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ABSTRACT. — The set of axes of hyperbolic elements in a Fuchsian group depends on the commensurability class of the group. In fact, it has been conjectured that it determines the commensurability class and this has been verified for groups of the second kind by G. Mess and for arithmetic groups by D. Long and A. Reid. Here we show that the conjecture holds for almost all Fuchsian groups and explain why our method fails for arithmetic groups.

RÉSUMÉ. — L'ensemble des axes d'éléments hyperboliques dans un groupe fuchsien dépend de la classe de commensurabilité du groupe. En effet, cet ensemble détermine la classe de commensurabilité pour les groupes du deuxième type, d'après G. Mess, et pour les groupes arithmétiques, d'après D. Long et A. Reid. Selon une veille conjecture, la classe de commensurabilité d'un groupe fuchsien non élémentaire est toujours déterminée par ses axes. Nous montrons ici que la conjecture est vraie pour presque tous les groupes fuchsiens et expliquons pourquoi notre méthode ne s'applique pas aux groupes arithmétiques.

1. Introduction

Let Σ be a closed orientable hyperbolic surface. The free homotopy classes of closed geodesics on Σ correspond to conjugacy classes of hyperbolic elements in Γ . If $\gamma \in \Gamma$ is a hyperbolic element, then associated to γ is an axis $\operatorname{ax}(\gamma) \subset \mathbb{H}$. The projection of $\operatorname{ax}(\gamma)$ to Σ determines a closed geodesic whose length is ℓ_{γ} . We shall denote the set of axes of all the hyperbolic elements in Γ by $\operatorname{ax}(\Gamma)$. It's easy to check that if $g \in \operatorname{PSL}(2, \mathbb{R})$ then we have the relation

$$\operatorname{ax}(g\Gamma g^{-1}) = g \operatorname{ax}(\Gamma).$$
(1.1)

Keywords: Fuchsian groups, commensurability.

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1.1. Isoaxial groups

Following Reid [12] we say that a pair of Fuchsian groups Γ_1 and Γ_2 are isoaxial iff $\operatorname{ax}(\Gamma_1) = \operatorname{ax}(\Gamma_2)$. One obtains a trivial example of an isoaxial pair by taking Γ_1 any Fuchsian group and $\Gamma_2 < \Gamma_1$ any finite index subgroup. This example can be extended to a more general setting as follows. Recall that a pair of subgroups Γ_1 and Γ_2 are commensurable if $\Gamma_1 \cap \Gamma_2$ is a subgroup of finite index in both Γ_1 and Γ_2 . Thus if Γ_1 and Γ_2 are commensurable then they are isoaxial because:

$$ax(\Gamma_1) = ax(\Gamma_1 \cap \Gamma_2) = ax(\Gamma_2).$$

It is natural to ask whether the converse is true:

If Γ_1 and Γ_2 are isoaxial then are they commensurable?

In what follows we shall say simply that the group Γ_1 is determined (up to commensurability) by its axes. We shall show that this conjecture holds for almost all Fuchsian groups:

THEOREM 1.1. — For almost every point ρ in the Teichmueller space of a hyperbolic surface Σ the corresponding Fuchsian representation of the fundamental group Γ is determined by its axes.

1.2. Spectra

We define the length spectrum of Σ to be the collection of lengths ℓ_{α} of primitive closed geodesics $\alpha \subset \Sigma$ counted with multiplicity. In fact, since Σ is compact, the multiplicity of any value in the spectrum is finite and moreover the set of lengths is discrete. Let α, β be primitive closed geodesics which meet at a point $z \in \Sigma$, we denote by $\alpha \angle_z \beta$, the angle measured in the counter-clockwise direction from α to β . We define the *angle spectrum* to be the collection of all such angles (counted with multiplicity) see Mondal [9, 10] who studies a related *marked* spectra. Note that we do not suppose that zis the sole intersection point of α, β and, as such, a pair of geodesics may contribute several angles to the spectrum.

1.2.1. Lengths, marked and unmarked

The length spectrum has proved useful in studying many problems concerning the geometry of hyperbolic surfaces. Fricke–Klein proved that a closed hyperbolic surface Σ is uniquely determined by its marked length

spectrum that is the set of pairs $([a], \ell_{\alpha})$ for all closed geodesics [a] is a non trivial conjugacy class of $\pi_1(\Sigma)$ and α is the unique oriented closed geodesic determined by [a]. Here the algebraic data is constitutes a marking and one obtains the length spectrum by forgetting the marking but keeping count of the multiplicities. It is natural to ask whether the unmarked spectrum determines the such a surface too. Generalising the result of Fricke–Klein is a difficult problem but Otal [11] and Croke [3] succeeded in proving that the marked spectrum determines a surface surfaces of variable negative curvature Vigneras [14] and Sunada [13] gave constructions which allow one to construct pairs of *isospectral surfaces*, that is surfaces which are not isometric though they share the same length spectrum.

1.2.2. Angles

The angle spectrum is very different from the length spectrum: the set of angles is obviously not discrete and, as we shall see, there are surfaces for which every value has infinite multiplicity. However, when considering the question of whether groups are isoaxial, the angle spectrum has a distinct advantage for it is easy to see that:

- There are isoaxial groups which do not have the same set of lengths, that is, the same angle spectrum without multiplicities.
- If two groups are isoaxial then they have the same set of angles, that is, the same angle spectrum without multiplicities.

Using properties of angles we will deduce Theorem 1.1 from the following lemma inspired by a result of G. Mess (see paragraph 2.1).

LEMMA 1.2. — Define the group of automorphisms of $\operatorname{ax}(\Gamma)$ to be the group of hyperbolic isometries which preserve $\operatorname{ax}(\Gamma)$. If Σ has a value in its angle spectrum with finite multiplicity then Γ is a subgroup of finite index in the group of automorphisms of $\operatorname{ax}(\Gamma)$.

It remains to prove that there are such points of $\mathcal{T}(\Sigma)$, we show in fact that they are generic:

THEOREM 1.3. — For almost every point $\rho \in \mathcal{T}(\Sigma)$ there is a value in the angle spectrum which has multiplicity exactly one.

