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A classification of degree 2 semi-stable rational maps $\mathbb{P}^2 \to \mathbb{P}^2$ with large finite dynamical automorphism group [(*)]

Michelle Manes [(1)] and Joseph H. Silverman [(2)]

Abstract. — Let K be an algebraically closed field of characteristic 0. In this paper we classify the $\mathrm{PGL}_3(K)$-conjugacy classes of semi-stable dominant degree 2 rational maps $f : \mathbb{P}^2_K \dashrightarrow \mathbb{P}^2_K$ whose automorphism group

$$\mathrm{Aut}(f) := \{\phi \in \mathrm{PGL}_3(K) : \phi^{-1} \circ f \circ \phi = f\}$$

is finite and of order at least 3. In particular, we prove that $\#\mathrm{Aut}(f) \leqslant 24$ in general, that $\#\mathrm{Aut}(f) \leqslant 21$ for morphisms, and that $\#\mathrm{Aut}(f) \leqslant 6$ for all but finitely many conjugacy classes of f.

Résumé. — Soit K un corps algébriquement clos de charactéristique 0. Dans cet article nous classifions les $\mathrm{PGL}_3(K)$-classes de conjugaison de fonctions rationelles f : $\mathbb{P}^2_K \dashrightarrow \mathbb{P}^2_K$ de degré 2 dominantes et semi-stables dont le groupe d'automorphismes

$$\mathrm{Aut}(f) := \{\phi \in \mathrm{PGL}_3(K) : \phi^{-1} \circ f \circ \phi = f\}$$

est fini et d'ordre au moins 3. En particulier, nous démontrons que $\#\mathrm{Aut}(f) \leqslant 24$ en général, que $\#\mathrm{Aut}(f) \leqslant 21$ pour les morphismes et que $\#\mathrm{Aut}(f) \leqslant 6$ pour toutes excepté un nombre fini de classes de conjugaisons de f.

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1. Introduction

Let $d \geqslant 1$ and $N \geqslant 1$ be integers and let

$$L = L(N,d) = \binom{N+d}{d}(N+1) - 1.$$

We identify $\mathbb{P}^L$ with the space of $(N+1)$-tuples of homogeneous polynomials of degree d in $N+1$ variables such that at least one polynomial is non-zero. Thus each $f = [f_0, \ldots, f_N] \in \mathbb{P}^L$ defines a rational map

$$f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N.$$

Although the map f need not be dominant, nor, if it is dominant, need it have degree d, we adopt the notation

$$\mathrm{Rat}_d^N := \mathbb{P}^L$$

and call Rat_d^N the parameter space of rational self-maps of $\mathbb{P}^N$ of *formal degree d*.

The group PGL_{N+1} acts on Rat_d^N via conjugation, i.e., the action of $\varphi \in \mathrm{PGL}_{N+1}$ on $f \in \mathrm{Rat}_d^N$ is

$$f^\varphi := \varphi^{-1} \circ f \circ \varphi.$$

This gives a homomorphism

$$\mathrm{PGL}_{N+1} \longrightarrow \mathrm{Aut}(\mathrm{Rat}_d^N) = \mathrm{Aut}(\mathbb{P}^L) \cong \mathrm{PGL}_{L+1}\,.$$

Geometric invariant theory [16] tells us that there are subsets $(\mathrm{Rat}_d^N)^{\mathrm{stab}}$ and $(\mathrm{Rat}_d^N)^{\mathrm{ss}}$ of stable and semi-stable points in Rat_d^N which admit good quotients for the action of PGL_{N+1}.(1) We denote these quotients by $(\mathcal{M}_d^N)^{\mathrm{stab}}$ and $(\mathcal{M}_d^N)^{\mathrm{ss}}$.

In this note we are interested in the locus in Rat_d^N of maps that admit a non-trivial automorphism.

Definition. — *The* automorphism group *of a map* $f \in \mathrm{Rat}_d^N$ *is*

$$\mathrm{Aut}(f) = \{\varphi \in \mathrm{PGL}_{N+1} : f^\varphi = f\}.$$

We note that $\mathrm{Aut}(f^\varphi) = \mathrm{Aut}(f)^\varphi$. *In particular, the isomorphism type of* $\mathrm{Aut}(f)$ *is a* PGL_{N+1}*-conjugation invariant.*

Remark 1.1. — It is known that $(\mathcal{M}_2^1)^{\mathrm{stab}} = (\mathcal{M}_2^1)^{\mathrm{ss}} \cong \mathbb{P}^2$ and that $\{f \in (\mathcal{M}_2^1)^{\mathrm{stab}} : \#\mathrm{Aut}(f) \geqslant 2\}$ is a cuspidal cubic curve in $\mathbb{P}^2$. More precisely, for f on this curve, $\mathrm{Aut}(f) \cong C_2$ for the non-cuspidal points and $\mathrm{Aut}(f) \cong S_3$ at the cuspidal point; see [23, Proposition 4.15]. We postpone

(1) More precisely, they admit good SL_{N+1}-quotients; cf. [23, §2.1].

to Section 2 an overview of our current knowledge of maps having non-trivial automorphism group.

Our primary goal in this paper is to describe the degree 2 maps in $(\mathcal{M}_2^2)^{\mathrm{ss}}$ having large finite automorphism group, i.e., we want to extend the above-mentioned classification of quadratic maps on $\mathbb{P}^1$ to quadratic maps on $\mathbb{P}^2$. We mention that a number of new phenomena appear, including semi-stable dominant rational quadratic maps $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ for which $\mathrm{Aut}(f)$ contains a copy of $\mathbb{G}_m$. For reasons that we explain later, we mostly exclude these maps from our analysis; see Section A. We also do not study maps with $\mathrm{Aut}(f) \cong C_2$, since they are too plentiful.

Before stating our main results, we need some additional notation. In general, for any finite subgroup $\mathcal{G} \subseteq \mathrm{PGL}_{N+1}$, we consider

$$\mathrm{Rat}_d^N(\mathcal{G}) := \big\{ f \in \mathrm{Rat}_d^N : \mathrm{Aut}(f) \supseteq \mathcal{G} \big\}.$$

If $\mathcal{G}^\varphi$ is a conjugate subgroup, then

$$\mathrm{Rat}_d^N(\mathcal{G}) \xrightarrow{\sim} \mathrm{Rat}_d^N(\mathcal{G}^\varphi), \quad f \xrightarrow{\sim} f^\varphi,$$

so it suffices to study $\mathrm{Rat}_d^N(\mathcal{G})$ for each conjugacy class of finite subgroups in PGL_{N+1}.

It is important to note that PGL_{N+1} generally does not act on $\mathrm{Rat}_d^N(\mathcal{G})$, since if $f \in \mathrm{Rat}_d^N(\mathcal{G})$ and $\varphi \in \mathrm{PGL}_{N+1}$, then $\mathrm{Aut}(f^\varphi) = \mathrm{Aut}(f)^\varphi \supseteq \mathcal{G}^\varphi$. Thus in order to ensure that f^φ is in $\mathrm{Rat}_d^N(\mathcal{G})$, we need $\mathcal{G}^\varphi = \mathcal{G}$, i.e., the map φ must be in the normalizer $N(\mathcal{G})$ of $\mathcal{G}$. We thus define(2)

$$\mathrm{Rat}_d^N(\mathcal{G})^{\mathrm{ss}} := \big\{ f \in \mathrm{Rat}_d^N(\mathcal{G}) : f \text{ is } N(\mathcal{G})\text{-semistable} \big\},$$

and similarly $\mathrm{Rat}_d^N(\mathcal{G})^{\mathrm{stab}}$ denotes the set of $N(\mathcal{G})$-stable maps. It turns out that if $f \in \mathrm{Rat}_d^N(\mathcal{G})$ is $N(\mathcal{G})$-semistable, then f is also PGL_{N+1}-semistable when viewed as a point in Rat_d^N, and further the natural map

$$\mathrm{Rat}_d^N(\mathcal{G})^{\mathrm{ss}}/N(\mathcal{G}) \longrightarrow (\mathrm{Rat}_d^N)^{\mathrm{ss}}/\,\mathrm{PGL}_{N+1} \tag{1.1}$$

is finite; see Proposition B.1 for a general result. However, the map (1.1) may fail to be injective due to the existence of f's that are PGL_{N+1}-conjugate, but are not $N(\mathcal{G})$-conjugate; see Example 1.7.

Our main results give a complete description of semistable dominant rational quadratic maps $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ satisfying $3 \leqslant \#\,\mathrm{Aut}(f) < \infty$.

Theorem 1.2. — *Let K be an algebraically closed field of characteristic 0, and let $\mathcal{G} \subset \mathrm{PGL}_3(K)$ be a finite subgroup with $\#\mathcal{G} \geqslant 3$. Suppose that there exists a map $f \in \mathrm{Rat}_2^2(\mathcal{G})^{\mathrm{ss}}(K)$ satisfying*

(2) Again, we are being somewhat informal in this introduction. To be rigorous, we lift $\mathcal{G}$ to an isomorphic subgroup $\tilde{\mathcal{G}}$ in SL_{N+1} and look at maps f that are $N(\tilde{\mathcal{G}})$-semistable.

- $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *is dominant.*
- $\deg(f) = 2$.
- $\mathrm{Aut}(f)$ *is finite.*

Then there is a $\mathrm{PGL}_3(K)$*-conjugate of* $\mathcal{G}$ *that contains one of the following groups, where* ζ_n *is a primitive n'th root of unity*:

$$\begin{aligned} &\mathcal{G}_3 = \left\langle \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{smallmatrix}\right) \right\rangle, \quad \mathcal{G}_4 = \left\langle \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right) \right\rangle, \quad \mathcal{G}_5 = \left\langle \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \zeta_5 & 0 \\ 0 & 0 & \zeta_5^3 \end{smallmatrix}\right) \right\rangle, \\ &\mathcal{G}_7 = \left\langle \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \zeta_7 & 0 \\ 0 & 0 & \zeta_7^3 \end{smallmatrix}\right) \right\rangle, \quad \mathcal{G}_{2,2} = \left\langle \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right) \right\rangle. \end{aligned} \tag{1.2}$$

THEOREM 1.3. — *Let* K *be an algebraically closed field of characteristic* 0*, and let* $\mathcal{G} \subset \mathrm{PGL}_3(K)$ *be one of the groups* (1.2) *listed in Theorem 1.2. Suppose that* $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *satisfies the following*:

- $f \in \mathrm{Rat}_2^2(\mathcal{G})^{\mathrm{ss}}(K)$.
- f *is dominant with* $\deg(f) = 2$.
- $\mathrm{Aut}(f)$ *is finite.*

Then f *is* $N(\mathcal{G})$*-conjugate to one of the maps listed in Table 1.1.* (*See Table 1.2 for an explanation of the entries in Table 1.1.*)

The next corollary catalogs the complete list of finite groups that appear as automorphism groups of semi-stable degee 2 maps of $\mathbb{P}^2$, as well as other information related to the maps in Table 1.1.

COROLLARY 1.4. — *Let* K *be an algebraically closed field of characteristic* 0*, and let* $f \in \mathrm{Rat}_2^2(K)$ *be a semi-stable dominant rational map of degree* 2 *with finite automorphism group.*

(a) $\mathrm{Aut}(f)$ *is isomorphic to one of the following nine groups*:

$$C_1, \quad C_2, \quad C_3, \quad C_4, \quad C_5, \quad C_2^2, \quad S_3, \quad S_4, \quad C_7 \rtimes C_3.$$

(b) *For each group* G *in* (a)*, there exists a group* $\mathcal{G} \subset \mathrm{PGL}_3$ *with* $\mathcal{G} \cong G$ *and a map* $f \in \mathrm{Rat}_2^2(\mathcal{G})^{\mathrm{ss}}$ *such that* f *is a dominant map of degree* 2 *satisfying* $\mathrm{Aut}(f) = \mathcal{G}$.

(c) *Let* $\mathcal{G} \subset \mathrm{PGL}_3$ *be a finite group that is* not *isomorphic to one of the following groups*:

$$C_1, \quad C_2, \quad C_3, \quad C_4, \quad C_2^2, \quad \text{or} \quad S_3.$$

Then $\mathcal{M}_2^2(\mathcal{G})^{\mathrm{ss}}$ *contains only finitely many dominant degree* 2 *maps.*

(d) *Let* $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *be a dominant degree* 2 *rational map such that* $\mathrm{Aut}(f)$ *contains a copy of* C_2^2 *or* C_5*. Then* f *is not a morphism.*

We briefly explain the strategy that we employ to classify maps with large automorphism group:

f	Coeffs	Aut	dim	St?	$\#I$	Crit	λ_1	λ_2
$\boldsymbol{\mathcal{G}_3}$: $f_{b,d,g} = [X^2+bYZ, Z^2+dXY, Y^2+gXZ]$, $f_{b,d,g} \sim f_{b,g,d}$								
1.1	$(2, 2\zeta_3, 2\zeta_3^2)$	$C_7 \rtimes C_3$	0	S	0	$L_1{\cdot}L_2{\cdot}L_3$	2	4
1.2	$(-1,-1,-1)$	S_4	0	S	3	$L_1{\cdot}L_2{\cdot}L_3$	1	1
1.3	$(0,0,0)$	S_3	0	S	0	$L_1{\cdot}L_2{\cdot}L_3$	2	4
1.4	$(b, b^{-1}, -1)$	S_3	1	S	3	$L_1{\cdot}L_2{\cdot}L_3$	$*^1$	$*^1$
1.5	(b,d,d)	S_3	2	S	0	Γ	2	4
1.6	$bdg=-1$	C_3	2	S	3	$L_1{\cdot}L_2{\cdot}L_3$	$*^2$	$*^2$
1.7	$bdg=8$	C_3	2	S	0	$L_1{\cdot}L_2{\cdot}L_3$	2	4
1.8	other	C_3	3	S	0	Γ	2	4
$\boldsymbol{\mathcal{G}_3}$: $f_{a,c,g} = [aX^2+YZ, cZ^2+XY, Y^2+gXZ]$, $f_{0,c,g} \sim f_{0,c/g,1/g}$ $(a,c,g) \neq (0,0,0)$								
2.1	$(0,0,g)$	C_3	1	SS	2	$C{\cdot}L$	$*^3$	2
2.2	$(0,c,1)$	S_3	1	S	1	Γ'	2	3
2.3	$(0,c,0)$	C_3	1	S	1	$3L$	2	1
2.4	$(a,0,0)$	C_3	1	SS	1	$3L$	2	1
2.5	$(a,0,g)$	C_3	2	SS	1	Γ'	2	3
2.6	$(0,c,g)$	C_3	2	S	1	Γ'	2	3
$\boldsymbol{\mathcal{G}_4}$: $f_{a,e} = [aX^2+Z^2, XY, Y^2+eXZ]$								
3.1	$(0,0)$	C_4	0	S	1	$2L_1{\cdot}L_2$	$\sqrt{2}$	2
3.2	$(0,e)$	C_4	1	S	1	$C{\cdot}L$	2	3
3.3	$(a,0)$	C_4	1	S	2	$2L_1{\cdot}L_2$	2	2
3.4	(a,e)	C_4	2	S	0	Γ	2	4
$\boldsymbol{\mathcal{G}_4}$: $f_c = [YZ, X^2+cZ^2, XY]$, $f_c \sim f_{1/c}$								
4.1	(-1)	$\mathbb{G}_m \rtimes C_2$	0	S	3	$L_1{\cdot}L_2{\cdot}L_3$	1	1
4.2	(1)	S_4	0	S	3	$L_1{\cdot}L_2{\cdot}L_3$	1	1
4.3	(0)	C_4	0	S	2	$2L_1{\cdot}L_2$	1	1
4.4	(c)	C_4	1	S	3	$L_1{\cdot}L_2{\cdot}L_3$	1	1
$\boldsymbol{\mathcal{G}_{2,2}}$: $f_{a,e} = [aX^2+Y^2-Z^2, XY, eXZ]$, $f_{0,e} \sim f_{0,1/e}$								
5.1	$(0,1)$	$\mathbb{G}_m \rtimes C_2$	0	S	3	$L_1{\cdot}L_2{\cdot}L_3$	1	1
5.2	$(0,-1)$	S_4	0	S	3	$L_1{\cdot}L_2{\cdot}L_3$	1	1
5.3	$(0,e)$	C_2^2	1	S	3	$L_1{\cdot}L_2{\cdot}L_3$	1	1
5.4	$(a,1)$	$\mathbb{G}_m \rtimes C_2$	1	S	2	$C{\cdot}L$	2	2
5.5	(a,e)	C_2^2	2	S	2	$C{\cdot}L$	2	2
$\boldsymbol{\mathcal{G}_{2,2}}$: $f = [YZ, XZ, XY]$								
6.1		S_4	0	S	3	$L_1{\cdot}L_2{\cdot}L_3$	1	1
$\boldsymbol{\mathcal{G}_5}$: $f = [YZ, X^2, Y^2]$								
7.1		C_5	0	SS	1	$2L_1{\cdot}L_2$	$\sqrt{2}$	2
$\boldsymbol{\mathcal{G}_7}$: $f = [Z^2, X^2, Y^2]$								
8.1		$C_7 \rtimes C_3$	0	S	0	$L_1{\cdot}L_2{\cdot}L_3$	2	4

Table 1.1. Dominant semistable degree 2 maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with large automorphism group

- For each family of Type $N.M$, Table 1.1 first gives a formula for the maps in the family $N.*$ and indicates by the notation $f \sim f'$ the $N(\mathcal{G})$-conjugacy equivalences between maps. It then lists subfamilies $M = 1, 2, \ldots$. The columns in Table 1.1 contain the following information:

Key for Columns in Table 1.1	
Coeffs	restrictions on the coefficients of f
Aut	the full automorphism group of f
dim	dimension of the familiy in $\mathcal{M}_2^2$
St?	stability, with S = stable and SS = semistable
$\#I$	number of points in the indeterminacy locus of f
Crit	geometry of the critical locus of f (see below for key)
λ_1	dynamical degree of f (see Remark 2.1)
λ_2	topological degree of f

- Within each type, the maps in a given line are understood to exclude the maps in all previous lines. So for example maps of Type 1.4 exclude the case $b = -1$, which is covered by Type 1.2, while Type 1.8 excludes maps satisfying $bdg \neq -1$ and $bdg \neq 8$ Further, each line includes the indicated PGL_3-equivalences, so for example Type 1.4 includes both $(b, b^{-1}, -1)$ and $(b, -1, b^{-1})$.
- Table 1.1 includes a few cases (Types 4.1, 5.1, 5.4) with $\mathrm{Aut}(f) \supset \mathbb{G}_m$. These help fill in the indicated family.
- The geometry of the critical locus is described by:

Key for Crit(f)	
Γ= smooth cubic curve	$L_1{\cdot}L_2{\cdot}L_3$= 3 distinct lines
Γ'= nodal cubic curve	$2L_1{\cdot}L_2$= double line $\cup$ line
$C{\cdot}L$= conic $\cup$ line	$3L$= triple line

$*^1$ We expect that maps of Type 1.4 satisfy $\lambda_1(f) = 2$ and $\lambda_2(f) = 4$.

$*^2$ For generic values of b, d, g satisfying $bdg = -1$, we expect that maps of Type 1.6 satisfy $\lambda_1(f) = 2$ and $\lambda_2(f) = 4$, but it seems likely that there is a countable collection of (b, d, g) triples satisfying $\lambda_1(f) < 2$ and $\lambda_2(f) < 4$.

$*^3$ Experiments suggest that $\deg(f^n)$ is the $(n+2)$'nd Fibonacci number, which would imply that $\lambda_1(f) = \frac{1}{2}(1 + \sqrt{5})$.

Table 1.2. Notes for Table 1.1

- Assume that $\mathrm{Aut}(f)$ contains a subgroup isomorphic to some finite group G satisfying $\#G \geqslant 3$.
- Classify the conjugacy classes $\mathcal{G}_1, \ldots, \mathcal{G}_k$ of finite subgroups of $\mathrm{PGL}_3(K)$ that are isomorphic to G and choose a (nice) representative group $G_i \subset \mathrm{PGL}_3(K)$ in $\mathcal{G}_i$ for each $1 \leqslant i \leqslant k$.(3)

(3) The complete classification of finite subgroups of $\mathrm{PGL}_3(K)$ is classical, but for completeness we include the short proof of the part that we need.

- For each G_i, decompose the set of $f \in \mathrm{Rat}_2^2$ satisfying $G_i \subseteq \mathrm{Aut}(f)$ into a disjoint union of irreducible families $\mathcal{F}_{i,1}, \mathcal{F}_{i,2}, \dots \subset \mathrm{Rat}_2^2$. For example, in the (typical) case that G_i is a group of diagonal matrices, the various $\mathcal{F}_{i,j}$ are characterized by the eigenvalues of the generators of G_i acting on the monomials in the coordinates of f.
- By inspection, determine which $f \in \mathcal{F}_{i,j}$ are dominant.
- Use the numerical criterion of Mumford–Hilbert to determine the set of semi-stable $f \in \mathcal{F}_{i,j}$.(4)
- It remains to determine the full automorphism group for dominant semi-stable maps $f \in \mathcal{F}_{i,j}$, or more generally, to determine

  $$\mathrm{Hom}(f, f') := \left\{\varphi \in \mathrm{PGL}_3(K) : f' = f^{\varphi}\right\} \quad \text{for } f, f' \in \mathcal{F}_{i,j}.$$

  (Taking $f' = f$ gives $\mathrm{Aut}(f)$.) A key tool in this endeavor is to exploit the fact that every $\varphi \in \mathrm{Hom}(f, f')$ induces an isomorphism of the associated indeterminacy and critical loci,

  $$I(f) \xrightarrow[\sim]{\varphi} I(f') \quad \text{and} \quad \mathrm{Crit}(f) \xrightarrow[\sim]{\varphi} \mathrm{Crit}(f').$$

  These isomorphisms impose restrictions on φ which can be used as the starting point of a case-by-case determination of $\mathrm{Hom}(f, f')$ and $\mathrm{Aut}(f)$.

