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Maximal radius of quaternionic hyperbolic manifolds (*)

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Abstract. — We derive an explicit lower bound on the radius of a ball embedded in a quaternionic hyperbolic manifold (the *maximal radius*). We then deduce a lower bound on the volume of a quaternionic hyperbolic manifold. Both those bounds decrease with the dimension, when it is not clear that it should be the behaviour of the maximal radius or of the minimal volume. Related to that question, we note however that the Margulis constant of the quaternionic hyperbolic space of dimension n is smaller than $C/\sqrt{n}$, so is decreasing as the dimension grows.

Résumé. — Nous donnons une borne inférieure explicite sur le rayon d'une boule plongée dans une variété hyperbolique quaternionique (le *rayon maximal)*. Nous en déduisons une minoration du volume de telles variétés. Les deux bornes exhibées décroissent avec la dimension, et il n'est pas clair que l'on doive s'attendre au même comportement pour le rayon maximal ou pour le volume minimal. En lien avec cette question, nous remarquons cependant que la constante de Margulis de l'espace hyperbolique quaternionique de dimension n est inférieure à $C/\sqrt{n}$, et décroit donc quand la dimension augmente.

1. Introduction

It has been known since the end of the 1960's with the work of Každan and Margulis [17], and the subsequent work of Wang [23], that any locally symmetric manifold of non-compact type contains an embedded ball of radius $r_G/2$ depending only on the group G of isometries of its universal cover. Given a symmetric space X, denoting by $G = I(X)$ its isometry group, a lower bound for $r_G/2$ provides geometric information on any manifold obtained as a quotient of X: for instance, one can then deduce a lower bound for the maximal injectivity radius of any such manifold, and information about its thick-thin decomposition.

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Article proposé par Vincent Guiradel.

In this article we focus on the case where X is the quaternionic hyperbolic space $H^n_{\mathbb{H}}$. We adapt techniques developed by Martin [19] in the real hyperbolic setting to obtain a bound λ_n for the maximal radius of a real hyperbolic n-manifold. These ideas where recently adapted to the complex hyperbolic case by Jiang, Wang and Xie [24].

The main result of this article is the following :

Main Theorem. — *Let $\Gamma \subset \mathbf{Sp}(n,1)$ be a discrete, torsion-free, non-elementary subgroup acting by isometries on the quaternionic hyperbolic space $H^n_{\mathbb{H}}$. There exists a point $p \in H^n_{\mathbb{H}}$ such that, for all $\gamma \in \Gamma$,*

$$\rho(p, \gamma(p)) \geqslant \lambda_n,$$

where $\lambda_n = \frac{0.05}{9^{n+1}}$. Any quaternionic hyperbolic manifold thus contains an embedded ball of radius $\lambda_n/2$.

No bounds have been previously known on this quantity, and further, our result somewhat improves the earlier bounds known in the real and complex case (see Remark 5.1).

Martin's work crucially relies on a Jørgensen-like inequality established in [20]. This inequality in turn depends on the explicit determination of a *Zassenhauss neighbourhood* of the isometry group of the hyperbolic real space. In [10], Friedland and Hersonsky slightly improved Martin's inequality, and used this new version to deduce a better bound for the maximal radius of real hyperbolic manifolds. It is this improved inequality that Jiang, Wang and Xie use in [24], and it is the one we shall use in this paper.

Section 3 is devoted to the presentation of these results: first we exhibit a Zassenhauss neighbourhood of $\mathbf{PSp}(n,1)$, the group of orientation preserving isometries of the quaternionic hyperbolic space. We then deduce the Martin-Jørgensen inequality and, following Martin, a stronger inequality satisfied by the torsion-free lattices in $\mathbf{PSp}(n,1)$ (Theorem 3.6).

In Section 4, we make explicit the fact that when A is an element of $\mathbf{PSp}(n,1)$, both $\|A\|$ and $\|A - I\|$ have to be small if A does not displace enough a given point $\mathbf{o}$ of $H^n_{\mathbb{H}}$. We finish by combining these results and Theorem 3.6 to reach our conclusion in Section 5.

The bounds for the maximal radius given by the authors we first mentionned, and the one presented here, both decrease exponentially with the dimension, though the methods employed do not allow us to discuss their optimality. The description of the behaviour of the maximal radius with the dimension (can it be uniformly bounded? Could it grow with dimension?) is a matter that does not seem well understood yet.

On the other hand, for *open* real hyperbolic manifold, Gendulphe [11] recently derived a bound for the maximal radius, which is dimension-free and optimal in dimension 3. His constructions greatly rely on packing theorems and cannot be obviously adapted to the case of spaces of non-constant sectional curvature.

Related to that matter, is the question of the behaviour of the *Margulis constant* with the dimension. This is the constant given by the following famous result, know as the *Margulis Lemma* (see for exemple [22, Lemma 5.10.1]):

Theorem (Margulis). — *For all n, there exists a positive constant $\mu_{H^n_{\mathbb{H}}} = \epsilon$ such that, for all discrete subgroup $\Gamma \subset \mathrm{Isom}(H^n_{\mathbb{H}})$, and for all point $x \in H^n_{\mathbb{H}}$, the group*

$$\Gamma_x(\epsilon) = \langle g \in \Gamma \mid d(x, g(x)) \leqslant \epsilon \rangle$$

is virtually nilpotent.

This theorem implies in particular that there is an embedded ball of radius $\mu_{H^n_{\mathbb{H}}}/2$ in any quotient of $H^n_{\mathbb{H}}$ by a discrete subgroup of its isometry group, so that

$$\mu_{H^n_{\mathbb{H}}} \leqslant r_{\mathrm{Isom}(H^n_{\mathbb{H}})}.$$

In the last section of this chapter, using a result of Kapovich mentionned in a paper of Belolipetsky [5], we present an argument showing that the Margulis constant of $H^n_{\mathbb{H}}$ (in fact, of $H^n_{\mathbb{K}}$, for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$) goes to 0 as the dimension n goes to infinity. This of course does not answer the question of the behaviour of the maximal radius with the dimension, but we believe that it further shows that this matter is of interest.

To conclude this introduction, note that our result does not cover the case of orbifolds. In this generality, to our knowledge, no bound for the maximal radius is known, in any of the real, complex, or quaternionic hyperbolic settings. In [21], Parker gave a bound for the maximal radius of an *open* complex hyperbolic orbifold O, obtained by a close study of the maximal embedded cusp of O. His methods are very different from the one we use in this paper (that is, the methods developed by Martin in his above mentioned work) and indeed, computing a bound for the maximal radius of an orbifold with the later does not seem to be easily achievable.

However, using those methods, Adeboye was able to derive a bound on the *volume* of a real hyperbolic orbifold, depending on the dimension and the order of torsion. He proved, in [1], that an upper bound for the maximal order of torsion of a discrete subgroup $\Gamma \subset \mathbf{SO}(n,1)$ leads to a uniform lower bound for $\|A - I\|$ for all A in $\Gamma - I$. This fact in turn allowed him to establish an upper bound for the number of elements of a discrete subgroup

$\Gamma \subset \mathbf{SO}(n,1)$ that fail to move a ball of radius r in $H^n_{\mathbb{R}}$ off itself. This last quantity allowed him to give an explicit bound for the volume of an orbifold covered by $H^n_{\mathbb{R}}$, obtained by bounding the volume of the image of such a ball in $\Gamma\backslash H^n_{\mathbb{R}}$. Note also that by different methods, Adeboye and Wei [2] and [3] were able to derive a lower bound for the volume of real or complex hyperbolic orbifolds, depending only on the dimension.

Aknowledgment. The author would like to thank the referee for very useful and interesting remarks. Prior to the referee's comments, the author had no knowledge of the existence of the article of Belolipetsky that contains Kapovich's Proposition on the behaviour of the Margulis constant in the real case, and she is very grateful for that piece of information.