Our method applies provided there is some value in the angle spectrum that has *finite* multiplicity. Unfortunately, for arithmetic surfaces, the multiplicity of every value is infinity (Lemma 2.5).

1.3. Sketch of proof

The method of proof of Theorem 1.1 follows the proof of the first part of Theorem 1.1 in [8]. This says that the set of surfaces in Teichmueller space where every value in the simple length spectrum has multiciplity exactly one is dense and its complement is measure zero (for the natural measure on Teichmueller space.)

1.3.1. Two properties of (simple) length functions

Recall that the *simple length spectrum* is defined to be the collection of lengths of simple closed geodesics counted with multiplicity.

There are two main ingredients used in [8]:

- The analyticity of the geodesic length ℓ_{α} as a function over Teichmueller space;
- The fact that if α, β are a pair of distinct simple closed geodesics then the difference $\ell_{\alpha} - \ell_{\beta}$ defines a non constant (analytic) function on the Teichmueller space $\mathcal{T}(\Sigma)$.

It is clear that the set of surfaces where every value in the simple length spectrum has multiciplity exactly one is the complement of

$$Z := \bigcup_{(\alpha,\beta)} \{\ell_{\alpha} - \ell_{\beta} = 0\},\$$

where the union is over all pairs α , β of distinct closed simple geodesics. Each of the sets on the left is nowhere dense and its intersection with any open set is measure zero. Since Z is countable union of such sets, its complement is dense and meets every open set in a set of full measure.

We note in passing that the second of these properties is not true without the hypothesis "simple". Indeed, there are pairs of distinct closed unoriented geodesics $\alpha \neq \beta$ such that $\ell_{\alpha} = \ell_{\beta}$ identically on $\mathcal{T}(\Sigma)$ (see [2] for an account of their construction).

1.3.2. Analogues for angles

We will deduce Theorem 1.1 using the same approach but instead of geodesic length functions we use angle functions. The most delicate point is to show that if α_1, α_2 are a pair of simple closed geodesics that meet in a

single point z and β_1, β_2 are a pair of closed geodesics that meet in a point z' then the difference $\alpha_1 \angle_z \alpha_2 - \beta_1 \angle_{z'} \beta_2$ defines a non constant function on Teichmueller space.

We do this by establishing the analogue of the following property of geodesic length functions:

FACT 1.4. — A closed geodesic $\alpha \subset \Sigma$ is simple if and only if the image of the geodesic length function ℓ_{α} is $]0, \infty[$.

Our main technical result (Theorem 6.3) is an analogue of this property. We consider pairs of simple closed geodesics α_1, α_2 which meet in a meet in a single point z. This configuration will be the analogue of a simple closed geodesic. Now, for any such pair we find a subset $X \subset \mathcal{T}(\Sigma)$ such that, for any other pair of closed geodesics β_1, β_2 which meet in $z' \neq z$:

- the image of X under $\beta_1 \angle_{z'} \beta_2$ is a proper subinterval of $]0, \pi[$
- whilst its image under $\alpha_1 \angle_z \alpha_2$ is the whole of $]0, \pi[$.

1.4. Further remarks

Since one objective of this work is to compare systematically the properties of geodesic length and angle functions we include an exposition of geodesic length functions, and give an account of the characterisation of simple geodesics, mentioned above, in our Proposition 3.2. Our argument is inspired by a treatment of a result of Yamada by Gendulphe [4].

Mondal [9] has obtained a rigidity result by using a richer collection of data than we use here. He defines a *length angle spectrum* and proves that this determines a surface up to isometry. However, the set of axes does not determine the lengths of closed geodesics and so commensurability is the best one can hope for in the context we consider here.

In paragraph 2.2.1 we answer a question of Mondal in [10] concerning multiplicities by observing that arithmetic surfaces are very special: the multiplicity of any angle in the angle spectrum is infinite.

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We also thank the refreree for pointing out that the case of a three holed sphere is completely settled by combining the results of Mess and Long–Reid.

2. Automorphisms and commensurators

To study this question we define, following Reid, two auxilliary groups. The first is the group of automorphisms of $ax(\Gamma)$:

$$\operatorname{Aut}(\operatorname{ax}(\Gamma)) := \{ \gamma \in \operatorname{PSL}(2, \mathbb{R}), \, \gamma(\operatorname{ax}(\Gamma)) = \operatorname{ax}(\Gamma) \}.$$

The second is the *commensurator of* Γ defined as:

 $\operatorname{Comm}(\Gamma) := \{ \gamma \in \operatorname{PSL}(2, \mathbb{R}) : \gamma \Gamma \gamma^{-1} \text{ is directly commensurable with } \Gamma \}.$

We leave it to the reader to check that $\operatorname{Aut}(\operatorname{ax}(\Gamma))$ and $\operatorname{Comm}(\Gamma)$ are indeed groups and that they contain Γ as a subgroup. In fact any element $\gamma \in \operatorname{Comm}(\Gamma)$ is an automorphism of $\operatorname{ax}(\Gamma)$. To see this, if $\gamma \in \operatorname{Comm}(\Gamma)$, then Γ and $\gamma \Gamma \gamma^{-1}$ are commensurable so are isoaxial. Now by (1.1) one has

$$\operatorname{ax}(\Gamma) = \operatorname{ax}(\gamma \Gamma x^{-1}) = \gamma \operatorname{ax}(\Gamma)$$

so $\gamma \in Aut(ax(\Gamma))$. In summary one has a chain of inclusions of subgroups:

$$\Gamma < \operatorname{Comm}(\Gamma) < \operatorname{Aut}(\operatorname{ax}(\Gamma)) < \operatorname{PSL}(2, \mathbb{R}).$$

We shall be concerned with two cases:

- (1) Γ is finite index in Aut(ax(Γ)).
- (2) Aut(ax(Γ)) is dense in PSL(2, ℝ) so that Γ is necessarily an infinite index subgroup.