Remark 1.5. — We offer some further brief comments on the final step. If $I(f) = I(f')$ is a finite set of points, then $\varphi \in \mathrm{Hom}(f, f')$ induces a permutation of these points, and similarly if $\mathrm{Crit}(f) = \mathrm{Crit}(f')$ is a union of (three) lines, or the union of a conic and a line, etc., then φ induces a permutation of these geometric configurations. However, there are three cases, Type 1.5, 1.8, and 3.4 in Table 1.1, for which $I(f) = \emptyset$ and $\mathrm{Crit}(f)$ is a smooth cubic. For Types 1.5 and 1.8 we exploit the fact that every $\varphi \in \mathrm{Aut}(f)$ permutes the 9 flex points of the smooth cubic curve $\mathrm{Crit}(f)$. This leads to several hundred cases, which we check by computer. For Type 3.4 we take a slightly different approach by first showing that $\mathrm{Crit}(f)$ is an elliptic curve with CM by $\mathbb{Z}[i]$, and that if $\varphi \in \mathrm{Aut}(f)$, then $\varphi : \mathrm{Crit}(f) \to \mathrm{Crit}(f)$ is translation by a 3-torsion point P_0. We next prove that if $P_0 \neq 0$, then $\mathrm{Aut}(f)$ would contain a copy of C_3^2, contradicting an earlier calculation. This allows us to conclude that $\mathrm{Aut}(f) \cong \mathbb{Z}[i]^* \cong C_4$.

(4) We remark that in many cases it turns out that $\mathcal{F}_{i,j}$ contains no dominant semi-stable maps. Indeed there are conjugacy classes with $\mathcal{G} \cong C_3$ and $\mathcal{G} \cong C_4$ that contain no dominant semi-stable maps.

Remark 1.6. — We take a moment to record some additional interesting properties of some of the maps in Table 1.1.

(a) The maps $f_c = [YZ, X^2 + cZ^2, XY]$ of Types 4.1–4.4 satisfy
$$\begin{aligned} \deg(f_c^n) &= n+1 \quad \text{if } c \ne \pm 1, \\ f_c^{2k} &= [X, c^k Y, Z] \quad \text{if } c = \pm 1. \end{aligned}$$
In all cases, the second iterate satisfies $\mathrm{Aut}(f_c^2) \supseteq \mathbb{G}_m$. This gives a family of examples of maps with $\mathrm{Aut}(f)$ finite and $\mathrm{Aut}(f^2)$ infinite. See Proposition 9.3.

(b) The maps $f_{0,e} = [Y^2 - Z^2, XY, eXZ]$ of Types 5.1–5.3 satisfy
$$\begin{aligned} &\deg(f_{0,e}^n) = n+1 \quad \text{if } e^2 \text{ is not an odd-order root of unity}, \\ &f_{0,e}^{4k+2} = [X, Y, Z] \quad \text{if } e^{4k+2} = 1. \end{aligned}$$
See Proposition 8.2.

(c) The map $f = [YZ, X^2, Y^2]$ of Type 7.1 with $C_5 \cong \mathrm{Aut}(f)$ has the property that $f^8 = [X^{16}, Y^{16}, Z^{16}]$.

(d) The map $f = [Z^2, X^2, Y^2]$ of Type 8.1 with $C_7 \subset \mathrm{Aut}(f)$ has the property that $f^3 = [X^8, Y^8, Z^8]$.

(e) The maps of Type 1.2, 4.2, 5.2, and 6.1 are $\mathrm{PGL}_3(K)$-conjugate to one another and have the property that $f^2 = [X, Y, Z]$. See Example 1.7.

Example 1.7. — Consider the following maps from Table 1.1:
$$\begin{aligned} f_{1.2} &:= [X^2 - YZ, Z^2 - XY, Y^2 - XZ], && \text{Type 1.1} \\ f_{4.2} &:= [YZ, X^2 + Z^2, XY], && \text{Type 4.2} \\ f_{5.2} &:= [Y^2 - Z^2, XY, -XZ], && \text{Type 5.2} \\ f_{6.1} &:= [YZ, XZ, XY], && \text{Type 6.1.} \end{aligned}$$
One easily checks that
$$f_{1.2} \in \mathrm{Rat}_2^2(\mathcal{G}_3), \quad f_{4.2} \in \mathrm{Rat}_2^2(\mathcal{G}_4), \quad f_{5.2}, f_{6.1} \in \mathrm{Rat}_2^2(\mathcal{G}_{2,2}).$$
Further, we find that all four maps are PGL_3-conjugate. Explicitly
$$f_{1.2}^{\alpha} = f_{4.2}^{\beta} = f_{5.2}^{\gamma} = f_{6.1}$$
for
$$\alpha = \begin{pmatrix} 1 & 1 & 1 \\ \zeta_3^2 & \zeta_2 & 1 \\ \zeta_3 & \zeta_3^2 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & \zeta_8^2 & 1 \\ -2\zeta_8 & 0 & 0 \\ 0 & 1 & \zeta_8^2 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix},$$
where ζ_n denotes a primitive n'th root of unity.

Theorem 1.3 says that $f_{5.2}$ and $f_{6.1}$ both satisfy $\mathrm{Aut}(f) = \mathcal{S}_3\mathcal{G}_{2,2}$, where $\mathcal{S}_3 \subset \mathrm{PGL}_3$ is the group of permutation matrices. In particular, $f_{5.2}$ and $f_{6.1}$

are $N(\mathcal{S}_3\mathcal{G}_{2,2})$-conjugate, since γ normalizes $\mathcal{S}_3\mathcal{G}_{2,2}$, but they are not $N(\mathcal{G}_{2,2})$-conjugate, since γ does not normalize $\mathcal{G}_{2,2}$. Thus $f_{5.2}$ and $f_{6.1}$ represent different points in $\mathrm{Rat}_2^2(\mathcal{G}_{2,2})^{\mathrm{ss}}$, but they define the same point in $\mathrm{Rat}_2^2(\mathcal{S}_3\mathcal{G}_{2,2})^{\mathrm{ss}}$.

Remark 1.8. — A referee has pointed out that for rational maps f that are not morphisms, it would be interesting, and possibly more natural, to compute the group of birational automorphisms of $\mathbb{P}^2$ that commute with f. Writing $\mathrm{BiRat}(\mathbb{P}^2)$ for the *Cremona group*, one might try to classify dominant, semi-stable degree 2 maps f for which the group

$$\mathrm{BiAut}(f) := \left\{\varphi \in \mathrm{BiRat}(\mathbb{P}^2) : f^{\varphi} = f\right\}$$

is finite and, say, has order at least 3. A starting point would be the known classification of the finite subgroups of the Cremona group [2, 18]. For example, a number of our maps with large $\mathrm{Aut}(f)$ are themselves elements of order 2 in the Cremona group, a typical example being the map $f = [YZ, XZ, XY]$ labeled (6.1) in Table 1.1. In these cases $\mathrm{BiAut}(f)$ is at least as large as $\mathrm{Aut}(f) \times C_2$, with the extra C_2 being generated by f. The analysis of maps with large finite $\mathrm{BiAut}(f)$ seems like an interesting problem that deserves further study, but in view of the length of the present paper, we will not address it at this time.

Remark 1.9. — A referee has pointed out that the present paper is close in spirit to the classification by Fornæss and Wu [5] of degree 2 polynomial automorphisms of $\mathbb{C}^3$, and work of Cerveau and Déserti [1] describing birational maps, especially of degrees 2 and 3, of $\mathbb{P}^2$, although we note that the latter paper studies the two-sided action of $\mathrm{PGL}_3 \times \mathrm{PGL}_3$, which leads to a different, albeit also very interesting, classification problem.

2. Background

We briefly summarize some of the existing literature on the study of $(\mathcal{M}_d^N)^{\mathrm{stab}}$ and $(\mathcal{M}_d^N)^{\mathrm{ss}}$. A fair amount is known in the case that $N = 1$. For example, it is known that $(\mathcal{M}_d^1)^{\mathrm{stab}}$ and $(\mathcal{M}_d^1)^{\mathrm{ss}}$ are rational varieties [10]. And for $N = 1$ and $d = 2$ there are natural isomorphisms $(\mathcal{M}_2^1)^{\mathrm{stab}} = (\mathcal{M}_2^1)^{\mathrm{ss}} \cong \mathbb{P}^2$, with the set of maps $\left\{f \in (\mathcal{M}_2^1)^{\mathrm{stab}} : \deg f = 2\right\}$ corresponding to $\mathbb{A}^2$. See [15] for the proof over $\mathbb{C}$ and [20] for the proof over $\mathrm{Spec}\,\mathbb{Z}$. For degree 2 morphisms $f : \mathbb{P}^1 \to \mathbb{P}^1$, the group $\mathrm{Aut}(f) \subset \mathrm{PGL}_2$ is isomorphic to either C_1, C_2, or S_3. The locus of $f \in \mathbb{A}^2$ with $C_2 \subset \mathrm{Aut}(f)$ is a cuspidal cubic curve, with the cusp corresponding to the only f having $\mathrm{Aut}(f) \cong S_3$; see [23, Proposition 4.15]. For similar results on $\mathcal{M}_3^1$, see [25]. More generally, for $N = 1$ and $d \geqslant 3$, the singular locus of $(\mathcal{M}_d^1)^{\mathrm{stab}}$ is exactly the set of f

with $\operatorname{Aut}(f) \neq 1$; see [14] for this result and for a calculation of the dimension and Picard and class groups of

$$\mathcal{M}_d^1(\mathcal{G})^{\mathrm{stab}} := \big\{ f \in (\mathcal{M}_d^1)^{\mathrm{stab}} : \mathcal{G} \subseteq \operatorname{Aut}(f) \big\} \quad \text{for } \mathcal{G} \subset \mathrm{PGL}_2.$$

(Note that here f represents a conjugacy class of maps, so $\operatorname{Aut}(f)$ is a conjugacy class of subgroups of PGL_2.)

For all $N \geqslant 1$ it is known that $\big\{ f \in \mathcal{M}_d^N : \operatorname{Aut}(f) = 1 \big\}$ is a non-empty Zariski open subset of $\mathcal{M}_d^N$; see [10]. Thus "most" maps f have no automorphisms. On the other hand, those f with $\operatorname{Aut}(f) \neq 1$ are of particular arithmetic interest, since they tend to have non-trivial $\bar{K}/K$-twists, i.e., families of maps that are $\mathrm{PGL}_{N+1}(\bar{K})$-conjugate, but not $\mathrm{PGL}_{N+1}(K)$-conjugate. There has been a considerable amount of work studying dynamical twist families and related problems having to do with fields of definition and fields of moduli; see for example [11, 13, 19, 24], [20, Chapter 7], [21, Sections 4.7–4.10].

For $N \geqslant 2$, there has been some progress. It is known that if $f : \mathbb{P}^N \to \mathbb{P}^N$ is a morphism of degree at least 2, then $\operatorname{Aut}(f)$ is finite; see [17]. For $N = 1$ and 2, there are explicit bounds. For example, a morphism $f : \mathbb{P}^2 \to \mathbb{P}^2$ satisfies $\#\operatorname{Aut}(f) \leqslant 6d^6$; see [3, Theorem 6.2]. It is also known that for every finite subgroup $\mathcal{G} \subset \mathrm{PGL}_{N+1}(\bar{K})$, there are infinitely many morphisms $f : \mathbb{P}^N_{\bar{K}} \to \mathbb{P}^N_{\bar{K}}$ of degree $\geqslant 2$ such that $\operatorname{Aut}(f) \supseteq \mathcal{G}$; see [3, Theorem 4.7]. However, the situation is more delicate if one or more of the following natural conditions is imposed:

- $\operatorname{Aut}(f)$ is exactly equal $\mathcal{G}$.
- The degree of f is specified.
- The map f is defined over a non-algebraically closed field K.

For example, every subgroup of PGL_2 except the tetrahedral group can be realized by a map $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined over $\mathbb{Q}$; see [3, Theorem 4.9]. And [3, Section 8.1] gives examples of morphisms $f : \mathbb{P}^2 \to \mathbb{P}^2$ defined over $\mathbb{Q}$ with very large automorphism groups. We also mention that [3, Section 5] contains a nice summary, in modern notation and with explicit generators, of the classical classification of finite subgroups of $\mathrm{PGL}_3(\mathbb{C})$.

Remark 2.1. — Two important invariants associated to dominant rational maps $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ are the *dynamical degree*

$$\lambda_1(f) := \lim_{k \to \infty} (\deg f^k)^{1/k}$$

and the *topological degree*

$$\lambda_2(f) := \# f^{-1}(P) \quad \text{for a generic point } P \in \mathbb{P}^n(K).$$

The map f is said to be *algebraically stable* if $\lambda_1(f) = \deg(f)$. There is a large literature studying dynamical degrees, algebraic stability, and the existence of invariant measures, of which we mention two articles. The first [7] gives a precise formula for the dynamical degree of a monomial map. The second [6] classifies degree two polynomial maps $f : \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ and shows that, up to affine conjugacy, there are 13 families of such maps, all of which extend to an algebraically stable map on either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.

3. Scope of This Paper and Further Questions

The original goal of this paper was to completely describe the moduli spaces $\mathcal{M}_2^2(\mathcal{G})^{\mathrm{ss}}$ and $\mathcal{M}_2^2(\mathcal{G})^{\mathrm{stab}}$ over an arbitrary field K, and more generally over $\operatorname{Spec} R$ for an appropriately chosen ring R. This analysis would have included giving normal forms for $N(\mathcal{G})$-conjugacy classes of maps, and it would have included classifying semi-stable maps that are not dominant or have degree 1. This turned out to be overambitious, as we realized when the analysis of the case $\mathcal{G} \cong C_2^2$ approached 50 pages and it became clear that the cases $\mathcal{G} \cong C_4$ and $\mathcal{G} \cong C_3$ were going to be even more complicated. Further, if K has positive characteristic, then one must also deal with finite cyclic subgroups of $\mathrm{PGL}_3(K)$ that are not diagonalizable, adding another level of complication.

We thus decided to restrict attention to algebraically closed fields of characteristic 0 and to restrict attention to maps that are dominant and have degree 2, since these are the maps whose iterates potentially have interesting dynamics. This curtailed goal ended up being sufficiently challenging, as the length of the present paper attests. However, we propose the following problems as deserving study in future papers and/or a monograph.

(1) Describe the geometry of the moduli spaces $\mathcal{M}_2^2(\mathcal{G})^{\mathrm{ss}}$ and $\mathcal{M}_2^2(\mathcal{G})^{\mathrm{stab}}$, and the geometry of the natural map $\mathcal{M}_2^2(\mathcal{G})^{\mathrm{ss}} \to \mathcal{M}_2^2(\mathcal{G}')^{\mathrm{ss}}$ for subgroups $\mathcal{G}' \subset \mathcal{G}$.

(2) Determine the field of moduli and minimal fields of definition for points in $\mathcal{M}_2^2(\mathcal{G})^{\mathrm{ss}}$, where $\mathcal{G} \subset \mathrm{PGL}_3(K)$ is a finite subgroup and K is minimal for the conjugacy class of $\mathcal{G}$.

(3) Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be defined over K. We recall that the set of $\bar{K}/K$-twists of f is the set of maps f' defined over K that are $\mathrm{PGL}_3(\bar{K})$-conjugate to f, modulo f' and f being considered equivalent if they are $\mathrm{PGL}_3(K)$-conjugate. The set of twists is classified by the kernel of the inflation map

$$H^1\big(\mathrm{Gal}(\bar{K}/K), \mathrm{Aut}(\varphi)\big) \longrightarrow H^1\big(\mathrm{Gal}(\bar{K}/K), \mathrm{PGL}_3(\bar{K})\big);$$

see [23, Section 7.1]. Find normal forms for the twists of the maps in Table 1.1.

(4) Classify rational maps f having large finite birational automorphism groups, as described in Remark 1.8.

Example 3.1. — We illustrate twisting with an example. Consider the map $f = [YZ, X^2, Y^2]$ of Type 7.1. Up to $\mathrm{PGL}_3(\bar{\mathbb{Q}})$-conjugacy, this is the only map in $(\mathcal{M}_2^2)^{\mathrm{ss}}$ whose automorphism group is finite and contains an element of order 5. The isomorphism

$$\boldsymbol{\mu}_5 \longrightarrow \mathrm{Aut}(f), \quad \zeta \longmapsto \varphi_\zeta = [X, \zeta Y, \zeta^3 Z],$$

is $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-invariant, and it turns out that every element of

$$H^1\big(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \boldsymbol{\mu}_5\big) \cong \mathbb{Q}^*/(\mathbb{Q}^*)^5$$

gives a twist of f. Precisely, let $b \in \mathbb{Q}^*$, let $\beta = b^{1/5} \in \bar{\mathbb{Q}}$, and let $\psi = [X, \beta Y, \beta^3 Z]$. Then the twist of f associated to b is

$$\begin{aligned} f_b &:= f^\psi(X,Y,Z) = \psi^{-1} \circ f \circ \psi(X,Y,Z) = \psi^{-1} \circ f(X, \beta Y, \beta^3 Z) \\ &= \psi^{-1}(\beta^4 YZ, X^2, \beta^2 Y^2) = [\beta^4 YZ, \beta^{-1} X^2, \beta^{-1} Y^2] = [bYZ, X^2, Y^2]. \end{aligned}$$

Note that f_b is defined over $\mathbb{Q}$, but that the map ψ conjugating f to f_b is only defined over $\mathbb{Q}(b^{1/5})$.

Similarly, the $\boldsymbol{\mu}_7$-twist of the map $f = [Z^2, X^2, Y^2]$ associated to $b \in \mathbb{Q}^*/(\mathbb{Q}^*)^7 \cong H^1\big(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \boldsymbol{\mu}_7\big)$ is $[bZ^2, X^2, Y^2]$. We leave the details of the computation to the reader.

4. Some Finite Subgroups of PGL_3

In this section we prove some elementary results concerning finite subgroups of PGL_3. This information may be gleaned from classical descriptions of all finite subgroups of PGL_3, but for completeness we shall prove what we need.

Lemma 4.1. — *Let K be an algebraically closed field of characteristic 0, and let $G \subset \mathrm{PGL}_3(K)$ be a finite subgroup.*

(a) *Suppose that $G \cong C_q$ with q a prime power. Then there is a $\varphi \in \mathrm{PGL}_3(K)$, a primitive q'th root of unity $\zeta \in K$, and an integer m such that*

$$G^\varphi = \left\langle \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^m \end{smallmatrix} \right) \right\rangle. \tag{4.1}$$

(b) *Suppose that $G \cong C_q$ with $q \in \{4,5,7\}$. Then there is a $\varphi \in \mathrm{PGL}_3(K)$ and a primitive q'th root of unity ζ such that*

$$\begin{aligned} G \cong C_4 \quad &\Longrightarrow \quad G^\varphi = \left\langle \left(\begin{smallmatrix}1&0&0\\0&\zeta&0\\0&0&1\end{smallmatrix}\right)\right\rangle \ \textit{or} \ \left\langle \left(\begin{smallmatrix}1&0&0\\0&\zeta&0\\0&0&-1\end{smallmatrix}\right)\right\rangle, \\ G \cong C_5 \quad &\Longrightarrow \quad G^\varphi = \left\langle \left(\begin{smallmatrix}1&0&0\\0&\zeta&0\\0&0&1\end{smallmatrix}\right)\right\rangle \ \textit{or} \ \left\langle \left(\begin{smallmatrix}1&0&0\\0&\zeta&0\\0&0&\zeta^3\end{smallmatrix}\right)\right\rangle, \\ G \cong C_7 \quad &\Longrightarrow \quad G^\varphi = \left\langle \left(\begin{smallmatrix}1&0&0\\0&\zeta&0\\0&0&1\end{smallmatrix}\right)\right\rangle \ \textit{or} \ \left\langle \left(\begin{smallmatrix}1&0&0\\0&\zeta&0\\0&0&\zeta^2\end{smallmatrix}\right)\right\rangle \ \textit{or} \ \left\langle \left(\begin{smallmatrix}1&0&0\\0&\zeta&0\\0&0&\zeta^3\end{smallmatrix}\right)\right\rangle. \end{aligned}$$

(c) *Suppose that $G \cong C_p \times C_p$ with p prime, and let ζ be a primitive p'th root of unity. Then there is a $\varphi \in \mathrm{PGL}_3(K)$ such that one of the following is valid*:

$$G^\varphi = \left\langle \left(\begin{smallmatrix}1&0&0\\0&\zeta&0\\0&0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&\zeta\end{smallmatrix}\right)\right\rangle, \qquad p \ \textit{arbitrary}. \tag{4.2}$$

$$G^\varphi = \left\langle \left(\begin{smallmatrix}1&0&0\\0&\zeta&0\\0&0&\zeta^2\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right)\right\rangle, \qquad p = 3 \ \textit{only}. \tag{4.3}$$

Further, the group (4.2) *with $p = 3$ is not* $\mathrm{GL}_3(K)$*-conjugate to the group* (4.3).

Proof. — We remark that if $\alpha \in \mathrm{PGL}_3(K)$ has finite order n, then we can lift it to an element $A \in \mathrm{GL}_3(K)$ having the same order. To see this, we start with an arbitrary lift A. Then $A^n = cI$ for some $c \in K^*$, so we can take $c^{-1/n}A$ as our lift of α. We also remark that since we have assumed that $\mathrm{char}(K) = 0$, every element in $\mathrm{GL}_3(K)$ of finite order is diagonalizable.

(a) Let $G = \langle \alpha \rangle$ with $\alpha \in \mathrm{PGL}_3(K)$ having order q. We lift α to an $A \in \mathrm{GL}_3(K)$ of order q. Conjugating A to put it into Jordan normal form, the fact that $A^q = 1$ implies that A is diagonal and its diagonal entries are q'th roots of unity. Replacing A by a scalar multiple, which we may do since we are really only interested in the image of A in $\mathrm{PGL}_3(K)$, we may assume that the upper left entry of A is 1. (Note that we still have $A^q = I$.) The fact that α has exact order q, where q is a prime power, implies that one of the other diagonal entries is a primitive q'th root of unity, which we denote ζ. Possibly after reversing the Y and Z coordinates, α is diagonal with entries $1, \zeta, \eta$, where η, being a q'th root of unity, is a power of ζ.