2. Preliminaries

2.1. Linear algebra on $\mathbb{H}$

In this text, $\mathbb{H}$ denotes the algebra of Hamilton quaternions $\mathbb{H} = \mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$, where i, j and k satisfy $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, and $\mathbb{H}^n$ the *right* vector space of dimension n over $\mathbb{H}$.

A quaternion of modulus 1 can be written $q = \cos(\theta) + \mu \sin(\theta)$, where $\mu^2 = -1$ is a purely imaginary quaternion of modulus 1:

$$\mu \in \{w \in \mathbb{H} \mid |w| = 1 \text{ and } \bar{w} = -w\} \simeq S^2,$$

and $\theta \in [0, \pi]$. To denote such a quaternion we shall use a more compact notation

$$\cos(\theta) + \sin(\theta)\mu = e^{\mu(\theta)}.$$

This notation satisfies $e^{\mu(a)}e^{\mu(b)} = e^{\mu(a+b)}$, and in particular,

$$\overline{e^{\mu(a)}} = \left(e^{\mu(a)}\right)^{-1} = e^{\mu(-a)}.$$

If q is any quaternion, denoting by r its modulus, q can then be written

$$r(\cos(\theta) + \sin(\theta)\mu) = re^{\mu(\theta)}, \quad r \in [0, +\infty[,\ \theta \in [0, \pi],\ \mu \in S^2,$$

and this writing is unique if q is not in $\mathbb{R}$, that is, if $r \neq 0$ and if $\theta \neq 0$ or π. We thence have the following decomposition of the non-real quaternions:

$$\mathbb{H} - \mathbb{R} =]0, +\infty[\times]0, \pi[\times S^2. \tag{2.1}$$

The group of unit quaternions $\mathbf{Sp}(1)$ acts by conjugacy on $\mathbb{H}$. This action is the identity on $Z(\mathbb{H}) = \mathbb{R}$ and restricts to an action on the non-real quaternions, which leaves invariant the first two factors of the decomposition (2.1) and is transitive on S^2. Thus, for $q \in \mathbb{H}$, the conjugacy class of q

$$O(q) = \left\{z^{-1}qz,\ z \in \mathbb{H}, z \neq 0\right\} = \{\bar{w}qw,\ w \in \mathbb{H}, |w| = 1\}$$

is reduced to a point if q is real, and is a sphere S^2 otherwise. Further, if we let

$$\mathbb{C} = \mathbb{R}[i] \subset \mathbb{H}$$

be a distinguished maximal subfield of $\mathbb{H}$, and

$$\mathbb{C}^+ = \{q \in \mathbb{C} \mid q - \bar{q} \geqslant 0\} \tag{2.2}$$

be the set of elements of $\mathbb{C}$ with positive or null imaginary part, then $\mathbb{C}^+$ is a set of representatives of the orbits of this action.

The capital roman letters $(A, B, \dots)$ denote matrices.

The letter I denotes the identity matrix.

Given $A \in \mathcal{M}_n(\mathbb{H})$, $A^* = {}^t\overline{A}$ denotes it's conjugate transpose.

Let $A \in \mathcal{M}_n(\mathbb{H})$. An *eigenvector* of A is a vector $\xi \in \mathbb{H}^n$ such that $A \cdot \xi = \xi\lambda$ for some $\lambda \in \mathbb{H}$. We call such a λ a *right eigenvalue* for A, or for short an *eigenvalue* of A. Observe that given ξ and λ as above,

$$A \cdot (\xi q) = \xi q(q^{-1}\lambda q), \quad \forall\, q \in \mathbb{H}, q \neq 0.$$

Consequently, if λ is an eigenvalue of A, so is any quaternion in the orbit $O(\lambda)$ of λ under the action of $\mathbf{Sp}(1)$ described above.

A *standard eigenvalue* of A is an eigenvalue of A belonging to the set $\mathbb{C}^+$ defined in (2.2). The set of standard eigenvalues of A thus bijectively corresponds to the conjugacy classes of eigenvalues of A. We call it the *spectrum* of A, and denote it by $\sigma(A)$.

The *spectral radius* of A is the number

$$r_\sigma(A) = \max_{\lambda \in \sigma(A)} |\lambda|.$$

The norm $\|\cdot\|$ denotes the *spectral norm* on $\mathcal{M}_n(\mathbb{H})$:

$$\|A\| = \sqrt{r_\sigma(A^*A)}.$$

The following remark will prove useful in the later sections:

Remark 2.1. — If $U \in \mathbf{Sp}(n)$, $\|UAU^{-1}\| = \|A\|$.

Let $\mathbf{Sp}(n)$ denote the group of *unitary matrices* of $\mathcal{M}_n(\mathbb{H})$:

$$\mathbf{Sp}(n) = \{A \in \mathcal{M}_n(\mathbb{H}), A^*A = I\}.$$

In this setting, one can formulate the following spectral theorem:

THEOREM 2.2. — *If A is a normal matrix (in particular, if A is unitary), there exists a unitary matrix U such that U^*AU is a diagonal matrix with diagonal elements in $\mathbb{C}^+$.*

(The reader may find more details on linear algebra on $\mathbb{H}$ in Zhang's survey [25] for example, or, regarding the spectral theory more specifically, in Farenick and Pidkowich's paper [9]).

Remark 2.3. — The orbit $O(q)$ of a quaternion q always contains its conjuguate $\bar{q}$. In particular, any complex number $\lambda \in \mathbb{C} = \mathbb{C}[i]$ is unitarily equivalent, in $\mathbb{H}$, to its conjugate $\bar{\lambda}$. Let us emphasize here the fact that in the spectral theorem over $\mathbb{H}$, one can chose the cœfficients of the diagonal matrix to have *positive* imaginary part, which need not be the case over $\mathbb{C}$.

The brackets $\langle \cdot , \cdot \rangle$ denote a hermitian form of signature $(n, 1)$ on $\mathbb{H}^{n+1}$.

Let $\mathbf{Sp}(n, 1)$ denote the subgroup of $GL_{n+1}(\mathbb{H})$ formed by the matrices (acting on $\mathbb{H}^{n+1}$ on the left) preserving $\langle \cdot , \cdot \rangle$.

The lower case roman letters $(f, g, h, \dots)$ denote the isometries of $H^n_{\mathbb{H}}$.

2.2. Quaternionic hyperbolic space and its isometries

2.2.1. The half-space model

Let $\mathbb{H}^{n,1}$ denote the quaternionic vector space $\mathbb{H}^{n+1}$ of dimension $n + 1$ endowed with a hermitian form of signature $(n, 1)$. The *quaternionic hyperbolic space* $H^n_{\mathbb{H}}$ is the grassmannian of negative lines with respect to such a form. Precisely, we consider the sets V_- and V_0 of negative and null vectors:

$$V_- = \left\{Z \in H^{n,1},\ \langle Z, Z\rangle < 0\right\};$$
$$V_0 = \left\{Z \in H^{n,1},\ \langle Z, Z\rangle = 0\right\};$$

denote by P the usual projection from $\mathbb{H}^{n+1}$ onto $P^n(\mathbb{H})$, and define

$$H^n_{\mathbb{H}} = P(V_-)$$
$$\text{and} \quad \partial H^n_{\mathbb{H}} = P(V_0).$$

To a choice of form corresponds a choice of model for $H^n_{\mathbb{H}}$. In this text, we will mainly work in the *half-space model*. This is the model given by the form

$$\langle Z, W\rangle = W^* J Z, \quad J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
$$= \overline{w_1} z_{n+1} + \overline{w_2} z_2 + \cdots + \overline{w_n} z_n + \overline{w_{n+1}} z_1,$$

where Z and W are two column vectors of $\mathbb{H}^{n,1}$.

