{g}$ is finite-dimensional). Consider a simple reflection α. The Pitman transformation $\mathcal{P}_\alpha : B(\pi_\kappa)^{\otimes\ell} \to B(\pi_\kappa)^{\otimes\ell}$ associated with α is defined by

$$\mathcal{P}_\alpha(\eta)(t) = \eta(t) - 2\inf_{s\in[0,t]}\langle\eta(s), \frac{\alpha}{\|\alpha\|^2}\rangle\alpha = \eta(t) - \inf_{s\in[0,t]}\langle\eta(s), \alpha^\vee\rangle\alpha \qquad (4.6)$$

for any $\eta \in B(\pi_\kappa)^{\otimes \ell}$ and any $t \in [0, \ell]$. Also define the dual transform $\mathcal{E}_\alpha := r\mathcal{P}_\alpha r$ on $B(\pi_\kappa)^{\otimes \ell}$. One verifies easily that we have in fact

$$\mathcal{E}_\alpha(\eta)(t) = \eta(t) - \inf_{s\in[t,\ell]} \langle \eta(s), \alpha^\vee \rangle \alpha + \inf_{s\in[0,\ell]} \langle \eta(s), \alpha^\vee \rangle \alpha \,. \tag{4.7}$$

Let w_0 be the maximal length element of W and fix a decomposition $w_0 = s_{i_1} \cdots s_{i_r}$ of w_0 as a product of reflections associated with simple roots.

Proposition 4.10 ([1]). — *For any path $\eta \in B(\pi_\kappa)^{\otimes \ell}$, we have*

$$\mathcal{P}(\eta) = \mathcal{P}_{\alpha_{i_1}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta) \text{ and } \mathcal{E}(\eta) = \mathcal{E}_{\alpha_{i_1}} \cdots \mathcal{E}_{\alpha_{i_r}}(\eta)\,. \tag{4.8}$$

Moreover, $\mathcal{P}$ and $\mathcal{E}$ do not depend on the decomposition of w_0 chosen.

Remarks 4.11.

(1) Since $\mathcal{P}(\eta)$ corresponds to the highest weight vertex of the crystal $B(\eta)$, we have $\mathcal{P}^2(\eta) = \mathcal{P}\ (\eta)$.
(2) One easily verifies that each transformation $\mathcal{P}_\alpha$ is continuous for the topology of uniform convergence on the space of continuous maps from $[0,\ell]$ to $\mathbb{R}$. Hence $\mathcal{P}$ is also continuous for this topology.
(3) Assume $\eta \in B(\eta_\lambda) \subset B(\pi_\kappa)^{\otimes \ell}$ where η_λ is the highest weight path of $B(\eta_\lambda)$. Then $\eta^\lambda = w_0(\eta_\lambda)$ (the action of W is that of Theorem 3.3) is the lowest weight path in $B(\eta_\lambda)$. In this particular case, one can show that we have in fact

$$\begin{aligned} \mathcal{P}_{i_{a+1}} \cdots \mathcal{P}_{i_r}(\eta^\lambda) &= s_{i_{a+1}} \cdots s_{i_r}(\eta^\lambda) \\ \text{and } \ \mathcal{E}_{i_{a+1}} \cdots \mathcal{E}_{i_r}(\eta_\lambda) &= s_{i_{a+1}} \cdots s_{i_r}(\eta_\lambda) \end{aligned} \tag{4.9}$$

for any $a = 1, \ldots, r-1$.

Let $\mathcal{W}$ be the random path of §4.3. We define the random process $\mathcal{H}$ setting

$$\mathcal{H} = \mathcal{P}(\mathcal{W}). \tag{4.10}$$

For any $\ell \geqslant 1$, we set $H_\ell := \mathcal{H}(\ell)$. The following Theorem was established in [9].

Theorem 4.12.

(1) *The random sequence $H := (H_\ell)_{\ell \geqslant 1}$ is a Markov chain with transition matrix*

$$\Pi(\mu, \lambda) = \frac{S_\lambda(\tau)}{S_\kappa(\tau) S_\mu(\tau)} \tau^{\kappa+\mu-\lambda} m_{\mu,\kappa}^\lambda \tag{4.11}$$

where $\lambda, \mu \in P_+$.
(2) *Assume $\eta \in B(\pi_\kappa)^{\otimes \ell}$ is a highest weight path of weight λ. Then*

$$\mathsf{P}(\mathcal{W}_\ell = \eta) = \frac{\tau^{\ell\kappa - \lambda} S_\lambda(\tau)}{S_\kappa(\tau)^\ell}$$

We shall also need the asymptotic behavior of the tensor product multiplicities established in [9].

THEOREM 4.13. — *Assume $\overline{m} \in \mathcal{D}_\kappa$ (see (4.3)). For any $\mu \in P$ and any sequence of dominant weights of the form $\lambda^{(\ell)} = \ell\overline{m} + o(\ell)$, we have*

(1) $\displaystyle\lim_{\ell\to+\infty} \frac{f^\ell_{\lambda^{(\ell)}-\gamma}}{f^\ell_{\lambda^{(\ell)}}} = \tau^{-\gamma}$ *for any $\gamma \in P$.*

(2) $\displaystyle\lim_{\ell\to+\infty} \frac{f^\ell_{\lambda^{(\ell)}/\mu}}{f^\ell_{\lambda^{(\ell)}}} = \tau^{-\mu} S_\mu(\tau)$.

COROLLARY 4.14. — *Under the assumptions of the previous theorem, we also have*

$$\lim_{\ell\to+\infty} \frac{f^{\ell-\ell_0}_{\lambda^{(\ell)}}}{f^\ell_{\lambda^{(\ell)}}} = \frac{1}{\tau^{-\ell_0\kappa} S^{\ell_0}_\kappa(\tau)}$$

for any nonnegative integer ℓ_0.

Proof. — We first consider the case where $\ell_0 = 1$. By definition of the tensor product multiplicities in (3.2) we have $s^\ell_\kappa = \sum_{\lambda\in P_+} f^\ell_\lambda s_\lambda$ but also $s^\ell_\kappa = s_\kappa \times s^{\ell-1}_\kappa = \sum_{\lambda\in P_+} f^{\ell-1}_{\lambda/\kappa} s_\lambda$. Therefore $f^\ell_\lambda = f^{\ell-1}_{\lambda/\kappa}$ for any $\ell \geqslant 1$ and any $\lambda \in P_+$. We get

$$\lim_{\ell\to+\infty} \frac{f^{\ell-1}_{\lambda^{(\ell)}}}{f^\ell_{\lambda^{(\ell)}}} = \lim_{\ell\to+\infty} \frac{f^{\ell-1}_{\lambda^{(\ell)}}}{f^{\ell-1}_{\lambda^{(\ell)}/\kappa}} = \frac{1}{\tau^{-\kappa} S_\kappa(\tau)} \tag{4.12}$$

by Assertion (2) of Theorem 4.13. Now observe that for any $\ell_0 \geqslant 1$ we have

$$\frac{f^{\ell-\ell_0}_{\lambda^{(\ell)}}}{f^\ell_{\lambda^{(\ell)}}} = \frac{f^{\ell-\ell_0}_{\lambda^{(\ell)}}}{f^{\ell-\ell_0+1}_{\lambda^{(\ell)}}} \times \cdots \times \frac{f^{\ell-1}_{\lambda^{(\ell)}}}{f^\ell_{\lambda^{(\ell)}}}.$$

By using (4.12) each component of the previous product tends to $\frac{1}{\tau^{-\kappa} S_\kappa(\tau)}$ when ℓ tends to infinity which gives the desired limit. □

The previous theorem also implies that the drift $\overline{m}$ determines the probability distribution on $B(\pi_\kappa)$. More precisely, consider p and p' two probability distributions defined on $B(\pi_\kappa)$ from $\tau \in]0,1[^n$ and $\tau' \in]0,1[^n$, respectively. Set $m = \sum_{\pi\in B(\pi_\kappa)} p_\pi \pi$ and $m' = \sum_{\pi\in B(\pi_\kappa)} p'_\pi \pi$.

PROPOSITION 4.15. — *We have $\overline{m} = \overline{m}'$ if and only if $\tau = \tau'$. Therefore, the map which associates to any $\tau \in]0,1[^n$ the drift $\overline{m} \in \mathcal{D}_\kappa$ is a one-to-one correspondence.*

Proof. — Assume $\overline{m} = \overline{m}'$. By applying Assertion (1) of Theorem 4.13, we get $\tau^\gamma = (\tau')^\gamma$ for any $\gamma \in P$. Consider $i \in \{1, \ldots, n\}$. For $\gamma = \alpha_i$, we obtain $\tau_i = \tau'_i$. Therefore $\tau = \tau'$. □

5. Some Limit theorems for the Pitman process

5.1. The law of large numbers and the central limit theorem for $\mathcal{W}$

We start by establishing two classical limit theorems for $\mathcal{W}$, deduced from the law of large numbers and the central limit theorem for the random walk $W = (W_\ell)_{\ell\geqslant 1} = (X_1 + \cdots + X_\ell)_{\ell\geqslant 1}$. Recall that $m = \sum_{\pi\in B(\pi_\kappa)} p_\pi \pi$ and $\overline{m} = m(1)$. Write $m^{\otimes\infty}$ for the random path such that

$$m^{\otimes\infty}(t) = \ell\overline{m} + m(t-\ell) \text{ for any } t > 0$$

where $\ell = \lfloor t \rfloor$.

Let $\Gamma = (\Gamma_{i,j})_{1\leqslant i,j\leqslant n} = {}^t X_\ell \cdot X_\ell$ be the common covariance matrix of each random variable X_ℓ.

Theorem 5.1. — *Let $\mathcal{W}$ be a random path defined on $(B(\pi_\kappa)^{\otimes\mathbb{Z}_{\geqslant 0}}, p^{\otimes\mathbb{Z}_{\geqslant 0}})$ with drift path m. Then, we have*

$$\lim_{\ell\to+\infty} \frac{1}{\ell} \sup_{t\in[0,\ell]} \left\|\mathcal{W}(t) - m^{\otimes\infty}(t)\right\| = 0 \textit{ almost surely.}$$

Furthermore, the family of random variables $\left(\dfrac{\mathcal{W}(t) - m^{\otimes\infty}(t)}{\sqrt{t}}\right)_{t>0}$ converges in law as $t\to+\infty$ towards a centered Gaussian law $\mathcal{N}(0,\Gamma)$.