(b) From (a), we can find φ so that G^φ is given by (4.1) with $0 \leqslant m < q$. For notational convenience, we let

$$\tau(m) := \left(\begin{smallmatrix}1&0&0\\0&\zeta&0\\0&0&\zeta^m\end{smallmatrix}\right).$$

We note that conjugation by a permutation matrix in PGL_3 has the effect of permuting the entries of a diagonal matrix. Writing $\sim$ to denote PGL_3-conjugation equivalence and using the fact that we are working in PGL_3, we have

$$\tau(m) \sim \left\langle \left(\begin{smallmatrix} \zeta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta^m \end{smallmatrix} \right) \right\rangle = \left\langle \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \zeta^{-1} & 0 \\ 0 & 0 & \zeta^{m-1} \end{smallmatrix} \right) \right\rangle = \tau(1-m \bmod q),$$

$$\tau(m) \sim \left\langle \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \zeta^m & 0 \\ 0 & 0 & \zeta \end{smallmatrix} \right) \right\rangle = \tau(m^{-1} \bmod q) \quad \text{if } \gcd(m,q)=1.$$

(We remark that the other three permutations do not give results that are useful for our purposes.) Hence after a further conjugation by a permutation matrix, we may take G^φ to be generated by any one of the following three matrices,

$$\tau(m), \quad \tau(1-m \bmod q), \quad \tau(m^{-1} \bmod q),$$

subject to $\gcd(m,q)=1$ for the last one.

Suppose first that $q=4$. Taking $m=0$ and $m=2$, we find that

$$\langle \tau(0) \rangle \sim \langle \tau(1) \rangle \quad \text{and} \quad \langle \tau(2) \rangle \sim \langle \tau(3) \rangle.$$

Hence we can find a φ so that G^φ is generated by either $\tau(0)$ or $\tau(2)$.

Next let $q=5$. Then

$$\langle \tau(0) \rangle \sim \langle \tau(1) \rangle \quad \text{and} \quad \langle \tau(2) \rangle \sim \langle \tau(4) \rangle \sim \langle \tau(3) \rangle.$$

Hence we can find a φ so that G^φ is generated by either $\tau(0)$ or $\tau(3)$.

Finally let $q=7$. Then

$$\langle \tau(0) \rangle \sim \langle \tau(1) \rangle, \quad \langle \tau(2) \rangle \sim \langle \tau(6) \rangle \sim \langle \tau(4) \rangle \quad \text{and} \quad \langle \tau(3) \rangle \sim \langle \tau(5) \rangle.$$

Hence we can find a φ so that G^φ is generated by either $\tau(0)$ or $\tau(2)$ or $\tau(3)$.

(c) Let $\alpha, \beta \in G$ be generators of G. We lift α and β, respectively, to matrices $A, B \in \mathrm{GL}_3(K)$ satisfying $A^p = B^p = I$. The fact that $\alpha\beta = \beta\alpha$ in $\mathrm{PGL}_3(K)$ tells us that there is an $\epsilon \in K^*$ such that $AB = \epsilon BA$ in $\mathrm{GL}_3(K)$.

We start with the case that $\epsilon = 1$, so we have diagonalizable matrices $A, B \in \mathrm{GL}_3(K)$ satisfying $AB = BA$. Standard linear algebra says that they can be simultaneously diagonalized, so after conjugation, we may assume that A and B are both diagonal. And since $A^p = B^p = I$ and the image of $\langle A, B \rangle$ in $\mathrm{PGL}_3(K)$ is of type C_p^2, we see that the group

$$\langle \zeta I, A, B \rangle = \{ \zeta^i A^j B^k : 0 \leqslant i,j,k \leqslant p-1 \} \subset \mathrm{GL}_3(K)$$

contains p^3 distinct diagonal elements of $\mathrm{GL}_3(K)$ of order dividing p. But $\mathrm{GL}_3(K)$ contains exactly p^3 diagonal matrices of order dividing p, namely the diagonal matrices with entries that are arbitrary p'th roots of unity. It follows that $G = \langle \alpha, \beta \rangle$, which is the image of $\langle \zeta I, A, B \rangle$ in $\mathrm{PGL}_3(K)$, is the group described in (4.2).

We next suppose that $\epsilon \neq 1$. Applying (a) to $\langle \alpha \rangle$, we can conjugate so that α lifts to a matrix of the form $A = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^m \end{smallmatrix} \right) \in \mathrm{GL}_3(K)$, where ζ is a primitive p'th root of unity and $0 \leqslant m < p$. Writing the lift B of β with generic entries, the relation $AB = \epsilon BA$ becomes

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = B = \epsilon A^{-1}BA = \epsilon \begin{pmatrix} a & \zeta b & \zeta^m c \\ \zeta^{-1}d & e & \zeta^{m-1}f \\ \zeta^{-m}g & \zeta^{1-m}h & i \end{pmatrix}.$$

The assumption that $\epsilon \neq 1$ forces $a = e = i = 0$. Since B is invertible, we see that either $b \neq 0$ or $c \neq 0$. For the former, we find that

$$b \neq 0 \quad\Longrightarrow\quad \epsilon = \zeta^{-1} \quad\Longrightarrow\quad \begin{cases} (1-\zeta^{m-1})c = 0, \\ (1-\zeta^{-2})d = 0, \\ (1-\zeta^{m-2})f = 0, \\ (1-\zeta^{-m-1})g = 0, \\ (1-\zeta^{-m})h = 0. \end{cases} \tag{4.4}$$

The invertibility of B also tells us that d and f are not both 0, and that g and h are not both 0. Therefore

$$(p = 2 \text{ or } m = 2) \quad\text{and}\quad (m = p-1 \text{ or } m = 0).$$

Hence either $p = 2$, or else $m = 2$ and $p = 3$. We consider these cases in turn.

If $p = 2$, then $m \in \{0, 1\}$, and (4.4) tells us that one of the following holds:

$$(p, m) = (2, 0) \Longrightarrow c = g = 0 \Longrightarrow B = \begin{pmatrix} 0 & b & 0 \\ d & 0 & f \\ 0 & h & 0 \end{pmatrix},$$
$$(p, m) = (2, 1) \Longrightarrow f = h = 0 \Longrightarrow B = \begin{pmatrix} 0 & b & c \\ d & 0 & 0 \\ g & 0 & 0 \end{pmatrix}.$$

Thus both m values with $p = 2$ lead to a matrix B that is not invertible.

If $m = 2$ and $p = 3$, then $c = d = h = 0$, so B has the form

$$B = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & f \\ g & 0 & 0 \end{pmatrix}.$$

We know that $B^3 = I$, so $bfg = 1$. Conjugating B by the matrix $\varphi := \begin{pmatrix} gf & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix}$ yields $B^\varphi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, while $A^\varphi = A$. This proves that G is conjugate to the group (4.3).

Next we assume that $b = 0$ and $c \neq 0$, which leads to

$$b = 0 \text{ and } c \neq 0 \quad\Longrightarrow\quad \epsilon = \zeta^{-m} \quad\Longrightarrow\quad \begin{cases} (1-\zeta^{-1-m})d = 0, \\ (1-\zeta^{-1})f = 0, \\ (1-\zeta^{-2m})g = 0, \\ (1-\zeta^{1-2m})h = 0. \end{cases} \tag{4.5}$$

Thus $f = 0$, and then the fact that $\det B = cdh \neq 0$ tells us that $d \neq 0$ and $h \neq 0$. Then (4.5) gives

$$d \neq 0 \quad\Longrightarrow\quad m = p-1 \quad\text{and}\quad h \neq 0 \quad\Longrightarrow\quad m = \frac{p+1}{2}.$$

Equating the values of m, we conclude that $p = 3$ and $m = 2$, and (4.5) forces $g = 0$. This shows that $B = \left(\begin{smallmatrix} 0&0&c\\ d&0&0\\ 0&h&0 \end{smallmatrix}\right)$, and using $B^3 = 1$ shows that $cdh = 1$. So if we conjugate by $\phi := \left(\begin{smallmatrix} c&0&0\\ 0&cd&0\\ 0&0&1 \end{smallmatrix}\right)$, we find that $A^\phi = A$ and $B^\phi = \left(\begin{smallmatrix} 0&0&1\\ 1&0&0\\ 0&1&0 \end{smallmatrix}\right)$. Hence the subgroup of PGL_3 generated by A^ϕ and B^ϕ is the group (4.3).

Finally, to prove that the groups in (4.2) with $p = 3$ and (4.3) are not $\mathrm{GL}_3(K)$-conjugate, we observe that (4.2) fixes three points in $\mathbb{P}^2$, while (4.3) has no fixed points. $\square$

Lemma 4.2. — *Let K be an algebraically closed field of characteristic* 0*, and let $\mathcal{G} \subset \mathrm{PGL}_3(K)$ be one of the groups* (1.2) *listed in Theorem 1.2. Then the identity component of the normalizer $\mathcal{G}$ is the group of diagonal matrices,*

$$N(\mathcal{G})^\circ = \mathcal{D} := \left\{ \left(\begin{smallmatrix} \alpha&0&0\\ 0&\beta&0\\ 0&0&\gamma \end{smallmatrix}\right) \in \mathrm{PGL}_3(K) \right\}.$$

More precisely, we have

$$N(\mathcal{G}_3) = N(\mathcal{G}_{2,2}) = \mathcal{S}_3\mathcal{D}, \qquad N(\mathcal{G}_4) = \left\langle \left(\begin{smallmatrix} 0&0&1\\ 0&1&0\\ 1&0&0 \end{smallmatrix}\right) \right\rangle \mathcal{D},$$

$$N(\mathcal{G}_5) = \left\langle \left(\begin{smallmatrix} 0&1&0\\ 1&0&0\\ 0&0&1 \end{smallmatrix}\right) \right\rangle \mathcal{D}, \qquad N(\mathcal{G}_7) = \left\langle \left(\begin{smallmatrix} 0&1&0\\ 0&0&1\\ 1&0&0 \end{smallmatrix}\right) \right\rangle \mathcal{D},$$

where $\mathcal{S}_3 \subset \mathrm{PGL}_3(K)$ is the group of permutation matrices.

Proof. — We recall that for any group G and subgroup $H \subseteq G$, the kernel of the standard homomorphism

$$N_G(H) \longrightarrow \mathrm{Aut}(H), \quad g \longmapsto (h \mapsto g^{-1}hg)$$

is the centralizer $C_G(H)$. We use this to simplify our calculations.

Let ζ be a primitive n'th root of unity with $n \geqslant 3$, let $2 \leqslant m < n$, and let $T = T_{n,m} := \left(\begin{smallmatrix} 1&0&0\\ 0&\zeta&0\\ 0&0&\zeta^m \end{smallmatrix}\right)$. Then $A = \left(\begin{smallmatrix} a&b&c\\ d&e&f\\ g&h&i \end{smallmatrix}\right) \in C(T)$ if and only if $A = T^{-1}AT$, so if and only if

$$\left(\begin{smallmatrix} a&b&c\\ d&e&f\\ g&h&i \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1&0&0\\ 0&\zeta&0\\ 0&0&\zeta^m \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} a&b&c\\ d&e&f\\ g&h&i \end{smallmatrix}\right) \left(\begin{smallmatrix} 1&0&0\\ 0&\zeta&0\\ 0&0&\zeta^m \end{smallmatrix}\right) = \left(\begin{smallmatrix} a&\zeta b&\zeta^m c\\ \zeta^{-1}d&e&\zeta^{m-1}f\\ \zeta^{-m}g&\zeta^{1-m}h&i \end{smallmatrix}\right).$$

Keeping in mind that we are working in PGL_3, we first note that if any of a, e, i is non-zero, then A is diagonal. Suppose that $a = e = i = 0$. If $b \neq 0$, then the fact that $\zeta^m, \zeta^{-1}, \zeta^{1-m}$ are distinct from ζ gives $c = d = h = 0$, and then the nonsingularity of A tells us that $fg \neq 0$, and $\zeta = \zeta^{m-1} = \zeta^{-m}$.

Hence $b \neq 0$ is allowed only if $n = 3$ and $m = 2$, in which case $C(T)$ contains the scaled cyclic permutation $\left(\begin{smallmatrix} 0 & b & 0 \\ 0 & 0 & f \\ g & 0 & 0 \end{smallmatrix}\right)$. A similar analysis for $c \neq 0$ yields the inverse scaled permutation for $n = 3$ and a contradiction for $n \geqslant 4$. This completes the proof that

$$C(T_{n,m}) = \begin{cases} \langle \pi \rangle \mathcal{D} & \text{if } n = 3 \text{ and } m = 2, \\ \mathcal{D} & \text{if } n \geqslant 4 \text{ and } 2 \leqslant m \leqslant n-1, \end{cases}$$

where $\pi \in \mathrm{PGL}_3$ is a cyclic permutation.

Suppose now that $(n, m) = (3, 2)$. Then the transposition $\alpha = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)$ satisfies $\alpha^{-1} T_{3,2} \alpha = T_{3,2}^2$, so $\alpha \in N(\mathcal{G}_3) \smallsetminus C(\mathcal{G}_3)$. Using the inclusion

$$N(\mathcal{G}_3)/C(\mathcal{G}_3) \hookrightarrow \mathrm{Aut}(\mathcal{G}_3) \cong (\mathbb{Z}/3\mathbb{Z})^* \cong \mathbb{Z}/2\mathbb{Z},$$

we conclude that $N(\mathcal{G}_3) = \langle \alpha \rangle C(\mathcal{G}_3) = \langle \alpha \rangle \langle \pi \rangle \mathcal{D} = \mathcal{S}_3 \mathcal{D}$.

Next let $(n, m) = (4, 2)$. Then the transposition $\beta = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$ satisfies $\beta^{-1} T_{4,2} \beta = T_{4,2}^3$, so $\beta \in N(\mathcal{G}_4) \smallsetminus C(\mathcal{G}_4)$. Using the inclusion

$$N(\mathcal{G}_4)/C(\mathcal{G}_4) \hookrightarrow \mathrm{Aut}(\mathcal{G}_4) \cong (\mathbb{Z}/4\mathbb{Z})^* \cong \mathbb{Z}/2\mathbb{Z},$$

we conclude that $N(\mathcal{G}_4) = \langle \beta \rangle C(\mathcal{G}_4) = \langle \beta \rangle \mathcal{D}$.

Next we consider $T_{m,n}$ with $m = 3$ and $n \geqslant 5$ prime. An element $A \in N(T) \smallsetminus C(T)$ needs to satisfy

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^3 \end{pmatrix}^{-1} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^3 \end{pmatrix}^{j} = \begin{pmatrix} a & \zeta^j b & \zeta^{3j} c \\ \zeta^{-1} d & \zeta^{j-1} e & \zeta^{3j-1} f \\ \zeta^{-3} g & \zeta^{j-3} h & \zeta^{3j-3} i \end{pmatrix}$$

for some $2 \leqslant j < n$ with $\gcd(j, n) = 1$. We have

$$\begin{aligned} a \neq 0 &\Longrightarrow b = d = e = g = 0 \Longrightarrow fh \neq 0 \\ &\Longrightarrow j \equiv 3 \pmod{n} \text{ and } 3j \equiv 1 \pmod{n} \Longrightarrow n \mid 8. \quad \rightarrow\leftarrow \\ e \neq 0 &\Longrightarrow a = b = d = h = 0 \Longrightarrow cg \neq 0 \\ &\Longrightarrow 3j \equiv j-1 \pmod{n} \text{ and } -3 \equiv j-1 \pmod{n} \\ &\Longrightarrow n \mid 3. \quad \rightarrow\leftarrow \\ i \neq 0 &\Longrightarrow c = e = f = g = h \Longrightarrow bd \neq 0 \\ &\Longrightarrow j \equiv 3j-3 \pmod{n} \text{ and } -1 \equiv 3j-3 \pmod{n} \\ &\Longrightarrow n \mid 5. \end{aligned}$$

So we find that the normalizer of $\mathcal{G}_5$ contains $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.

We next look for maps with $a = e = i = 0$, so

$$\begin{pmatrix} 0 & b & c \\ d & 0 & f \\ g & h & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^3 \end{pmatrix}^{-1} \begin{pmatrix} 0 & b & c \\ d & 0 & f \\ g & h & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^3 \end{pmatrix}^{j} = \begin{pmatrix} 0 & \zeta^j b & \zeta^{3j} c \\ \zeta^{-1} d & 0 & \zeta^{3j-1} f \\ \zeta^{-3} g & \zeta^{j-3} h & 0 \end{pmatrix}$$

This gives

$$\begin{aligned}
b \neq 0 &\Longrightarrow c = h = 0 \Longrightarrow g \neq 0\\
&\Longrightarrow j \equiv -3 \pmod{n} \Longrightarrow d = 0 \Longrightarrow f \neq 0\\
&\Longrightarrow j \equiv -3 \equiv 3j - 1 \pmod{n} \Longrightarrow n \mid 7.\\
c \neq 0 &\Longrightarrow b = f = 0 \Longrightarrow d \neq 0\\
&\Longrightarrow -1 \equiv 3j \pmod{n} \Longrightarrow g = 0 \Longrightarrow h \neq 0\\
&\Longrightarrow -1 \equiv 3j \equiv j - 3 \pmod{n} \Longrightarrow n \mid 7.
\end{aligned}$$

So we find that the normalizer of $\mathcal{G}_7$ contains $\left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}\right)$.

This completes the computation of $N(\mathcal{G}_n)$ with $n = 3, 4, 5, 7$.

Finally, consider an element $A \in C(\mathcal{G}_{2,2})$ of the centralizer of $\mathcal{G}_{2,2}$. It satisfies the two equations

$$\begin{aligned}
\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix} &= \begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}^{-1}\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix} = \begin{pmatrix}a&-b&c\\-d&e&-f\\g&-h&i\end{pmatrix}\\
\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix} &= \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}^{-1}\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix} = \begin{pmatrix}a&b&-c\\d&e&-f\\-g&-h&i\end{pmatrix}.
\end{aligned}$$

If any of a, e, i is non-zero, then these two equations combine to tell us that A is diagonal. On the other hand, if $a = e = i = 0$, then $b \neq 0$ forces $g = h = 0$, contradicting the non-singularlity of A, and similarly $c \neq 0$ forces $b = h = 0$, giving the same contradiction. Hence $C(\mathcal{G}_{2,2}) = \mathcal{D}$. Next we observe that every permutation in $\mathcal{S}_3$ is in $N(\mathcal{G}_{2,2})$, so

$$\mathcal{S}_3 \longrightarrow N(\mathcal{G}_{2,2})/C(\mathcal{G}_{2,2}) \hookrightarrow \operatorname{Aut}(\mathcal{G}_{2,2}) \cong \operatorname{GL}_2(\mathbb{F}_2) \cong \mathcal{S}_3.$$

A quick calculation shows that the map $\mathcal{S}_3 \to \operatorname{Aut}(\mathcal{G}_{2,2})$ is an isomorphism, and hence $N(\mathcal{G}_{2,2}) = \mathcal{S}_3 C(\mathcal{G}_{2,2})$. $\square$

5. Diagonal Stability and Maps of Finite Order

In this section we set notation that is used throughout the rest of this paper, we remind the reader of the Hilbert–Mumford criterion for GIT stability, and we create two tables that we will use to determine the stability of elements of Rat_2^2.

For a fixed root of unity ζ and integer m, we define

$$\tau_m = \tau_{\zeta,m} \in \operatorname{GL}_3(K), \quad \tau_m(X, Y, Z) = (X, \zeta Y, \zeta^m Z).$$

Further, for each pair of integers (k, ℓ) we define a 1-parameter subgroup of SL_3 by

$$L_{k,\ell} : \mathbb{G}_m \to \mathrm{SL}_3, \quad L_{k,\ell}(t) = \begin{pmatrix} t^k & 0 & 0 \\ 0 & t^\ell & 0 \\ 0 & 0 & t^{-k-\ell} \end{pmatrix}.$$

We now compute the effect of applying τ_m and $L_{k,\ell}$ to each of the quadratic monomials in a degree 2 map of $\mathbb{A}^3$.

Table 5.1 gives the effect of applying the map $\tau_m = (X, \zeta Y, \zeta^m Z)$ to each quadratic monomial, where an integer entry ϵ in Table 5.1 means that the monomial is multiplied by ζ^ϵ. Similarly, Table 5.2 gives the effect of applying $L_{k,\ell}(t)$ to each quadratic monomial, where an integer entry δ in Table 5.2 means that the monomial is multiplied by t^δ.