In our setting we thus have

$$P(V_-) = P\left(\left\{Z \in \mathbb{H}^{n,1},\ 2\Re(\overline{z_{n+1}} z_1) + |z_2|^2 + \cdots + |z_n|^2 < 0\right\}\right),$$

i.e., in the chart $\{z_{n+1} = 1\}$,

$$H^n_{\mathbb{H}} = \left\{2\Re(z_1) + |z_2|^2 + \cdots + |z_n|^2 < 0\right\}.$$

The boundary consists of the points

$$Z = {}^t\begin{bmatrix} z_1 & \ldots & z_n & 1 \end{bmatrix}, \quad 2\Re(z_1) + |z_2|^2 + \cdots + |z_n|^2 = 0.$$

together with a distinguished point at infinity $q_\infty = {}^t\left[\begin{smallmatrix} 1 & 0 & \ldots & 0 \end{smallmatrix}\right]$ (the unique point of $P(V_0)$ not contained in the chart $\{z_{n+1} = 1\}$).

We define the *horospherical height* of a point $Z \in H^n_{\mathbb{H}}$:

$$u_Z = -(2\Re(z_1) + |z_2|^2 + \cdots + |z_n|^2),$$

and then the *horospherical coordinates* of a point $Z = {}^t\left[\begin{smallmatrix} z_1 & \ldots & z_n & 1 \end{smallmatrix}\right] \in H^n_{\mathbb{H}}$:

$$(\xi_Z, v_Z, u_Z) = ((z_2, \ldots, z_n), 2\Im(z_1), -(2\Re(z_1) + |z_2|^2 + \cdots + |z_n|^2)).$$

These coordinates may be thought of as a generalization of the cartesian coordinates on $H^2_{\mathbb{R}}$.

The *vertical geodesics* are the lines $\{(\xi_0, v_0, u), u \in \mathbb{R}^+\}$ joining a point $(\xi_0, v_0, 0)$ on the boundary to q_∞. In particular, we will denote by $(0, \infty)$ the vertical geodesic $\{(0, 0, u), u \in \mathbb{R}^+\}$ joining $(0, 0, 0)$ and q_∞.

We choose an origin in $H^n_{\mathbb{H}}$, namely the point $\mathbf{o} = {}^t\left[\begin{smallmatrix} -1 & 0 & \ldots & 0 & 1 \end{smallmatrix}\right]$, or $(0, 0, 2)$ in horospherical coordinates. It belongs to the vertical geodesic $(0, \infty)$.

Remark 2.4. — Another classical model is the *ball model* that comes with the choice of the form $J_1 = \left[\begin{smallmatrix} I_n & 0 \\ 0 & -1 \end{smallmatrix}\right]$. The Cayley transform from one model to the other is given by the change of basis from J to J_1, namely the unitary matrix

$$C = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & I_{n-1} & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix}.$$

In this model, the hyperbolic space

$$\begin{aligned} H^n_{\mathbb{H}} = P(V_-) &= P\left(\left\{Z \in \mathbb{H}^{n,1},\ |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 - |z_{n+1}|^2 < 0\right\}\right) \\ &\simeq \left\{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 < 1\right\} \end{aligned}$$

is identified with the unit ball in $\mathbb{H}^n$, and the origin $\mathbf{o}$ of the half-space model is carried by the Cayley transform onto the origin 0 of the ball (the point ${}^t[\,0\ \dots\ 0\ 1\,]$ in the inhomogenous coordinates).

We shall use this model when describing the maximal compact subgroup of the isometry group of $H^n_{\mathbb{H}}$, that is the stabilizer of a point in $H^n_{\mathbb{H}}$. The computations will prove to be more elegant in this setting. However, this concerns only two small parts of our text (the description of the elliptic elements in the next paragraph, and the proof of Lemma 4.5) so unless otherwise explicitly stated, the reader should always think that we are working in the half-space model.

2.2.2. Classification of the isometries

We shall now present a couple of facts regarding the isometries of $H^n_{\mathbb{H}}$ that will be needed in the rest of the text. A more detailed account can be found in the article of Kim and Parker mentioned in the introduction [18].

The group of orientation preserving isometries of $H^n_{\mathbb{H}}$ is the group

$$\mathbf{PSp}(n,1) = \mathbf{Sp}(n,1)/\{\pm I\}.$$

As in the real and complex hyperbolic cases, these isometries can be of one of the following three kinds:

1. *loxodromic*, if they fix exactly two points in $\partial H^n_{\mathbb{H}}$ (and have no fixed-point in $H^n_{\mathbb{H}}$);
2. *parabolic*, if they fix exactly one point in $\partial H^n_{\mathbb{H}}$ (and have no fixed-point in $H^n_{\mathbb{H}}$);
3. *elliptic*, if they fix a point in $H^n_{\mathbb{H}}$.

In the ball model, a direct computation, using the fact that we are working with elements perserving the form J_1, shows that elliptic elements fixing the origin $0 = {}^t[\,0\ \dots\ 0\ 1\,]$ correspond to matrices of $\mathbf{Sp}(n,1)$ of the form

$$A = \begin{bmatrix} \Theta & 0 \\ 0 & e^{\mu(\theta)} \end{bmatrix},\ \Theta \in \mathbf{Sp}(n).$$

We thus see that

$$\mathrm{Stab}(0) \simeq \mathbf{P}(\mathbf{Sp}(n) \times \mathbf{Sp}(1)).$$

Remark 2.5. — Elliptic elements stabilizing 0 thus have norm equal to 1. These elements correspond, under the Cayley transform C, to those stabilizing the origin $\mathbf{o}$ in the half-space model. Since C is unitary, using Remark 2.1, we see that elliptic elements stabilizing $\mathbf{o}$ in the half-space model also have norm 1. This will prove usefull in the sequel.

Remark 2.6. — In the real or complex hyperbolic cases, after projectivizing, we can assume that an elliptic element has the form

$$A = \begin{bmatrix} \Theta & 0 \\ 0 & 1 \end{bmatrix}, \ \Theta \in U_n.$$

In our case however, scalar matrices are not central (except for $\pm I$), and we can no longer make this assumption. This fact is responsible for a slight difference between our results and their analogue in the real and complex cases (compare Lemma 4.2 of [24] and Lemma 4.1 of [10] with Lemma 4.5).

2.2.3. Elementary groups of isometries

The *limit set* of a discrete subgroup Γ of isometries of $H^n_{\mathbb{H}}$ is the set of accumulation points of the orbit of an arbitrary point $x \in H^n_{\mathbb{H}}$, denoted by $L(\Gamma)$. A discrete group Γ is called *non-elementary* if its limit set contains strictly more than two points, *elementary* otherwise.

In case Γ is elementary, one of the three following holds (see e.g. [13]):

(1) $L(\Gamma) = \emptyset$. Then Γ is finite.
(2) $L(\Gamma) = \{x_0\}$. Then every infinite order element of Γ is parabolic with fixed point x_0.
(3) $L(\Gamma) = \{x_0, y_0\}$. Then every infinite order element of Γ is loxodromic with fixed points x_0 and y_0.

In particular, if Γ is discrete, elementary, and *torsion free*, the elements of Γ are either *all* parabolic or *all* loxodromic. This is the only fact about elementary groups that we need in this paper (in the proof of our main inequality, Theorem 3.6).