More precisely, setting $\mathcal{W}^{(\ell)}(t) := \dfrac{\mathcal{W}(\ell t) - m^{\otimes\infty}(\ell t)}{\sqrt{\ell}}$ for any $0\leqslant t\leqslant 1$ and $\ell\geqslant 1$, the sequence of random processes $(\mathcal{W}^{(\ell)}(t))_{\ell\geqslant 1}$ converges to a n-dimensional Brownian motion $(B_\Gamma(t))_{0\leqslant t\leqslant 1}$ with covariance matrix Γ.

Proof. — Fix $\ell \geqslant 1$ and observe that

$$\sup_{t\in[0,\ell]} \left\|\mathcal{W}(t) - m^{\otimes\infty}(t)\right\| = \sup_{0\leqslant k\leqslant \ell-1}\ \sup_{t\in[k,k+1]} \left\|\mathcal{W}(t) - k\overline{m} - m(t-k)\right\|.$$

For any $0\leqslant k\leqslant \ell$ and $t\in[k,k+1]$, we have $\mathcal{W}(t) = W_k + X_{k+1}(t-k)$ so that

$$\mathcal{W}(t) - m^{\otimes\infty}(t) = W_k - k\overline{m} + \big(X_{k+1}(t-k) - m(t-k)\big) \tag{5.1}$$

with $\sup_{t\in[k,k+1]} \|X_{k+1}(t-k) - m(t-k)\| = \sup_{t\in[0,1]} \|X_{k+1}(t) - m(t)\| \leqslant +2L$, since both paths in $B(\kappa)$ and m have length L, by Proposition 4.6. It readily follows that

$$\sup_{t\in[0,\ell]} \left\|\mathcal{W}(t) - m^{\otimes\infty}(t)\right\| \leqslant \sup_{0\leqslant k\leqslant \ell} \|W_k - k\overline{m}\| + 2L. \tag{5.2}$$

By the law of large number for the random walk $W = (W_k)_{k\geqslant 1}$, one gets $\lim_{k\to+\infty} \frac{1}{k}\left\|W_k - k\overline{m}\right\| = 0$ almost surely; this readily implies

$$\lim_{\ell\to+\infty} \frac{1}{\ell} \sup_{t\in[0,\ell]} \left\|\mathcal{W}(t) - m^{\otimes\infty}(t)\right\| = 0 \text{ almost surely.}$$

Let us now prove the central limit theorem; for any $t > 0$, set $k_t := \lfloor t \rfloor$ and notice that decomposition (5.1) yields

$$\frac{\mathcal{W}(t) - m^{\otimes\infty}(t)}{\sqrt{t}} = \sqrt{\frac{k_t}{t}} \times \frac{W_{k_t} - k_t\overline{m}}{\sqrt{k_t}} + \frac{X_{k_t+1}(t-k_t) - m(t-k_t)}{\sqrt{t}} \tag{5.3}$$

By the central limit theorem in $\mathbb{R}^n$, one knows that the sequence of random variables $\left(\dfrac{W_k - k\overline{m}}{\sqrt{k}}\right)_{k\geqslant 1}$ converges in law as $k \to +\infty$ towards a centered Gaussian law $\mathcal{N}(0,\Gamma)$; on the other hand, one gets $\lim_{t\to+\infty}\sqrt{\dfrac{k_t}{t}} = 1$ and $\limsup_{t\to+\infty} \left\|\dfrac{X_{k_t+1}(t-k_t) - m(t-k_t)}{\sqrt{t}}\right\| \leqslant \limsup_{t\to+\infty} \dfrac{2L}{\sqrt{t}} = 0$, so one may conclude using Slutsky's theorem.

The convergence of the sequence $(\mathcal{W}_\ell(t))_{\ell\geqslant 1}$ towards a Brownian motion goes along the same line. One sets

$$W^{(\ell)}(t) := \frac{W_{\lfloor \ell t\rfloor} + (\ell t - \lfloor \ell t\rfloor)X_{\lfloor \ell t\rfloor+1}(1) - \ell t\overline{m}}{\sqrt{\ell}} \quad \text{for all } \ell \geqslant 1 \text{ and } 0 \leqslant t \leqslant 1$$

and observes that $\left\|\mathcal{W}^{(\ell)}(t) - W^{(\ell)}(t)\right\| \leqslant \dfrac{2}{\sqrt{\ell}}(\|m\|_\infty + \|X_{\lfloor nt\rfloor+1}\|_\infty)$. □

5.2. The law of large numbers and the central limit theorem for $\mathcal{H}$

To prove the law of large numbers and the central limit theorem for $\mathcal{H}$, we need the two following preparatory lemmas. Consider a simple root α and a trajectory $\eta \in \Omega$ such that $\frac{1}{\ell}\langle \eta(\ell), \alpha^\vee\rangle$ converges to a *positive* limit when ℓ tends to infinity.

Lemma 5.2. — *There exists a nonnegative integer ℓ_0 such that for any $\ell \geqslant \ell_0$*

$$\inf_{t\in[0,\ell]} \langle \eta(t), \alpha^\vee\rangle = \inf_{t\in[0,\ell_0]} \langle \eta(t), \alpha^\vee\rangle .$$

Proof. — Since $\frac{1}{\ell}\langle \eta(\ell), \alpha^\vee\rangle$ converges to a positive limit, we have in particular that $\lim_{\ell\to+\infty} \langle \eta(\ell), \alpha^\vee\rangle = +\infty$. Consider $t > 0$ and set $\ell = \lfloor t \rfloor$. We can write by definition of $\eta \in \Omega$, $\eta(t) = \eta(\ell) + \pi(t-\ell)$ where π is a path of $B(\pi_\kappa)$. So $\langle \eta(t), \alpha^\vee \rangle = \langle \eta(\ell), \alpha^\vee \rangle + \langle \pi(t-\ell), \alpha^\vee \rangle$. Since $\pi \in B(\pi_\kappa)$, we have

$$\|\pi(t-\ell)\| \leqslant L$$

where L is the common length of the paths in $B(\pi_\kappa)$. So the possible values of $\langle \pi(t-\ell), \alpha^\vee \rangle$ are bounded. Since $\lim_{\ell \to +\infty} \langle \eta(\ell), \alpha^\vee \rangle = +\infty$, we also get $\lim_{t \to +\infty} \langle \eta(t), \alpha^\vee \rangle = +\infty$. Recall that $\eta(0) = 0$. Therefore $\inf_{t \in [0,\ell]} \langle \eta(t), \alpha^\vee \rangle \leqslant 0$. Since $\lim_{t \to +\infty} \langle \eta(t), \alpha^\vee \rangle = +\infty$ and the path η is continuous, there should exist an integer ℓ_0 such that $\inf_{t \in [0,\ell_0]} \langle \eta(t), \alpha^\vee \rangle = \inf_{t \in [0,\ell_0]} \langle \eta(t), \alpha^\vee \rangle$ for any $\ell \geqslant \ell_0$. □

Lemma 5.3.

(1) *Consider a simple root α and a trajectory $\eta \in \Omega$ such that $\frac{1}{\ell} \langle \eta(\ell), \alpha^\vee \rangle$ converges to a* positive *limit when ℓ tends to infinity. We have for any simple root α*

$$\sup_{t \in [0,+\infty[} \|\mathcal{P}_\alpha(\eta)(t) - \eta(t)\| < +\infty,$$

in particular, $\frac{1}{\ell} \langle \mathcal{P}_\alpha(\eta)(\ell), \alpha^\vee \rangle$ also converges to a positive limit.

(2) *More generally, let $\alpha_{i_1}, \cdots, \alpha_{i_r}, r \geqslant 1$, be simple roots of $\mathfrak{g}$, and η a path in Ω satisfying $\lim_{t \to +\infty} \langle \eta(t), \alpha_{i_j}^\vee \rangle = +\infty$ for $1 \leqslant j \leqslant r$. Then one has*

$$\sup_{t \in [0,+\infty[} \|\mathcal{P}_{\alpha_{i_1}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta)(t) - \eta(t)\| < +\infty.$$

Proof. — (1) By definition of the transform $\mathcal{P}_\alpha$, we have

$$\|\mathcal{P}_\alpha(\eta)(t) - \eta(t)\| = \left| \inf_{t \in [0,t]} \langle \eta(s), \alpha^\vee \rangle \right| \|\alpha^\vee\|$$

for any $t \geqslant 0$. By the previous lemma, there exists an integer ℓ_0 such that for any $t \geqslant \ell_0$,

$$\|\mathcal{P}_\alpha(\eta)(t) - \eta(t)\| = \left| \inf_{s \in [0,t]} \langle \eta(s), \alpha^\vee \rangle \right| \|\alpha^\vee\| = \left| \inf_{s \in [0,\ell_0]} \langle \eta(s), \alpha^\vee \rangle \right| \|\alpha^\vee\|.$$

Since the infimum $\inf_{s \in [0,\ell_0]} \langle \eta(s), \alpha^\vee \rangle$ does not depend on ℓ, we are done. Now $\frac{1}{\ell} \langle \mathcal{P}_\alpha(\eta(\ell)), \alpha^\vee \rangle$ and $\frac{1}{\ell} \langle \eta(\ell), \alpha^\vee \rangle$ admit the same limit.

(2) Consider $a \in \{2, \ldots, r-1\}$ and assume by induction that we have

$$\sup_{t \in [0,+\infty[} \left\| \mathcal{P}_{\alpha_{i_a}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta)(t) - m^{\otimes \infty}(t) \right\| < +\infty. \tag{5.4}$$

We then deduce

$$\lim_{\ell \to +\infty} \frac{1}{\ell} \langle \mathcal{P}_{\alpha_{i_a}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta)(\ell), \alpha_{i_{a-1}}^\vee \rangle = \langle \overline{m}, \alpha_{i_{a-1}}^\vee \rangle > 0. \tag{5.5}$$

This allows us to apply Lemma 5.3 with $\eta' = \mathcal{P}_{\alpha_{i_a}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta)$ and $\alpha = \alpha_{i_{a-1}}$. We get

$$\sup_{t\in[0,+\infty[} \left\| \mathcal{P}_{\alpha_{i_{a-1}}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta)(t) - \mathcal{P}_{\alpha_{i_a}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta)(t) \right\| < +\infty \, .$$

By using (5.4), this gives

$$\sup_{t\in[0,+\infty[} \left\| \mathcal{P}_{\alpha_{i_{a-1}}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta)(t) - m^{\otimes\infty}(t) \right\| < +\infty \, . \tag{5.6}$$

We thus have proved by induction that (5.6) holds for any $a = 2, \ldots, r-1$. ☐

Theorem 5.4. — *Let $\mathcal{W}$ be a random path defined on $\Omega = (B(\pi_\kappa)^{\otimes \mathbb{Z}_{\geqslant 0}}, p^{\otimes \mathbb{Z}_{\geqslant 0}})$ with drift path m and let $\mathcal{H} = \mathcal{P}(\mathcal{W})$ be its Pitman transform. Assume $\overline{m} \in \mathcal{D}_\kappa$. Then we have*

$$\lim_{\ell\to+\infty} \frac{1}{\ell} \sup_{t\in[0,\ell]} \left\| \mathcal{H}(t) - m^{\otimes\infty}(t) \right\| = 0 \text{ almost surely.}$$

Furthermore, the family of random variables $\left(\frac{\mathcal{H}(t) - m^{\otimes\infty}(t)}{\sqrt{t}} \right)_{t>0}$ converges in law as $t \to +\infty$ towards a centered Gaussian law $\mathcal{N}(0, \Gamma)$.