	X^2	Y^2	Z^2	XY	XZ	YZ
X-coord	0	2	$2m$	1	m	$m+1$
Y-coord	-1	1	$2m-1$	0	$m-1$	m
Z-coord	$-m$	$2-m$	m	$1-m$	0	1

Table 5.1. Effect of $\tau_m = (X, \zeta Y, \zeta^m Z)$ on monomials

	X^2	Y^2	Z^2	XY	XZ	YZ
X-coord	k	$-k+2\ell$	$-3k-2\ell$	ℓ	$-k-\ell$	$-2k$
Y-coord	$2k-\ell$	ℓ	$-2k-3\ell$	k	-2ℓ	$-k-\ell$
Z-coord	$3k+\ell$	$k+3\ell$	$-k-\ell$	$2k+2\ell$	k	ℓ

Table 5.2. Effect of $L_{k,\ell}(t) = (t^k X, t^\ell Y, t^{-k-\ell} Z)$ on monomials

We are going to use the numerical criterion of Hilbert–Mumford [16, Chapter 2, Theorem 2.1] to determine the stability of maps. We recall the general setup. (See [23, Section 2.2] or [10] for similar calculations.) Let $\mathcal{G} \subseteq \mathrm{SL}_{n+1}$ be an algebraic subgroup of SL_{n+1} and let $\boldsymbol{a} \in \mathbb{P}^n$. For any given one-parameter subgroup $L : \mathbb{G}_m \to \mathcal{G}$, choose coordinates on $\mathbb{P}^{n+1}$ so that the image of L is contained in the group of diagonal matrices. Write $\boldsymbol{a} = [a_1, \ldots, a_n] \in \mathbb{P}^n$ in these coordinates, let $\hat{\boldsymbol{a}} = (\hat{a}_1, \ldots, \hat{a}_n)$ be a lift of $\boldsymbol{a}$ to $\mathbb{A}^n$, and write the action of L on the lift $\hat{\boldsymbol{a}}$ as

$$L(t) \cdot \hat{\boldsymbol{a}} = (t^{r_1} \hat{a}_1, t^{r_2} \hat{a}_2, \ldots, t^{r_n} \hat{a}_n),$$

where $r_1, \ldots, r_n \in \mathbb{Z}$. The numerical factor associated to L at $\boldsymbol{a}$ is the quantity

$$\mu^{\mathcal{O}(1)}(\boldsymbol{a}, L) = \max\{-r_i : i \text{ satisfies } \hat{a}_i \neq 0\}.$$

Then the Hilbert–Mumford numerical criterion says that

$$\begin{aligned} \boldsymbol{a} \text{ is } \mathcal{G}\text{-unstable} \quad &\Longleftrightarrow \quad \mu^{\mathcal{O}(1)}(L,\boldsymbol{a}) < 0 \text{ for some } L, \\ \boldsymbol{a} \text{ is } \mathcal{G}\text{-not stable} \quad &\Longleftrightarrow \quad \mu^{\mathcal{O}(1)}(L,\boldsymbol{a}) \leqslant 0 \text{ for some } L. \end{aligned}$$

Equivalently, $\boldsymbol{a}$ is $\mathcal{G}$-stable if $\mu^{\mathcal{O}(1)}(L,\boldsymbol{a}) > 0$ for all L, and it is $\mathcal{G}$-semistable if $\mu^{\mathcal{O}(1)}(L,\boldsymbol{a}) \geqslant 0$ for all L.

We write $f_{m,\epsilon}$ to denote a generic element of Rat_2^2 whose affine lift $\hat{f}_{m,\epsilon} : \mathbb{A}^3 \to \mathbb{A}^3$ satisfies

$$\hat{f}_{m,\epsilon}^{\tau_m} = \zeta^\epsilon \hat{f}_{m,\epsilon}.$$

Since the one-parameter subgroup $L_{k,\ell}$ is already diagonalized, the following two-step procedure computes $\mu^{\mathcal{O}(1)}(f_{m,\epsilon}, L_{k,\ell})$.

- Look at Table 5.1 and check off all of the boxes whose entry is congruent to ϵ mod p.
- Then $\mu^{\mathcal{O}(1)}(f_{m,\epsilon}, L_{k,\ell})$ is equal to the maximum of the *negatives* of the corresponding entries in Table 5.2.

We note that every diagonalized one-parameter subgroup of SL_3 is conjugate to $L_{k,\ell}$ for some $(k,\ell) \neq (0,0)$. We set the notation

$$\mathcal{D} = \left\{ \left(\begin{smallmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{smallmatrix}\right) \in \mathrm{SL}_3(K) \right\}$$

for the group of diagonal matrices. By abuse of notation, we may sometimes also write $\mathcal{D}$ for the diagonal subgroup of PGL_3. Similarly, we set the notation

$$\mathcal{S}_3 := \left\langle \left(\begin{smallmatrix} 1&0&0\\0&1&0\\0&0&1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&0&1\\1&0&0\\0&1&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&1&0\\0&0&1\\1&0&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&1&0\\1&0&0\\0&0&1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&0&1\\0&1&0\\1&0&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&0&0\\0&0&1\\0&1&0 \end{smallmatrix}\right) \right\rangle$$

for the group of permutation matrices in PGL_3 or SL_3.

Numerical Criterion for $\mathcal{D}$-Stability. Let $\mathcal{G} \subset \mathrm{PGL}_3$ be a finite subgroup such that $N(\mathcal{G})^\circ = \mathcal{D}$, and let $f \in \mathrm{Rat}_2^2(\mathcal{G})$.

$$\begin{aligned} f \text{ is } N(\mathcal{G})\text{-unstable} &\iff \mu^{\mathcal{O}(1)}(f, L_{k,\ell}) < 0 \text{ for some } (k,\ell). \\ f \text{ is } N(\mathcal{G})\text{-semistable} &\iff \mu^{\mathcal{O}(1)}(f, L_{k,\ell}) \geqslant 0 \text{ for all } (k,\ell). \\ f \text{ is } N(\mathcal{G})\text{-stable} &\iff \mu^{\mathcal{O}(1)}(f, L_{k,\ell}) > 0 \text{ for all } (k,\ell) \neq (0,0). \end{aligned}$$

6. Maps with an Automorphism of Prime Order $p \geqslant 5$

In this section we analyze maps having an automorphism of prime order $p \geqslant 5$.

Proposition 6.1. — *Let K be an algebraically closed field of characteristic 0, let $p \geqslant 5$ be prime, let ζ be a primitive p'th root of unity, let $m \in \mathbb{Z}/p\mathbb{Z}$, and let $f \in \mathrm{Rat}_2^2$ be a dominant rational map such that $\tau_m \in \mathrm{Aut}(f)$. Choose $0 \leqslant \epsilon < p$ so that the lifts $\hat{f} : \mathbb{A}^3 \to \mathbb{A}^3$ of f satisfy $\hat{f}^{\tau_m} = \zeta^\epsilon \hat{f}$. Then one of the following is true*:

(a) *f is $\mathcal{D}$-unstable.*
(b) *f is $\mathcal{D}$-semistable, but not $\mathcal{D}$-stable, and is* $\mathrm{PGL}_3(K)$*-conjugate to a rational map of the form $[aX^2 + YZ, bXY, cXZ]$ with $bc \neq 0$. Further $(m, \epsilon) = (-1, 0)$.*
(c) *$p = 5$, and f is $\mathcal{D}$-stable and* $\mathrm{PGL}_3(K)$*-conjugate to the rational map $[YZ, X^2, Y^2]$ with $(m, \epsilon) = (3, 4)$.*
(d) *$p = 7$, and f is* PGL_3*-stable and* $\mathrm{PGL}_3(K)$*-conjugate to the morphism $[Z^2, X^2, Y^2]$ with $(m, \epsilon) = (3, 6)$.*

Proof. — We write $f = f_{m,\epsilon}$ to help keep track of the dependence on m and ϵ. The assumption that f is dominant implies that its coordinate functions are non-zero, so it necessarily includes at least three monomials. Looking at Table 5.1, we see that each of the quantities 0, 1, and m appears three times, while each of the quantities

$$-m,\ 1-m,\ 2-m,\ -1,\ 2,\ m-1,\ m+1,\ 2m-1,\ 2m \tag{6.1}$$

appears exactly once.

We suppose first that $\epsilon \notin \{0, 1, m\}$. Then the only way for $f_{m,\epsilon}$ to have at least three monomials is for at least three of the quantities in the list (6.1) to be equal. Our assumption that $p \geqslant 5$ means that the elements in each of the subsets

$$\{-m, 1-m, 2-m\}, \quad \{-1, 2\}, \quad \{m-1, m+1\}, \quad \{2m-1, 2m\},$$

remain distinct when reduced modulo p, so in order to obtain three equal values modulo p, we first choose three of these four sets, then choose an element from each set, then equate the three quantities and solve for m modulo p. This gives a total of 44 possibilities, although many of them give no value for m, since each choice yields two equations for the one quantity m. Further, some choices give $\epsilon \in \{0, 1, m\}$, which we are not presently considering. We do not know a clever way to do this computation, but working through the complete set of possibilities, we find that exactly 10 choices yield values of m, and all but two of these require either $p = 5$ or $p = 7$. The data and resulting maps are listed in Table 6.1.

In Table 6.1, we require $abc \neq 0$, since we need at least three monomials. We start by noting that the maps $f_{0,-1}$ and $f_{1,2}$, which work for all p, are clearly non-dominant (indeed, they are constant maps), so they may be discarded. (It is also easy to check that they are $\mathcal{D}$-unstable.)

Values from Table 5.1			m	ϵ	p	$f_{m,\epsilon}$
-1	$m-1$	$2m-1$	0	-1	all p	$[0, aX^2+bZ^2+cXZ, 0]$
2	$m+1$	$2m$	1	2	all p	$[aY^2+bZ^2+cYZ, 0, 0]$
-1	$m+1$	$-m+2$	3	-1	5	$[aYZ, bX^2, cY^2]$
-1	$-m+1$	$2m$	2	-1	5	$[aZ^2, bX^2, cXY]$
2	$m-1$	$-m$	3	2	5	$[aY^2, bXZ, cX^2]$
2	$-m+1$	$2m-1$	4	2	5	$[aY^2, bZ^2, cXY]$
$m+1$	$-m$	$2m-1$	2	3	5	$[aYZ, bZ^2, cX^2]$
$m-1$	$-m+2$	$2m$	4	3	5	$[aZ^2, bXZ, cY^2]$
-1	$-m+2$	$2m$	3	-1	7	$[aZ^2, bX^2, cY^2]$
2	$-m$	$2m-1$	5	2	7	$[aY^2, bZ^2, cX^2]$

Table 6.1. Values of m mod p and $\epsilon \notin \{0,1,m\}$ such that $f_{m,\epsilon}$ has at least 3 monomials

We next note that the six families of maps for $p=5$ are $\mathcal{S}_3$-conjugates, i.e., they may be obtained from one another by permuting the variables. It thus suffices to consider $f_{3,-1} = [aYZ, bX^2, cY^2]$, which is a dominant rational map having a single point $[0,0,1]$ of indeterminacy. Using Table 5.2, we find that

$$\mu^{\mathcal{O}(1)}(f_{3,-1}, L_{k,\ell}) = \max\{2k, -2k+\ell, -k-3\ell\}.$$

The identity

$$7\cdot(2k) + 6(-2k+\ell) + 2(-k-3\ell) = 0$$

shows that at least one of the quantities in parentheses is non-negative, and indeed unless $k=\ell=0$, one of them is positive. Hence

$$\inf_{(k,\ell)\neq(0,0)} \mu^{\mathcal{O}(1)}(f_{3,-1}, L_{k,\ell}) = \inf_{(k,\ell)\neq(0,0)} \max\{2k, -2k+\ell, -k-3\ell\} > 0,$$

which proves that $f_{3,-1}$ is $\mathcal{D}$-stable.

In order to obtain the map in (c), we observe that the a,b,c coefficients of $f_{3,-1}$ are twist parameters. To see this, let $\sigma(X,Y,Z) = [uX, vY, wZ]$. Then

$$f_{3,-1}^{\sigma} = [v^2w^2aYZ, u^3wbX^2, uv^3cY^2],$$

so setting $u^{20} = a^3b^{-6}c^{-2}$, $v^{20} = a^{-1}b^2c^{-6}$, and $w^{20} = a^{-9}b^{-2}c^6$ (with an appropriate choice of 20'th roots) yields $f_{3,-1}^{\sigma} = [YZ, Z^2, X^2]$. Thus the family $f_{3,-1}$ for $p=5$ is a single $\mathcal{D}$-orbit.

Similarly, the two families of maps for $p=7$ are conjugate via a cyclic permutation of the variables. It is also clear that they are morphisms, so in particular they are stable [10]. Further, just as in (c), the coefficients are twist parameters. Thus for $p=7$ and $\sigma(X,Y,Z) = [uX, vY, wZ]$ we have

$$f_{3,-1}^{\sigma} = [vw^3aZ^2, u^3wbX^2, uv^3cY^2],$$

so setting $u^{28} = a^3b^{-9}c^{-1}$, $v^{28} = a^{-1}b^3c^{-9}$, and $w^{28} = a^{-9}b^{-1}c^3$ (with an appropriate choice of 28'th roots) yields $f^\sigma_{3,-1} = [Z^2, X^2, Y^2]$. Thus the family $f_{3,-1}$ for $p = 7$ is also a single $\mathcal{D}$-orbit.

This completes the classification of dominant semistable maps $f_{m,\epsilon}$ with $\epsilon \notin \{0, 1, m\}$. We next observe that if $\epsilon \in \{0, 1, m\}$ and $f_{m,\epsilon}$ has exactly three monomials, then Tables 5.1 and 5.2 give the following three maps and their numerical invariants:

ϵ	$f_{m,\epsilon}$	$\mu^{\mathcal{O}(1)}(f_{m,0}, L_{k,\ell})$
0	$[aX^2, bXY, cXZ]$	$-k$
1	$[aXY, bY^2, cYZ]$	$-\ell$
m	$[aXZ, bYZ, cZ^2]$	$k + \ell$

Thus in all cases $f_{m,\epsilon}$ induces the linear map $[aX, bY, cZ]$, and we can find a $(k, \ell) \neq (0, 0)$ making $\mu^{\mathcal{O}(1)}(f_{m,0}, L_{k,\ell}) < 0$, so all of these maps are $\mathcal{D}$-unstable.

We now assume that $\epsilon \in \{0, 1, m\}$ and that $f_{m,\epsilon}$ has at least four monomials.

$\boldsymbol{\epsilon = 0}$. Since $f_{m,0}$ has four or more monomials, Table 5.1 tells us that p must divide one of the quantities in the set

$$\{m - 2, m - 1, m, m + 1, 2m - 1\}.$$

Since τ_m depends only on m modulo p, this gives five possibilities:

$m \bmod p$	$f_{m,0}(X, Y, Z)$
2	$[aX^2, bXY, cY^2 + dXZ]$
1	$[aX^2, bXY + cXZ, dXY + eXZ]$
0	$[aX^2 + bZ^2 + cXZ, dXY + eYZ, fX^2 + gZ^2 + hXZ]$
-1	$[aX^2 + bYZ, cXY, dXZ]$
2^{-1}	$[aX^2, bZ^2 + cXY, dXZ]$

For each of these families we use Table 5.2 to compute

$$\begin{aligned}
\mu^{\mathcal{O}(1)}(f_{2,0}, L_{k,\ell}) &\leqslant \max\{-k, -k - 3\ell\} \xrightarrow{(k,\ell)=(1,0)} -1,\\
\mu^{\mathcal{O}(1)}(f_{1,0}, L_{k,\ell}) &\leqslant \max\{-k, 2\ell, -2k - 2\ell\} \xrightarrow{(k,\ell)=(2,-1)} -2,\\
\mu^{\mathcal{O}(1)}(f_{0,0}, L_{k,\ell}) &\leqslant \max\{-k, 3k + 2\ell, k + \ell, -3k - \ell\} \xrightarrow{(k,\ell)=(1,-2)} -1,\\
\mu^{\mathcal{O}(1)}(f_{-1,0}, L_{k,\ell}) &= \max\{-k, 2k\} \geqslant 0 \text{ for all } (k, \ell) \neq (0, 0),\\
\mu^{\mathcal{O}(1)}(f_{1/2,0}, L_{k,\ell}) &\leqslant \max\{-k, 2k + 3\ell\} \xrightarrow{(k,\ell)=(1,-1)} -1.
\end{aligned}$$

Thus $f_{-1,0}$ is $\mathcal{D}$-semi-stable provided $b \neq 0$ and a, c, d are not all 0, while the maps in the other four families are $\mathcal{D}$-unstable. Making a change of variables

$[X,Y,Z] \to [X, b^{-1}Y, Z]$, we can make the coefficient of YZ in $f_{-1,0}$ equal to 1. It is also clear that if c or d is 0, then $f_{-1,0}$ is not dominant. This gives the family of maps in (b).

$\boldsymbol{\epsilon = 1.}$ The straightforward approach is to use the assumption that $f_{m,1}$ has four or more monomials and Table 5.1 to deduce that

$$m \in \{2^{-1}, 1, 0, 2, -1\} \bmod p,$$

which leads to the five families of maps:

$m \bmod p$	$f_{m,1}(X,Y,Z)$
2	$[aXY, bY^2 + cXZ, dYZ]$
1	$[aXY + bXZ, cY^2 + dZ^2 + eYZ, eY^2 + fZ^2 + gYZ]$
0	$[aXY + bYZ, cY^2, dXY + eYZ]$
-1	$[aXY, bY^2, cX^2 + dYZ]$
2^{-1}	$[aZ^2 + bXY, cY^2, cYZ]$

Up to $\mathcal{S}_3$-conjugation, these are exactly the five families that we found for $\epsilon = 0$, so we obtain nothing new.

An alternative is to let $\sigma(X,Y,Z) = (Y,X,Z)$ and to observe that

$$(\hat{f}^{\sigma}_{m,1})^{\tau_{1-m}} = \hat{f}^{\sigma}_{m,1}.$$

Thus the map $f^{\sigma}_{m,1}$ is in the family of maps $f_{1-m,0}$, and since $f^{\sigma}_{m,1}$ and $f_{m,1}$ have the same number of non-zero monomials, the set of maps with $\epsilon = 1$ is equal to the set of σ-conjugates of the maps with $\epsilon = 0$. (It's also amusing to note that the set of m values that we obtained for $\epsilon = 0$ is invariant under $m \to 1 - m$.)

$\boldsymbol{\epsilon = m.}$ Again using the assumption that $f_{m,1}$ has four or more monomials, Table 5.1 tells us that

$$m \in \{2^{-1}, 1, 0, 2, -1\} \bmod p.$$

We are currently dealing with the case $\epsilon = m$, and we have already analyzed the cases $\epsilon = 0$ and $\epsilon = 1$, so it remains to consider $m \in \{2^{-1}, 2, -1\}$. This gives three families of maps:

$m \bmod p$	$f_{m,1}(X,Y,Z)$
2	$[aY^2 + bXZ, cYZ, dZ^2]$
-1	$[aXZ, bX^2 + cYZ, dZ^2]$
2^{-1}	$[aXZ, bYZ, cZ^2 + dXY]$

These three families are $\mathcal{S}_3$-conjugate to three of the families that we found for $\epsilon = 0$. Hence up to PGL_3 equivalence, we again obtain nothing new. $\square$

The next step is to compute the full automorphism groups of the maps appearing in Proposition 6.1.

PROPOSITION 6.2. — (a) *Let* $f = [YZ, X^2, Y^2]$*, and let* ζ *be a primitive* 5*'th root of unity. Then*

$$\mathrm{Aut}(f) = \left\langle \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^3 \end{smallmatrix}\right) \right\rangle \cong C_5.$$

(b) *Let* $f = [Z^2, X^2, Y^2]$*, and let* ζ *be a primitive* 7*'th root of unity. Then*

$$\mathrm{Aut}(f) = \left\langle \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right) \right\rangle \cong C_7 \rtimes C_3.$$

(c) *Let* $(a, b, c) \in K^3$ *with* $bc \neq 0$*, and let* $f = [aX^2 + YZ, bXY, cXZ]$*. Then*

$$\mathrm{Aut}(f) \supset \left\{ \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-1} \end{smallmatrix}\right) : t \in \mathbb{G}_m \right\} \cong \mathbb{G}_m.$$

Proof. — (a) We see by inspection that $\mathrm{Aut}(f) \supseteq \langle \tau_3 \rangle \cong C_5$. We claim that this is the full automorphism group. The indeterminacy and critical loci of f are

$$I(f) = \{[0, 0, 1]\} \quad \text{and} \quad \mathrm{Crit}(f) = \{4XY^2 = 0\}.$$

Since any $\varphi \in \mathrm{Aut}(f)$ preserves both of these sets, with their multiplicities, we see that φ leaves both of the lines $X = 0$ and $Y = 0$ invariant. Hence φ necessarily has the form $\varphi = \left(\begin{smallmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ \gamma & \delta & \epsilon \end{smallmatrix}\right)$. Comparing the first coordinates of $f \circ \varphi$ and $\varphi \circ f$ (note $\alpha\beta \neq 0$),

$$\begin{aligned} f \circ \varphi(X, Y, Z) &= [\beta Y(\gamma X + \delta Y + \epsilon Z), \ldots], \\ \varphi \circ f(X, Y, Z) &= [\alpha YZ, \ldots], \end{aligned}$$

we see that $\gamma = \delta = 0$, i.e., the map φ is diagonal. Without loss of generality, we write $\varphi \in \mathrm{PGL}_3$ as $\varphi(X, Y, Z) = [X, vY, wZ]$, and then

$$f^\varphi = [v^2w^2YZ, wX^2, v^3Z^2].$$

Hence $f^\varphi = f$ if and only if $v^2w^2 = w = v^3$. Substituting $w = v^3$ into $v^2w^2 = v^3$ gives $v^8 = v^3$, so $v^5 = 1$. Therefore v is a 5'th root of unity and $w = v^3$, so $\varphi \in \langle \tau_3 \rangle$.

(b) We see by inspection that $\mathrm{Aut}(f) \supseteq \langle \tau_3 \rangle \cong C_7$, but it turns out that $\mathrm{Aut}(f)$ is strictly larger than this. Precisely, if we let $\pi(X, Y, Z) = [Z, X, Y]$, then it is easy to check that $f^\pi = f$, so $\pi \in \mathrm{Aut}(f)$. Also, we compute $\pi^{-1}\tau_3\pi = \tau_3^2$, so $\mathrm{Aut}(f)$ contains the semi-direct product $\langle \tau_3 \rangle \rtimes \langle \pi \rangle \cong C_7 \rtimes C_3$. We claim that this is the full automorphism group of f.