The first case arises for the class of Fuchsian groups of the second kind studied by G. Mess and the second for arithmetic groups.

2.1. Fuchsian groups of the second kind

In an IHES preprint, G. Mess studied a variety of questions relating to $ax(\Gamma)$ notably proving the following result:

THEOREM 2.1 (Mess). — If Γ_1 and Γ_2 are isoaxial Fuchsian groups of the second kind then they are commensurable.

In fact, any hyperbolic surface homeomorphic to a pair of pants is determined by its set of axes. This is a special case of Mess' theorem with one exceptional case, namely the three punctured sphere, and that case is settled by Long–Reid theorem in the next section.

The proof of Mess' result is a consequence of the fact that, under the hypotheses, $\operatorname{Aut}(\operatorname{ax}(\Gamma))$ is a discrete, convex cocompact Fuchsian group. It is easy to deduce from this that Γ is finite index in $\operatorname{ax}(\Gamma)$.

To show that $\operatorname{Aut}(\operatorname{ax}(\Gamma))$ is discrete it suffices to find a discrete subset of \mathbb{H} , containing at least two points, on which it acts. Recall that the *convex hull* of the limit set of Γ , is a convex subset $C(\Lambda) \subset \mathbb{H}$. If Γ is a Fuchsian groups of the second kind then its limit set Λ is a proper subset of $\partial \mathbb{H}$ and $C(\Lambda)$ is a proper subset of \mathbb{H} whose frontier $\partial C(\Lambda)$ consists of countably many complete geodesics which we call sides. By definition $\operatorname{ax}(\Gamma)$ is $\operatorname{Aut}(\operatorname{ax}(\Gamma))$ -invariant and so $C(\Lambda)$ is too since, in fact, it is the minimal convex set containing $\operatorname{ax}(\Gamma)$. Now choose a minimal length perpendicular λ between edges of $C(\Lambda)$; such a minimising perpendicular exists because the double of $C(\Lambda)/\Gamma$ is a compact surface without boundary, every perpendicular between edges of $C(\Lambda)$ gives rise to a closed geodesic on the double and the length spectrum of the double is discrete. Let L be the $\operatorname{Aut}(\operatorname{ax}(\Gamma))$ -orbit of λ and observe that $L \cap \partial C(\Lambda)$ is a discrete set which contains at least two points.

2.2. Arithmetic groups

In the case of Fuchsian groups of the first kind Long and Reid [6] proved the conjecture for arithmetic groups.

THEOREM 2.2 (Long-Reid). — If a Fuchsian group is arithmetic then its commensurator is exactly the group of automorphisms of the group.

Also notice that if Γ_1 and Γ_2 are isoaxial Fuchsian groups, then for any $\gamma \in \Gamma_2$

$$\operatorname{ax}(\Gamma_1) = \operatorname{ax}(\gamma \Gamma_1 \gamma^{-1}),$$

and therefore $\gamma \in \operatorname{Aut}(\operatorname{ax}(\Gamma))$. Hence $\Gamma_2 < \operatorname{Aut}(\operatorname{ax}(\Gamma))$.

So by the above discussion $\Gamma_2 < \text{Comm}(\Gamma_1)$, and if Γ_2 is also arithmetic, then Γ_1 and Γ_2 are commensurable. Thus they obtain as a corollary:

COROLLARY 2.3. — Any pair of isoaxial arithmetic Fuchsian groups is commensurable.

2.2.1. Multiplicities for arithmetic groups

Let Γ be an arithmetic Fuchsian group and since its commensurator is dense in $SL(2, \mathbb{R})$, the set of geodesic intersctions is "locally homogenous" in the following sense:

LEMMA 2.4. — Let $\theta = \alpha \angle_z \beta$ be an angle of intersection of closed geodesics, then for any open subset $U \subset \Sigma$ there is a pair of closed geodesics α_u, β_u such that:

$$\alpha_u \angle_{z_u} \beta_u, \, z_u \in U.$$

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Proof. — Choose hyperbolic elements $a, b \in \Gamma$ such that the axis of a (resp. b) is a lift of α (resp β) to \mathbb{H} and so that the axes meet in a lift $\hat{z} \in \mathbb{H}$ of z. Since Comm(Γ) is dense in SL(2, \mathbb{R}), there is some element $g \in \text{Comm}(\Gamma)$ so that $g(\hat{z}) \in \hat{U}$ for some lift of U to \mathbb{H} . By the commensurability of the groups Γ and $g\Gamma g^{-1}$ there is a positive integer m such that $(gag^{-1})^m, (gbg^{-1})^m \in \Gamma$ so that the axes of these elements project to closed geodesics α_u, β_u on Σ meeting in a point z_u as required.

An immediate corollary is:

COROLLARY 2.5. — The multiplicity of any angle θ in the spectrum of an arithmetic surface Σ/Γ is infinite.

3. Functions on Teichmueller space

Recall that the Teichmueller of a surface Σ , $\mathcal{T}(\Sigma)$, is the set of marked complex structures and that, by Riemann's Uniformization Theorem, this is identified with a component of the character variety of $\mathrm{PSL}(2,\mathbb{R})$ -representations of $\pi_1(\Sigma)$. Thus we think of a point $\rho \in \mathcal{T}(\Sigma)$ as an equivalence class of $\mathrm{PSL}(2,\mathbb{R})$ -representations of $\pi_1(\Sigma)$. We remark that $\mathrm{PSL}(2,\mathbb{R}) :=$ $\mathrm{SL}(2,\mathbb{R})/\langle -I_2 \rangle$ so that although the trace $\mathrm{tr} \,\rho(a)$ is not well defined for $a \in \pi_1(\Sigma)$, the square of the trace $\mathrm{tr}^2 \,\rho(a)$ is and so is $|\mathrm{tr} \,\rho(a)|$. In fact, there is a natural topology $\mathcal{T}(\Sigma)$ such that for each $a \in \pi_1(\Sigma)$, $\rho \mapsto \mathrm{tr}^2 \,\rho(a)$ is a real analytic function.