Proof. — Recall we have $\mathcal{P} = \mathcal{P}_{\alpha_{i_1}} \cdots \mathcal{P}_{\alpha_{i_r}}$ by Proposition 4.10. Consequently, by Theorem 5.1 and Lemma 5.3, the random variable $\mathcal{H} - \mathcal{W} = \mathcal{P}(\mathcal{W}) - \mathcal{W}$ is finite almost surely. It follows that

$$\begin{aligned} &\limsup_{\ell\to+\infty} \frac{1}{\ell} \sup_{t\in[0,\ell]} \left\| \mathcal{H}(t) - m^{\otimes l}(t) \right\| \\ &\quad \leqslant \limsup_{\ell\to+\infty} \frac{1}{\ell} \sup_{t\in[0,\ell]} \left\| \mathcal{W}(t) - m^{\otimes l}(t) \right\| + \limsup_{\ell\to+\infty} \frac{1}{\ell} \sup_{t\geqslant 0} \left\| \mathcal{H}(t) - \mathcal{W}(t) \right\| = 0 \end{aligned}$$

almost surely. To get the central limit theorem for the process $\mathcal{H}(t)$, we write similarly

$$\frac{\mathcal{H}(t) - m^{\otimes l}(t)}{\sqrt{t}} = \frac{\mathcal{W}(t) - m^{\otimes l}(t)}{\sqrt{t}} + \frac{\mathcal{H}(t) - \mathcal{W}(t)}{\sqrt{t}}.$$

By Theorem 5.1, the first term in this decomposition satisfies the central limit theorem; on the other hand the second one tends to 0 almost surely and one concludes using Slutsky theorem. ☐

6. Harmonic functions on multiplicative graphs associated to a central measure

Harmonic functions on the Young lattice are the key ingredients in the study of the asymptotic representation theory of the symmetric group. In fact, it was shown by Kerov and Vershik that these harmonic functions completely determine the asymptotic characters of the symmetric groups. We refer the reader to [6] for a detailed review. The Young lattice is an oriented graph with set of vertices the set of all partitions (each partition is conveniently identified with its Young diagram). We have an arrow $\lambda \to \Lambda$ between the partitions λ and Λ when Λ can be obtained by adding a box to λ. The Young lattice is an example of branching graph in the sense that its structure reflects the branching rules between the representation theory of the groups S_ℓ and $S_{\ell+1}$ with $\ell > 0$. One can also consider harmonic functions on other interesting graphs.

Here we focus on graphs defined from the weight lattice of $\mathfrak{g}$. These graphs depend on a fixed $\kappa \in P_+$ and are multiplicative in the sense that a positive integer, equal to a tensor product multiplicity, is associated to each arrow. We call them the multiplicative tensor graphs. We are going to associate a Markov chain to each multiplicative tensor graph $\mathcal{G}$. The aim of this section is to determine the harmonic functions on $\mathcal{G}$ when this associated Markov chain is assumed to have a drift. We will show this is equivalent to determine the central probability measure on the subset $\Omega_{\mathcal{C}}$ containing all the trajectories which remain in $\mathcal{C}$. When $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and $\kappa = \omega_1$ (that is $V(\kappa) = \mathbb{C}^{n+1}$ is the defining representation of $\mathfrak{sl}_{n+1}$), $\mathcal{G}$ is the subgraph of the Young lattice obtained by considering only the partitions with at most $n+1$ parts and we recover the harmonic functions as specializations of Schur polynomials. Observe nevertheless that the definition of $\mathcal{G}$ is related to tensor products of representations of $\mathfrak{sl}_{n+1}$ and not to an induction procedure on the irreducible representations of the symmetric group.

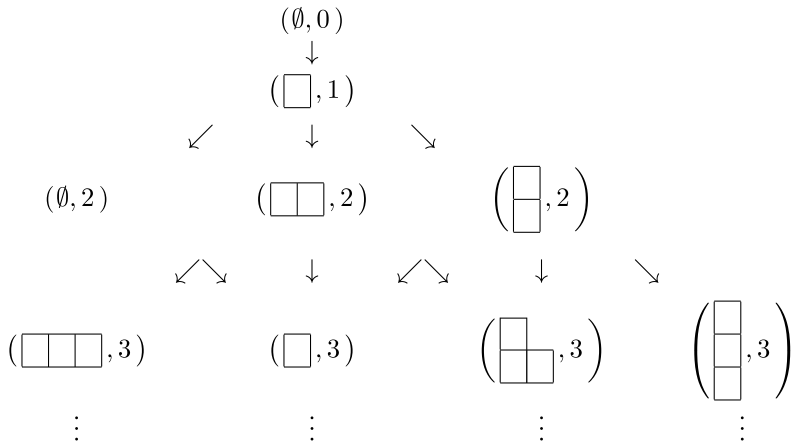
6.1. Multiplicative tensor graphs, harmonic functions and central measures

So assume $\kappa \in P_+$ is fixed. We denote by $\mathcal{G}$ the oriented graph with set of vertices the pairs $(\lambda, \ell) \in P_+ \times \mathbb{Z}_{\geqslant 0}$ and arrows

$$(\lambda, \ell) \overset{m_{\lambda,\kappa}^{\Lambda}}{\to} (\Lambda, \ell + 1)$$

with multiplicity $m_{\lambda,\kappa}^{\Lambda}$ when $m_{\lambda,\kappa}^{\Lambda} > 0$. In particular there is no arrows between (λ, ℓ) and $(\Lambda, \ell + 1)$ when $m_{\kappa,\kappa}^{\Lambda} = 0$.

Example 6.1. — Consider $\mathfrak{g} = \mathfrak{sp}_{2n}$. Then $P = \mathbb{Z}^n$ and P_+ can be identified with the set of partitions with at most n parts. For $\kappa = \omega_1$ the graph $\mathcal{G}$ contains the arrow $(\lambda, \ell) \to (\Lambda, \ell + 1)$ with $m_{\lambda,\kappa}^{\Lambda} = 1$ if and only if the Young diagram of Λ is obtained from that of λ by adding or deleting one box. We have drawn below the connected component of $(\emptyset, 0)$ up to $\ell \leqslant 3$.



Observe that in the case $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and $\kappa = \omega_1$, we have $m_{\lambda,\kappa}^{\Lambda} = 1$ if and only if of the Young diagram of Λ is obtained by adding one box to that of λ and $m_{\lambda,\kappa}^{\Lambda} = 0$ otherwise. So in this very particular case, it is not useful to keep the second component ℓ since it is equal to the rank of the partition λ. The vertices of $\mathcal{G}$ are simply the partitions with at most n parts (i.e. whose Young diagram has at most n rows).

Now return to the general case. Our aim is now to relate the harmonic functions on $\mathcal{G}$ and the central probability distributions on the set $\Omega_{\mathcal{C}}$ of infinite trajectories with steps in $B(\pi_\kappa)$ which remain in $\mathcal{C}$. We will identify the elements of $P_+ \times \mathbb{Z}_{\geqslant 0}$ as elements of the $\mathbb{R}$-vector space $P_{\mathbb{R}} \times \mathbb{R}$ (recall $P_{\mathbb{R}} = \mathbb{R}^n$). For any $\ell \geqslant 0$, set $H^\ell = \{\pi \in B(\pi_\kappa)^{\otimes \ell} \mid \operatorname{Im} \pi \subset \mathcal{C}\}$. Also if $\lambda \in P_+$, set $H_\lambda^\ell = \{\pi \in H^\ell \mid \mathrm{wt}(\pi) = \lambda\}$. Given $\pi \in H^\ell$, we denote by

$$C_\pi = \{\omega \in \Omega_{\mathcal{C}} \mid \Pi_\ell(\omega) = \pi\}$$

the cylinder defined by π. We have $C_\varnothing = \Omega_{\mathcal{C}}$. Each probability distribution Q on $\Omega_{\mathcal{C}}$ is determined by its values on the cylinders and we must have

$$\sum_{\pi \in H^\ell} \mathsf{Q}(C_\pi) = 1$$

for any $\ell \geqslant 0$.

Definition 6.2. — *A central probability distribution on* $\Omega_{\mathcal{C}}$ [3] *is a probability distribution* Q *on* $\Omega_{\mathcal{C}}$ *such that*

$$\mathsf{Q}(C_\pi) = \mathsf{Q}(C_{\pi'})$$

provided that $\mathrm{wt}(\pi) = \mathrm{wt}(\pi')$ *and* π, π' have the same length.

Consider a central probability distribution Q on $\Omega_{\mathcal{C}}$. For any $\ell \geqslant 1$, we have $\sum_{\pi \in H^\ell} \mathbb{Q}(C_\pi) = 1$, so it is possible to define a probability distribution q on H^ℓ by setting $q_\pi = \mathsf{Q}(C_\pi)$ for any $\pi \in H^\ell$. Since Q is central, we can also define the function

$$\varphi : \begin{cases} \mathcal{G} & \to \ [0,1] \\ (\lambda, \ell) & \longmapsto \mathsf{Q}(C_\pi) \end{cases} \tag{6.1}$$

where π is any path of H^ℓ. Now observe that

$$C_\pi = \bigsqcup_{\eta \in B(\pi_\kappa) | \mathrm{Im}(\pi \otimes \eta) \subset \mathcal{C}} C_{\pi \otimes \eta} \,.$$

This gives

$$\mathsf{Q}(C_\pi) = \sum_{\eta \in B(\pi_\kappa) | \mathrm{Im}(\pi \otimes \eta) \subset \mathcal{C}} \mathsf{Q}(C_{\pi \otimes \eta}) \,. \tag{6.2}$$

Assume $\pi \in H^\ell_\lambda$. By Theorem 3.3, the cardinality of the set $\{\eta \in B(\pi_\kappa) \mid \mathrm{Im}(\pi \otimes \eta) \subset \mathcal{C} \text{ and } \mathrm{wt}(\pi \otimes \eta) = \Lambda\}$ is equal to $m^\Lambda_{\lambda,\kappa}$. Therefore, we get

$$\varphi(\lambda, \ell) = \sum_\Lambda m^\Lambda_{\lambda,\kappa} \varphi(\Lambda, \ell + 1) \,. \tag{6.3}$$

This means that the function φ is *harmonic* on the multiplicative graph $\mathcal{G}$.