2.2.4. Formulae

The distance in $H^n_{\mathbb{H}}$ can be explicitly described in terms of the hermitian structure on $\mathbb{H}^{n,1}$ (see for example Chen and Greenberg's artcile, [7]). If X and Y are two points in $H^n_{\mathbb{H}}$ and $\tilde{X}$, $\tilde{Y}$ two corresponding vectors of $\mathbb{H}^{n,1}$,

$$\cosh\left(\frac{\rho(X,Y)}{2}\right) = \frac{\langle \tilde{X}, \tilde{Y}\rangle\langle \tilde{Y}, \tilde{X}\rangle}{\langle \tilde{X}, \tilde{X}\rangle\langle \tilde{Y}, \tilde{Y}\rangle}. \tag{2.3}$$

Remark 2.7. — We did not choose the same normalization as Chen and Greenberg, and in their paper, the $\frac{1}{2}$ factor does not appear on the left hand side of the equation. In our text, the metric is normalized so that the sectional curvature is -1 on planes contained in quaternionic lines (and is thus globally pinched between -1 and $-1/4$).

In order to obtain a lower bound for the volume of a quaternionic hyperbolic manifold, we need to be able to compute the volume of a ball of given radius. This is given by the following lemma which can be found, for instance, in an article of Alfred Gray [12]:

Lemma 2.8. — *The volume of a ball of radius R in the quaternionic hyperbolic space is*

$$\mathrm{Vol}(B(R)) = \frac{(4\pi)^{2n}}{(2n+1)!} \sinh^{4n}\left(\frac{R}{2}\right)\left(2n\cosh^2\left(\frac{R}{2}\right)+1\right).$$

3. Jørgensen-like inequality and consequences

As we announced, we begin by giving a Zassenhauss neighbourhood for $\mathbf{Sp}(n,1)$, that is a neighbourhood of the identity in $\mathbf{Sp}(n,1)$ such that any discrete subgroup of $\mathbf{Sp}(n,1)$ generated by elements of this neighbourhood is nilpotent.

Theorem 3.1. — *$\Omega = B(I,\tau)$ is a Zassenhauss neighbourhood for $\mathbf{Sp}(n,1)$, where $\tau \simeq 0.2971..$ is the positive root of the equation $2\tau(1+\tau)^2=1$.*

This result, with a slightly worse bound, was established by Martin in [20]: he obtained the Zassenhauss neighbourhood $\Omega_{O^+(1,n)} = B(I, 2-\sqrt{3})$. It was then improved and generalized by Friedland and Hersonsky in [10] who obtained $\Omega_G = B(I,\tau)$ for a large class of Lie groups G. Friedland and Hersonsky's improvement comes from an elementary remark which in our setting can be stated in this way:

$$\text{for } A \in \mathbf{Sp}(n,1), \quad \|A^{-1}\| = \|A\|.$$

We give the proof of Theorem 3.1, for the reader convenience, and for it reveals a crucial inequality (Inequality (3.1)) which we shall use again and again.

Proof. — Let then A, B be in $\Omega = \{M \in \mathbf{Sp}(n,1),\ \|M-I\| < \tau\}$. We have

$$\begin{aligned}[A,B]-I &= ABA^{-1}B^{-1} - I\\ &= (AB-BA)A^{-1}B^{-1}\\ &= ((A-I)(B-I)-(B-I)(A-I))A^{-1}B^{-1}.\end{aligned}$$

Hence

$$\begin{aligned}\|[A,B]-I\| &\leqslant 2\|A-I\|\|B-I\|\|A^{-1}\|\|B^{-1}\| \\ &< 2\tau^2(1+\tau)^2 = \tau.\end{aligned} \tag{3.1}$$

Now, if $\Gamma \subset \mathbf{Sp}(n,1)$ is a discrete subgroup, $\Gamma \cap \Omega = \{A_1, \dots, A_n\}$ is finite, and there exists an $r < \tau < 1$ such that $\|A - I\| < r$ for all $A_i \in \Gamma \cap \Omega$.

From Inequality (3.1), we thus have, for all elements $A_{i_0}, \dots, A_{i_k}$ of $\Gamma \cap \Omega$,

$$\|[A_{i_1}, A_{i_0}] - I\| < 2r(1+r)^2\|A_{i_0} - I\| < r\|A_{i_0} - I\|$$

and

$$\|[A_{i_k}, \dots, [A_{i_1}, A_{i_0}] \dots] - I\| < r^k\|A_{i_0} - I\|.$$

Hence, Γ being discrete, there exists an integer m such that for all sequence $(B_k)_{k\in\mathbb{N}}$ given by

$$B_k = [A_{i_k}, \dots, [A_{i_1}, A_{i_0}] \dots],$$

we have $B_j = I \ \forall\, j \geqslant m$. The group $\langle A_1, A_2, \dots, A_n\rangle$ is thus nilpotent. □

Remark 3.2. — A discrete and non-elementary group being non-nilpotent, we immediately see that if two elements A and B of $\mathbf{Sp}(n,1)$ generate a discrete non-elementary subgroup, necessarily $\max\{\|A-1\|, \|B-1\|\} \geqslant \tau$. Furthermore, if one demands $\langle A, B\rangle$ to be torsion-free, A must be parabolic or loxodromic, and it is easily seen that if $\langle A, B^{-1}AB\rangle$ stabilizes one or two points of the boundary of $H^n_{\mathbb{H}}$, then so does the group $\langle A, B\rangle$. Therefore, when $\langle A, B\rangle$ is discrete and torsion-free, $\langle A, B^{-1}AB\rangle = \langle A, [A,B]\rangle$ is elementary if and only if $\langle A, B\rangle$ is. Theorem 3.1 thus has an (almost) immediate corollary:

Corollary 3.3. — *Let $\Gamma \subset \mathbf{Sp}(n,1)$ be a discrete torsion-free subgroup, and A and B be two elements of Γ. We have the following alternative:*

(1) *either A and B generate an elementary subgroup of Γ;*
(2) *or $\max\{\|A-I\|, \|B-I\|\} \geqslant \tau$ and $\max\{\|A-I\|, \|[A,B]-I\|\} \geqslant \tau$.*

We are now ready to established a Jørgensen-like inequality. This inequality is originally due to Martin, in [19]. We state it here in it's improved version as derived by Friedland and Hersonsky in [10].

Corollary 3.4 (Jørgensen–Martin inequality). — *Let $\Gamma \subset \mathbf{Sp}(n,1)$ be a discrete torsion-free subgroup and A and B be two elements of Γ. Then, either A and B generate an elementary subgroup of Γ, or*

$$\max\{\|B\|\|B-I\|, \|A\|\|A-I\|\} \geqslant \omega$$

where $\omega = (\frac{\tau}{2})^{1/2} \simeq 0.3854..$ is the positive root of the equation $2\omega(2\omega^2+1) = 1$.

Proof. — Suppose that A and B are two elements of Γ that do not generate an elementary subgroup of Γ and such that

$$\|A\|\|A-I\| < \omega \quad \text{and} \quad \|B\|\|B-I\| < \omega.$$

Using Inequality (3.1) derived in the proof of Theorem 3.1, we get

$$\|[A,B]-I\| \leqslant 2\|A\|\|A-I\|\|B\|\|B-I\| < 2\omega^2 = \tau.$$

Next, since $\langle [A,B],A\rangle$ cannot be elementary (see Remark 3.2), by Corollary 3.3 we must have

$$\|[A,[A,B]]-1\| \geqslant \tau.$$

Therefore, using Inequality (3.1) again,

$$2\|A\|\|A-1\|\|[A,B]\|\|[A,B]-1\| \geqslant \tau,$$

so

$$2\omega\|[A,B]\| \geqslant 1.$$

However

$$\|[A,B]\| \leqslant 1+\|[A,B]-1\| < 1+\tau = \frac{1}{2\omega}$$

which is a contradiction. □

Remark 3.5. — Friedland and Hersonsky's improvement is an immediate consequence of their bettering of the Zassenhauss' neighbourhood. Martin considers the neighbourhood $B(1, 2-\sqrt{3})$ and obtains the bound $(\frac{2-\sqrt{3}}{2})^{1/2}$.