3.1. Geodesic length

If $a \in \pi_1(\Sigma)$ is non trivial then there is a unique oriented closed simple geodesic α in the conjugacy class [a] determined by a. The length of α , measured in the Riemannian metric on $\Sigma = \mathbb{H}/\rho(\pi_1(\Sigma)))$, can be computed from tr $\rho(a)$ using the well-known formula

$$|\operatorname{tr} \rho(a)| = 2\cosh(\ell_{\alpha}/2). \tag{3.1}$$

There is a natural function,

$$\ell : \mathcal{T}(\Sigma) \times \{\text{homotopy classes of loops}\} \to]0, +\infty[$$

which takes the pair ρ , [a] to the length ℓ_{α} of the geodesic in the homotopy class [a]. It is an abuse, though common in the literature, to refer merely to the length of the geodesic α (rather than, more properly, the length of the geodesic in the appropriate homotopy class).

We define the length spectrum of Σ to be the collection of lengths ℓ_{α} of closed geodesics $\alpha \subset \Sigma$ counted with multiplicity. In fact, since Σ is compact, the multiplicity of any value in the spectrum is finite and moreover the set of lengths is discrete.

3.1.1. Analyticity

A careful study of properties of length functions was made in [8] where one of the key ingredients is the analyticity of this class of functions:

FACT 3.1. — For each closed geodesic α , the function

 $\mathcal{T}(\Sigma) \to]0, +\infty[, \rho \mapsto \ell_{\alpha}]$

is a non constant, real analytic function.

See [1] for a proof of this. Note that, to prove that such a function is non constant, it is natural to consider two cases according to whether the geodesic α is simple or not:

- (1) if α is simple then by including it as a curve in a pants decomposition one can view ℓ_{α} as one of the Fenchel–Nielsen coordinates so it is obviously non constant and, moreover, takes on any value in $]0, +\infty[$
- (2) if α is not simple then it suffices to find a closed simple geodesic β such that α and β meet and use the inequality (see Buser [2])

$$\sinh(\ell_{\alpha}/2)\sinh(\ell_{\beta}/2) \ge 1 \tag{3.2}$$

to see that if $\ell_{\beta} \to 0$ then $\ell_{\alpha} \to \infty$ and so is non constant.

3.1.2. Characterization of simple geodesics

There is always a simple closed geodesic shorter than any given closed geodesic. More precisely, if $\beta \subset \Sigma$ is a closed geodesic which is not simple then by doing surgery at the double points one can construct a simple closed geodesic $\beta' \subset \Sigma$ with $\ell_{\beta'} < \ell_{\beta}$.

For $\epsilon > 0$ define the ϵ -thin part of the Teichmueller space $\mathcal{T}(\Sigma)$ to be the set

 $\mathcal{T}_{<\epsilon}(\Sigma) := \{\ell_{\beta} < \epsilon, \forall \beta \text{ closed simple}\} \subset \mathcal{T}(\Sigma).$

By definition, on the complement of the thin part $\ell_{\beta} \ge \epsilon$ for all simple closed geodesics and since, by the preceding remark, there is always a simple closed geodesic shorter than any given closed geodesic, $\ell_{\beta} \ge \epsilon$ for all closed geodesics.

PROPOSITION 3.2. — Let Σ be a finite volume hyperbolic surface. Then a closed geodesic $\alpha \subset \Sigma$ is simple if and only if the infimum over $\mathcal{T}(\Sigma)$ of the geodesic length function ℓ_{α} is zero.

Proof. — In one direction, if α is simple then ℓ_{α} is one of the Fenchel– Nielsen coordinates for some pants decomposition of Σ so there is some (non convergent) sequence $\rho_n \in \mathcal{T}(\Sigma)$ such that $\ell_{\alpha} \to 0$.

Now suppose that α is not simple and we seek a lower bound for its length. There are two cases depending on whether there exists a closed simple geodesic β disjoint from α or not. If there is no such geodesic then α meets every simple closed geodesic $\beta \subset \Sigma$ and it is cusomary to call such a curve a *filling curve*. Choose $\epsilon > 0$ and consider the decomposition of the Teichmueller space into the ϵ -thin part and its complement. On the thick part $\ell_{\alpha} \geq \epsilon$ whilst on the thin part, by the inequality (3.2), it is bounded below by $\operatorname{arcsinh}(1/\sinh(\epsilon/2))$.

If there is an essential simple closed geodesic disjoint from α then we cut along this curve to obtain a possibly disconnected surface with geodesic boundary. We repeat this process to construct a compact surface $C(\alpha)$ such that α is a filling curve in $C(\alpha)$. By construction $C(\alpha)$ embeds isometrically as a subsurface of Σ and since α is not simple $C(\alpha)$ is not an annulus. On the other hand, by taking the Nielsen extension of $C(\alpha)$ then capping off with a punctured disc we obtain a conformal embedding $C(\alpha) \hookrightarrow C(\alpha)^*$ where $C(\alpha)^*$ is a punctured surface with a natural Poincaré metric. By the Ahlfors–Pick–Schwarz Lemma there is a contraction between the metrics induced on $C(\alpha)$ from the metric on Σ and from the Poincaré metric on $C(\alpha)^*$. A consequence of this is that the geodesic in the homotopy class determined by α on $C(\alpha)$ is longer than the one in $C(\alpha)^*$. So, to bound ℓ_{α} it suffices to bound the length of every filling curve on a punctured surface. There are two cases.

- If $C(\alpha)$ has an essential simple closed curve then we have already treated this case above.
- If $C(\alpha)$ has no essential simple closed curves then it is a 3 punctured sphere and the bound is trivial since the Teichmueller space consists of a point.

4. Fenchel–Nielsen twist deformation

Whilst make no claim as to the originality of the material in this section it is included to set up notation give an exposition of two results which we use in Section 6.1.

4.1. The Fenchel–Nielsen twist

We choose a simple closed curve $\alpha \subset \Sigma$. Following [5], cut along this curve, and take the completion of the resulting surface with respect to the path metric to obtain a possibly disconnected surface with geodesic boundary Σ' .