Conversely, if φ' is harmonic on the multiplicative graph $\mathcal{G}$, for any cylinder C_π in $\Omega_{\mathcal{C}}$ with $\pi \in H^\ell_\lambda$, we set $\mathsf{Q}'(C_\pi) = \varphi'(\lambda, \ell)$. Then Q' is a probability distribution on $\Omega_{\mathcal{C}}$ since it verifies (6.2) and is clearly central. Therefore, *a central probability distribution on* $\Omega_{\mathcal{C}}$ *is characterized by its associated harmonic function on* $\mathcal{G}$ defined by (6.1).

6.2. Harmonic function on a multiplicative tensor graph

Let Q be a central probability distribution on $\Omega_{\mathcal{C}}$. Consider $\pi = \pi_1 \otimes \cdots \otimes \pi_\ell \in H^\ell_\lambda$ and $\pi^\# = \pi_1 \otimes \cdots \otimes \pi_\ell \otimes \pi_{\ell+1} \in H^{\ell+1}_\Lambda$. Since we have the inclusion of events $C_{\pi^\#} \subset C_\pi$, we get

$$\mathsf{Q}(C_{\pi^\#} \mid C_\pi) = \frac{\mathsf{Q}(C_{\pi^\#})}{\mathsf{Q}(C_\pi)} = \frac{\varphi(\Lambda, \ell+1)}{\varphi(\lambda, \ell)}$$

[3] Alternatively, one can also consider central probability distributions $\mathbb{M}$ on the multiplicative graph $\mathcal{G}$. The paths in $\mathcal{C}$ should then be replaced by the weighted paths in $\mathcal{G}$ which slightly modifies the definition of central distributions (see [5]).

where the last equality is by definition of the harmonic function φ (which exists since Q is central). Let us emphasize that $\mathsf{Q}(C_{\pi^\#})$ and $\mathsf{Q}(C_\pi)$ do not depend on the paths π and $\pi^\#$ but only on their lengths and their ends λ and Λ. We then define a Markov chain $Z = (Z_\ell)_{\ell \geqslant 0}$ from $(\Omega_{\mathcal{C}}, \mathsf{Q})$ with values in $\mathcal{G}$ and starting at $Z_0 = (\mu, \ell_0) \in \mathcal{G}$ by

$$Z_\ell(\omega) = (\mu + \omega(\ell), \ell + \ell_0)\,.$$

Its transition probabilities are given by

$$\Pi_Z((\lambda, \ell), (\Lambda, \ell+1)) = \sum_{\pi^\#} \mathsf{Q}(C_{\pi^\#} \mid C_\pi)$$

where π is any path in H^ℓ_λ and the sum runs over all the paths $\pi^\# \in H^{\ell+1}_\Lambda$ such that $\pi^\# = \pi \otimes \pi_{\ell+1}$. Observe, the above sum does not depend on the choice of π in H^ℓ_λ because Q is central. Since there are $m^\Lambda_{\lambda,\kappa}$ such pairs, we get

$$\Pi_Z((\lambda, \ell), (\Lambda, \ell+1)) = \frac{m^\Lambda_{\lambda,\kappa}\varphi(\Lambda, \ell+1)}{\varphi(\lambda, \ell)} \tag{6.4}$$

and by (6.3) $Z = (Z_\ell)_{\ell \geqslant 0}$ is indeed a Markov chain. We then write $\mathsf{Q}_{(\mu,\ell_0)}(Z_\ell = (\lambda, \ell))$ for the probability that $Z_\ell = (\lambda, \ell)$ when the initial value is $Z_0 = (\mu, \ell_0)$. When $Z_0 = (0, 0)$, we simply write $\mathsf{Q}(Z_\ell = (\lambda, \ell)) = \mathsf{Q}_{(0,0)}(Z_\ell = (\lambda, \ell))$.

Lemma 6.3. — *For any $\mu, \lambda \in P_+$ and any integer $\ell_0 \geqslant 1$, we have*

$$\mathsf{Q}_{(\mu,\ell_0)}(Z_{\ell-\ell_0} = (\lambda, \ell)) = f^{(\ell-\ell_0)}_{\lambda/\mu} \frac{\varphi(\lambda, \ell)}{\varphi(\mu, \ell_0)} \text{ for any } \ell \geqslant \ell_0\,.$$

Proof. — By (6.4), the probability $\mathsf{Q}_{(\mu,\ell_0)}(Z_{\ell-\ell_0} = (\lambda, \ell))$ is equal to the quotient $\frac{\varphi(\lambda,\ell)}{\varphi(\mu,\ell_0)}$ times the number of paths in $\mathcal{C}$ of length $\ell - \ell_0$ starting at μ and ending at λ. The lemma thus follows from the fact that this number is equal to $f^{(\ell-\ell_0)}_{\lambda/\mu}$ by Theorem 3.3. $\square$

We will say that the family of Markov chains Z with transition probabilities given by (6.4) and initial distributions of the form $Z_0 = (\mu, \ell_0) \in \mathcal{G}$ admits a drift $\overline{m} \in P_{\mathbb{R}}$ when

$$\lim_{\ell \to +\infty} \frac{Z_\ell}{\ell} = (\overline{m}, 1) \text{ } \mathsf{Q}\text{-almost surely}$$

for any initial distributions $Z_0 = (\mu, \ell_0) \in \mathcal{G}$.

Theorem 6.4. — *Let Q be a central probability distribution on $\Omega_{\mathcal{C}}$ such that Z admits the drift $\overline{m} \in \mathcal{D}_\kappa$ (see (4.3)).*

(1) *The associated harmonic function φ on $\mathcal{G}$ satisfies $\varphi(\mu, \ell_0) = \frac{\tau^{-\mu} S_\mu(\tau)}{\tau^{-\ell_0\kappa} S^{\ell_0}_\kappa(\tau)}$ for any $\mu \in P_+$ and any $\ell_0 \geqslant 0$ where τ is determined by $\overline{m}$ as prescribed by Proposition 4.15.*

(2) *The probability transitions* (6.4) *do not depend on* ℓ.

(3) *For any* $\pi \in H_\mu^{\ell_0}$, *we have* $\mathsf{Q}(C_\pi) = \frac{\tau^{-\mu} S_\mu(\tau)}{\tau^{-\ell_0\kappa} S_\kappa^{\ell_0}(\tau)}$. *In particular,* Q *is the unique central probability distribution on* $\Omega_{\mathcal{C}}$ *such that* Z *admits the drift* $\overline{m}$. *We will denote it by* $\mathsf{Q}_{\overline{m}}$.

Proof. — (1) Consider a sequence of random dominant weights of the form $\lambda^{(\ell)} = \ell\overline{m} + o(\ell)$. We get by using Lemma 6.3

$$\frac{f_{\lambda^{(\ell)}/\mu}^{(\ell-\ell_0)}}{f_{\lambda^{(\ell)}}^{(\ell)}} \times \frac{1}{\varphi(\mu,\ell_0)} = f_{\lambda^{(\ell)}/\mu}^{(\ell-\ell_0)} \times \frac{\varphi(\lambda^{(\ell)},\ell)}{\varphi(\mu,\ell_0)} \times [f_{\lambda^{(\ell)}}^{(\ell)} \times \varphi(\lambda^{(\ell)},\ell)]^{-1}$$

$$= \frac{\mathsf{Q}_{(\mu,\ell_0)}(Z_{\ell-\ell_0} = (\lambda^{(\ell)},\ell))}{\mathsf{Q}(Z_\ell = (\lambda^{(\ell)},\ell))} = \frac{\mathsf{Q}_{(\mu,\ell_0)}(\frac{Z_{\ell-\ell_0}}{\ell-\ell_0} = (\frac{\lambda^{(\ell)}}{\ell-\ell_0}, \frac{\ell}{\ell-\ell_0}))}{\mathsf{Q}(\frac{Z_\ell}{\ell} = (\frac{\lambda^{(\ell)}}{\ell}, 1))}.$$

Since Z admits the drift $\overline{m}$, we obtain

$$\lim_{\ell\to+\infty} \frac{\mathsf{Q}_{(\mu,\ell_0)}(\frac{Z_{\ell-\ell_0}}{\ell-\ell_0} = (\frac{\lambda^{(\ell)}}{\ell-\ell_0}, \frac{\ell}{\ell-\ell_0}))}{\mathsf{Q}(\frac{Z_\ell}{\ell} = (\frac{\lambda^{(\ell)}}{\ell}, 1))} = \frac{1}{1} = 1$$

$$\text{and} \lim_{\ell\to+\infty} \frac{f_{\lambda^{(\ell)}/\mu}^{(\ell-\ell_0)}}{f_{\lambda^{(\ell)}}^{(\ell)}} \times \frac{1}{\varphi(\mu,\ell_0)} = 1\,.$$

This means that

$$\varphi(\mu,\ell_0) = \lim_{\ell\to+\infty} \frac{f_{\lambda^{(\ell)}/\mu}^{(\ell-\ell_0)}}{f_{\lambda^{(\ell)}}^{(\ell)}}\,.$$

Now by Theorem 4.13 and since $\overline{m} \in \mathcal{D}_\kappa$ we can write

$$\lim_{\ell\to+\infty} \frac{f_{\lambda^{(\ell)}/\mu}^{(\ell-\ell_0)}}{f_{\lambda^{(\ell)}}^{(\ell)}} = \lim_{\ell\to+\infty} \frac{f_{\lambda^{(\ell)}/\mu}^{(\ell-\ell_0)}}{f_{\lambda^{(\ell)}}^{(\ell-\ell_0)}} \times \lim_{\ell\to+\infty} \frac{f_{\lambda^{(\ell)}}^{(\ell-\ell_0)}}{f_{\lambda^{(\ell)}}^{(\ell)}} = \frac{\tau^{-\mu} S_\mu(\tau)}{\tau^{-\ell_0\kappa} S_\kappa^{\ell_0}(\tau)}$$

where $\tau \in]0,1[^n$ is determined by the drift $\overline{m}$ as prescribed by Proposition 4.15. We thus obtain $\varphi(\mu,\ell_0) = \frac{\tau^{-\mu} S_\mu(\tau)}{\tau^{-\ell_0\kappa} S_\kappa^{\ell_0}(\tau)}$.

(2) We have $\Pi_Z((\lambda,\ell),(\Lambda,\ell{+}1)) = \frac{m_{\lambda,\kappa}^{\Lambda}\varphi(\Lambda,\ell+1)}{\varphi(\lambda,\ell)} = \frac{S_\Lambda(\tau)}{S_\kappa(\tau)S_\lambda(\tau)}\tau^{\kappa+\lambda-\Lambda} m_{\lambda,\kappa}^{\Lambda}$ which does not depend on ℓ.