The main result of this section is the following:

THEOREM 3.6. — *Let Γ be a discrete, torsion-free, non-elementary subgroup of $\mathbf{Sp}(n,1)$. There exists an $H \in \mathbf{Sp}(n,1)$ such that*

$$\|A\|\|A-1\| \geqslant \omega \text{ for all } A \in H\Gamma H^{-1}. \tag{3.2}$$

Proof. — Let us assume, without loss of generality, that no element of Γ fixes q_∞ or 0 (the point of $\partial H^n_{\mathbb{H}}$ with horospherical coordinates $(0,0,0)$). Denote by h_t the loxodromic flow from 0 to q_∞, and by H_t the corresponding elements of $\mathbf{Sp}(n,1)$. h_t converges to q_∞ locally uniformly on $\overline{H^n_{\mathbb{H}}}\backslash\{0,q_\infty\}$ as t goes to $+\infty$, and $h_t^{-1} = h_{-t}$ converges to 0 locally uniformly on $\overline{H^n_{\mathbb{H}}}\backslash\{0,q_\infty\}$.

We argue by contradiction and suppose that there is no $t \in \mathbb{R}$ for which $H_t\Gamma H_t^{-1}$ satisfies (3.2).

Firstly, remark that for a fixed element A of $\mathbf{Sp}(n,1)$, $\|H_tAH_t^{-1}\|$ goes to infinity as t does. Indeed, denote by γ the isometry corresponding to A. By assumption, γ does not fix 0, so

$$\gamma h_t^{-1}(o) \underset{t\to+\infty}{\longrightarrow} \gamma(0) \in \partial H^2_{\mathbb{H}} - \{0\}.$$

Consequently, the convergence being locally uniform,

$$h_t\gamma h_t^{-1}(o) \underset{t\to+\infty}{\longrightarrow} q_\infty.$$

Put

$$h_t\gamma h_t^{-1} = (a_{i,j}(t))_{1\leqslant i,j\leqslant n+1},$$

then

$$h_t \gamma h_t^{-1}(o) = \begin{bmatrix} a_{1,1}(t) + a_{1,n+1}(t) \\ a_{2,1}(t) + a_{2,n+1}(t) \\ \vdots \\ a_{n+1,1}(t) + a_{n+1,n+1}(t) \end{bmatrix}.$$

Hence,

$$h_t \gamma h_t^{-1}(o) \underset{t\to+\infty}{\longrightarrow} q_\infty$$
$$\text{if and only if} \quad a_{1,1}(t) \underset{t\to+\infty}{\longrightarrow} +\infty \text{ or } a_{1,n+1}(t) \underset{t\to+\infty}{\longrightarrow} +\infty,$$

so that $\|H_t A H_t^{-1}\|_\infty \underset{t\to+\infty}{\longrightarrow} \infty$, and finally, all norms being equivalent,

$$\|H_t A H_t^{-1}\| \underset{t\to+\infty}{\longrightarrow} \infty.$$

Naturally, a similar argument using the fact that γ does not fix q_∞ shows that $\|H_t A H_t^{-1}\|$ goes to infinity as t goes to $-\infty$.

Next, we exhibit a sequence t_i going to infinity and a sequence of distinct elements A_i of Γ such that

$$\|H_{t_i} A_i H_{t_i}^{-1}\| \|H_{t_i} A_i H_{t_i}^{-1} - I\| < \omega \tag{3.3}$$

and

$$\|H_{t_i} A_{i+1} H_{t_i}^{-1}\| \|H_{t_i} A_{i+1} H_{t_i}^{-1} - I\| < \omega. \tag{3.4}$$

To make the notation less cluttered, for $t \in \mathbb{R}$ and $A \in \Gamma$, we put

$$N(t, A) = \|H_t A H_t^{-1}\| \|H_t A H_t^{-1} - I\|.$$

We are thus looking for two sequences satisfying $N(t_i, A_i) < \omega$ and $N(t_i, A_{i+1}) < \omega$. To achieve this, for any element A of Γ put

$$V_A = \{t \in \mathbb{R}, N(t, A) < \omega\}.$$

Since $N(t, A)$ goes to infinty as t does, if V_A is non-empty, V_A is a bounded open set. Further, by assumption, for all $t \in \mathbb{R}$ there is an element $A \in \Gamma$ contradicting (3.2), and the set $\{V_A, A \in \Gamma\}$ thus forms an open cover of $\mathbb{R}$ by bounded sets.

Now, choose a locally finite open refinement of that cover, $\mathcal{V} = \{V'\}$. Put $t_0 = 0$. Then t_0 is in some $V' \in \mathcal{V}$ which is in turn contained in some V_B. Put $A_0 = B$.

We then construct the sequences by induction. Suppose t_i and A_i are constructed. We want to exhibit an element $A_{i+1} \neq A_i$ such that $t_i \in V_{A_{i+1}}$ (so that (3.4) is satisfied). t_i is in some set $V' \subset V_{A_i}$ of $\mathcal{V}$. Any real number close enough to the supremum of V' is contained in another set V'' of $\mathcal{V}$. If $V'' \subset V_B$ with $B \neq A_i$, choose any such real number for t_{i+1} and put $A_{i+1} = B$. If this is not the case, do the same procedure with the supremum of V''. Since $\mathcal{V}$ is locally finite, we are ensure to exit V_{A_i} after a finite number of steps. The sequence t_i constructed in this way is strictly increasing and further, by local finiteness of $\mathcal{V}$ it can not accumulate and consequently goes to infinity. Also by construction, $t_i \in V_{A_i} \cap V_{A_{i+1}}$ and $A_i \neq A_{i+1}$ for all i.

Finally, (3.3) and (3.4) are satisfied, and from the Jørgensen–Martin inequality (Corollary 3.4), we see that the group generated by $H_{t_i} A_{i+1} H_{t_i}^{-1}$ and $H_{t_i} A_i H_{t_i}^{-1}$ must be elementary, hence its conjugate $\langle A_i, A_{i+1}\rangle$ must be too.

Consequently (see Section 2.2.3), either A_i and A_{i+1} are both parabolic and then we fix the same point x_0 of the boundary, or they are both loxodromic and then we fix the two same points x_0 and y_0 of the boundary. That being true for all i, we see that the A_i either are all parabolic or are all loxodromic, and have a common fix point x_0 on the boundary. Further, denoting by f_i the isometries corresponding to the A_i, we may assume, extracting a subsequence if necessary,

$$f_i(x) \longrightarrow x_0$$

locally uniformly on $\overline{H^n_{\mathbb{H}}} \backslash \{x_0\}$ if all the f_i are parabolic, and locally uniformly on $\overline{H^n_{\mathbb{H}}} \setminus \{x_0, y_0\}$ if they are all loxodromic.