The critical locus of f is

$$\mathrm{Crit}(f) = \{8XYZ = 0\},$$

so $\operatorname{Crit}(f)$ consists of the three lines $XYZ = 0$. Any $\sigma \in \operatorname{Aut}(f)$ must permute these lines and their intersection points. The map π is a cyclic permutation of the intersection points, so replacing σ by $\pi^{\pm 1}\sigma$ if necessary, we may assume that σ fixes $[1,0,0]$ and either fixes or swaps $[0,1,0]$ and $[0,0,1]$. If σ fixes all three points, then σ is a diagonal map, say $\sigma(X,Y,Z) = [X, vY, wZ]$, and we have

$$f^\sigma = [vw^3Z^2, wX^2, v^3Y^2].$$

Hence $f^\sigma = f$ if and only if $vw^3 = w = v^3$. Substituting $w = v^3$ into $vw^3 = v^3$ gives $v^{10} = v^3$, so $v \in \boldsymbol{\mu}_7$ and $w = v^3$. Hence $\sigma \in \langle \tau_3 \rangle$.

It remains to deal with the case that σ fixes $[1,0,0]$ and permutes $[0,1,0]$ and $[0,0,1]$. But then we would have $f^\sigma(1,0,0) = [0,0,1]$, while $f(1,0,0) = [0,1,0]$, so f^σ cannot equal f. (More precisely, the maps f and f^σ are inverses in their action on the three points.) This completes the proof that $\operatorname{Aut}(f)$ is generated by τ_3 and π.

(c) It is trivial to check that the indicated copy of $\mathbb{G}_m$ is contained in $\operatorname{Aut}(f)$. A more detailed analysis, which we leave to the interested reader, can be used to show that $\operatorname{Aut}(f) \cong \mathbb{G}_m$. $\square$

7. Maps with Automorphism Group Containing $C_p \times C_p$ with $p \geqslant 3$

Our goal in this section is essentially a non-existence result. Somewhat surprisingly, the case $p = 3$ will be crucial to our analysis of maps whose automorphism group contains a copy of C_4. We also note that the proposition is wildly incorrect for $p = 2$, and indeed we devote a long section (Section 8) to classifying maps whose automorphism group contains a copy of C_2^2.

Proposition 7.1. — *Let K be an algebraically closed field of characteristic 0. Let $p \geqslant 3$ be prime, and let $f \in \operatorname{Rat}_2^2(K)$ have the property that $\operatorname{Aut}(f)$ contains a copy of C_p^2. Then either f is a linear map or else f is not dominant.*

Proof. — Let $G \subset \operatorname{PGL}_3(K)$ be a subgroup of type C_p^2 that is contained in $\operatorname{Aut}(f)$. Lemma 4.1 tells us that after an appropriate conjugation, we may assume that

$$G = \langle \alpha, \beta \rangle \quad \text{with} \quad \alpha = \begin{pmatrix} 1&0&0\\0&\zeta&0\\0&0&1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&\zeta \end{pmatrix},$$

where ζ is a primitive p'th root of unity. The following table describes the action of α and β on quadratic monomials that might appear in f. An entry (i,j) in the table means that α multiplies the monomial by ζ^i and that β multiplies the monomial by ζ^j.

	X^2	Y^2	Z^2	XY	XZ	YZ
X-coordinate	$(0,0)$	$(2,0)$	$(0,2)$	$(1,0)$	$(0,1)$	$(1,1)$
Y-coordinate	$(-1,0)$	$(1,0)$	$(-1,2)$	$(0,0)$	$(-1,1)$	$(0,1)$
Z-coordinate	$(0,-1)$	$(2,-1)$	$(0,1)$	$(1,-1)$	$(0,0)$	$(1,0)$

There are a number of possible families of maps invariant for G, indexed by the pairs (i,j) modulo p. The most interesting case is $p=3$, so $-1\equiv 2$, in which case there are 9 families of maps as given in the following table:

(i,j)	0	1	2
0	$[aX^2,bXY,cXZ]$	$[aXY,bY^2,cYZ]$	$[aY^2,bX^2,0]$
1	$[aXZ,bYZ,cZ^2]$	$[aYZ,0,0]$	$[0,aXZ,0]$
2	$[aZ^2,0,bX^2]$	$[0,0,aXY]$	$[0,aZ^2,bY^2]$

Three of these families coincide with the linear map $[aX,bY,cZ]$, while the other six families clearly give non-dominant maps. And if $p\geqslant 5$, then we obtain the same three linear maps, plus nine additional maps defined by a single monomial.

According to Lemma 4.1, it remains to deal with the case that $p=3$ and, after appropriate conjugation,

$$G=\langle\tau_2,\pi\rangle\quad\text{with}\quad \tau_2=\left(\begin{smallmatrix}1&0&0\\0&\zeta&0\\0&0&\zeta^2\end{smallmatrix}\right)\quad\text{and}\quad \pi=\left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right),$$

where ζ is a primitive cube root of unity and τ_2 is as in Section 5. Suppose that $f\in\mathrm{Rat}_2^2$ with $\tau_2\in\mathrm{Aut}(f)$. Using Table 5.1 with $m=2$ and entries reduced modulo 3, we find that f has one of the following forms:

$$\begin{aligned}F &:= [aX^2+bYZ, cZ^2+dXY, eY^2+gXZ],\\ G &:= [aZ^2+bXY, cY^2+dXZ, eX^2+gYZ],\\ H &:= [aY^2+bXZ, cX^2+dXZ, eZ^2+gXY].\end{aligned}$$

Conjugating by the cyclic permutation $\pi(X,Y,Z)=[Y,Z,X]$ yields

$$\begin{aligned}F^\pi &= [eZ^2+gXY, aY^2+bXZ, cX^2+dYZ],\\ G^\pi &= [eY^2+gXZ, aX^2+bYZ, cZ^2+dXY],\\ H^\pi &= [eX^2+gYZ, aZ^2+bXY, cY^2+dXZ].\end{aligned}$$

Since F and F^π have no non-zero monomials in common, it follows that $\pi\notin\mathrm{Aut}(F)$, and similarly for G and H. This completes the proof that there are no maps $f\in\mathrm{Rat}_2^2$ with $G\subseteq\mathrm{Aut}(f)$. $\square$

8. Maps with Automorphism Group Containing $C_2 \times C_2$

In this section we classify maps with $\mathcal{G}_{2,2} \subseteq \operatorname{Aut}(f)$.

Proposition 8.1. — *Let K be an algebraically closed field of characteristic 0, let $\mathcal{G}_{2,2}$ be the group described in Theorem 1.2, and let $f \in \operatorname{Rat}_2^2(\mathcal{G}_{2,2})^{\mathrm{ss}}$ be a dominant map of degree 2. Then f is $N(\mathcal{G}_{2,2})$-stable and one of the following holds*:

(a) *f is $N(\mathcal{G}_{2,2})$-conjugate to a map of the form*

$$[X^2+Y^2-Z^2, dXY, eXZ] \quad \text{with } d,e \in K^*.$$

The $N(\mathcal{G}_{2,2})$-conjugacy class of the map f is uniquely determined by the unordered pair $\{d,e\}$. The automorphism group of f is

$$\operatorname{Aut}(f) = \begin{cases} \mathcal{G}_{2,2} & \text{if } d \neq e, \\ \mathbb{G}_m \rtimes C_2 & \text{if } d = e. \end{cases}$$

Further, $\deg(f^n) = 2^n$.

(b) *f is $N(\mathcal{G}_{2,2})$-conjugate to a map of the form*

$$[Y^2 - Z^2, XY, eXZ] \quad \text{with } e \in K^*.$$

The $N(\mathcal{G}_{2,2})$-conjugacy class of the map f is uniquely determined by the unordered pair $\{e, e^{-1}\}$. The automorphism group of f is

$$\operatorname{Aut}(f) = \begin{cases} \mathcal{G}_{2,2} & \text{if } e \neq \pm 1, \\ \mathcal{S}_3\mathcal{G}_{2,2} & \text{if } e = -1, \\ \mathbb{G}_m \rtimes C_2 & \text{if } e = 1. \end{cases}$$

Further, if e^2 is not an odd-order root of unity, then $\deg f^n = n+1$, *while if $e^{4k+2} = 1$, then $f^{4k+2} = [X,Y,Z]$.*

(c) *f is $N(\mathcal{G}_{2,2})$-conjugate to the map $[YZ, XZ, XY]$. The automorphism group of f is $\operatorname{Aut}(f) = \mathcal{S}_3\mathcal{G}_{2,2} \cong \mathcal{S}_4$. Further, $f^2 = [X,Y,Z]$.*

Proof. — Let

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

so $\mathcal{G}_{2,2} = \langle \alpha, \beta \rangle \cong C_2^2$ is the group that we assume is contained in $\operatorname{Aut}(f)$. The following table describes the action of α and β on quadratic monomials that might appear in f. An entry (i,j) in the table means that α multiplies the monomial by $(-1)^i$ and that β multiplies the monomial by $(-1)^j$.

	X^2	Y^2	Z^2	XY	XZ	YZ
X-coordinate	$(0,0)$	$(0,0)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
Y-coordinate	$(1,0)$	$(1,0)$	$(1,0)$	$(0,0)$	$(1,1)$	$(0,1)$
Z-coordinate	$(0,1)$	$(0,1)$	$(0,1)$	$(1,1)$	$(0,0)$	$(1,0)$

There are thus four families of $\mathcal{G}_{2,2}$-invariant maps, indexed by pairs (i, j) modulo 2 (or equivalently, by characters $\mathcal{G}_{2,2} \to \mathbb{C}^*$),

$$\begin{aligned}
f_{0,0} &= [aX^2 + bY^2 + cZ^2, dXY, eXZ],\\
f_{1,0} &= [aXY, bX^2 + cY^2 + dZ^2, eYZ],\\
f_{0,1} &= [aXZ, bYZ, cX^2 + dY^2 + eZ^2],\\
f_{1,1} &= [aYZ, bXZ, cXY].
\end{aligned}$$

Lemma 4.1 (b) tells us that the normalizer $N(\mathcal{G}_{2,2})$ of $\mathcal{G}_{2,2}$ contains all of the permutation matrices. In particular, the permutation $\pi(X, Y, Z) = [Y, Z, X]$ is in $N(\mathcal{G}_{2,2})$, and applying π and π^2 to $f_{0,0}$ yields

$$\begin{aligned}
f_{0,0}^{\pi} &= [eXY, cX^2 + aY^2 + bZ^2, dYZ],\\
f_{0,0}^{\pi^2} &= [dXZ, eYZ, bX^2 + cY^2 + aZ^2].
\end{aligned}$$

Hence the families $f_{1,0}$ and $f_{0,1}$ are $N(\mathcal{G}_{2,2})$-conjugate to the family $f_{0,0}$.

We next observe that a map in the $f_{1,1}$ family is dominant if and only if $abc \neq 0$. And under this assumption, conjugation by $[uX, vY, wZ] \in \mathcal{D} \subset N(\mathcal{G}_{2,2})$ with $u^4 = a^{-1}bc$, $v^4 = ab^{-1}c$, and $w^4 = abc^{-1}$ transforms $f_{1,1}$ into the map $[YZ, XZ, XY]$, i.e., the family $f_{1,1}$ with $abc \neq 0$ consists of a single $N(\mathcal{G}_{2,2})$-conjugacy class. This gives the map in (c), which by abuse of notation we continue to denote by $f_{1,1}$. The critical locus of $f_{1,1}$ is the union of the coordinate axes,

$$\mathrm{Crit}(f_{1,1}) = \{X = 0\} \cup \{Y = 0\} \cup \{Z = 0\}.$$

Any $\varphi \in \mathrm{Aut}(f_{1,1})$ thus leaves the union of the coordinate axes invariant, from which we conclude that φ has the form

$$\varphi = \pi \circ \sigma \quad \text{for some } \pi \in \mathcal{S}_3 \text{ and some } \sigma = \left(\begin{smallmatrix} 1 & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \gamma \end{smallmatrix}\right) \in \mathcal{D}.$$

But one easily checks that $\mathcal{S}_3 \subset \mathrm{Aut}(f_{1,1})$, so it suffices to determine which diagonal matrices $\sigma \in \mathcal{D}$ are in $\mathrm{Aut}(f_{1,1})$. Letting $\sigma = [X, \beta Y, \gamma Z]$, we find that

$$f_{1,1}^{\sigma} = [\beta\gamma YZ, \beta^{-1}\gamma XZ, \beta\gamma^{-1}XY],$$

so

$$\sigma \in \mathrm{Aut}(g) \iff \beta\gamma = \beta^{-1}\gamma = \beta\gamma^{-1} \iff \beta^2 = \gamma^2 = 1 \iff \sigma \in \mathcal{G}_{2,2}.$$

This completes the proof that

$$\mathrm{Aut}(f_{1,1}) = \mathcal{S}_3\mathcal{G}_{2,2} \cong S_4,$$

which also completes the proof of (c).

We now concentrate on maps in the family $f_{0,0}$, and to ease notation, we drop the subscript and simply write

$$f = [aX^2 + bY^2 + cZ^2, dXY, eXZ].$$

It is clear that f is dominant if and only if $de \neq 0$ and at least one of a, b, c is non-zero. Further, if $b = c = 0$, then $f = [aX, dY, eZ]$ has degree 1, so we may assume that one of b and c is non-zero. And since the involution $\sigma(X, Y, Z) = [X, Z, Y] \in \mathcal{S}_3 \subset N(\mathcal{G}_{2,2})$ has the effect

$$f^\sigma = [aX^2 + cY^2 + bZ^2, eXY, dXZ]$$

of switching the roles of b and c, after another $N(\mathcal{G}_{2,2})$ conjugacy we may assume that $cde \neq 0$. Using this assumption and Table 5.2, the action of the one-parameter subgroup $L_{k,\ell}(t)$ on f is

$$\mu^{\mathcal{O}(1)}(f, L_{k,\ell}) = \begin{cases} \max\{-k, k - 2\ell, 3k + 2\ell\} & \text{if } b \neq 0, \\ \max\{-k, 3k + 2\ell\} & \text{if } b = 0. \end{cases}$$

Thus if $b = 0$, then $\mu^{\mathcal{O}(1)}(f, L_{1,-2}) = -1$, so f is $\mathcal{D}$-unstable.

On the other hand, if $b \neq 0$, then the identity

$$4(-k) + (k - 2\ell) + (3k + 2\ell) = 0$$

shows that $\mu^{\mathcal{O}(1)}(f, L_{k,\ell}) > 0$ for all $(k, \ell) \neq (0, 0)$, so f is $\mathcal{D}$-stable.

We are reduced to studying f with $bcde \neq 0$. We conjugate by a diagonal map $\delta = [uX, vY, wZ] \in \mathcal{D} \subset N(\mathcal{G}_{2,2})$ to obtain

$$f^\delta = [u^2aX^2 + v^2bY^2 + w^2cZ^2, u^2dXY, u^2eXZ].$$

Thus taking $v^2 = b^{-1}$ and $w^2 = -c^{-1}$, we may assume that $b = 1$ and $c = 1$. Further, if $a \neq 0$, then we may take $u^2 = a^{-1}$ to reduce to maps with $a = 1$, while if $a = 0$, then we may take $u^2 = d^{-1}$ to reduce to maps satisfying $d = 1$. It thus suffices to anaylze the maps in the following two families:

$$\begin{aligned} &f = [X^2 + Y^2 - Z^2, dXY, eXZ] \quad \text{with } d, e \in K^*, \\ &f = [Y^2 - Z^2, XY, eXZ] \quad \text{with } e \in K^*. \end{aligned}$$

$\boldsymbol{f = [X^2 + Y^2 - Z^2, dXY, eXZ],\ de \neq 0.}$ The indeterminacy and critical loci of f are

$$I(f) = \bigl\{[0, 1, \pm 1]\bigr\} \quad \text{and} \quad \operatorname{Crit}(f) = \{X = 0\} \cup \{X^2 - Y^2 + Z^2 = 0\}.$$

We consider two maps $f = [X^2 + Y^2 - Z^2, dXY, eXZ]$ and $f' = [X^2 + Y^2 - Z^2, d'XY, e'XZ]$ and compute

$$\operatorname{Hom}(f, f') := \bigl\{\varphi \in \mathrm{PGL}_3(K) : f^\varphi = f'\bigr\}.$$

Note that by taking $f' = f$, we will obtain $\operatorname{Aut}(f)$.

Every $\varphi \in \mathrm{Hom}(f, f')$ stabilizes the line $\{X = 0\}$ and either fixes or permutes the two point $[0, 1, \pm 1]$. Thus φ has the form

$$\varphi = \left(\begin{smallmatrix} 1 & 0 & 0 \\ \gamma & \alpha & \beta \\ \delta & \pm\beta & \pm\alpha \end{smallmatrix}\right) \quad \text{with } \alpha^2 \neq \beta^2,$$

where choosing the plus sign fixes $[0, 1, \pm 1]$ and choosing the minus sign swaps them.

We compare the second and third coordinates of $f \circ \varphi$ and $\varphi \circ f'$,

$$\begin{aligned} f \circ \varphi &= [*,\ \gamma d X^2 + \alpha d XY + \beta d XZ, \delta e X^2 \pm \beta e XY \pm \alpha e XZ], \\ \varphi \circ f' &= [*,\ \gamma X^2 + \gamma Y^2 - \gamma Z^2 + \alpha d' XY + \beta e' XZ, \\ &\qquad\qquad \delta X^2 + \delta Y^2 - \delta Z^2 \pm \beta d' XY \pm \alpha e' XZ]. \end{aligned}$$

Since the second and third coordinates of $f \circ \varphi$ have no Y^2 term, we conclude that $\gamma = \delta = 0$. Under this assumption, we find that

$$\begin{aligned} f \circ \varphi &= [X^2 + (\alpha^2 - \beta^2)(Y^2 - Z^2), X(\alpha d Y + \beta d Z), \pm X(\beta e Y + \alpha e Z)], \\ \varphi \circ f' &= [X^2 + Y^2 - Z^2, X(\alpha d' Y + \beta e' Z), \pm X(\beta d' Y + \alpha e' Z)]. \end{aligned}$$

Hence $\varphi \in \mathrm{Hom}(f, f')$ if and only if

$$\alpha^2 - \beta^2 = 1 \quad \text{and} \quad \alpha(d - d') = \beta(d - e') = \beta(e - d') = \alpha(e - e') = 0.$$

This leads to three cases.

$\boldsymbol{\alpha\beta \neq 0}$. Then we must have $d = e = d' = e'$, i.e., $f' = f$ and $d = e$. The automorphism group of these maps is the set of all matrices of the form $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \pm\beta & \pm\alpha \end{smallmatrix}\right)$ satisfying $\alpha^2 - \beta^2 = 1$, which is easily seen to be isomorphic to $\mathbb{G}_m \rtimes C_2$.

$\boldsymbol{\beta = 0,\ \alpha = \pm 1}$. Then $d' = d$ and $e' = e$, i.e., $f' = f$, and we obtain exactly four possible maps φ, namely the four maps in $\mathcal{G}_{2,2}$ that we already know are in $\mathrm{Aut}(f)$.

$\boldsymbol{\alpha = 0,\ \beta = \pm 1}$. Then $d' = e$ and $e' = d$, so the maps $[X^2 + Y^2 - Z^2, dXY, eXZ]$ and $[X^2 + Y^2 - Z^2, eXY, dXZ]$ are $N(\mathcal{G}_{2,2})$-conjugate via the permutation $[X, Y, Z] \to [X, Z, Y]$.

It remains to prove that $\deg(f^n) = 2^n$, a task that we postpone to Section 12, where we study the degree sequences of all of the maps in this paper. This completes our analysis for maps of the form $f = [X^2 + Y^2 - Z^2, dXY, eXZ]$.

$\boldsymbol{f = [Y^2 - Z^2, XY, eXZ],\ e \neq 0.}$ The iterates of f are explicitly described later in Proposition 8.2, which in particular gives the stated results for $\deg(f^n)$. To make our computation of $\operatorname{Aut}(f)$ easier, we instead work with a PGL_3-conjugate of f. Thus we let

$$\lambda(X,Y,Z) = [2X, Y-Z, Y+Z]$$

and define

$$g(X,Y,Z) := f^\lambda(X,Y,Z) = \bigl[YZ, X(AY+BZ), X(BY+AZ)\bigr]$$

with

$$A = \frac{1+e}{2} \quad\text{and}\quad B = \frac{1-e}{2}.$$

We note that $A^2 - B^2 = e$. The advantage of g over f is the fact that the critical locus of g is the union of the coordinate axes,

$$\operatorname{Crit}(g) = \{X=0\} \cup \{Y=0\} \cup \{Z=0\}.$$

As usual, we let g' be another map of this form, with e' in place of e, and we let $\varphi \in \operatorname{Hom}(g,g')$. Then φ leaves the union of the coordinate axes invariant, from which we conclude that φ has the form

$$\varphi = \pi \circ \sigma \quad \text{for some } \pi \in \mathcal{S}_3 \text{ and some } \sigma = \left(\begin{smallmatrix} 1&0&0\\ 0&\beta&0\\ 0&0&\gamma\end{smallmatrix}\right) \in \mathcal{D}.$$

For each of the six elements of $\mathcal{S}_3$ we need to compute the effect of $\pi\circ\sigma$ on g. We let

$$S(X,Y,Z) = [X,Z,Y] \quad\text{and}\quad T(X,Y,Z) = [Y,Z,X]$$

be generators for $\mathcal{S}_3$. Our task is simplified by the observation that $S \in \operatorname{Aut}(g)$ and $S \in \operatorname{Aut}(g')$, so it suffices to take $\pi \in \{I, T, T^2\}$. For each of these choices we compute the action on g,

$$\begin{aligned}
g^\sigma &= \left[YZ, X\left(\frac{1}{\beta\gamma}AY + \frac{1}{\beta^2}BZ\right), X\left(\frac{1}{\gamma^2}BY + \frac{1}{\beta\gamma}AZ\right)\right],\\
g^{T\sigma} &= [AXY + \gamma BYZ, *, *],\\
g^{T^2\sigma} &= [AXZ + \beta BYZ, *, *].
\end{aligned}$$

We consider three cases:

$\boldsymbol{e \neq \pm 1,\ AB \neq 0.}$ In this case the fact that g' has no XY or XZ in its first coordinate rules out $\pi = T$ or $\pi = T^2$.