(3) This follows from the fact that $\mathsf{Q}(C_\pi) = \varphi(\lambda,\ell)$ for any $\pi \in H_\lambda^\ell$. □

Consider $\overline{m} \in \mathcal{D}_\kappa$ and write τ for the corresponding n-tuple in $]0,1[^n$. Let W be the random walk starting at 0 defined on P from κ and τ as in §4.3.

Corollary 6.5. — *Let* Q *be a central probability distribution on* $\Omega_{\mathcal{C}}$ *such that* Z *admits the drift* $\overline{m} \in \mathcal{D}_\kappa$. *Then, the processes* $(Z_\ell)_\ell$ *and* $((\mathcal{P}(W_\ell),\ell))_\ell$ *have the same law.*

Proof. — By the previous theorem, the transitions of the Markov chain Z on $\mathcal{G}$ are given by $\Pi_Z((\lambda,\ell),(\Lambda,\ell+1)) = \frac{m_{\lambda,\kappa}^{\Lambda}\varphi(\Lambda,\ell+1)}{\varphi(\lambda,\ell)}$. By Theorem 4.12, the transition matrix Π_Z thus coincides with the transition matrix of $\mathcal{P}(W)$ as desired. □

Let $\mathsf{P}_{\overline{m}}$ and $\mathsf{Q}_{\overline{m}}$ be the probability distributions associated to $\overline{m}$ (recall $\overline{m}$ determines $\tau \in]0,1[^n$) defined on the spaces Ω and $\Omega_{\mathcal{C}}$, respectively.

Corollary 6.6. — *The Pitman transform $\mathcal{P}$ is a homomorphism of probability spaces between $(\Omega, \mathsf{P}_{\overline{m}})$ and $(\Omega_{\mathcal{C}}, \mathsf{Q}_{\overline{m}})$, that is we have*

$$\mathsf{Q}_{\overline{m}}(C_\pi) = \mathsf{P}_{\overline{m}}(\mathcal{P}^{-1}(C_\pi))$$

for any $\ell \geqslant 1$ and any $\pi \in H^\ell$.

Proof. — Assume $\pi \in H_\lambda^\ell$. We have $\mathsf{Q}_{\overline{m}}(C_\pi) = \varphi(\lambda,\ell) = \frac{\tau^{-\lambda}S_\lambda(\tau)}{\tau^{-\ell\kappa}S_\kappa^\ell(\tau)}$. By definition of the generalized Pitman transform $\mathcal{P}$, $\mathcal{P}^{-1}(C_\pi) = \{\omega \in \Omega \mid \mathcal{P}(\Pi_\ell(\omega)) = \pi\}$, that is $\mathcal{P}^{-1}(C_\pi)$ is the set of all trajectories in Ω which remain in the connected component $B(\pi) \subset B(\pi_\kappa)^{\otimes \ell}$ for any $t \in [0,\ell]$. We thus have $\mathsf{P}_{\overline{m}}(\mathcal{P}^{-1}(C_\pi)) = p^{\otimes \ell}(B(\pi)) = \frac{\tau^{-\lambda}S_\lambda(\tau)}{\tau^{-\ell\kappa}S_\kappa^\ell(\tau)}$ by Assertion (2) of Theorem 4.12. Therefore we get $\mathsf{P}_{\overline{m}}(\mathcal{P}^{-1}(C_\pi)) = \mathsf{Q}_{\overline{m}}(C_\pi)$ as desired. □

7. Isomorphism of dynamical systems

In this section, we first explain how the trajectories in Ω and $\Omega_{\mathcal{C}}$ can be equipped with natural shifts S and J, respectively. We then prove that the generalized Pitman transform $\mathcal{P}$ intertwines S and J. When $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and $\kappa = \omega_1$, we recover in particular some analogous results from [16].

7.1. The shift operator

Let $S : \Omega \to \Omega$ be the shift operator on Ω defined by

$$S(\pi) = S(\pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \cdots) := (\pi_2 \otimes \pi_3 \otimes \ldots)$$

for any trajectory $\pi = \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \cdots \in \Omega$. Observe that S is measure preserving for the probability distribution $\mathsf{P}_{\overline{m}}$. We now introduce the map $J : \Omega_{\mathcal{C}} \to \Omega_{\mathcal{C}}$ defined by

$$J(\pi) = \mathcal{P} \circ S(\pi)$$

for any trajectory $\pi \in \Omega_{\mathcal{C}}$. Observe that $S(\pi)$ does not belong to $\Omega_{\mathcal{C}}$ in general so we need to apply the Pitman transform $\mathcal{P}$ to ensure that J takes values in $\Omega_{\mathcal{C}}$.

7.2. Isomorphism of dynamical systems

Theorem 7.1.

(1) *The Pitman transform is a factor map of dynamical systems, i.e. the following diagram commutes:*

$$\begin{array}{ccc} \Omega & \overset{S}{\rightarrow} & \Omega \\ \mathcal{P}\downarrow & & \downarrow\mathcal{P} \\ \Omega_{\mathcal{C}} & \underset{J}{\rightarrow} & \Omega_{\mathcal{C}} \end{array}$$

(2) *For any $\overline{m} \in \mathcal{D}_\kappa$, the transformation $J : \Omega_{\mathcal{C}} \to \Omega_{\mathcal{C}}$ is measure preserving with respect to the (unique) central probability distribution $\mathsf{Q}_{\overline{m}}$ with drift $\overline{m}$.*

Proof. — (1) To prove this assertion, it suffices to establish that the above diagram commutes on trajectories of any finite length $\ell > 0$. So consider $\pi = \pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_\ell \in B(\pi_\kappa)^{\otimes \ell}$ and set $\mathcal{P}(\pi) = \pi_1^+ \otimes \pi_2^+ \otimes \cdots \otimes \pi_\ell^+$. We have to prove that

$$\mathcal{P}(\pi_2 \otimes \cdots \otimes \pi_\ell) = \mathcal{P}(\pi_2^+ \otimes \cdots \otimes \pi_\ell^+)$$

which means that both vertices $\pi_2 \otimes \cdots \otimes \pi_\ell$ and $\pi_2^+ \otimes \cdots \otimes \pi_\ell^+$ belong to the same connected component of $B(\pi_\kappa)^{\otimes \ell-1}$. We know that $\mathcal{P}(\pi)$ is the highest weight vertex of $B(\pi)$. This implies that there exists a sequence of root operators $\tilde{e}_{i_1}, \ldots, \tilde{e}_{i_r}$ such that

$$\pi_1^+ \otimes \pi_2^+ \otimes \cdots \otimes \pi_\ell^+ = \tilde{e}_{i_1} \cdots \tilde{e}_{i_r}(\pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_\ell)\,. \tag{7.1}$$

By (3.6), we can define a subset $X := \{k \in \{1, \ldots, r\}$ such that $\tilde{e}_{i_k}$ acts on the first component of the tensor product $\tilde{e}_{i_{k+1}} \cdots \tilde{e}_{i_r}(\pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_\ell)\}$. We thus obtain

$$\pi_2^+ \otimes \cdots \otimes \pi_\ell^+ = \prod_{k \in \{1,\ldots,r\} \setminus X} \tilde{e}_{i_k}(\pi_2 \otimes \cdots \otimes \pi_\ell)$$

which shows that $\pi_2 \otimes \cdots \otimes \pi_\ell$ and $\pi_2^+ \otimes \cdots \otimes \pi_\ell^+$ belong to the same connected component of $B(\pi_\kappa)^{\otimes \ell-1}$. They thus have the same highest weight path as desired.

(2) Let $A \subset \Omega_{\mathcal{C}}$ be a Q-measurable set. We have $\mathsf{Q}(J^{-1}(A)) = \mathsf{P}(\mathcal{P}^{-1}(J^{-1}(A))$ since $\mathcal{P}$ is a homomorphism. Using the fact that the previous diagram commutes and S preserves P, we get $\mathsf{Q}(J^{-1}(A)) = \mathsf{P}(S^{-1}(\mathcal{P}^{-1}(A))) = \mathsf{P}(\mathcal{P})$, so $\mathsf{Q}(J^{-1}(A)) = \mathsf{Q}(A)$ since $\mathcal{P}$ is a homomorphism. $\square$

8. Dual random path and the inverse Pitman transform

It is well known (see [15]) that the Pitman transform on the line is reversible. The aim of this paragraph is to establish that $\mathcal{E}$, restricted to a relevant set of infinite trajectories with measure 1, can be regarded as a similar inverse for the generalized Pitman transform $\mathcal{P}$. We *assume in the remaining of the paper that* $\overline{m} \in D_\kappa$. This permits to define a random walk $\mathcal{W}$ and a Markov chain $\mathcal{H} = \mathcal{P}(\mathcal{W})$ as in Section 4. Since $\overline{m}$ is fixed, we will denote for short by P and Q the probability distributions $\mathsf{P}_{\overline{m}}$ and $\mathsf{Q}_{\overline{m}}$, respectively.

8.1. Typical trajectories

Consider $\overline{m} \in \mathcal{D}_\kappa$ and the associated distributions $\mathsf{P}_{\overline{m}}$ and $\mathsf{Q}_{\overline{m}}$ defined on Ω and $\Omega_{\mathcal{C}}$, respectively. We introduce the subsets of typical trajectories $\Omega^{\mathrm{typ}}, \Omega^{\iota\mathrm{typ}}$ and $\Omega_{\mathcal{C}}^{\mathrm{typ}}$ as

$$\Omega^{\mathrm{typ}} = \{\pi \in \Omega \mid \lim_{\ell \to +\infty} \frac{1}{\ell} \langle \pi(\ell), \alpha_i^\vee \rangle = \langle \overline{m}, \alpha_i^\vee \rangle \in \mathbb{R}_{>0} \quad \forall i = 1, \dots, n\},$$

$$\Omega^{\iota\mathrm{typ}} = \{\pi \in \Omega \mid \lim_{\ell \to +\infty} \frac{1}{\ell} \langle \pi(\ell), \alpha_i^\vee \rangle = \langle w_0(\overline{m}), \alpha_i^\vee \rangle \in \mathbb{R}_{<0} \quad \forall i = 1, \dots, n\},$$

$$\Omega_{\mathcal{C}}^{\mathrm{typ}} = \{\pi \in \Omega_{\mathcal{C}} \mid \lim_{\ell \to +\infty} \frac{1}{\ell} \langle \pi(\ell), \alpha_i^\vee \rangle = \langle \overline{m}, \alpha_i^\vee \rangle \in \mathbb{R}_{>0} \quad \forall i = 1, \dots, n\}.$$

By Theorems 5.1 and 5.4, we have

$$\mathsf{P}_{\overline{m}}(\Omega^{\mathrm{typ}}) = 1 \ \text{ and } \ \mathsf{Q}_{\overline{m}}(\Omega_{\mathcal{C}}^{\mathrm{typ}}) = 1\,.$$

We are going to see that the relevant Pitman inverse coincides with $\mathcal{E}$ acting on the trajectories of $\Omega_{\mathcal{C}}^{\mathrm{typ}}$ and we will show that $\mathcal{E}(\mathcal{H})$ is then a random trajectory with drift $w_0(\overline{m})$ where w_0 is the longest element of the Weyl group W.