Now, consider the sequence $h_i f_i h_i^{-1}$ with $h_i = h_{t_i}$. Since 0 and q_∞ are not fixed by any element of Γ, we have $\{0, q_\infty\} \cap \{x_0, y_0\} = \emptyset$, and, the convergence being locally uniform,

$$h_i f_i h_i^{-1}(0, q_\infty) = h_i f_i(0, q_\infty) \longrightarrow h_i(x_0) \longrightarrow q_\infty.$$

But if a sequence $\{g_i\}$ of isometries of $H^n_{\mathbb{H}}$ satisfies $|g_i(x) - g_i(y)| \to 0$ for two distinct points x and y in $\overline{H^n_{\mathbb{H}}}$, denoting by B_i the corresponding elements of $\mathbf{Sp}(n,1)$, necessarily $\|B_i\| \to \infty$. However here, we see from (3.3) that $\|H_i A_i H_i^{-1}\|$ is bounded (indeed, if X satisfies $\|X\|\|X - I\| < \omega$, then $|\|X\|| \, |\|X\| - 1| < \omega$, and $\|X\|$ has to be smaller than $\frac{1+\sqrt{1+4\omega}}{2}$). We thus obtain a contradiction, which concludes the proof of the theorem. □

4. Intermediate results

We now want to use Theorem 3.6 to derive the Main Theorem. To this end, given an element f of a discrete torsion-free subgroup of the isometry group of $H^n_{\mathbb{H}}$ and denoting the corresponding matrix by A, we seek to bound the quantity

$$\|A\|\|A - I\|$$

from above by a function of the distance $\rho(\mathbf{o}, f(\mathbf{o}))$, in order to obtain a contradiction if f does not displace the point $\mathbf{o}$ enough.

In the rest of this section, f is an isometry of $H^n_{\mathbb{H}}$ and $A \in \mathbf{Sp}(n,1)$ is the corresponding matrix. We put

$$\delta = \rho(\mathbf{o}, f(\mathbf{o}))$$

and

$$r = \exp(\delta/2).$$

We also put $K = \mathrm{Stab}(\mathbf{o}) \simeq \mathbf{P}(\mathbf{Sp}(n) \times \mathbf{Sp}(1))$. Recall (see Remark 2.5) that elements of K have norm 1.

After conjugating A by an element of $\mathbf{Sp}(n,1)$, we can assume that f sends $\mathbf{o}$ to a point on the vertical geodesic $(0,\infty)$ at distance δ from $\mathbf{o}$. We thus suppose that

$$f(\mathbf{o}) = {}^t\begin{bmatrix}-r^2 & 0 & \dots & 0 & 1\end{bmatrix} \sim {}^t\begin{bmatrix}-r & 0 & \dots & 0 & 1/r\end{bmatrix}.$$

The *dilatation associated to f* is the loxodromic element fixing 0 and q_∞ sending $\mathbf{o}$ to $f(\mathbf{o})$, with corresponding matrix

$$D = \begin{bmatrix} r & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & 1/r \end{bmatrix}.$$

In particular, this element satisfies

$$AD^{-1} \in K,$$

and an immediate computation shows that

$$\|D\| = r \quad \text{and} \quad \|D - I\| = \|D^{-1} - I\| = r - 1.$$

We easily bound $\|A\|$ from above:

Lemma 4.1. — $\|A\| \leqslant r$.

Proof. — Since $AD^{-1} \in K$, we have $\|AD^{-1}\| = 1$ and

$$\|A\| = \|AD^{-1}D\| \leqslant \|AD^{-1}\|\|D\| = r. \qquad \square$$

Remark 4.2. — Also, note that this implies that, for all integer q,

$$\|A^q\| \leqslant r^q.$$

This very simple remark is responsible for a slight improvement in the final bound we derive in Main Theorem compared to those previously known for the real and complex cases (compare this equation to the one at the bottom of p. 772 of the article of Xie, Wang and Jiang [24] for instance).

Bounding $\|A - I\|$ from above turns out to be more subtle: for some given element in $\mathbf{Sp}(n,1)$, it is not *a priori* clear weather it is close to the identity or not. For an element R of K, however, either R is of finite order, or it is an *irrational rotation*: it is therefore possible to approach I arbitrarily close by some power of R, and this is what we make explicit in Lemma 4.5. We then use the triangle inequality to bound $\|A - I\|$ (actually $\|A^q - I\|$) from above:

$$\|A^q - I\| \leqslant \|A^q - R^q\| + \|R^q - I\|. \tag{4.1}$$

The following lemma gives a bound for the first part of the right side of this expression:

Lemma 4.3. — *There exists an element R of K such that*

$$\|A^q - R^q\| \leqslant r(r^q - 1).$$

Proof. — Let $R \in K$. Recall the identity

$$A^q - R^q = (A - R)R^{q-1} + R(A - R)A^{q-2} + \cdots + R^{q-1}(A - R). \tag{4.2}$$

Using the fact that $\|A\| = r$ and that $\|R\| = 1$ we then obtain, for all $R \in K$,

$$\|A^q - R^q\| \leqslant \frac{r^q - 1}{r - 1}\|A - R\|.$$

Set $R = AD^{-1}$. Then

$$\|A - AD^{-1}\| \leqslant \|A\|\|1 - D^{-1}\| \leqslant r(r - 1),$$

and finally we get

$$\|A^q - (AD^{-1})^q\| \leqslant r(r^q - 1). \qquad \square$$

Next, we have to bound the second part of the right side of (4.1) from above. We shall do it by using the Dirichlet's pigeon-hole principle, which we recall (see for example [15, Chapter 3, Section 3]):

Lemma 4.4 (Dirichlet's pigeon-hole principle). — *Given n real numbers $\theta_i \in [0, 1]$, $i = 1, \ldots, n$, for all $Q \geqslant 1$, there exists an integer $q \leqslant Q^n$ and integers p_i, $i = 1, 2 \ldots, n$ such that*

$$\left|\theta_i - \frac{p_i}{q}\right| \leqslant \frac{1}{qQ}.$$

We deduce:

Lemma 4.5. — *Let R be in K. Then, for all $Q > 1$, there exists an integer q, $1 \leqslant q \leqslant Q^{n+1}$, such that*

$$\|R^q - I\| \leqslant \frac{\pi}{Q}.$$

Proof. — For this proof, we place ourselves in the ball model. Recall that $K \simeq K'$, where K' is the stabilizer of the origin 0 of the ball, the isomorphism being given by the conjugation by the Cayley transform C, which is unitary. By Remark 2.1, we thus see that proving Lemma 4.5 for elements of K' amounts to proving it for elements of K.

Let then R be an element of K', and write $R = \begin{bmatrix} R' & 0 \\ 0 & e^{\mu_1(2\pi\theta_{n+1})} \end{bmatrix}$. Without loss of generality, we can actually assume that

$$R = \begin{bmatrix} R' & 0 \\ 0 & e^{i(\pi\theta_{n+1})} \end{bmatrix}, \qquad R' \in \mathbf{Sp}(n), \theta_{n+1} \in [0,1].$$

We diagonalize R' (by the spectral theorem, see Section 2.1):

$$R' = P' \ \mathrm{Diag}(e^{i(\pi\theta_1)}, \dots, e^{i(\pi\theta_n)}) \ P'^{-1},$$

with $P' \in \mathbf{Sp}(n)$. Then

$$R = PR_1P^{-1},$$

with

$$R_1 = \mathrm{Diag}(e^{i(\pi\theta_1)}, \dots, e^{i(\pi\theta_n)}, e^{i\pi(\theta_{n+1})})$$

and

$$P = \begin{bmatrix} P' & 0 \\ 0 & 1 \end{bmatrix} \in K'.$$

Let $Q > 1$ be an integer, and let q, $p_i, i = 1 \dots n+1$, be the integer corresponding to the θ_i as in Lemma 4.4. Put

$$B = PB_1P^{-1} \in K',$$

where

$$B_1 = \mathrm{Diag}(e^{i(\pi\frac{p_1}{q})}, e^{i(\pi\frac{p_2}{q})}, \dots, e^{i(\pi\frac{p_n}{q})}, e^{i(\pi\frac{p_{n+1}}{q})}).$$