Obviously, one can recover the original surface from Σ' by identifying pairs of points of one from each of the boundary components. More generally, if $t \in \mathbb{R}$ then a *(left) Fenchel–Nielsen twist along* α allows one to construct a new surface Σ_t , homeomorphic to Σ by identifying the two boundary components with a left twist of distance t, i.e. the pair of points which are identified to obtain Σ are now separated by distance t along the image of α in Σ_t . Thus this construction gives rise to a map, which we will call the *time* t twist along α ,

$$\tau^t_{\alpha}: \Sigma \to \Sigma_t,$$

discontinuous for $t \neq 0$ and mapping $\Sigma \setminus \alpha$ isometrically onto $\Sigma_t \setminus \alpha$. Note that τ^t_{α} is not unique but this will not be important for our analysis, what is important, and easy to see from the construction, is that the geometry of $\Sigma_t \setminus \alpha$ does not vary with t as we will exploit this to obtain our main result.

4.2. The lift of the twist to \mathbb{H}

Let Γ be Fuchsian group such that $\Sigma := \mathbb{H}/\Gamma$ is a closed surface, $\alpha \subset \Sigma$ a non separating simple closed geodesic and $x \notin \alpha$ a basepoint for Σ . Now let $A \subset \mathbb{H}$ denote the set of all lifts of α and $\hat{x} \in \mathbb{H}$ a lift of x. Then the complement of A consists of an infinite collection of pairwise congruent, convex sets. Moreover, if P denotes the connected component of the complement of Acontaining \hat{x} , then P can be identified with the universal cover of the surface $\Sigma \setminus \alpha$ and the subgroup $\Gamma^P < \Gamma$ that preserves P is isomorphic to the fundamental group of this subsurface. Since the geometry of Σ_t does not change with $t \in \mathbb{R}$ the geometry of P does not change either. This observation is the key to establishing uniform bounds in the proof of Theorem 6.3.

Each of the other connected components of $\mathbb{H} \setminus P$ can be viewed as a translate of $g_i(P)$ for some element g_i of Γ and so \mathbb{H} is *tiled* by copies of P. Let us consider how this tiling evolves under the time t twist τ^t_{α} along α . There is a unique lift $\hat{\tau}^t_{\alpha} : \mathbb{H} \to \mathbb{H}$ which fixes \hat{x} and hence P. We can calculate the image of a translate of P under the lift of $\hat{\tau}^t_{\alpha}$ by a recursive procedure. Suppose that for some $g_1, \ldots, g_n \in \Gamma$;

- $\cup g_i(\bar{P})$ is connected,
- we have determined the images of $g_1(P), \ldots g_n(P)$.

Let $g_{n+1}(P)$ be a translate of P such that $g_{n+1}(\bar{P}) \cap g_n(\bar{P}) = \hat{\alpha}$, and we consider two cases:

- (1) If $g_n(P) = P$ then the image of $g_{n+1}(P)$ is $\phi^t(g_{n+1}(P))$ where ϕ^t is a hyperbolic translation of length t with axis $\hat{\alpha}$.
- (2) If $g_n(P) \neq P$ and its image under $\hat{\tau}^t_{\alpha}$ is h(P) then the image of $g_{n+1}(P)$ is $h \circ \phi^t \circ g_n^{-1}(g_{n+1}(P))$ where ϕ^t is a hyperbolic translation of length t with axis $g_n^{-1}(\hat{\alpha}) \subset A$.

This procedure allows us to prove the following:

LEMMA 4.1. — Let $\Lambda^P \subset \partial \mathbb{H}$ denote the limit set of Γ^P . Then $\hat{\tau}^t_{\alpha}$ admits a canonical extension $\hat{\tau}^t_{\alpha} : \mathbb{H} \sqcup \partial \mathbb{H} \to \mathbb{H} \sqcup \partial \mathbb{H}$ which is continuous on $\partial \mathbb{H}$. Further:

- (1) For any $w \in \Gamma^P$ one has $\hat{\tau}^t_{\alpha}(w) = w$;
- (2) For any $w \in \partial \mathbb{H}$ one has $\lim_{t \to \pm \infty} \hat{\tau}^t_{\alpha}(w) \in \Lambda^P$ and further this is an endpoint of an edge of $\partial \overline{P}$.

Proof. — It is standard from the theory of negatively curved groups that the lift admits a unique extension to $\mathbb{H} \sqcup \partial \mathbb{H}$, continuous on the boundary $\partial \mathbb{H}$, since \mathbb{H}/Γ is compact. It follows too from compactness of the quotient that the restriction of this lift to the set of lifts of the base point $x \in \Sigma$ is Lipschitz.

Since the extension is continuous, to prove (1) it suffices to note that the lift of the Fenchel–Nielsen deformation fixes the endpoints of the edges of $\partial \overline{P}$ and these are dense in Λ^{P} .

For (2) let $w \in \partial \mathbb{H}$ and suppose that it is not a point of $\partial \overline{P}$. Then there is an edge $\hat{\alpha}$ of $\partial \overline{P}$ such that w is a point of the interval determined by the endpoints of this geodesic. It is easy to check using our recursive description of the action of $\hat{\tau}^t_{\alpha}$ on \mathbb{H} that w converges to the appropriate endpoint of $\hat{\alpha}$.

We note that (2) can also be proved as follows. For $t = n\ell_{\alpha}, n \in \mathbb{Z}$ the Fenchel–Nielsen twist coincides with a Dehn twist. If β is a loop, disjoint from α then (up to homotopy) it is fixed by the Dehn twist. If β is a loop which crosses α then under iterated Dehn twists $\operatorname{tw}_{\alpha}^{n}$ it limits to a curve on Σ that spirals to α . That is, lifting to \mathbb{H} and considering the extension of the lift of the Dehn twist $\operatorname{tw}_{\alpha}^{n} : \mathbb{H} \sqcup \partial \mathbb{H} \to \mathbb{H} \sqcup \partial \mathbb{H}$, an endpoint of $\operatorname{tw}_{\alpha}^{n}(\beta)$ converges to an endpoint of some lift of α . It is not difficult to pass to general t using the fact that the $\hat{\tau}_{\alpha}^{t}$ extends to a homeomorphism on $\mathbb{H} \sqcup \partial \mathbb{H}$.