On the other hand, for $\pi = I$ we have

$$\sigma \in \operatorname{Hom}(g,g') \iff \beta\gamma = A/A' \quad\text{and}\quad \beta^2 = \gamma^2 = B/B'.$$

In particular, this can occur only if

$$\begin{aligned} 0 = (\beta\gamma)^2 - \beta^2\gamma^2 &= \left(\frac{A}{A'}\right)^2 - \left(\frac{B}{B'}\right)^2 = \frac{(AB')^2 - (A'B)^2}{(A'B')^2} \\ &= \frac{1}{(A'B')^2}\left(\left(\frac{1+e}{2}\cdot\frac{1-e'}{2}\right)^2 - \left(\frac{1+e'}{2}\cdot\frac{1-e}{2}\right)^2\right) \\ &= \frac{(e-e')(1-ee')}{(A'B')^2}. \end{aligned}$$

If $e' = e$, i.e., if $g' = g$, then we find that $\sigma \in \mathrm{Aut}(g)$ if and only if $\beta\gamma = \beta^2 = \gamma^2 = 1$, so if and only if $\beta = \gamma = \pm 1$. This gives two elements of $\mathrm{Aut}(g)$, and composing with S gives two additional elements. These elements form the copy of C_2^2 that we already know exists in $\mathrm{Aut}(g)$. Further, if $e' = e^{-1}$, then $A = eA'$ and $B = -eB'$, so we find that $\sigma \in \mathrm{Hom}(g, g')$ if we take $\beta = \sqrt{-e}$ and $\gamma = -\beta$.

To recapitulate, we have shown that if $e \neq \pm 1$, then

$$\mathrm{Aut}(g) = \left\langle \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{smallmatrix}\right) \right\rangle \cong C_2^2,$$

and that g' is $\mathrm{PGL}_3(K)$-conjugate to g if and only if $e' \in \{e, e^{-1}\}$. Undoing the conjugation by λ, we find that if $e \neq \pm 1$, then $\mathrm{Aut}(f) = \mathcal{G}_{2,2}$, and that f is $N(\mathcal{G}_{2,2})$-conjugate to f' if and only if $e' \in \{e, e^{-1}\}$.

$\boldsymbol{e = 1,\ A = 1,\ B = 0.}$ In this case the map g is simply

$$g = [YZ, XY, XZ].$$

It satisfies $g^2 = [X, Y, Z]$, and $\mathrm{Aut}(g)$ contains a copy of $\mathbb{G}_m$ in the form of all maps $[X, tY, t^{-1}Z]$, and it contains S, so $\mathbb{G}_m \rtimes C_2 \subseteq \mathrm{Aut}(g)$. Since it is not needed for the proof of our main theorem, we leave for the reader the proof that this inclusion is an equality.

$\boldsymbol{e = -1,\ A = 0,\ B = 1.}$ In this case the map g has the simple form

$$g = [YZ, XZ, XY],$$

and we observe that g is the map that we already analyzed in (c). In particular, $\mathrm{Aut}(g) = \mathcal{S}_3\mathcal{G}_{2,2} \cong S_4$. This proves that $\mathrm{Aut}(f) \cong S_4$, in fact one can easily check that λ normalizes the group $\mathcal{S}_3\mathcal{G}_{2,2}$, so $\mathrm{Aut}(f) = \mathcal{S}_3\mathcal{G}_{2,2}$. □

We next give an explicit formula for the iterates of the family of maps in Proposition 8.1 (b).

Proposition 8.2. — *Let $e \in K^*$, and for $k \geqslant 0$, let*

$$U_k(Y, Z) = Y^2 - e^{2k} Z^2.$$

Then the iterates of the map $f = [Y^2 - Z^2, XY, eXZ]$ *are given by the formulas*

$$f^n(X,Y,Z) = \begin{cases} [XU_1U_3\cdots U_{n-1}, YU_0U_2\cdots U_{n-2}, e^nZU_0U_2\cdots U_{n-2}] \\ \qquad\qquad\qquad\qquad \text{if } n \text{ is even,} \\ [U_0U_2\cdots U_{n-1}, XYU_1U_3\cdots U_{n-2}, e^nXZU_1U_3\cdots U_{n-2}] \\ \qquad\qquad\qquad\qquad \text{if } n \text{ is odd.} \end{cases}$$

(1) *If* $e^{2k} \neq 1$ *for all odd integers* k, *then*

$$\deg(f^n) = n+1 \quad \text{for all } n \geqslant 0.$$

(2) *If* $e^{2k} = 1$ *for some odd integer* k, *then*

$$f^{2k}(X,Y,Z) = [X,Y,Z].$$

Proof. — We note that $f = \big[U_0(Y,Z), XY, eXZ\big]$. The proof of the formulas for f^n is an easy induction on n, using the identity

$$U_k(WY, eWZ) = W^2 U_{k+1}(Y,Z). \tag{8.1}$$

This allows us to compute

$$\begin{aligned} f^{2k+2}(X,Y,Z) &= f^{2k+1}\big(U_0(Y,Z), XY, eXZ\big) \quad \text{definition of } f, \\ &= \big[(U_0U_2\cdots U_{2k})(XY,eXZ), \\ &\qquad U_0(Y,Z)\cdot XY\cdot (U_1U_3\cdots U_{2k-1})(XY,eXZ), \\ &\qquad e^{2k+1}\cdot U_0(Y,Z)\cdot eXZ\cdot (U_1U_3\cdots U_{2k-1})(XY,eXZ)\big] \\ &\qquad\qquad \text{induction hypothesis,} \\ &= \big[X^{2k+2}(U_1U_3\cdots U_{2k+1})(Y,Z), \\ &\qquad U_0(Y,Z)\cdot X^{2k+1}\cdot Y(U_2U_4\cdots U_{2k})(Y,Z), \\ &\qquad e^{2k+2}\cdot U_0(Y,Z)\cdot X^{2k+1}\cdot Z(U_2U_4\cdots U_{2k})(Y,Z)\big] \\ &\qquad\qquad \text{using (8.1).} \end{aligned}$$

Canceling X^{2k+1} gives the desired formula. The computation of f^{2k+1} using the formula for f^{2k} is similar. This completes the proof of the formulas for f^n.

Since $\deg U_k(Y,Z) = 2$, we see immediately from the formulas for f^n that $\deg(f^n) \leqslant n+1$, with equality if and only if the coordinate functions have no common factor. Since $e \neq 0$, we see that a common factor occurs if and only if some odd index $U_{2\ell+1}(Y,Z)$ has a factor in common with some even index $U_{2m}(Y,Z)$. But

$$\begin{aligned}\mathrm{Res}\big(U_{2\ell+1}(Y,Z), U_{2m}(Y,Z)\big) &= \mathrm{Res}(Y^2 - e^{4\ell+2}Z^2, Y^2 - e^{4m}Z^2)\\ &= e^{8m}(e^{2(2\ell-2m+1)} - 1)^2.\end{aligned}$$

Hence if e^2 is not an odd-order root of unity, then there is no cancelation and $\deg(f^n) = n+1$. Finally, if e^2 is an odd order root of unity, say $e^{4\ell+2} = 1$, then for all $k \geqslant 0$ we have

$$U_{k+2\ell+1}(Y,Z) = Y^2 - e^{2(k+2\ell+1)}Z^2 = Y^2 - e^{2k}Z^2 = U_k(Y,Z).$$

This allows us to switch even index U_k's with odd index U_k's. In particular, using this identity in the formula for $f^{4\ell+2}$, we find that all of the U_k factors cancel, leaving $f^{4\ell+2} = [X,Y,Z]$. $\square$

9. Maps with an Automorphism of Order 4

In this section we classify maps in Rat_2^2 that admit an automorphism of order 4.

Proposition 9.1. — *Let K be an algebraically closed field of characteristic 0, let $\mathcal{G}_4$ be the group described in Theorem 1.2, and let $f \in \mathrm{Rat}_2^2(\mathcal{G}_4)^{\mathrm{ss}}$ be a dominant map of degree 2 with finite automorphism group. Then f is $N(\mathcal{G}_4)$-stable and one of the following holds*:

(a) *f is $N(\mathcal{G}_4)$-conjugate to a map of the form*

$$f_{a,e} := [aX^2 + Z^2, XY, Y^2 + eXZ] \quad \text{with } a, e \in K.$$

The automorphism group of $f_{a,e}$ is given by

$$\mathrm{Aut}(f_{a,e}) = \mathcal{G}_4.$$

Two maps $f_{a,e}$ and $f_{a',e'}$ are $N(\mathcal{G}_4)$-conjugate if and only $(a,e) = (a',e')$.

(b) *f is $N(\mathcal{G}_4)$-conjugate to a map of the form*[5]

$$f_c := [YZ, X^2 + cZ^2, XY] \quad \text{with } c \in K \smallsetminus \{-1\}.$$

The automorphism group of f_c is given by[6]

$$\begin{aligned}\mathrm{Aut}(f_c) &= \mathcal{G}_4 \quad \text{if } c \neq \pm 1,\\ \mathrm{Aut}(f_c) &\cong S_4 \quad \text{if } c = 1.\end{aligned}$$

Two maps f_c and $f_{c'}$ are $N(\mathcal{G}_4)$-conjugate if and only if $cc' = 1$.

[5] See Proposition 9.3 for explicit formulas for the iterates of f_c and a detailed description of its geometry. In particular, although $\mathrm{Aut}(f)$ is finite for $c \neq -1$, it turns out that $\mathrm{Aut}(f_c^2)$ always contains a copy of $\mathbb{G}_m$.

[6] In the excluded case $c = -1$, we have $\mathrm{Aut}(f_{-1}) \supseteq \mathbb{G}_m \rtimes C_2$.

Proposition 9.2. — *Let K be an algebraically closed field of characteristic* 0*, and let $f \in \mathrm{Rat}_2^2$ satisfy*

$$\mathrm{Aut}(f) \supseteq \left\langle \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \right\rangle .$$

Then f is $\mathcal{D}$-unstable.

Proof of Proposition 9.1. — Let $\zeta = i$ to be a primitive 4'th root of unity, so the map $[X, iY, -Z]$ corresponds to the matrix τ_2 defined in Section 5. Table 5.1 with $m = 2$ and entries reduced modulo 4 is

	X^2	Y^2	Z^2	XY	XZ	YZ
X-coord	0	2	0	1	2	3
Y-coord	3	1	3	0	1	2
Z-coord	2	0	2	3	0	1

Hence the assumption that $\tau_2 \in \mathrm{Aut}(f)$ leads to the following four families of maps:

$$\begin{aligned} f_{2,0} &:= [aX^2 + bZ^2, cXY, dY^2 + eXZ], \\ f_{2,1} &:= [aXY, bY^2 + cXZ, dYZ], \\ f_{2,2} &:= [aY^2 + bXZ, cYZ, dX^2 + eZ^2], \\ f_{2,3} &:= [aYZ, bX^2 + cZ^2, dXY]. \end{aligned}$$

Conjugation by the permutation $\pi(X, Y, Z) = [Z, Y, X] \in N(\mathcal{G}_4)$ gives

$$f_{2,0}^{\pi} = [aX^2 + bZ^2, cXY, dY^2 + eXZ]^{\pi} = [dY^2 + eXZ, cYZ, bX^2 + aZ^2],$$

so π identifies the families $f_{2,0}$ and $f_{2,2}$. (One also easily checks that π stabilizes each of the families $f_{2,1}$ and $f_{2,3}$.) Further, the family $f_{2,1}$ has infinite automorphism group,

$$\mathrm{Aut}(f_{2,1}) \supseteq \left\{ \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^2 \end{smallmatrix}\right) : t \in \mathbb{G}_m \right\} \cong \mathbb{G}_m.$$

It remains to consider the families $f_{2,0}$ and $f_{2,3}$.

In order for $f_{2,0}$ to be dominant, we need $c \neq 0$. Table 5.2 tells us that

$$\mu^{\mathcal{O}(1)}(f_{2,0}, L_{k,\ell}) \leqslant \max\{-k, 3k + 2\ell, -k - 3\ell\}, \tag{9.1}$$

with equality if $bd \neq 0$. Since

$$\begin{aligned} b = 0 &\Longrightarrow \mu^{\mathcal{O}(1)}(f_{2,0}, L_{k,\ell}) \leqslant \max\{-k, -k - 3\ell\} \xrightarrow{(k,\ell)=(1,0)} -1, \\ d = 0 &\Longrightarrow \mu^{\mathcal{O}(1)}(f_{2,0}, L_{k,\ell}) \leqslant \max\{-k, 3k + 2\ell\} \xrightarrow{(k,\ell)=(1,-2)} -1, \end{aligned}$$

our assumption that f is $N(\mathcal{G}_4)$-semistable tells us that we also have $bd \neq 0$. We may thus conjugate $f_{2,0}$ by

$$\sigma = [uX, vY, wZ] \text{ with } (u^6, v^{12}, w^3) = (bc^{-3}, b^{-1}c^3d^{-6}, b^{-1}),$$

which with appropriate choice of roots puts $f_{2,0}$ into the form $f_{2,0} = [aX^2 + Z^2, XY, Y^2 + eXZ]$. And since (9.1) is an equality, the identity

$$14 \cdot (-k) + 6 \cdot (3k + 2\ell) + 4 \cdot (-k - 3\ell) = 0$$

shows that $\mu^{\mathcal{O}(1)}(f_{2,0}, L_{k,\ell}) > 0$ for all $(k,\ell) \neq (0,0)$. This completes the proof that $f_{2,0}$ of this form are $N(\mathcal{G}_4)$-stable.

In order for $f_{2,3}$ to be dominant, we need $ad \neq 0$ and at least one of b, c non-zero. Since $f_{2,3}^{\pi} = [dYZ, cX^2 + bY^2, aXY]$ has the effect of switching b and c (as well as switching a and d), we may assume without loss of generality that $b \neq 0$. Then conjugation by $[uX, vY, wZ]$ with $(u^8, v^8, w^8) = (ab^{-2}c^{-1}, a^{-1}b^2d^{-3}, a^{-3}b^{-2}d^3)$ puts $f_{2,3}$ in the form $f_{2,3} = [YZ, X^2 + cZ^2, XY]$. Table 5.2 tells us that

$$\mu^{\mathcal{O}(1)}(f_{2,3}, L_{k,\ell}) = \begin{cases} \max\{2k, -2k+\ell, 2k+3\ell, -2k-2\ell\} & \text{if } c \neq 0, \\ \max\{2k, -2k+\ell, -2k-2\ell\} & \text{if } c = 0. \end{cases}$$

In both cases the identity

$$3(2k) + 2(-2k + \ell) + (-2k - 2\ell) = 0$$

shows that $\mu^{\mathcal{O}(1)}(f_{2,3}, L_{k,\ell}) > 0$ for all $(k,\ell) \neq (0,0)$. This completes the proof that $f_{2,3}$ of this form are $N(\mathcal{G}_4)$-stable.

Computation of $\operatorname{Aut}(f_{2,0})$ for $f_{2,0} = [aX^2 + Z^2, XY, Y^2 + eXZ]$

To ease notation, we are going to drop the subscript on f. We consider two maps

$$f = [aX^2 + Z^2, XY, Y^2 + eXZ] \quad \text{and} \quad f' = [a'X^2 + Z^2, XY, Y^2 + e'XZ]$$

and compute

$$\operatorname{Hom}(f, f') := \bigl\{\varphi \in \operatorname{PGL}_3(K) : f^{\varphi} = f'\bigr\}.$$

The critical locus of f is

$$\operatorname{Crit}(f) = \{aeX^3 - eXZ^2 + 2Y^2Z = 0\},$$

and the indeterminacy locus of f is

$$I(f) = \begin{cases} \emptyset & \text{if } ae \neq 0, \\ \bigl\{[1,0,0]\bigr\} & \text{if } a = 0, \\ \bigl\{[1,0,\pm\sqrt{-a}]\bigr\} & \text{if } e = 0 \text{ and } a \neq 0. \end{cases}$$

In particular, if $ae \neq 0$, then f is a morphism.

We assume that $\operatorname{Hom}(f, f') \neq \emptyset$, and we let $\varphi \in \operatorname{Hom}(f, f')$.

We begin by computing $\operatorname{Hom}(f,f')$ in the generic case $ae \neq 0$. Since φ sends $I(f)$ to $I(f')$, it follows that also $a'e' \neq 0$. The critical locus of f is an irreducible cubic curve. Indeed, setting

$$E : aeX^3 - eXZ^2 + 2Y^2Z = 0 \quad \text{and} \quad \mathcal{O} = [0,1,0],$$

we see that $\mathcal{O}$ is a flex point of the cubic, so $(E,\mathcal{O})$ is an elliptic curve with group law specified by the usual rule that distinct points $P,Q,R \in E$ sum to $\mathcal{O}$ if and only if P,Q,R are colinear. Further, the elliptic curve $(E,\mathcal{O})$ has CM by $\mathbb{Z}[i]$, so $\operatorname{Aut}(E,\mathcal{O}) \cong \mathbb{Z}[i]^* = \{\pm 1, \pm i\}$, and since the four maps in $\mathcal{G}_4$ induce automorphisms of $(E,\mathcal{O})$, we see that

$$\operatorname{Aut}(E,\mathcal{O}) = \mathcal{G}_4.$$

And similarly for the elliptic curve $(E',\mathcal{O}')$ associated to f'.

We next observe that there are exactly four isomorphisms from $(E,\mathcal{O})$ to $(E',\mathcal{O}')$, since if ψ_1 and ψ_2 are any two such isomorphisms, then $\psi_2^{-1}\circ\psi_1 \in \operatorname{Aut}(E,\mathcal{O}) = \mathcal{G}_4$. Explicitly, if we fix $u,v \in K$ satisfying $u^4 = a'/a$ and $v^2 = e/e'$, then the four elements of $\operatorname{Isom}\big((E,\mathcal{O}),(E',\mathcal{O}')\big)$ are

$$\begin{aligned} \psi_0(X,Y,Z) &= [u^2X, uvY, Z], & \psi_1(X,Y,Z) &= [u^2X, iuvY, -Z], \\ \psi_2(X,Y,Z) &= [u^2X, -uvY, Z], & \psi_3(X,Y,Z) &= [u^2X, -iuvY, -Z]. \end{aligned}$$

Since these are diagonal maps in PGL_3, we also note that

$$\psi\circ\alpha = \alpha\circ\psi \quad \text{for all } \alpha \in \mathcal{G}_4 \text{ and all } \psi \in \operatorname{Isom}\big((E,\mathcal{O}),(E',\mathcal{O}')\big). \tag{9.2}$$

The map $\varphi \in \operatorname{Hom}(f,f')$ sends $\operatorname{Crit}(f)$ to $\operatorname{Crit}(f')$, so $\varphi(E) = E'$. In other words, $\varphi|_E$ induces an isomorphism of genus 1 curves $E \to E'$. (There is, however, no *a priori* reason that φ needs to send $\mathcal{O}$ to $\mathcal{O}'$.) Standard properties of elliptic curves [22, III.4.7] tell us that there is an isogeny $\psi : (E,\mathcal{O}) \to (E',\mathcal{O}')$ and a point $P_0 \in E$ so that

$$\varphi(P) = \psi(P+P_0) = \psi\circ T_{P_0}(P) \quad \text{for all } P \in E,$$

where $T_{P_0} : E \to E$ denotes the translation-by-P_0 map. Further, since φ is invertible on all of $\mathbb{P}^2$ and since translation by P_0 is invertible on E, we see that ψ is bijective, hence $\psi \in \operatorname{Isom}\big((E,\mathcal{O}),(E',\mathcal{O}')\big)$ is one of the fours maps listed earlier.

We next exploit the fact that $\varphi \in \mathrm{PGL}_3(K)$ maps lines to lines. Thus if $P,Q,R \in E$ are distinct points satisfying $P+Q+R = \mathcal{O}$, then $\varphi(P),\varphi(Q)$, $\varphi(R)$ are also colinear, so $\varphi(P)+\varphi(Q)+\varphi(R) = \mathcal{O}'$. Using the fact that ψ is a group isomorphism, we compute

$$\begin{aligned} \mathcal{O}' = \varphi(P)+\varphi(Q)+\varphi(R) &= \psi(P+P_0)+\psi(Q+P_0)+\psi(R+P_0) \\ &= \psi(P+Q+R)+3\psi(P_0) = \psi(\mathcal{O})+3\psi(P_0) = 3\psi(P_0). \end{aligned}$$

Hence $\psi(P_0)$ is a 3-torsion point of E', and since ψ is a group isomorphism, we conclude that P_0 is a 3-torsion point of E.

We claim that $P_0 = \mathcal{O}$. To prove this claim, we assume that $P_0 \ne \mathcal{O}$ and derive a contradiction. For an arbitrary $\alpha \in \mathcal{G}_4 \subseteq \mathrm{Aut}(f)$, we note that the composition $\alpha \circ \varphi^{-1} \circ \alpha^{-1} \circ \varphi$ is in $\mathrm{Aut}(f)$. On the other hand, we can write this composition explicitly as

$$\begin{aligned} \alpha \circ \varphi^{-1} \circ \alpha^{-1} \circ \varphi &= \alpha \circ (\psi \circ T_{P_0})^{-1} \circ \alpha^{-1} \circ (\psi \circ T_{P_0}) \\ &= \alpha \circ T_{-P_0} \circ \psi^{-1} \circ \alpha^{-1} \circ \psi \circ T_{P_0} \\ &= \alpha \circ T_{-P_0} \circ \alpha^{-1} \circ \psi^{-1} \circ \psi \circ T_{P_0} \quad \text{from (9.2)}, \\ &= T_{-\alpha(P_0)} \circ T_{P_0} \\ &= T_{P_0 - \alpha(P_0)}. \end{aligned}$$

Our assumption that P_0 is a non-trivial 3-torsion point implies that $P_0 \ne \alpha(P_0)$ for all $\alpha \in \mathcal{G}_4 \smallsetminus \{1\}$, since for such α, the kernel of $\alpha - 1$ consists of 2-torsion points. Hence the set

$$\bigl\{P_0 - \alpha(P_0) : \alpha \in \mathcal{G}_4\bigr\}$$

contains four distinct elements of $E[3]$; in particular, it contains generators of $E[3]$. We saw above that all of the translations $T_{P_0-\alpha(P_0)}$ are in $\mathrm{Aut}(f)$, so using the fact that $\mathrm{Aut}(f)$ is a group, we have proven that

$$\mathrm{Aut}(f) \supset \bigl\{T_Q : Q \in E[3]\bigr\}.$$

Thus $\mathrm{Aut}(f)$ contains a subgroup of type $C_3 \times C_3$. This and the fact that f is a degree 2 morphism contradicts Proposition 7.1, which concludes the proof that $P_0 = \mathcal{O}$.