8.2. An involution on the trajectories

We have seen that the reverse map r on paths defined in (3.4) flips the actions of the operators $\tilde{e}_i$ and $\tilde{f}_i$ on any connected crystal $B(\pi_\kappa)$ of highest path π_κ. Nevertheless, we have

$$r(B(\pi_\kappa) \neq B(\pi_\kappa)$$

in general. So $r(\Omega) \neq \Omega$. To overcome this difficulty we can replace our space of trajectories Ω by the set $\mathcal{L}_\infty$ of all infinite paths defined from the set $\mathcal{L}$ of

§3.2. But $\mathcal{L}_\infty$ has not a probability space structure neither a simple algebraic interpretation. Rather, it is interesting to give another definition of $\mathcal{E}$ where the involution r is replaced by the Lusztig involution ι which stabilizes $B(\pi_\kappa)$ (see for example [10]). The longest element w_0 of the Weyl group W (which is an involution) induces an involution $*$ on the set of simple roots defined by $\alpha_{i^*} = -w_0(\alpha_i)$ for any $i = 1, \ldots, n$. Write π_κ^{low} for the lowest weight vertex of $B(\pi_\kappa)$, that is π_κ^{low} is the unique vertex of $B(\pi_\kappa)$ such that $\tilde{f}_i(\pi_\kappa^{low}) = 0$ for any $i = 1, \ldots, n$. The involution ι is first defined on the crystal $B(\pi_\kappa)$ by

$$\iota(\pi_\kappa) = \pi_\kappa^{low} \text{ and } \iota(\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \pi_\kappa) = \tilde{e}_{i_1^*} \cdots \tilde{e}_{i_r^*}(\pi_\kappa^{low})$$

for any sequence of crystal operators $\tilde{f}_{i_1}, \ldots, \tilde{f}_{i_r}$ with $r > 0$. This means that ι flips the orientation of the arrows of $B(\pi_\kappa)$ and each label i is changed to i^*. In particular, we have $\mathrm{wt}(\iota(\pi)) = w_0(\mathrm{wt}(\pi))$ for any $\pi \in B(\pi_\kappa)$. We extend ι by linearity on the linear combinations of paths in $B(\pi_\kappa)$.

We next define the involution ι on $B(\pi_\kappa)^{\otimes \ell}$ by setting

$$\iota(\pi_1 \otimes \cdots \otimes \pi_\ell) = \iota(\pi_\ell) \otimes \cdots \otimes \iota(\pi_1)$$

for any $\pi_1 \otimes \cdots \otimes \pi_\ell \in B(\pi_\kappa)^{\otimes \ell}$. It then follows from (3.6) that for any $i = 1, \ldots, n$ we have

$$\iota \tilde{f}_i \iota(\pi_1 \otimes \cdots \otimes \pi_\ell) = \tilde{e}_{i^*}(\pi_1 \otimes \cdots \otimes \pi_\ell)\,. \tag{8.1}$$

Thus the involution ι flips the lowest and highest weight paths, reverses the arrows and changes each label i to i^*. In particular, for any connected component $B(\eta)$ of $B(\pi_\kappa)^{\otimes \ell}$, the set $\iota(B(\eta))$ is also a connected component of $B(\pi_\kappa)^{\otimes \ell}$. In addition, we have

$$\mathrm{wt}(\iota(\pi_1 \otimes \cdots \otimes \pi_\ell)) = w_0(\mathrm{wt}(\pi_1 \otimes \cdots \otimes \pi_\ell)). \tag{8.2}$$

Remark 8.1. — Observe that ι is very close to r. The crucial difference is that the crystals $\iota(B(\pi_\kappa))$ and $B(\pi_\kappa)$ coincide whereas $r(B(\pi_\kappa))$ is not a crystal in general.

Example 8.2. — We resume Example 4.7 and consider $\mathfrak{g} = \mathfrak{sp}_4$ and $\kappa = \omega_1$. In this particular case we get $w_0 = -\mathrm{id}$ and $\iota = r$ on $B(\pi_{\omega_1})$. We then have $\iota(\pi_1) = \pi_{\overline{1}}$ and $\iota(\pi_2) = \pi_{\overline{2}}$. In the picture below we have drawn the path η and $\iota(\eta)$ where

$$\begin{aligned}\eta &= 112111\bar{2}\bar{1}\bar{2}111222\bar{1}\bar{2}111222111\bar{2}\bar{1}22211,\\ \iota(\eta) &= \bar{1}\bar{1}\bar{2}\bar{2}\bar{2}12\bar{1}\bar{1}\bar{1}\bar{2}\bar{2}\bar{2}\bar{1}\bar{1}\bar{1}21\bar{2}\bar{2}\bar{2}\bar{1}\bar{1}\bar{1}212\bar{1}\bar{1}\bar{1}\bar{2}\bar{1}\bar{1}.\end{aligned}$$

Here we simply write $a \in \{\bar{2}, \bar{1}, 1, 2\}$ instead of π_a and omit for short the symbols $\otimes$.

The following proposition shows we can replace the involution r by ι in the definition of the dual Pitman transform.

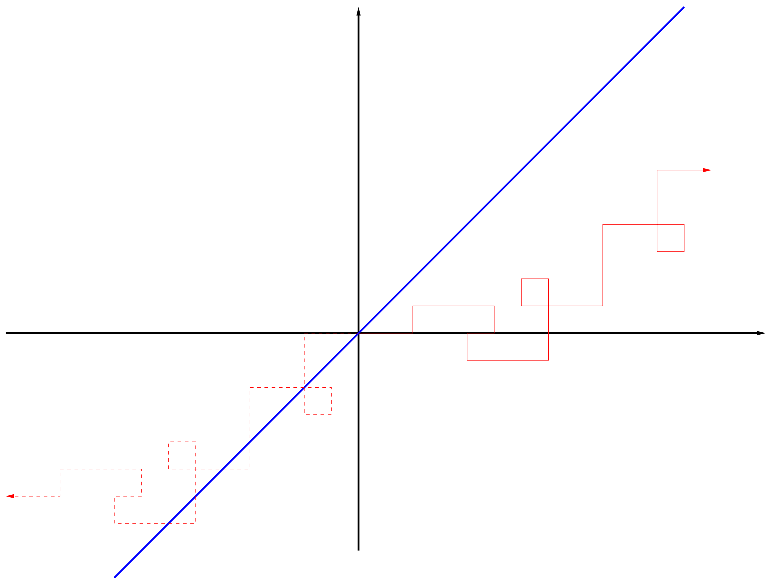


Figure 8.1. The paths η (in red) and $\iota(\eta)$ (in dashed read)

Proposition 8.3. — *We have*

$$\mathcal{E} = r\mathcal{P}r = \iota\mathcal{P}\iota\,.$$

Proof. — Observe first that for any simple root α_i, and any path $\eta \in B(\pi_\kappa)^{\otimes \ell}$, we have by (8.1) $\mathcal{E}_{\alpha_i}(\eta) = \iota\mathcal{P}_{\alpha_{i^*}}\iota(\eta)$ because the action of $\mathcal{E}_{\alpha_i}$ on any path reduces to a product of operators $\tilde{f}_i$. Since $\mathcal{E} = \mathcal{E}_{\alpha_1}\cdots\mathcal{E}_{\alpha_r}$, we get $\mathcal{E} = \iota\mathcal{P}_{\alpha_{1^*}}\cdots\mathcal{P}_{\alpha_{r^*}}\iota$. But $\mathcal{P}_{\alpha_{1^*}}\cdots\mathcal{P}_{\alpha_{r^*}} = \mathcal{P}_{\alpha_1}\cdots\mathcal{P}_{\alpha_r} = \mathcal{P}$ by Proposition 4.10 because $w_0 = s_{\alpha_{1^*}}\cdots s_{\alpha_{r^*}}$ is also a minimal length decomposition of w_0. We therefore obtain $\mathcal{E} = \iota\mathcal{P}\iota$ as desired. $\square$

8.3. Dual random path

Let us define the probability distribution p^ι on $B(\pi_\kappa)$ by setting

$$p^\iota_\pi = p_{\iota(\pi)} = \frac{\tau^{\kappa - w_0 \mathrm{wt}(\pi)}}{S_\kappa(\tau)} \text{ for any } \pi \in B(\pi_\kappa) \tag{8.3}$$

and consider a random variable Y with values in $B(\pi_\kappa)$ and probability distribution p^ι. Set $m^\iota = E(Y)$, $\overline{m}^\iota = m^\iota(1)$ and $\mathcal{D}^\iota_\kappa = w_0(\mathcal{D}_\kappa)$.

Lemma 8.4. — *We have*

(1) $m^\iota = \iota(m)$

(2) $\overline{m}^\iota = w_0(\overline{m})$. *In particular,* $\overline{m} \in \mathcal{D}_\kappa$ *if and only if* $\overline{m}^\iota \in \mathcal{D}_\kappa^\iota$.