4.3. Separated geodesics

We say that a pair of geodesics $\hat{\gamma}_1, \hat{\gamma}_2 \subset \mathbb{H}$ are separated by a geodesic $\hat{\gamma}$ with end points $\hat{\gamma}^{\pm} \in \partial \mathbb{H}$ if the ideal points of $\hat{\gamma}_1, \hat{\gamma}_2$ are in different connected components of $\partial \mathbb{H} \setminus {\{\hat{\gamma}^{\pm}\}}$. Note that $\hat{\gamma}_1, \hat{\gamma}_2$ are necessarily disjoint.

If $\gamma_1, \gamma_2 \subset \mathbb{H}$ are a pair of simple closed geodesics, such that $\alpha, \gamma_1, \gamma_2$ are disjoint and we choose an arc β between γ_1 and γ_2 that meets α transversely in a single point then this configuration lifts to \mathbb{H} as $\hat{\gamma}_1, \hat{\gamma}_2$ separated by a lift $\hat{\alpha}$ of α . It is easy to convince oneself that, as we deform by the Dehn twist $\operatorname{tw}^n_{\alpha}$, the length of β goes to infinity. Essentially, our next lemma says that this is true for any pair of geodesics γ_1, γ_2 in Σ admitting an arc that meets α in an essential way.

LEMMA 4.2. — Let $\hat{\gamma}_1, \hat{\gamma}_2 \subset \mathbb{H}$ be a pair of geodesics which are separated by some lift of α then the distance between $\hat{\tau}^t_{\alpha}(\hat{\gamma}_1)$ and $\hat{\tau}^t_{\alpha}(\hat{\gamma}_2)$ tends to infinity as $t \to \pm \infty$.

Proof. — Let $\hat{\alpha}$ be a lift of α which separates $\hat{\gamma}_1, \hat{\gamma}_2 \subset \mathbb{H}$. Let P_1 and P_2 be the pair of complementary regions which have $\hat{\alpha}$ as a common edge and we label these so that $\hat{\gamma}_i$ is on the same side of $\hat{\alpha}$ as P_i for i = 1, 2. We choose the lift of the base point to be in P_1 and lift the Fenchel–Nielsen deformation.

First consider the orbit $\hat{\tau}_{\alpha}^{t}(y)$ of an ideal endpoint y of $\hat{\gamma}_{2}$ as $t \to \infty$. Since $x \in P_{1}$, the region P_{2} gets translated and so, for any side β of P_{2} , the sequence $\hat{\tau}_{\alpha}^{t}(\beta)$ converges to the endpoint $\hat{\alpha}^{+}$. Now there is a pair of edges β_{1}, β_{2} such that the endpoints of $\hat{\gamma}_{2}$ are contained in the closed interval containing the endpoints of β_{1}, β_{2} . Since each of the β_{i} converges to $\hat{\alpha}^{+}$ under the deformation it is easy to see that $\hat{\tau}_{\alpha}^{t}(\hat{\gamma}_{2})$ must converge to $\hat{\alpha}^{+}$ too.

Now consider the orbit of an endpoint y of $\hat{\gamma}_1$ under the deformation. It suffices to show that, under this deformation, y does not converge to $\hat{\alpha}^+$. There are two cases according to whether or not y belongs to the limit set Λ^{P_1} of the subgroup of Γ which stabilises P_1 .

- (1) If $y \in \Lambda^{P_1}$ then it is invariant under the Fenchel–Nielsen deformation.
- (2) If $y \notin \Lambda^{P_1}$ then it limits to a point in $y_{\infty} \in \Lambda^{P_1}$ which is an endpoint of one of the edges of P_1 . By hypothesis $\hat{\gamma}_1$ does not meet $\hat{\alpha}$ and so y_{∞} is not $\hat{\alpha}^+$.

5. Geodesic angle functions

We present two methods for computing (functions of) the angle $\alpha_1 \angle_z \alpha_2$ between α_1, α_2 at z. The first method, just like the formula (3.1) for geodesic length, is a closed formula in terms of traces (equation (5.1) whilst the second is in terms of end points of lifts of α_1, α_2 to the Poincaré disk (equation (5.2)). This second formula will prove useful for obtaining estimates for the variation of angles along a Fenchel–Nielsen deformation. In either case, we start as before by identifying Σ with the quotient \mathbb{H}/Γ where $\Gamma = \rho(\pi_1(\Sigma)), \rho \in \mathcal{T}(\Sigma)$. We choose z as a basepoint for Σ and associate elements $a_1, a_2 \in \pi_1(\Sigma, z)$ such that α_i is the unique oriented closed geodesic in the conjugacy class $[a_i]$ in the obvious way.

5.1. Traces and analyticity

As explained in the introduction we shall need an analogue of Fact 3.1 so we give a brief account of the analyticity of the angle functions:

PROPOSITION 5.1. — Let α_1, α_2 be a pair of (not necessarily simple) closed geodesics meeting at z. If $\rho \in \mathcal{T}(\Sigma)$ is a point in Teichmüller space then

$$\mathcal{T}(\Sigma) \to]0, 2\pi[, \rho \mapsto \alpha_1 \angle_z \alpha_2,$$

is a real analytic function.

Proof. — With the notation above we have the following expression for the angle:

$$\sin^2(\alpha_1 \angle_z \alpha_2) = \frac{4(2 - \operatorname{tr}[\rho(a_1), \rho(a_2)])}{(\operatorname{tr}^2 \rho(a_1) - 4)(\operatorname{tr}^2 \rho(a_2) - 4)}.$$
(5.1)

This equation is actually implicit in [7] but it is not claimed to be new there and seems to have been well known. The left hand side of (5.1) is clearly an analytic function on $\mathcal{T}(\Sigma)$ and it follows from elementary real analysis that the angle varies real analytically too.