Then

$$\begin{aligned} \|R - B\| = \|R_1 - B_1\| &= \sqrt{r_\sigma\left((R_1^* - B_1^*)(R_1 - B_1)\right)} \\ &= \max \sqrt{\left|e^{i(\pi\theta_i)} - e^{i(\pi\frac{p_i}{q})}\right|^2} \\ &= \max \sqrt{|2 - 2\cos(\pi(\theta_i - \frac{p_i}{q})|} \\ &= \max \sqrt{|4\sin^2(\frac{\pi}{2}(\theta_i - \frac{p_i}{q}))|} \\ &= 2\max|\sin(\frac{\pi}{2}(\theta_i - \frac{p_i}{q}))| \\ &\leqslant \pi \max|\theta_i - \frac{p_i}{q}| \leqslant \frac{\pi}{qQ}. \end{aligned}$$

Finally, we use Identity (4.2) stated in Lemma 4.3 and the fact that $\|R\| = \|B\| = 1$ to obtain:

$$\begin{aligned} \|R^q - I\| &= \|R^q - B^q\| \\ &\leqslant q\|R - B\| \\ &\leqslant \frac{\pi}{Q}. \end{aligned}$$

□

Remark 4.6. — Let us emphasize: the fact that the eigenvalues of R' can be all chosen to have *positive* imaginary part is specific to the quaternionic setting (see Remark 2.3). This is responsible for an improvement of the constant λ_n bounding the maximal radius from below in the quaternionic setting (compare with [24, Lemma 4.2] in the complex setting).

Let us summarize the results obtained in this section:

Lemma 4.7. — *Let f be an isometry of $H^n_{\mathbb{H}}$, and $A \in \mathbf{Sp}(n,1)$ be the corresponding matrix. Put $\delta = \rho(\mathbf{o}, f(\mathbf{o}))$ and $r = \exp(\delta/2)$. Then, for all $Q > 1$, there exists an integer q, $1 < q \leqslant Q^{n+1}$, such that*

$$\|A^q\|\|A^q - I\| \leqslant r^q \left(r(r^q - 1) + \frac{\pi}{Q} \right).$$

Proof. — According to Lemma 4.1, $\|A\| \leqslant r$, therefore $\|A^q\| \leqslant r^q$. Combining (4.1), Lemma 4.3, and Lemma 4.5 we thus obtain

$$\|A^q - I\| \leqslant r(r^q - 1) + \frac{\pi}{Q}. \qquad \square$$

5. Conclusion

5.1. Proof of Main Theorem

We are now ready to give a proof of the main theorem of this article, which we state here again for convenience:

Main Theorem. — *Let $\Gamma \subset \mathbf{Sp}(n,1)$ be a discrete, torsion-free, non-elementary subgroup acting by isometries on the quaternionic hyperbolic space $H^n_{\mathbb{H}}$. There exists a point $p \in H^n_{\mathbb{H}}$ such that, for all $A \in \Gamma$, denoting by γ the corresponding isometry,*

$$\rho(p, \gamma(p)) \geqslant \lambda_n,$$

where $\lambda_n = \frac{0.05}{9^{n+1}}$. Every quaternionic hyperbolic manifold thus contains an embedded ball of radius $\lambda_n/2$.

Proof. — Firstly, from Theorem 3.6, we know that there exists an $H \in \mathbf{Sp}(n,1)$ such that, for all $C \in H\Gamma H^{-1}, C \neq I$,

$$\|C\|\|C - I\| \geqslant \omega \simeq 0.3854\ldots \tag{5.1}$$

Denoting by h the isometry of $H^n_{\mathbb{H}}$ corresponding to H, we shall prove the theorem with $p = h^{-1}(\mathbf{o})$.

Assume on the contrary that there is an isometry γ not satisfying the inequality of the theorem. Denote by A the corresponding matrix in $\mathbf{Sp}(n,1)$, put $\hat{A} = HAH^{-1}$, and let $\hat{\gamma}$ be the corresponding isometry. Also, put $r = \exp(\rho(\mathbf{o}, \hat{\gamma}(\mathbf{o}))/2)$.

Next, apply Lemma 4.7 with $Q = 9$: there exists an integer $q \leqslant 9^{n+1}$ such that

$$\|\hat{A}^q\| \|\hat{A}^q - I\| \leqslant r^q \left(r(r^q - 1) + \frac{\pi}{9} \right). \tag{5.2}$$

By assumption $r < e^{\lambda_n/2}$, so (since $n \geqslant 2$)

$$r < e^{\frac{0.025}{9^{n+1}}} \leqslant e^{\frac{0.025}{9^3}}$$
$$\text{and} \quad r^q \leqslant r^{9^{n+1}} < e^{\frac{\lambda_n}{2} \cdot 9^{n+1}} = e^{0.025}.$$

Consequently,

$$\|\hat{A}\| \|\hat{A} - I\| < e^{0.025} \left(e^{\frac{0.025}{9^3}} (e^{0.025} - 1) + \frac{\pi}{9} \right) \simeq 0.3838.. < 0.3854..$$

which contradicts (5.1). □

Remark 5.1. — We pointed out earlier, in Remarks 4.2 and 4.6, two facts that allowed us to improve the previously known bound for the maximal radius in the real and in the complex case. Precisely, in those two cases, instead of our inequality (5.2), the estimates in [24] are

$$\|\hat{A}^q\| \|\hat{A}^q - I\| \ \leqslant \ (r(r^q - 1) + 1) \left(r(r^q - 1) + \frac{2\pi}{Q} \right)$$

(with Q as in Lemma 4.4), and ultimately, this explains the differences between those results and the ours.

6. Related quantities

6.1. Volume

An immediate corollary of Main Theorem is that one can bound below the volume of a quaternionic hyperbolic manifold $\Gamma \backslash H^n_{\mathbb{H}}$ by the volume of such a ball. We compute the later using Lemma 2.8.

Corollary 6.1. — *Let M be a quaternionic hyperbolic manifold of dimension n. Then*

$$\mathrm{Vol}(M) \geqslant \frac{(4\pi)^{2n}}{(2n+1)!} \sinh^{4n} \left(\frac{0.0175}{9^{n+1}} \right).$$

However, as was shown by Corlette [8] and Gromov and Schoen [14], all lattices in $\mathbf{Sp}(n,1)$ are arithmetic. Hence, one could likely give a lower bound for the volume of (finite volume) quaternionic hyperbolic manifolds *via* arithmetic methods. Those methods have proven to be fruitful in the real hyperbolic case (see for example the work of Belolipetsky [4] and Belolipetsky and Emery [6]) and could probably be used to improve Corollary 6.1.

Nevertheless, information about the maximal radius of a manifold is not the same as, though related to, information on the volume of such a manifold. In particular, in this paper, the subgroup Γ need not be a lattice, but only a discrete subgroup of isometries, and the result of Main Theorem remains valid if M is of *infinite volume*.

6.2. Margulis constant

As we mentioned in the introduction, another quantity closely related to the maximal radius is the *Margulis constant* μ_n of a hyperbolic manifold of dimension n. For this number too, an interesting problem is to understand how it depends on the dimension. If we restrict our attention to the arithmetic case, conjecturally, $\mu_n^{\text{arithm.}}$ is uniformly bounded from below. On the other hand, there is an argument for why the Margulis constant of an *infinite volume* hyperbolic manifold can behave differently than in the finite volume case, due to Mikhail Kapovich, which can be found in Belolipetsky's 2014 ICM address.

To conclude this chapter, we show that Belolipetsky's arguments imply that if we take n large enough, we can exhibit a discrete subgroup $\Gamma \subset \mathbf{Sp}(n,1)$, *of infinite covolume*, with Margulis constant arbitrarily small, for it further motivates the methods we employed over the arithmetic ones.