Proof. — By using that ι is an involution on $B(\pi_\kappa)$, we get

$$m^\iota = \sum_{\pi \in B(\pi_\kappa)} p_\pi^\iota \pi = \sum_{\pi \in B(\pi_\kappa)} p_{\iota(\pi)} \pi = \iota\left(\sum_{\pi \in B(\pi_\kappa)} p_{\iota(\pi)} \iota(\pi)\right) = \iota(m)$$

which proves Assertion (1). In particular, if we set $\overline{m}^\iota = m^\iota(1)$, we have $\overline{m}^\iota = w_0(\overline{m})$ and Assertion (2) follows. □

Similarly, we may consider the probability measure $(p^\iota)^{\otimes \ell}$ on $B(\pi_\kappa)^{\otimes \ell}$ defined by

$$\begin{aligned}(p^\iota)^{\otimes \ell}(\pi_1 \otimes \cdots \otimes \pi_\ell) = p^\iota(\pi_1) \cdots p^\iota(\pi_\ell) &= \frac{\tau^{\ell\kappa - w_0(\pi_1(1) + \cdots \pi_\ell(1))}}{S_\kappa(\tau)^\ell} \\ &= \frac{\tau^{\ell\kappa - w_0(\mathrm{wt}(b))}}{S_\kappa(\tau)^\ell}.\end{aligned}$$ [4]

By the Kolmogorov extension theorem, the family of probability mesures $((p^\iota)^{\otimes \ell})_\ell$ admits a unique extension $\mathsf{P}^\iota := (p^\iota)^{\otimes \mathbb{Z}_{\geqslant 0}}$ to the space $B(\pi_\kappa)^{\otimes \mathbb{Z}_{\geqslant 0}}$. For any $\ell \geqslant 1$, let $Y_\ell : B(\pi_\kappa)^{\otimes \mathbb{Z}_{\geqslant 0}} \longrightarrow B(\pi_\kappa)$ be the canonical projection on the ℓ^{th} coordinate; by construction, the variables $Y_1, Y_2, \cdots$ are independent and identically distributed with the same law as Y. We denote by $\mathcal{W}^\iota$ the random path defined by

$$\mathcal{W}^\iota(t) := Y_1(1) + Y_2(1) + \cdots + Y_{\ell-1}(1) + Y_\ell(t - \ell + 1) \text{ for } t \in [\ell - 1, \ell].$$

Then $\mathcal{W}^\iota$ is defined on the probability space $\Omega^\iota = (B(\pi_\kappa)^{\otimes \mathbb{Z}_{\geqslant 0}}, \mathsf{P}^\iota)$; notice that the set of trajectories of Ω^ι is the same as the one of Ω but the probability P^ι is defined from p^ι. We have in particular

$$\mathsf{P}^\iota(\Omega^{\iota\mathrm{typ}}) = 1.$$

We also define the random walk $W^\iota = (W_\ell^\iota)_{\ell \geqslant 1}$ such that $W_\ell^\iota = \mathcal{W}^\iota(\ell)$ for any $\ell \geqslant 1$. Let $\mathcal{H}^\iota$ be the random process $\mathcal{H}^\iota = \mathcal{P}(\mathcal{W}^\iota)$ and define $H^\iota = (H_\ell^\iota)_{\ell \geqslant 1}$ such that $H_\ell^\iota = \mathcal{H}^\iota(\ell)$ for any $\ell \geqslant 1$. We then have (see [9, Proposition 4.6])

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[4] We now have two probability measures $p^{\otimes \ell}$ and $(p^\iota)^{\otimes \ell}$ on $B(\pi_\kappa)^{\otimes \ell}$. Observe that for any event $E \subset B(\pi_\kappa)^{\otimes \ell}$, we get

$$(p^\iota)^{\otimes \ell}(E) = p^{\otimes \ell}(\iota(E)). \tag{8.4}$$

Theorem 8.5.

(1) *For any $\beta, \eta \in P$, one has*

$$\mathsf{P}^\iota(W_{\ell+1}^\iota = \beta \mid W_\ell^\iota = \eta) = K_{\kappa,\beta-\eta} \frac{\tau^{\kappa - w_0(\beta-\eta)}}{S_\kappa(\tau)}.$$

(2) *The random sequence H^ι is a Markov chain with the same law as H, that is with transition matrix*

$$\Pi(\mu, \lambda) = \frac{S_\lambda(\tau)}{S_\kappa(\tau) S_\mu(\tau)} \tau^{\kappa+\mu-\lambda} m_{\mu,\kappa}^\lambda$$

where $\lambda, \mu \in P_+$.

(3) *For any path $\pi \in H_\lambda^\ell$,we have*

$$\mathsf{P}^\iota(\mathcal{H}^\iota = \pi) = \mathsf{P}(\mathcal{H} = \pi) = \frac{\tau^{\ell\kappa - \lambda} S_\lambda(\tau)}{S_\kappa(\tau)^\ell}.$$

8.4. Asymptotic behavior in a fixed component

Consider $\pi \in B(\pi_\kappa)^{\otimes \ell}$ and $\eta \in \Omega$ such that $\frac{1}{L}\langle \eta(L), \alpha_i^\vee \rangle$ converges to a positive limit for any positive root $\alpha_i, i = 1, \ldots n$. For any L, set $\Pi_L(\eta) = \eta_L$ so that we have $\eta_L \in B(\pi_\kappa)^{\otimes L}$. Since $\pi \in B(\pi_\kappa)^{\otimes \ell}$, the path $\eta_L \otimes \pi$ is defined on $[0, \ell + L]$. More precisely, we have $\eta_L \otimes \pi(t) = \eta_L(t)$ for $t \in [0, L[$ and $\eta_L \otimes \pi(t) = \eta_L(L) + \pi(t - L)$ for $t \in [L, \ell + L]$.

Lemma 8.6. — *With the previous notation, we get*

$$\mathcal{P}(\eta_L \otimes \pi) = \mathcal{P}(\eta_L) \otimes \pi$$

for L sufficiently large.

Proof. — Recall that $\mathcal{P}$=$\mathcal{P}_{\alpha_{i_1}} \cdots \mathcal{P}_{\alpha_{i_r}}$. One proves by induction that for any $s = 1, \ldots, r$, there exists a nonnegative integer L_s such that

$$\mathcal{P}_{\alpha_{i_s}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta_L \otimes \pi) = \mathcal{P}_{\alpha_{i_s}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta_L) \otimes \pi$$

for any $L > L_s$ and $\lim_{L \to +\infty} \langle \mathcal{P}_{\alpha_{i_s}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta)(L), \alpha^\vee \rangle = +\infty$ for any simple root α. The lemma then follows by putting $s = 1$. ☐

Let $\mathcal{H} = (\mathcal{H}_\ell)_{\ell \geqslant 1}$ be a random process in $\Omega_{\mathcal{C}} \subset \Omega$ with distribution $\mathsf{Q}_{\overline{m}}$. Since $\mathcal{H}$ takes value in Ω, we can write $\mathcal{H}_\ell = T_1 \otimes \cdots \otimes T_\ell$ for any $\ell \geqslant 1$, where the random variable T_i takes values in $B(\pi_\kappa)$ for any $i \geqslant 1$. By Corollary 6.6, there exists a random process $\mathcal{W}$ with values in Ω and distribution $\mathsf{P}_{\overline{m}}$ such that $\mathcal{H}$ and $\mathcal{P}(\mathcal{W})$ coincide $\mathsf{P}_{\overline{m}}$-almost surely. Notice that we also have $\mathcal{W}_\ell = X_1 \otimes \cdots \otimes X_\ell$ for any $\ell \geqslant 1$, where X_ℓ is a random variable with values in $B(\pi_\kappa)$ with the law defined in (4.5).

Proposition 8.7. — *$\mathsf{P}_{\overline{m}}$-almost surely, the random variables T_ℓ and X_ℓ coincide for any large enough ℓ.*

Proof. — Consider a trajectory $\omega \in \Omega^{\text{typ}}$. For any $\ell \geqslant 1$ and set $\Pi_\ell(\omega) = \pi_1 \otimes \cdots \otimes \pi_\ell$. We can apply Lemma 8.6 to $\pi_1 \otimes \cdots \otimes \pi_{\ell-1} \otimes \pi_\ell$ since we have $\omega \in \Omega^{\text{typ}}$. Hence, for ℓ sufficiently large, we have

$$\mathcal{P}(\pi_1 \otimes \cdots \otimes \pi_{\ell-1} \otimes \pi_\ell) = \mathcal{P}(\pi_1 \otimes \cdots \otimes \pi_{\ell-1}) \otimes \pi_\ell .$$

We thus have $\lim_{\ell \to +\infty}(T_\ell - X_\ell) = 0$ on Ω^{typ}. We are done since $\mathsf{P}_{\overline{m}}(\Omega^{\text{typ}}) = 1$. □

8.5. The transformations $\mathcal{P}$ and $\mathcal{E}$ on infinite paths

The transformations $\mathcal{P}$ and $\mathcal{E}$ defined on $B(\pi_\kappa)^{\otimes \ell}$ can be extended to Ω and $\Omega^{\text{typ}}_{\mathcal{C}}$, respectively. For any $\eta \in \Omega$ and any simple root α, set

$$\mathcal{P}_\alpha(\eta)(t) = \eta(t) - \inf_{s \in [0,t]} \langle \eta(s), \alpha^\vee \rangle \alpha \quad \text{and} \quad \mathcal{P}(\eta) = \mathcal{P}_{\alpha_{i_1}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta) .$$

Similarly, for any $\eta \in \Omega$ and any simple root α such that $\lim_{t \to \infty} \langle \eta(t), \alpha^\vee \rangle = +\infty$, the path $\mathcal{E}_\alpha(\eta)$ such that

$$\mathcal{E}_\alpha(\eta)(t) = \eta(t) - \inf_{s \in [t,+\infty[} \langle \eta(s), \alpha^\vee \rangle \, \alpha + \inf_{s \in [0,+\infty[} \langle \eta(s), \alpha^\vee \rangle \, \alpha$$

for any $t \geqslant 0$ is well defined.

Proposition 8.8. — *Consider η in $\Omega^{\text{typ}}_{\mathcal{C}}$. Then $\mathcal{E}(\eta) = \mathcal{E}_{\alpha_{i_1}} \cdots \mathcal{E}_{\alpha_{i_r}}(\eta)$ is well defined and belongs to $\Omega^{\iota \text{typ}}$.*

Proof. — We proceed by induction and show that $\mathcal{E}(\eta) = \mathcal{E}_{\alpha_{i_a}} \cdots \mathcal{E}_{\alpha_{i_r}}(\eta)$ is well-defined for any $a = 1, \ldots, r$. It suffices to prove that

$$\lim_{t \to \infty} \langle \eta(t), \alpha_r \rangle = +\infty \text{ and } \lim_{t \to \infty} \langle \mathcal{E}_{\alpha_{i_{a+1}}} \cdots \mathcal{E}_{\alpha_{i_r}} \eta(t), \alpha_a \rangle = +\infty$$

for any $a = 1, \ldots r-1$. We get $\lim_{t \to \infty} \langle \eta(t), \alpha_r \rangle = +\infty$ directly from the definition of $\Omega^{\text{typ}}_{\mathcal{C}}$. Now for any $a = 1, \ldots, r-1$, and any integer $\ell \geqslant 0$, we have that $\mathcal{E}_{\alpha_{i_{a+1}}} \cdots \mathcal{E}_{\alpha_{i_r}} \eta(\ell)$ is the weight of the path $\Pi_\ell(\eta)$. So we obtain by (4.9)

$$\langle \mathcal{E}_{\alpha_{i_{a+1}}} \cdots \mathcal{E}_{\alpha_{i_r}} \eta(\ell), \alpha_a \rangle = \langle s_{i_{a+1}} \cdots s_{i_r} \eta(\ell), \alpha_a \rangle = \langle \eta(\ell), s_{i_r} \cdots s_{i_{a+1}}(\alpha_a) \rangle .$$