Note that, though we will not need this, (5.1) shows that the square of the sine is in fact a rational function of traces (see Mondal [10] for applications of this).

5.1.1. Cross ratio formula

It will prove useful to have another formula for the angle in terms of a cross ratio. This formula is well-known, see for example, *The Geometry*

of Discrete Groups, by A.F. Beardon but since we will use it extensively to obtain bounds we give a short exposition. If θ is the angle between two hyperbolic geodesics $\hat{\alpha}, \hat{\beta} \subset \mathbb{H}$ then $\tan^2(\theta/2)$ can be expressed as a cross ratio. One can prove this directly by taking $\hat{\alpha}$ to have endpoints $\alpha^{\pm} = \pm 1$ and $\hat{\beta}$ endpoints $\beta^{\pm} = \pm e^{i\theta}$ in the Poincaré disc model. Then

$$\left(\frac{\alpha^+ - \beta^+}{\alpha^+ - \beta^-}\right) \left(\frac{\alpha^- - \beta^-}{\alpha^- - \beta^+}\right) = \left(\frac{1 - e^{i\theta}}{1 + e^{i\theta}}\right) \left(\frac{-1 + e^{i\theta}}{-1 - e^{i\theta}}\right) = \left(\frac{1 - e^{i\theta}}{1 + e^{i\theta}}\right)^2$$
$$= \tan^2(\theta/2). \quad (5.2)$$

6. Angles defined by closed geodesics

6.1. Variation of angles

In this paragraph we give an improved version of the following well known fact:

FACT 6.1. — Let $\alpha, \beta \subset \Sigma$ be a pair of simple closed simple geodesics that meet in a point $z \in \Sigma$. Then for any $\theta \in]0, p_i[$ there exists $\rho \in \mathcal{T}(\Sigma)$ such that

$$\alpha \angle_x \beta = \theta.$$

Under the hypothesis, there is a convex subsurface $\Sigma' \subset \Sigma$ homeomorphic to a one holed torus which contains $\alpha \cup \beta$. The fact follows by presenting Σ' as the quotient of \mathbb{H} by a Schottky group.

Using the preceding discussion of the Fenchel–Nielsen deformation we can relax the hypothesis on β even whilst taking the restriction of the angle function to a one dimensional submanifold of $\mathcal{T}(\Sigma)$. The proof will should also serve to familiarise the reader with the notation and provide intuition as to why this case is different to that of an intersection of a generic pair of closed geodesics treated in Theorem 6.3

LEMMA 6.2. — Let $\alpha, \beta \subset \Sigma$ be a pair of closed geodesics that meet in a point $z \in \Sigma$. If α is simple then for any $\theta \in]0, \pi[$ and any $\rho_0 \in \mathcal{T}(\Sigma)$ there exists $\rho_t \in \mathcal{T}(\Sigma)$ obtained from ρ_0 by a (finite) Fenchel–Nielsen twist along α such that

 $\alpha \angle_x \beta = \theta.$

Moreover,

$$\lim_{t \to \pm \infty} \alpha \angle_x \beta \in \{0, \pi\}.$$

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Proof. — With the notation of Subsection 4.2, there is a convex region P in \mathbb{H} bounded by lifts of α as before. Let $\hat{\alpha}$ be an edge of $\partial \overline{P}$, and choose a corresponding lift $\hat{\beta}$ which intersects $\hat{\alpha}$. There is an element of the covering group $g \in \Gamma$ such that

$$\hat{\alpha} = \partial \overline{P} \cap g(\partial \overline{P})$$

We lift the Fenchel-Nielsen deformation and consider, as before, its extension

$$\hat{\tau}^t_{\alpha} : \mathbb{H} \sqcup \partial \mathbb{H} \to \mathbb{H} \sqcup \partial \mathbb{H}.$$

Now, arguing as in Lemma 4.1, we see that:

- the endpoints of $\hat{\alpha}$ are fixed by $\hat{\tau}^t_{\alpha}$,
- the endpoint of $\hat{\beta}$ on the same side of $\hat{\alpha}$ as P converges to a point $z \neq \alpha^+$ as $t \to -\infty$,
- the other endpoint of $\hat{\beta}$ converges to α^+ as $t \to -\infty$.

It follows that, after possibly changing the orientation of β , that the angle between $\hat{\alpha}$ and $\hat{\beta}$, and hence $\alpha \angle_x \beta$, tends to 0. Likewise, as $t \to +\infty$ the angle between $\hat{\alpha}$ and $\hat{\beta}$, and hence $\alpha \angle_x \beta$, tends to π .

Thus, by continuity, the range of the angle function is $]0, \pi[$.

THEOREM 6.3. — Let β_1, β_2 be a pair of closed geodesics and $y \in \beta_1 \cap \beta_2$. Then for any simple closed geodesic α , different from both β_1 and β_2 , the angle function $\beta_1 \angle_y \beta_2$ is bounded away from π along the Fenchel–Nielsen orbit of $\rho \in \mathcal{T}(\Sigma)$.

Proof. — If α and $\beta_1 \cup \beta_2$ are disjoint then $\beta_1 \angle_y \beta_2$ is constant along the $\hat{\tau}^t_{\alpha}$ -orbit so the result is trivial.

Suppose now that α and $\beta_1 \cup \beta_2$ are not disjoint and choose x as a basepoint of Σ . Then, with the notation of paragraph 4.2, there is a convex region P in \mathbb{H} bounded by lifts of α . We now consider three cases according to the number of edges of $\partial \overline{P}$ that $\hat{\beta}_1 \cup \hat{\beta}_2$ meets.