Proposition 6.2 (Kapovich, see [5, Proposition 5.2]). — *There exists a constant $C > 0$ such that $\mu_{H^n_{\mathbb{R}}} \leqslant \frac{C}{\sqrt{n}}$.*

Which can be reformulated

Proposition 6.2′. — *For all n, one can construct a discrete subgroup Γ of $\mathrm{Isom}(H^n_{\mathbb{R}})$ and a point $x \in H^n_{\mathbb{R}}$ such that*

$$\Gamma_x(\epsilon) = \langle g \in \Gamma \mid d(x, g(x)) \leqslant \epsilon \rangle$$

is not virtually nilpotent, where $\epsilon = \frac{C}{\sqrt{n}}$.

Remark 6.3. — Further, the group Γ exhibited in the proof of Proposition 6.2′ has infinite covolume in $\mathrm{Isom}(H^n_{\mathbb{R}})$. The idea of the construction is as follows: consider the free group on two generators $F_2 = \langle f, g \rangle$, and $\mathcal{G}$

its Cayley graph (relatively to the generating system $\{f, g\}$). Construct an F_2-equivariant map σ of $\mathcal{G}$ into $H^n_{\mathbb{R}}$, sending the root of $\mathcal{G}$ and its two first edges f and g on a point $x = p_0$ in $H^n_{\mathbb{R}}$ and two geodesic segments of length ϵ emanating from p_0. Then, if σ is injective, the group

$$\sigma(F_2)_\epsilon(p_0) = \langle \gamma \in \sigma(F_2) \mid d(p_0, \gamma(p_0)) \leqslant \epsilon \rangle$$

is F_2 itself, highly non virtually nilpotent. Therefore, once one has constructed such a map an proved that it is an embedding, Proposition 6.2′ is proved, provided the image of F_2 in $\mathrm{Isom}(H^n_{\mathbb{H}})$ is discrete.

At first, Kapovich thus constructs a local F_2-equivariant embedding $\sigma : \mathcal{G} \to H^n_{\mathbb{R}}$, sending the first two edges of the graph on two geodesic segments of length ϵ.

Secondly, he shows that for $\epsilon \geqslant \frac{C}{\sqrt{n}}$, this map is a quasi-isometric embedding. This ensures that the image of F_2 is discrete (see for instance his article [16, Lemma 2.2]) which completes the proof.

Now, this proposition in turn implies

Proposition 6.4. — *There exists a constant $C > 0$ such that $\mu_{H^n_{\mathbb{H}}} \leqslant \frac{C}{\sqrt{n}}$.*

Proof. — Indeed, fix an n, and let $\Gamma \subset \mathbf{SO}(n,1)$ and $x \in H^n_{\mathbb{R}}$ be the discrete group of isometries and point of $H^n_{\mathbb{R}}$ given by Proposition 6.2′. The group $\mathbf{SO}(n,1)$ is embedded in $\mathbf{Sp}(n,1)$,

$$\iota : \mathbf{SO}(n,1) \hookrightarrow \mathbf{Sp}(n,1),$$

and acts on a totally geodesic isometric copy $\iota_*(H^n_{\mathbb{R}})$ of $H^n_{\mathbb{R}}$. So Γ itself is a discrete subgroup of $\mathbf{Sp}(n,1)$, acting on $\iota_*(H^n_{\mathbb{R}})$, and the group

$$\Gamma_{\iota_*(x)}(\epsilon) = \langle g \in \Gamma \mid d(\iota_*(x), g(\iota_*(x))) \leqslant \epsilon \rangle$$

is not virtually nilpotent. □

Observe that the proof above also works for $H^n_{\mathbb{C}}$, so that ultimately Proposition 6.2′ implies

Proposition 6.5. — *There exists a constant $C > 0$ such that*

$$\mu_{H^n_{\mathbb{K}}} \leqslant \frac{C}{\sqrt{n}}$$

for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Bibliography

[1] I. Adeboye, “Lower bounds for the volume of hyperbolic n-orbifolds”, *Pac. J. Math.* **237** (2008), no. 1, p. 1-19.

[2] I. Adeboye & G. Wei, "On volumes of hyperbolic orbifolds", *Algebr. Geom. Topol.* **12** (2012), no. 1, p. 215-233.

[3] ———, "On volumes of complex hyperbolic orbifolds", *Mich. Math. J.* **63** (2014), no. 2, p. 355-369.

[4] M. Belolipetsky, "On volumes of arithmetic quotients of $\mathrm{SO}(1,n)$", *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* **3** (2004), no. 4, p. 749-770.

[5] ———, "Hyperbolic orbifolds of small volume", https://arxiv.org/abs/1402.5394, 2014.

[6] M. Belolipetsky & V. Emery, "On volumes of arithmetic quotients of $\mathrm{PO}(n,1)^{\circ}$, n odd", *Proc. Lond. Math. Soc.* **105** (2012), no. 3, p. 541-570.

[7] S.-S. Chen & L. Greenberg, "Hyperbolic spaces", in *Contributions to analysis (a collection of papers dedicated to Lipman Bers)*, Academic Press, 1974, p. 49-87.

[8] K. Corlette, "Archimedean superrigidity and hyperbolic geometry", *Ann. Math.* **135** (1992), no. 1, p. 165-182.

[9] D. R. Farenick & B. A. F. Pidkowich, "The spectral theorem in quaternions", *Linear Algebra Appl.* **371** (2003), p. 75-102.

[10] S. Friedland & S. Hersonsky, "Jorgensen's inequality for discrete groups in normed algebras", *Duke Math. J.* **69** (1993), no. 3, p. 593-614.

[11] M. Gendulphe, "Systole et rayon interne des variétés hyperboliques non compactes", *Geom. Topol.* **19** (2015), no. 4, p. 2039-2080.

[12] A. Gray, *Tubes*, 2nd ed., Progress in Mathematics, vol. 221, Birkhäuser, 2004, xiii+280 pages.

[13] M. Gromov, "Hyperbolic groups", in *Essays in group theory*, Mathematical Sciences Research Institute Publications, vol. 8, Springer, 1987, p. 75-263.

[14] M. Gromov & R. Schoen, "Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one", *Publ. Math., Inst. Hautes Étud. Sci.* (1992), no. 76, p. 165-246.

[15] M. Hindry, *Arithmétique*, Tableau Noir, vol. 102, Calvage et Mounet, 2008, xvi+328 pages.

[16] M. Kapovich, "Representations of polygons of finite groups", *Geom. Topol.* **9** (2005), p. 1915-1951.

[17] D. A. Každan & G. A. Margulis, "A proof of Selberg's conjecture", *Math. USSR, Sb.* **4** (1968), no. 1, p. 147-152.

[18] I. Kim & J. R. Parker, "Geometry of quaternionic hyperbolic manifolds", *Math. Proc. Camb. Philos. Soc.* **135** (2003), no. 2, p. 291-320.

[19] G. J. Martin, "Balls in hyperbolic manifolds", *J. Lond. Math. Soc.* **40** (1989), no. 2, p. 257-264.

[20] ———, "On discrete Möbius groups in all dimensions: a generalization of Jørgensen's inequality", *Acta Math.* **163** (1989), no. 3-4, p. 253-289.

[21] J. R. Parker, "On the volumes of cusped, complex hyperbolic manifolds and orbifolds", *Duke Math. J.* **94** (1998), no. 3, p. 433-464.

[22] W. P. Thurston, "The geometry and topology of 3-manifolds", 1980, electronic edition of the 1980 lecture notes, distributed by Princeton University. Available at http://library.msri.org/books/gt3m/.

[23] H.-c. Wang, "Discrete nilpotent subgroups of Lie groups", *J. Differ. Geom.* **3** (1969), p. 481-492.

[24] B. Xie, J. Wang & Y. Jiang, "Balls in complex hyperbolic manifolds", *Sci. China, Math.* **57** (2014), no. 4, p. 767-774.

[25] F. Zhang, "Quaternions and matrices of quaternions", *Linear Algebra Appl.* **251** (1997), p. 21-57.