Since w_0 is an involution, $w_0 = s_{i_r} \cdots s_{i_1}$ is also a minimal length decomposition. By (3.1), we know that $s_{i_r} \cdots s_{i_{a+1}}(\alpha_a) = \alpha$ is a positive root. It follows that

$$\lim_{\ell \to \infty} \langle \eta(\ell), s_{i_r} \cdots s_{i_{a+1}}(\alpha_a) \rangle = \lim_{\ell \to \infty} \langle \mathcal{E}_{\alpha_{i_{a+1}}} \cdots \mathcal{E}_{\alpha_{i_r}} \eta(\ell), \alpha_a \rangle = +\infty .$$

We finally get $\lim_{t\to\infty}\langle \mathcal{E}_{\alpha_{i_{a+1}}}\cdots\mathcal{E}_{\alpha_{i_r}}\eta(t),\alpha_a\rangle = +\infty$ because

$$\left\| \mathcal{E}_{\alpha_{i_{a+1}}}\cdots\mathcal{E}_{\alpha_{i_r}}(\eta(t)) - \mathcal{E}_{\alpha_{i_{a+1}}}\cdots\mathcal{E}_{\alpha_{i_r}}(\eta(\ell))\right\| \text{ with } \ell = \lfloor t \rfloor$$

is bounded by the common length of the elementary paths of $B(\pi_\kappa)$, uniformly in ℓ. This proves that $\mathcal{E}(\eta)$ is well-defined. Since $\eta\in\Omega_{\mathcal{C}}^{\mathrm{typ}}$, the path $\eta_\ell = \Pi_\ell(\eta)$ is of highest weight. Thus, the path $\mathcal{E}(\eta_\ell)$ is of lowest weight. Comparing their weights, we get $\mathcal{E}(\eta)(\ell) = w_0(\eta(\ell))$ which implies that $\mathcal{E}(\eta)\in\Omega^{\iota\mathrm{typ}}$. □

Observe we have $\mathcal{P}(\eta) = \lim_{\ell\to+\infty}\mathcal{P}(\eta_\ell)$ and $\mathcal{E}(\eta) = \lim_{\ell\to+\infty}\mathcal{E}(\eta_\ell)$ where $\eta_\ell = \Pi_\ell(\eta)$.

8.6. Composition of the transformations $\mathcal{P}$ and $\mathcal{E}$

Consider $\pi\in B(\pi_\kappa)^{\otimes\ell}, \eta\in\Omega_{\mathcal{C}}^{\mathrm{typ}}$, and $\xi\in\Omega^{\iota\mathrm{typ}}$. For any positive integer L, set $\Pi_L(\eta)=\eta_L$ and $\Pi_L(\xi)=\xi_L$.

Lemma 8.9. — *With the above notation we have for L sufficiently large*

(1) $\mathcal{P}\mathcal{E}(\pi\otimes\eta_L) = \pi\otimes\eta_L$ *when $\pi\otimes\eta_L$ is a highest weight path,*
(2) $\mathcal{E}\mathcal{P}(\pi\otimes\xi_L) = \pi\otimes\mathcal{E}(\xi_L)$.

Proof. — (1) Since $\pi\otimes\eta_L$ is a highest weight path, $\mathcal{E}(\pi\otimes\eta_L)$ is the lowest weight path of $B(\pi\otimes\eta_L)$, the crystal associated to $\pi\otimes\eta_L$. Therefore $\mathcal{P}\mathcal{E}(\pi\otimes\eta_L) = \pi\otimes\eta_L$ is the highest weight path of $B(\pi\otimes\eta_L)$.

(2) Since $\xi\in\Omega^{\iota\mathrm{typ}}$, we have for any $i=1,\dots,n$, $\lim\limits_{L\to+\infty}\langle\xi_L(L),\alpha_i^\vee\rangle = -\infty$. We get by (8.2)

$$\langle\iota(\xi_L)(L),\alpha_i^\vee\rangle = \langle w_0(\xi_L(L)),\alpha_i^\vee\rangle = \langle \xi_L(L), w_0(\alpha_i^\vee)\rangle = -\langle\xi_L(L),\alpha_{i^*}^\vee\rangle$$

for any $i=1,\dots,n$. So $\lim\limits_{L\to+\infty}\langle\iota(\xi_L)(L),\alpha_i^\vee\rangle = +\infty$ for any $i=1,\dots,n$. Recall the $\iota\mathcal{P}=\mathcal{E}\iota$ and $\iota\mathcal{E}=\mathcal{P}\iota$ by Lemma 8.3. We have the equivalences

$$\begin{aligned}\mathcal{E}(\pi\otimes\xi_L) = \pi\otimes\mathcal{E}(\xi_L) &\iff \iota\mathcal{E}(\pi\otimes\xi_L) = \iota(\pi\otimes\mathcal{E}(\xi_L))\\ &\iff \mathcal{P}(\iota(\xi_L)\otimes\iota(\pi)) = \mathcal{P}(\iota(\xi_L))\otimes\iota(\pi).\end{aligned}$$

But the last equality holds by Lemma 8.6 for L sufficiently large. This proves that $\mathcal{E}(\pi\otimes\xi_L)=\pi\otimes\mathcal{E}(\xi_L)$ for L sufficiently large. Now, observe that $\pi\otimes\xi_L$ and $\mathcal{E}(\pi\otimes\xi_L)=\pi\otimes\mathcal{E}(\xi_L)$ both belong to the crystal $B(\pi\otimes\xi_L)$. In this crystal the transforms $\mathcal{P}$ and $\mathcal{E}$ return the highest and lowest paths, respectively. Therefore, we have $\mathcal{E}\mathcal{P}(\pi\otimes\xi_L) = \mathcal{E}\mathcal{P}(\pi\otimes\mathcal{E}(\xi_L))$. But $\pi\otimes\mathcal{E}(\xi_L)=\mathcal{E}(\pi\otimes\xi_L)$ is the lowest path of $B(\pi\otimes\xi_L)$. This implies that $\mathcal{E}\mathcal{P}(\pi\otimes\xi_L) = \pi\otimes\mathcal{E}(\xi_L)$ for L sufficiently large as desired. □

Theorem 8.10.

(1) *For any* $\eta \in \Omega_{\mathcal{C}}^{\mathrm{typ}}$, *we have* $\mathcal{PE}(\eta) = \eta$.
(2) *For any* $\xi \in \Omega^{\iota\mathrm{typ}}$, *we have* $\mathcal{EP}(\xi) = \xi$.

Proof. — Consider a positive integer ℓ. For any integer $L \geqslant \ell$ we can write $\Pi_L(\eta) = \Pi_\ell(\eta) \otimes \eta_L$ and $\Pi_L(\xi) = \Pi_\ell(\xi) \otimes \xi_L$ with η_L and ξ_L in $B(\pi_\kappa)^{\otimes L-\ell}$. Since $\eta \in \Omega_{\mathcal{C}}^{\mathrm{typ}}$ and $\xi \in \Omega^{\iota\mathrm{typ}}$, we have for any simple root α_i,

$$\lim_{L\to+\infty} \langle \eta_L(L), \alpha_i^\vee \rangle = +\infty \quad \text{and} \quad \lim_{L\to+\infty} \langle \xi_L(L), \alpha_i^\vee \rangle = -\infty\,.$$

So by applying Lemma 8.9, we get for L sufficiently large (depending on ℓ)

$$\mathcal{PE}(\Pi_L(\eta)) = \Pi_\ell(\eta) \otimes \eta_L \quad \text{and} \quad \mathcal{EP}(\Pi_L(\xi)) = \Pi_\ell(\xi) \otimes \mathcal{E}(\xi_L)$$

for any $\ell \leqslant L$. This shows that $\mathcal{PE}(\eta) = \eta$ and $\mathcal{EP}(\xi) = \xi$ by taking the limit when ℓ tends to infinity. □

Remark 8.11. — It is possible to state a slightly stronger statement of the previous theorem where Ω is replaced by $\mathcal{L}_\infty$ (see §8.2) in the definition of $\Omega_{\mathcal{C}}^{\mathrm{typ}}$ and $\Omega^{\iota\mathrm{typ}}$.

Write $\mathcal{W}^\iota = Y_1 \otimes Y_2 \cdots$ the dual random path with drift $\iota(\overline{m})$. The following theorem shows that the transformation $\mathcal{E}$ defined on $\Omega_{\mathcal{C}}^{\mathrm{typ}}$ can be regarded as the inverse of the generalized Pitman transform $\mathcal{P}$. Recall that for both random trajectories $\mathcal{W}^\iota$ and $\mathcal{W}$, we have $\mathcal{H} = \mathcal{P}(\mathcal{W}) = \mathcal{P}(\mathcal{W}^\iota)$.

Theorem 8.12. — *Assume* $\overline{m} \in \mathcal{D}_\kappa$. *Then we have*

(1) $\mathcal{EP}(\mathcal{W}^\iota) = \mathcal{W}^\iota$ P^ι*-almost surely,*
(2) *We have* $\mathcal{E}(\mathcal{H}) = Y_1 \otimes Y_2 \otimes \cdots$ *where the sequence of random variables* $(Y_\ell)_{\ell \geqslant 1}$ *is i.i.d. and each variable* $Y_\ell, \ell \geqslant 1$, *has law* Y *as defined in* (8.3).
(3) $\mathcal{PE}(\mathcal{H}) = \mathcal{H}$ Q*-almost surely.*

Proof. — (1) Write $\mathcal{W}^\iota = Y_1 \otimes Y_2 \cdots$. Since $\mathsf{P}^\iota(\Omega^{\iota\mathrm{typ}}) = 1$, we get $\mathcal{EP}(\mathcal{W}^\iota) = \mathcal{W}^\iota$ P^ι-almost surely by Assertion (2) of Theorem 8.10. Since $\mathcal{P}(\mathcal{W}^\iota) = \mathcal{H}$, we have $\mathcal{E}(\mathcal{H}) = \mathcal{EP}(\mathcal{W}^\iota)$. By Assertion (1), this means that $\mathcal{E}(\mathcal{H}) = \mathcal{W}^\iota$ which proves Assertion (2).

To obtain Assertion (3), it suffices to observe that $\mathcal{PE}(\mathcal{H}) = \mathcal{H}$ Q-almost surely by Assertion (1) of Theorem 8.10 since we have $\mathsf{Q}(\Omega_{\mathcal{C}}^{\mathrm{typ}}) = 1$. □

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