We first deal with the simplest case. Suppose that $\hat{\beta}_1 \cup \hat{\beta}_2$ meets $\partial \overline{P}$ in four distinct edges denoted $C_1, C_2, C_3, C_4 \subset \mathbb{H}$, and, after possibly relabelling these, $\hat{\beta}_1$ meets C_1, C_2 whilst $\hat{\beta}_2$ meets C_3, C_4 as in Figure 6.1. Now we deform ρ_0 by a Fenchel–Nielsen twist along α to obtain a 1-parameter family of $\rho_t \in \mathcal{T}(\Sigma), t \in \mathbb{R}$. As we have seen above, under such a deformation the length of α does not change nor does the geometry of $\partial \overline{P}$ in particular the positions of the C_i remain unchanged. From our discussion of the $\hat{\tau}^t_{\alpha}$ and its extension to $\mathbb{H} \sqcup \partial \mathbb{H}$ it is clear that, $\forall t \in \mathbb{R}, \hat{\tau}^t_{\alpha}(\beta_1)$ meets C_1, C_2 whilst $\hat{\tau}^t_{\alpha}(\beta_2)$ meets C_3, C_4 . Thus, if the diameters of the circles were small, the angle at \hat{z} cannot not vary much from its value at ρ_0 since the radii of the



Figure 6.1. Case of 4 intersections.

circles are small. More generally, we can bound the size of the angle using the cross ratio formula. Labeling the endpoints as in Figure 6.1 one has:

$$\tan^{2}(\theta/2) = \left|\frac{\beta_{1}^{+} - \beta_{2}^{+}}{\beta_{1}^{+} - \beta_{2}^{-}}\right| \left|\frac{\beta_{1}^{-} - \beta_{2}^{-}}{\beta_{1}^{-} - \beta_{2}^{+}}\right|$$

Note first that each of the four points lies on the unit circle and so that its diameter, that is 2, is a trivial upper bound for each of the four distances appearing on the left hand side of this equation. Now under the deformation each of the endpoints $\hat{\tau}^t_{\alpha}(\beta^{\pm}_i)$ stays in one of four disjoint euclidean discs defined by one of the C_j . In particular, there exists $\delta_4 > 0$ such that for all $t \in \mathbb{R}$

$$\delta_4 \leqslant |\hat{\tau}^t_{\alpha}(\beta_1^{\pm}) - \hat{\tau}^t_{\alpha}(\beta_2^{\pm})| \leqslant 2, \\ \delta_4 \leqslant |\hat{\tau}^t_{\alpha}(\beta_1^{\pm}) - \hat{\tau}^t_{\alpha}(\beta_2^{\pm})| \leqslant 2,$$

and this is sufficient to obtain bounds on the cross ratio:

$$1/2\delta_4 \leqslant \tan(\theta/2) \leqslant 2/\delta_4. \tag{6.1}$$

Let $\hat{\beta}_1 \cup \hat{\beta}_2$ meet $\partial \overline{P}$ in just two edges, $C_1, C_2 \subset \mathbb{H}$ say. Although we no longer have a uniform lower bound for $|\hat{\tau}^t_{\alpha}(\beta_1^{\pm}) - \hat{\tau}^t_{\alpha}(\beta_2^{\pm})|$ in this case, there still exists $\delta_2 > 0$ such that for all $t \in \mathbb{R}$,

$$\delta_2 \leqslant |\hat{\tau}^t_\alpha(\beta_1^{\pm}) - \hat{\tau}^t_\alpha(\beta_2^{\pm})|.$$

Thus, for all $t \in \mathbb{R}$,

$$0 \leqslant \tan(\theta/2) \leqslant 2/\delta_2. \tag{6.2}$$

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Figure 6.2. Case of 2 intersections.

Finally, if $\hat{\beta}_1 \cup \hat{\beta}_2$ meets ∂P in exactly three edges then it is easy to see that, using the same reasoning as for the two edge case, there is δ_3 such that:

$$\delta_3 \leqslant |\hat{\tau}^t_{\alpha}(\beta_1^{\pm}) - \hat{\tau}^t_{\alpha}(\beta_2^{\mp})|.$$

COROLLARY 6.4. — Let α_1, α_2 be a pair of simple closed geodesics which meet in a single point z and β_1, β_2 primitive closed geodesics which meet in z'. If the difference

$$\alpha_1 \angle_z \alpha_2 - \beta_1 \angle_{z'} \beta_2$$

is constant then the angles are equal and, after possibly relabelling the geodesics, $\alpha_i = \beta_i$ and z = z'.

Proof. — We first consider the case where four geodesics are distinct then, under the Fenchel Nielsen twist along α_1 , the image of $\alpha_1 \angle_z \alpha_2$ is $]0, \pi[$ whilst, by Theorem 6.3, the image $\beta_1 \angle_{z'} \beta_2$ is a strict subinterval. It is easy to see $\alpha_1 \angle_z \alpha_2 - \beta_1 \angle_{z'} \beta_2$ cannot be a constant.

Now suppose, $\alpha_1 = \beta_1$. If $\alpha_2 = \beta_2$ then, since α_1 and α_2 meet in a single point, we must have z = z' and the angles must be the same.

On the other hand, if $\alpha_2 \neq \beta_2$ then z, z' may or may not be distinct:

- If z = z' then, by Lemma 6.2, both $\alpha_1 \angle_z \alpha_2$ and $\beta_1 \angle_{z'} \beta_2$ tend to 0 or π as the Fenchel–Nielsen parameter $t \to \pm \infty$. Therefore, if the difference is constant it must be 0 or π and so, up to switching orientation, $\beta_1 = \beta_2$.
- If z = z' then, by Lemma 6.2 and Theorem 6.3 $\beta_1 \angle_{z'} \beta_2$ is a proper subinterval of the range of $\alpha_1 \angle_z \alpha_2$ so the difference cannot be constant.

Proof of Lemma 1.2. — Suppose that Σ has a value in its angle spectrum, say θ , with finite multiplicity. Let $x_1, x_2, \ldots x_n \in \Sigma$ be a complete list of points such that there are pair of closed geodesics meeting at x_i at angle θ . Then the set of preimages of the x_i under the covering map $\mathbb{H} \to \Sigma$ is a discrete set which is invariant under $\operatorname{Aut}(\operatorname{ax}(\Gamma))$. Thus $\operatorname{Aut}(\operatorname{ax}(\Gamma))$ is discrete and has Γ as a finite index subgroup.

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