We now know that every $\varphi \in \mathrm{Hom}(f, f')$ has the form $\varphi = \psi$ for some $\psi \in \mathrm{Isom}\bigl((E,\mathcal{O}),(E',\mathcal{O}')\bigr)$, i.e., $\mathrm{Hom}(f,f')$ consists of the four maps listed earlier, so there is an integer m such that

$$\varphi(X,Y,Z) = [u^2X, i^m uvY, (-1)^m Z],$$

where we recall that u and v satisfy $u^4 = a'/a$ and $v^2 = e/e'$. We compute

$$\begin{aligned} f^\varphi = \varphi^{-1} \circ f \circ \varphi(X,Y,Z) &= [u^2aX^2 + u^{-2}Z^2, u^2XY, u^2v^2Y^2 + u^2eXZ] \\ &= \left[a'X^2 + Z^2, \frac{a'}{a}XY, \frac{a'e}{ae'}(Y^2 + e'XZ)\right]. \end{aligned}$$

Comparing this to $f' = [a'X^2 + Z^2, XY, Y^2 + e'XZ]$, we see that $f^\varphi = f'$ if and only if $a = a'$ and $e = e'$, i.e., if and only if $f' = f$. This completes the proof that

$$\mathrm{Hom}(f,f') = \begin{cases} \emptyset & \text{if } f \ne f', \\ \mathcal{G}_4 & \text{if } f = f', \end{cases}$$

which completes the proof of (a) in the case that $ae \ne 0$.

We next consider the case of maps with $a = 0$, i.e., maps of the form

$$f = [Z^2, XY, Y^2 + eXZ].$$

These maps satisfy

$$I(f) = \{[1,0,0]\} \quad \text{and} \quad \mathrm{Crit}(f) = \{Z = 0\} \cup \{eXZ = 2Y^2\}.$$

Letting $f' = [Z^2, XY, Y^2 + e'XZ]$ be another such map, we conclude that any $\varphi \in \mathrm{Hom}(f, f')$ fixes the point $[1,0,0]$ and stabilizes the line $Z = 0$. (This is true regardless of whether $e = 0$ and/or $e' = 0$, since φ preserves the multiplicities of components of $\mathrm{Crit}(f)$.) Thus φ has the form $\varphi = \left(\begin{smallmatrix}1&\alpha&\beta\\0&\gamma&\delta\\0&0&\epsilon\end{smallmatrix}\right)$. Equating

$$\begin{aligned} f \circ \varphi &= [\epsilon^2 Z^2, \gamma XY + \delta XZ + \alpha\gamma Y^2 + (\alpha\delta + \beta\gamma)YZ + \beta\delta Z^2, *], \\ \varphi \circ f' &= [Z^2 + \alpha XY + \beta Y^2 + \beta e' XZ, \gamma XY + \delta Y^2 + \delta e' XZ, *], \end{aligned}$$

we see from the XY and Y^2 terms in the first coordinate that $\alpha = \beta = 0$, and then the Y^2 term in the second coordinate gives $\delta = 0$. Hence φ is a diagonal matrix, and we have

$$[Z^2, XY, Y^2 + e'XZ] = f' = f^\varphi = [\epsilon^2 Z^2, XY, \gamma^2\epsilon^{-1}Y^2 + eXZ].$$

Therefore $\varphi \in \mathrm{Hom}(f, f')$ if and only if $e = e'$ and $\epsilon^2 = 1$ and $\gamma^2 = \epsilon$. So if $f = f'$, then we get the four maps in $\mathcal{G}_4$, and if $f \neq f'$, then f and f' are not conjugate.

It remains to consider the case of maps with $a \neq 0$ and $e = 0$, i.e., maps of the form

$$f = [aX^2 + Z^2, XY, Y^2] \quad \text{with } a \neq 0.$$

These maps satisfy

$$\mathrm{Crit}(f) = \{Z = 0\} \cup \{Y^2 = 0\},$$

i.e., the critical locus of f consists of two lines, one with multiplicity 1 and one with multiplicity 2. Letting $f' = [a'X^2 + Z^2, XY, Y^2]$ be another such map, we conclude that any $\varphi \in \mathrm{Hom}(f, f')$ stabilizes the lines $\{Z = 0\}$ and $\{Y = 0\}$, so φ has the form $\varphi = \left(\begin{smallmatrix}\alpha&\beta&\gamma\\0&\delta&0\\0&0&1\end{smallmatrix}\right)$. Equating the middle coordinates of

$$\begin{aligned} f \circ \varphi &= [*\,, \alpha\delta XY + \beta\delta Y^2 + \gamma\delta YZ, *], \\ \varphi \circ f' &= [*\,, \delta XY, *], \end{aligned}$$

and using the fact that $\delta \neq 0$ (since φ is invertible), we see that $\beta = \gamma = 0$. Hence φ is a diagonal matrix, and we have

$$[a'X^2 + Z^2, XY, Y^2] = f' = f^\varphi = [\alpha^2 aX^2 + Z^2, \alpha^2 XY, \alpha\delta^2 Y^2].$$

Therefore $\varphi \in \mathrm{Hom}(f, f')$ if and only if $a = a'$ and $\alpha^2 = 1$ and $\delta^2 = \alpha^{-1}$. So if $f = f'$, then we get the four maps in $\mathcal{G}_4$, and if $f \neq f'$, then f and f' are not conjugate.

Computation of $\mathrm{Aut}(f_{2,3})$ for $f_{2,3} = [YZ, X^2 + cZ^2, XY]$

To ease notation, we again drop the subscript on f. We consider two such maps

$$f = [YZ, X^2 + cZ^2, XY] \quad \text{and} \quad f' = [YZ, X^2 + c'Z^2, XY],$$

and we compute $\mathrm{Hom}(f, f')$, where we suppose that $\mathrm{Hom}(f, f') \neq \emptyset$. Let $\varphi \in \mathrm{Hom}(f, f')$, and let $\gamma = \sqrt{-c}$ and $\gamma' = \sqrt{-c'}$.

The indeterminacy locus of f consists of three points and the critical locus of f consists of three lines (with multiplicity if $c = 0$),

$$\begin{aligned} I(f) &= \big\{[0,1,0], [\pm\gamma, 0, 1]\big\}, \\ \mathrm{Crit}(f) &= \{Y = 0\} \cup \{X = \gamma Z\} \cup \{X = -\gamma Z\}. \end{aligned}$$

The orbit portrait of $\mathrm{Crit}(f)$ is

$$\{X = \pm\gamma Z\} \xrightarrow{f} [1, 0, \pm\gamma] \in \{Y = 0\} \xrightarrow{f} [0, 1, 0] \in I(f),$$

and similarly for f'. Suppose first that $c \neq \pm 1$. Then $[1, 0, \pm\gamma] \notin I(f)$, so the fact that the map $\varphi \in \mathrm{Hom}(f, f')$ sends the orbit portrait of $\mathrm{Crit}(f)$ to the orbit portrait of $\mathrm{Crit}(f')$ implies that φ fixes the line $\{Y = 0\}$ and the point $[0, 1, 0]$. This means that φ has the form

$$\varphi = \left(\begin{smallmatrix} s & 0 & t \\ 0 & 1 & 0 \\ u & 0 & v \end{smallmatrix}\right) \in \mathrm{PGL}_3(K).$$

We are assuming that $f^\varphi = f'$. We start by comparing the first and third coordinates of $f \circ \varphi$ and $\varphi \circ f'$,

$$\begin{aligned} f \circ \varphi &= [uXY + vYZ, *, sXY + tYZ], \\ \varphi \circ f' &= [tXY + sYZ, *, vXY + uYZ]. \end{aligned}$$

Thus there is an $\epsilon \in K^*$ such that $(u, v, s, t) = (\epsilon t, \epsilon s, \epsilon v, \epsilon u)$, and this in turn implies that $u = \epsilon t = \epsilon^2 u$ and $v = \epsilon s = \epsilon^2 v$. The invertibility of φ implies that u and v are not both 0, so $\epsilon = \pm 1$. Further, setting $u = \epsilon t$ and $v = \epsilon s$, the middle coordinates of $f \circ \varphi$ and $\varphi \circ f'$ look like

$$\begin{aligned} f \circ \varphi &= [*, (s^2 + ct^2)X^2 + 2st(1 + c)XZ + (t^2 + cs^2)Z^2, *], \\ \varphi \circ f' &= [*, X^2 + c'Z^2, *]. \end{aligned}$$

Hence we must have

$$s^2 + ct^2 = \epsilon, \quad st(1 + c) = 0, \quad t^2 + cs^2 = \epsilon c'.$$

This leads to two cases (since we are assuming for the present that $c \neq -1$):

$$\begin{aligned} s = 0 &\quad\Longrightarrow\quad \epsilon c^{-1} = t^2 = \epsilon c' \quad\Longrightarrow\quad cc' = 1, \\ t = 0 &\quad\Longrightarrow\quad s^2 = \epsilon \quad\Longrightarrow\quad c = c'. \end{aligned}$$

We see that if $f' \neq f$, i.e., $c' \neq c$, then we must have $c' = c^{-1}$ and $s = v = 0$, and in this case we find that the permutation $[Z, Y, X] \in N(\mathcal{G}_4)$ is in $\operatorname{Hom}(f, f')$.

If $c' = c$, i.e., we are computing $\operatorname{Aut}(f)$, then our assumption that $c^2 \neq 1$ means that we must have $t = u = 0$ and $s^2 = \epsilon = \pm 1$. This proves that

$$\operatorname{Aut}(f) = \left\{ \left(\begin{smallmatrix} s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon s \end{smallmatrix} \right) \in \mathrm{PGL}_3(K) : \epsilon = \pm 1 \text{ and } s = \pm\sqrt{\epsilon} \right\} = \mathcal{G}_4.$$

Next we consider the case that $c = 1$. This gives the map that is labeled $f_{4.2}$ in Example 1.7. It is shown in that example that $f_{4.2}$ is PGL_3-conjugate to the map $f_{6.1} := [YZ, XZ, XY]$. We proved in Proposition 8.1(c) that $\operatorname{Aut}(f_{6.1}) = \mathcal{S}_3\mathcal{G}_{2,2} \cong S_4$, and hence we find that $\operatorname{Aut}(f_{4.2}) \cong S_4$. Explicitly, $\operatorname{Aut}(f_{4.2})$ is the subgroup of PGL_3 given by conjugating $\mathcal{S}_3\mathcal{G}_{2,2}$ by the inverse of the map β given in Example 1.7.

Finally, if $c = -1$, then a similar calculation shows that s and t need only satisfy the single relation $s^2 - t^2 = \epsilon$, so

$$\operatorname{Aut}(f) \supseteq \left\{ \left(\begin{smallmatrix} s & 0 & t \\ 0 & 1 & 0 \\ \epsilon t & 0 & \epsilon s \end{smallmatrix} \right) \in \mathrm{PGL}_3(K) : \begin{matrix} \epsilon = \pm 1 \text{ and} \\ s^2 - t^2 = \epsilon \end{matrix} \right\} \cong \mathbb{G}_m \rtimes C_2.$$

□

Proof of Proposition 9.2. — Taking $\zeta = i$ to be a primitive 4'th root of unity, we see that the map $[X, iY, Z]$ is τ_0. Then Table 5.1 with $m = 0$ and entries reduced modulo 4 is

	X^2	Y^2	Z^2	XY	XZ	YZ
X-coord	0	2	0	1	0	1
Y-coord	3	1	3	0	3	0
Z-coord	0	2	0	1	0	1

Hence $\tau_2 \in \operatorname{Aut}(f)$ leads to the following four families of maps:

$$\begin{aligned} f_{0,0} &:= [aX^2 + bZ^2 + cXZ, dXY + eYZ, fX^2 + gZ^2 + hXZ], \\ f_{0,1} &:= [aXY + bYZ, cY^2, dXY + eYZ], \\ f_{0,2} &:= [aY^2, 0, bY^2], \\ f_{0,3} &:= [0, aX^2 + bZ^2 + cXZ, 0]. \end{aligned}$$

For each family we use Table 5.2 to compute

$$\begin{aligned}
\mu^{\mathcal{O}(1)}(f_{0,0}, L_{k,\ell}) &\leqslant \max\{-k, 3k+2\ell, k+\ell, -3k-\ell\} \xrightarrow{(k,\ell)=(1,-2)} -1,\\
\mu^{\mathcal{O}(1)}(f_{0,1}, L_{k,\ell}) &\leqslant \max\{-\ell, 2k, -\ell, -2k-2\ell, -k\} \xrightarrow{(k,\ell)=(0,1)} -1,\\
\mu^{\mathcal{O}(1)}(f_{0,2}, L_{k,\ell}) &\leqslant \max\{k-2\ell, -k-3\ell\} \xrightarrow{(k,\ell)=(0,1)} -3,\\
\mu^{\mathcal{O}(1)}(f_{0,3}, L_{k,\ell}) &\leqslant \max\{-3k-\ell, k+\ell, -k\} \xrightarrow{(k,\ell)=(1,-2)} -1.
\end{aligned}$$

This shows that all of these maps are $\mathcal{D}$-unstable. □

We now investigate more closely one of the families of maps appearing in Proposition 9.1.

Proposition 9.3. — *Let $f = [YZ, X^2 + cZ^2, XY]$ be the map from Proposition 9.1(b). Let $R(X,Z) = X^2 + cZ^2$ and $S(X,Z) = cX^2 + Z^2$. Then the iterates of f are given by the explicit formulas*

$$\begin{aligned}
f^{2k}(X,Y,Z) &= \big[R(X,Y)^k X, S(X,Y)^k Y, R(X,Y)^k Z\big],\\
f^{2k+1}(X,Y,Z) &= \big[S(X,Y)^k YZ, R(X,Y)^{k+1}, S(X,Y)^k XY\big].
\end{aligned}$$

In particular, if $c \neq \pm 1$, then $\deg(f^n) = 2n+1$, while if $c = \pm 1$, then $f^{2k} = [X, (-1)^k Y, Z]$.

Let $p(X,Y,Z) = [X,Z]$. Then there is a commutative diagram

$$\begin{array}{ccc}
\mathbb{P}^2 & \xrightarrow{\quad f \quad} & \mathbb{P}^2 \\
\Big\downarrow p & & \Big\downarrow p \\
\mathbb{P}^1 & \xrightarrow{[U,V]\to[V,U]} & \mathbb{P}^1
\end{array}$$

The second iterate $f^2 = [X^3 + cXZ^2, cX^2Y + YZ^2, X^2Z + cZ^3]$ has infinte automorphism group,

$$\mathrm{Aut}(f^2) \supset \left\{ \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) : t \in \mathbb{G}_m \right\}.$$

Proof. — We need to prove that

$$\begin{aligned}
f^{2k}(X,Y,Z) &= \big[R^k X, S^k Y, R^k Z\big],\\
f^{2k+1}(X,Y,Z) &= \big[S^k YZ, R^{k+1}, S^k XY\big].
\end{aligned}$$

The proof is by induction on k. The formulas are visibly correct for $k=0$. Assuming that the formula for f^{2k} is correct, we compute

$$\begin{aligned} f^{2k+1}(X,Y,Z) &= f\bigl(f^{2k}(X,Y,Z)\bigr) \\ &= f(R^kX, S^kY, R^kZ) \\ &= [R^kS^kYZ, R^{2k}X^2 + cR^{2k}Z^2, R^kS^kXY] \\ &= [S^kYZ, R^{k+1}, S^kXY], \end{aligned}$$

which shows that the formula for f^{2k+1} is correct. Similarly, assuming that the formula for f^{2k+1} is correct, we compute

$$\begin{aligned} f^{2k+2}(X,Y,Z) &= f\bigl(f^{2k+1}(X,Y,Z)\bigr) \\ &= f(S^kYZ, R^{k+1}, S^kXY) \\ &= [R^{k+1}S^kXY, S^{2k}Y^2Z^2 + cS^{2k}X^2Y^2, R^{k+1}S^kYZ] \\ &= [R^{k+1}X, S^{k+1}Y, R^{k+1}Z], \end{aligned}$$

which shows that the formula for f^{2k+2} is correct. This completes the proof of the formulas for the iterates of f.

We now ask when the coordinates of f^n have a common factor. It is clear that neither X nor Y nor Z is a common factor, so any non-trivial common factor must be a common factor of R and S. Since

$$\operatorname{Res}(R,S) = \operatorname{Res}(X^2+cZ^2, cX^2+Z^2) = (c^2-1)^2,$$

there is no common factor if $c \ne \pm 1$. Hence if $c \ne \pm 1$, then $\deg(f^n) = n+1$. On the other hand, if $c = \pm 1$, then $S = \pm R$, so $f^{2k} = [X, (-1)^kY, Z]$.

Finally, the commutativity of the diagram and verification that the indicated matrices are in $\operatorname{Aut}(f^2)$ are trivial calculations. This completes the proof of Proposition 9.3. $\square$

10. Maps with an Automorphism of Order 3

In this section we classify maps in Rat_2^2 that admit an automorphism of order 3. The classification that we give in Table 10.1 arises naturally during the proof, but we note that later in Section 11 we will use somewhat different normal forms in order to create the families as described in Table 1.1.

Proposition 10.1. — *Let K be an algebraically closed field of characteristic 0, let $\mathcal{G}_3$ be the group described in Theorem 1.2, and let $f \in \mathrm{Rat}_2^2(\mathcal{G}_3)^{\mathrm{ss}}$ be a dominant map of degree 2 with finite automorphism group. Further define a subgroup $\mathcal{G}_{3,2} \subset \mathrm{PGL}_3(K)$ by*

$$\mathcal{G}_{3,2} := \left\langle \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right) \right\rangle \cong S_3.$$

Then f is $N(\mathcal{G}_3)$-conjugate to one of the maps in Table 10.1, where the penultimate column indicates if the given maps are morphisms and the last column describes when two maps of the given form are $N(\mathcal{G}_3)$-conjugate to one another.[7] *Further, maps of Type $\boldsymbol{C}_3(n)$ and $\boldsymbol{C}_3(n')$ for $n \neq n'$ are not $N(\mathcal{G}_3)$-conjugate.*

Proposition 10.2. — *Let K be an algebraically closed field of characteristic 0, let ζ be a primitive cube root of unity, and let $f \in \mathrm{Rat}_2^2$ be a dominant rational map satisfying*

$$\mathrm{Aut}(f) \supseteq \left\langle \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \right\rangle.$$

Then f is $\mathcal{D}$-unstable.

Proof of Proposition 10.1. — During the proof we will frequently use the fact that $N(\mathcal{G}_3) = \mathcal{S}_3\mathcal{D}$; see Lemma 4.2. The map $[X, \zeta Y, \zeta^2 Z]$ generating $\mathcal{G}_3$ is the map defined by the matrix τ_2 in Section 5. Using Table 5.1 with $m = 2$ and entries reduced modulo 3, we find that

	X^2	Y^2	Z^2	XY	XZ	YZ
X-coord	0	2	1	1	2	0
Y-coord	2	1	0	0	1	2
Z-coord	1	0	2	2	0	1

Hence assuming that $\tau_2 \in \mathrm{Aut}(f)$ leads to the following three families of maps:

$$\begin{aligned} f &:= [aX^2 + bYZ, cZ^2 + dXY, eY^2 + gXZ], \\ f' &:= [aZ^2 + bXY, cY^2 + dXZ, eX^2 + gYZ], \\ f'' &:= [aY^2 + bXZ, cX^2 + dXZ, eZ^2 + gXY]. \end{aligned}$$

Conjugating by the cyclic permutation $\pi(X, Y, Z) = [Y, Z, X] \in N(\mathcal{G}_3)$ gives

$$\begin{aligned} f^\pi &= [eZ^2 + gXY, aY^2 + bXZ, cX^2 + dYZ], \\ f^{\pi^2} &= [cY^2 + dXZ, eX^2 + gYZ, aZ^2 + bXY], \end{aligned}$$

[7] In some cases we have given only the isomorphism class of $\mathrm{Aut}(f)$. But during the proof of the proposition, we give an explicit description of $\mathrm{Aut}(f)$ as a subgroup of $\mathrm{PGL}_3(K)$.

f	Coeffs	$\mathrm{Aut}(f)$	Mor?	$f' \sim f$
$\boldsymbol{C_3(1)}$: $f_b = [X^2 + bYZ, Z^2, Y^2]$, $b \in K$				
		$\mathcal{G}_{3,2}$	Yes	$b' = b$
$\boldsymbol{C_3(2)}$: $f_{a,b} = [aX^2 + YZ, XY, Y^2 + gXZ]$, $a, g \in K$ not both 0				
		$\mathcal{G}_3$	No	–
$\boldsymbol{C_3(3)}$: $f_b = [bYZ, Z^2 + XY, Y^2]$, $b \in K^*$				
		$\mathcal{G}_3$	No	–
$\boldsymbol{C_3(4)}$: $f_{b,g} = [bYZ, Z^2 + XY, Y^2 + gXZ]$, $b, g \in K^*$				
	$g \neq 1$	$\mathcal{G}_3$	No	$(b', g') = (bg, g^{-1})$
	$g = 1$	$\mathcal{G}_{3,2}$	No	–
$\boldsymbol{C_3(5)}$: $f_{a,b} = [aX^2 - aYZ, Z^2 - XY, bY^2 - bXZ]$, $a, b \in K^*$				
	$a \neq 1$ and $b \neq 1$	$\cong C_3$	No	$(a', b') = (b, a)$
	exactly one of $a, b = 1$	$\cong C_3 \rtimes C_2$	No	$(a', b') = (b, a)$
	$a = b = 1$	$\cong S_4$	No	–
$\boldsymbol{C_3(6)}$: $f_{a,b} := [aX^2 + bYZ, Z^2 + XY, Y^2]$, $a, b \in K^*$				
		$\mathcal{G}_3$	Yes	–
$\boldsymbol{C_3(7)}$: $f_{b,d,g} = [X^2 + bYZ, Z^2 + dXY, Y^2 + gXZ]$, $b, d, g \in K^*$, $bdg \neq -1, 8$				
	$g \neq d$	$\mathcal{G}_3$	Yes	–
	$g = d$	$\mathcal{G}_{3,2}$	Yes	–
$\boldsymbol{C_3(8)}$: $f_{c,e} = [X^2 + 2YZ, cZ^2 + 2cXY, eY^2 + 2eXZ]$, $c, e \in K^*$				
	$(c, e) \neq (\zeta_3, \zeta_3^2)$ and (ζ_3^2, ζ_3)	$\mathcal{G}_3$	Yes	$(c', e') = (e, c)$
	$(c, e) = (\zeta_3, \zeta_3^2)$ or (ζ_3^2, ζ_3)	$\cong C_7 \rtimes C_3$	Yes	$(c', e') = (e, c)$

Table 10.1. Dominant degree 2 maps $f \in \mathrm{Rat}_2^2(\mathcal{G}_3)^{\mathrm{ss}}$

which shows that our three families are $N(\mathcal{G}_3)$-conjugates. It thus suffices to analyze one of them, so we concentrate on

$$f(X, Y, Z) = [aX^2 + bYZ, cZ^2 + dXY, eY^2 + gXZ]. \tag{10.1}$$

Using Table 5.2, we see that

$$\mu^{\mathcal{O}(1)}(f, L_{k,\ell}) = \max\Bigl\{ \overbrace{-k}^{a,d,g\neq 0}, \overbrace{2k}^{b\neq 0}, \overbrace{2k + 3\ell}^{c\neq 0}, \overbrace{-k - 3\ell}^{e\neq 0} \Bigr\},$$

where $-k$ appears in the max if one or more of a, d, g is non-zero.

In Table 10.2, maps marked as being semi-stable are not stable. Also, in the column marked a, d, g, the symbol $\neq 0$ means that at least one of a, d, g is non-zero, while 0 means that all three values are 0.

To justify our assertion that the maps in Case 1 are stable, we use the identity

$$(-k) + (2k + 3\ell) + (-k - 3\ell) = 0,$$

Case	a, d, g	b	c	e	stability
1	$\neq 0$	$*$	$\neq 0$	$\neq 0$	stable
2	$\neq 0$	$\neq 0$	$\neq 0$	0	semi-stable
3	$\neq 0$	$\neq 0$	0	$\neq 0$	semi-stable
4	$\neq 0$	$\neq 0$	0	0	semi-stable
5	0	$*$	$*$	$*$	unstable

Table 10.2. Semi-stability and stability conditions

which shows that

$$\mu^{\mathcal{O}(1)}(f, L_{k,\ell}) = \max\{-k, 2k + 3\ell, -k - 3\ell\} > 0 \quad \text{for all } (k, \ell) \neq (0, 0).$$

For Cases 2, 3 and 4 it is clear that $\mu^{\mathcal{O}(1)} \geqslant 0$ due to the $-k$ and $2k$ terms in the max, but taking $k = 0$ and $\ell = \pm 1$ gives a non-zero (k, ℓ) pair with $\mu^{\mathcal{O}(1)}(f, L_{k,\ell}) = 0$. Hence these cases give maps that are semi-stable, but not stable. Finally, in Case 5 we have

$$\mu^{\mathcal{O}(1)}(f, L_{k,\ell}) \leqslant \max\{2k, 2k + 3\ell, -k - 3\ell\} \xrightarrow{(k,\ell)=(-2,1)} -1,$$

which shows that these maps are unstable. (We also remark that the Case 5 map $f = [bYZ, cZ^2, eY^2]$ is not dominant, since its image is contained in the conic $b^2YZ = ceX^2$.)

Conjugating (10.1) by $\sigma = [uX, vY, wZ] \in N(\mathcal{G}_3)$ yields the twist

$$f^\sigma = \left[uaX^2 + \frac{vw}{u}bYZ, \frac{w^2}{uv}cZ^2 + udXY, \frac{v^2}{uw}eY^2 + ugXZ\right]. \tag{10.2}$$

If $e = 0$ and $c \neq 0$, we can use the permutation $[X, Z, Y] \in N(\mathcal{G}_3)$ that swaps c and e. The remainder of the proof is a case-by-case analysis that depends on properties of the coefficients.

$\boldsymbol{c = e = 0.}$ The map

$$f(X, Y, Z) = [aX^2 + bYZ, dXY, gXZ]$$

has the property that $[X, tY, t^{-1}Z] \in \mathrm{Aut}(f)$ for every t, so $\mathrm{Aut}(f)$ contains a copy of $\mathbb{G}_m$.

$\boldsymbol{c = 0}$ **and** $\boldsymbol{e \neq 0.}$ We have

$$f(X, Y, Z) = [aX^2 + bYZ, dXY, eY^2 + gXZ].$$

The dominance of f implies that $d \neq 0$, and the semi-stability of f implies that $b \neq 0$. Using the fact that $bde \neq 0$, we see from the twisting formula (10.2) that an appropriate twist lets us take $b = d = e = 1$. So

$$f(X, Y, Z) = [aX^2 + YZ, XY, Y^2 + gXZ].$$

The indeterminacy locus is

$$I(f) = \begin{cases} \{[0,0,1]\} & \text{if } a \neq 0, \\ \{[0,0,1],[1,0,0]\} & \text{if } a = 0. \end{cases}$$

Suppose that $\varphi \in \operatorname{Hom}(f, f')$ fixes $[0,0,1]$, which is forced if $a \neq 0$, and is one of two possibilities if $a = 0$. Thus $\varphi = \left(\begin{smallmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ \lambda & \mu & 1 \end{smallmatrix}\right)$. Comparing

$$\begin{aligned} \varphi \circ f' &= [*,\ \gamma(a'X^2 + YZ) + \delta XY,\ *], \\ f \circ \varphi &= [*,\ (\alpha X + \beta Y)(\gamma X + \delta Y),\ *], \end{aligned}$$

we see that $\gamma = 0$ because $f \circ \varphi$ has no YZ term, and that $\beta\delta = 0$ because $\varphi \circ f'$ has no Y^2 term. But $\gamma = \delta = 0$ contradicts the invertibility of φ, so $\beta = 0$. Hence $\varphi = \left(\begin{smallmatrix} \alpha & 0 & 0 \\ 0 & \delta & 0 \\ \lambda & \mu & 1 \end{smallmatrix}\right)$. Next we look at the first coordinates,

$$\begin{aligned} \varphi \circ f' &= [\alpha(a'X^2 + YZ),\ *,\ *], \\ f \circ \varphi &= [a(\alpha X)^2 + (\delta Y)(\lambda X + \mu Y + Z),\ *,\ *]. \end{aligned}$$

Since $\delta \neq 0$, the lack of an XY term gives $\lambda = 0$ and the lack of a Y^2 term gives $\mu = 0$. Hence φ is diagonal. Then

$$f' = f^\varphi = [\alpha^2 aX^2 + \delta YZ, \alpha^2 XY, \alpha\delta^2 Y^2 + \alpha^2 gXZ].$$

Normalizing on the XY term, this formula holds if and only if

$$(a, \alpha^{-2}\delta, \alpha^{-1}\delta^2, g) = (a', 1, 1, g').$$

So if and only if $f' = f$ and $\delta = \alpha^2$ and $\delta^2 = \alpha$. Hence α is a cube root of unity and $\delta = \alpha^2$, which gives the copy of $\mathcal{G}_3$ that we already know is in $\operatorname{Aut}(f)$.

Next suppose that $a = 0$ and that $\varphi \in \operatorname{Hom}(f, f')$ swaps $[0,0,1]$ and $[1,0,0]$. Then $f = [YZ, XY, Y^2 + gXZ]$ and $\varphi = \left(\begin{smallmatrix} 0 & \alpha & \beta \\ 0 & 1 & 0 \\ \gamma & \delta & 0 \end{smallmatrix}\right)$, and we have

$$\varphi \circ f' = [*,\ XY,\ *], \quad f \circ \varphi = [*,\ (\alpha Y + \beta Z)Y,\ *].$$

This gives a contradiction, so even in the case that $a = 0$, we obtain no new elements.

***ce* $\neq$ 0.** We see from the twisting formula (10.2) that an appropriate twist lets us take $c = e = 1$, so

$$f(X, Y, Z) = [aX^2 + bYZ, Z^2 + dXY, Y^2 + gXZ].$$

Further, the semi-stablity of f tells us that at least one of a, d, g is non-zero. Further, the permutation $\pi = [X, Z, Y] \in N(\mathcal{G}_3)$ conjugates f to

$$f^\pi = [aX^2 + bYZ, Z^2 + gXY, Y^2 + dXZ],$$

i.e., it swaps d and g. This gives two subcases: (1) $d \neq 0$; (2) $d = g = 0$.

***ce* $\neq$ 0 and *d* = *g* = 0.** Semi-stablity of f tells us that $a \neq 0$, and then a twist (10.2) lets us set $a = 1$, so

$$f(X, Y, Z) = [X^2 + bYZ, Z^2, Y^2].$$

We observe that f is a morphism with critical set

$$\mathrm{Crit}(f) = \{XYZ = 0\}.$$

We also observe that the permutation $\pi = [X, Z, Y] \in \mathrm{Aut}(f)$. Let $\varphi \in \mathrm{Hom}(f, f')$, so φ permutes the three lines in $\mathrm{Crit}(f)$. If φ swaps the lines $Y = 0$ and $Z = 0$, then $\pi\varphi$ fixes them, so it suffices to analyze the maps φ that fix the lines $Y = 0$ and $Z = 0$ and the maps φ that satisfy $\{X = 0\} \to \{Y = 0\} \to \{Z = 0\} \to \{X = 0\}$.

The maps fixing the lines $Y = 0$ and $Z = 0$ have the form $\varphi = \left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & \delta & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Then

$$\begin{aligned} \varphi \circ f' &= [\alpha(X^2 + b'YZ) + \beta Z^2 + Y^2, \delta Z^2, Y^2], \\ f \circ \varphi &= [(\alpha X + \beta Y + \gamma Z)^2 + b\delta YZ, Z^2, \delta^2 Y^2]. \end{aligned}$$

Looking at the XY and XZ terms in the first coordinate tells us that $\alpha\beta = \alpha\gamma = 0$. The invertibility of φ implies that $\alpha \neq 0$, so $\beta = \gamma = 0$, i.e., φ is diagonal. Then

$$f' = f^\varphi = [\alpha\delta X^2 + \alpha^{-1}\delta^2 bYZ, Z^2, \delta^3 Y^2],$$

so we need $\delta^3 = \alpha\delta = 1$ and $b' = \alpha^{-1}\delta^2 b$. But the first conditions imply that $\alpha^{-1}\delta^2 = \delta^3 = 1$, so $b' = b$, and we recover the three maps in $\mathcal{G}_3$ that we already knew were in $\mathrm{Aut}(f)$. Composing with π gives a copy of $\mathcal{G}_{3,2} \cong S_3$ sitting in $\mathrm{Aut}(f)$.

Next suppose that φ cyclically permutes the lines $XYZ = 0$ as described earlier. Then $\varphi = \left(\begin{smallmatrix} 0 & 0 & \alpha \\ \beta & 0 & 0 \\ 0 & \gamma & 0 \end{smallmatrix}\right)$, and

$$f^\varphi = [\gamma^3 Y^2, \beta^3 X^2, *].$$

This cannot possibly equal $f' = [X^2 + bYZ, *, *]$.

***dce* $\neq$ 0.** In this case we can twist using (10.2) to make $d = 1$, so

$$f(X, Y, Z) = [aX^2 + bYZ, Z^2 + XY, Y^2 + gXZ],$$

and the assumed dominance of f tells us that a and b are not both 0.

We first determine when f is a morphism. Suppose that $[x, y, z] \in I(f)$. Then

$$ax^2 = -byz, \quad z^2 = -xy, \quad y^2 = -gxz, \tag{10.3}$$

and multiplying these three equations yields $a(xyz)^2 = -bg(xyz)^2$. So either $xyz = 0$ or $a = -bg$. We start with the former and observe:

$$\begin{aligned}
&x = 0 \Longrightarrow z = 0 \Longrightarrow y = 0 \quad \rightarrow\leftarrow, \\
&y = 0 \Longrightarrow z = 0 \Longrightarrow ax^2 = 0 \Longrightarrow a = 0, \\
&z = 0 \Longrightarrow y = ax^2 = 0 \Longrightarrow a = 0.
\end{aligned}$$

Hence $I(f) \cap \{XYZ = 0\}$ is empty if $a \neq 0$, and it contains the point $\bigl\{[1, 0, 0]\bigr\}$ if $a = 0$.

Next suppose that $xyz \neq 0$ and $a = -bg$. If $a = 0$, then necessarily $g = 0$ (since we cannot have a and b both 0). But then $y = 0$, which contradicts the case that we are studying. On the other hand, if $g \neq 0$, then the solutions to (10.3) with $xyz \neq 0$ are the points of the form $[1, -\gamma^2, \gamma]$ with $\gamma^3 = -g$. This completes the proof that

$$I(f) = \begin{cases} \emptyset & \text{if } a \neq 0 \text{ and } a \neq -bg, \\ \bigl\{[1, 0, 0]\bigr\} & \text{if } a = 0, \\ \bigl\{[1, -\gamma^2, \gamma] : \gamma^3 = -g\bigr\} & \text{if } a = -bg \neq 0. \end{cases}$$

We also compute the critical locus

$$\operatorname{Crit}(f) = \{agX^3 + bY^3 + bgZ^3 - (4a + bg)XYZ = 0\}.$$

***dce* $\neq$ 0 and *a* = 0.** We note that since $a = 0$, we must have $b \neq 0$. The map f has the form

$$f(X, Y, Z) = [bYZ, Z^2 + XY, Y^2 + gXZ],$$

and its indeterminacy locus is a single point, $I(f) = \bigl\{[1, 0, 0]\bigr\}$. Further, the critical locus of f is the cubic curve

$$\operatorname{Crit}(f) = \{Y^3 + gZ^3 - gXYZ = 0\}.$$

If $g \neq 0$, then $\operatorname{Crit}(f)$ is a nodal cubic with node $[1, 0, 0]$, while if $g = 0$, then $\operatorname{Crit}(f)$ is the triple line $Y^3 = 0$. We consider these cases separately.

dce $\boldsymbol{\neq 0}$ **and** $\boldsymbol{a=g=0}$**.** We are now in the case that $f(X,Y,Z)=[bYZ,Z^2+XY,Y^2]$, $I(f)=\{[1,0,0]\}$, and $\mathrm{Crit}(f)=\{Y=0\}$. Any $\varphi\in\mathrm{Hom}(f,f')$ maps $I(f)$ and $\mathrm{Crit}(f)$ to $I(f')$ and $\mathrm{Crit}(f')$, so must have the form $\varphi=\left(\begin{smallmatrix}1&\mu&\lambda\\0&\alpha&0\\0&\gamma&\delta\end{smallmatrix}\right)$. Comparing the first coordinates of

$$\begin{aligned}\varphi\circ f'&=\bigl[b'YZ+\mu(Z^2+XY)+\lambda Y^2,\,*,\,*\bigr],\\ f\circ\varphi&=\bigl[b(\alpha Y)(\gamma Y+\delta Z),\,*,\,*\bigr],\end{aligned}$$

the lack of a Z^2 term in the latter forces $\mu=0$. We also see that $b'=\alpha\delta b$ and $\lambda=b\delta$. Setting $\mu=0$ gives

$$\begin{aligned}\varphi\circ f'&=\bigl[*,\,\alpha(Z^2+XY),\,*\bigr],\\ f\circ\varphi&=\bigl[*,\,(\gamma Y+\delta Z)^2+(X+\lambda Z)(\alpha Y),\,*\bigr].\end{aligned}$$

The lack of a Y^2 term in the former forces $\gamma=0$, and thus

$$\begin{aligned}\varphi\circ f'&=\bigl[b'YZ+\lambda Y^2,\alpha Z^2+\alpha XY,\delta Y^2\bigr],\\ f\circ\varphi&=\bigl[\alpha\delta bYZ,\delta^2Z^2+\alpha XY+\alpha\lambda YZ,\alpha^2Y^2\bigr].\end{aligned}$$

The lack of a Y^2 term in the first coordinate forces $\lambda=0$, so φ is diagonal, and then comparing the remaining terms, we see that

$$\varphi\circ f'=f\circ\varphi\quad\Longleftrightarrow\quad[b',\alpha,\alpha,\delta]=[\alpha\delta b,\delta^2,\alpha,\alpha^2].$$

Hence $\varphi\circ f'=f\circ\varphi$ if and only if $\alpha^3=\delta^3=1$ and $\delta=\alpha^2$ and $b'=b$, which completes the proof (in this case) that b is an $N(\mathcal{G}_3)$ invariant and that $\mathrm{Aut}(f)\cong C_3$.

dceg $\boldsymbol{\neq 0}$ **and** $\boldsymbol{a=0}$**.** We are working with maps of the form $f(X,Y,Z)=[bYZ,Z^2+XY,Y^2+gXZ]$. As in the previous case, we have $I(f)=\{[1,0,0]\}$, but now the critical locus is a nodal cubic curve,

$$C:Y^3+gZ^3-gXYZ=0,$$

with node at $[1,0,0]$. Let $\varphi\in\mathrm{Hom}(f,f')$. Then φ fixes $[1,0,0]$, and also induces an isomorphism $C\to C'$. In particular, the two tangent lines at the nodes form a φ-invariant set, so φ either leaves each of the lines $YZ=0$ invariant, or it swaps them.

Suppose first that it leaves them invariant. Then φ has the form $\varphi=\left(\begin{smallmatrix}\alpha&\beta&\gamma\\0&\delta&0\\0&0&1\end{smallmatrix}\right)$ and we find that

$$\begin{aligned}\varphi\circ f'&=[*,\,*,\,Y^2+g'XZ],\\ f\circ\varphi&=\bigl[*,\,*,\,\delta^2Y^2+g(\alpha X+\beta Y+\gamma Z)Z\bigr].\end{aligned}$$

The lack of YZ and Z^2 monomials and the assumption that $g \neq 0$ gives $\beta = \gamma = 0$. Hence φ is diagonal, which yields

$$f^\varphi = [\alpha^{-1}\delta bYZ, \delta^{-1}Z^2 + \alpha XY, \delta^2 Y^2 + \alpha gXZ].$$

Therefore

$$f' = f^\varphi \quad \Longleftrightarrow \quad [\alpha^{-1}\delta b, \delta^{-1}, \alpha, \delta^2, \alpha g] = [b', 1, 1, 1, g'].$$

The middle three coordinates give $\delta^{-1} = \alpha = \delta^2$, which is equivalent to $\alpha^3 = \delta^3 = 1$ and $\delta = \alpha^2$. Then the other coordinates force $b' = b$ and $g' = g$, so we obtain only the three maps in $\mathrm{Aut}(f)$ that we already knew.

Next we consider the case that φ swaps the nodal tangent lines, which means that φ has the form $\varphi = \left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 0 & \delta \\ 0 & 1 & 0 \end{smallmatrix}\right)$. Then

$$\begin{aligned} \varphi \circ f' &= [*,\ *,\ Z^2 + XY], \\ f \circ \varphi &= \bigl[*,\ *,\ (\delta Z)^2 + g(\alpha X + \beta Y + \gamma Z)Y\bigr]. \end{aligned}$$

The lack of Y^2 and YZ monomials (and the fact that $g \neq 0$) tells us that $\beta = \gamma = 0$. Then

$$f^\varphi = [\alpha^{-1}\delta bYZ, \delta^2 Z^2 + \alpha gXY, \delta^{-1}Y^2 + \alpha XZ],$$

so $f' = f^\varphi$ if and only if

$$[\alpha^{-1}\delta b, \delta^2, \alpha g, \delta^{-1}, \alpha] = [b', 1, 1, 1, g'].$$

A bit of algebra shows that this last equality is equivalent to the following four conditions:

$$g' = g^{-1}, \quad b' = bg, \quad \delta^3 = 1, \quad \alpha = (\delta g)^{-1}.$$

So first we find that $f' \neq f$ is $N(\mathcal{G}_3)$-conjugate to f if and only if $(b', g') = (bg, g^{-1})$. And second, we find that $\mathrm{Aut}(f)$ has elements of this form if and only if $g = 1$, in which case taking $\delta^3 = 1$ and $\alpha = g\delta^2$ gives three additional elements, making $\mathrm{Aut}(f)$ isomorphic to $C_3 \rtimes C_2$.

***$adce \neq 0$* and *$a = -bg$*.** In this case our maps look like

$$f(X,Y,Z) = [-bgX^2 + bYZ, Z^2 + XY, Y^2 + gXZ].$$

To make our computation notationally less cumbersome, we twist by $[uX, vY, wZ]$ with $w = g^{-1/3}$, $v = 1$, and $u = -w^2$. Then f has the form

$$f(X,Y,Z) = [bgX^2 - bgYZ, Z^2 - XY, gY^2 - gXZ].$$

The indeterminacy locus consists of three points,

$$I(f) = \bigl\{[1, \rho, \rho^2] : \rho \in \boldsymbol{\mu}_3\bigr\}.$$

We make another change of variables to move the points in $I(f)$ to the standard basis vectors. Thus we let ζ be a primitive cube root of unity and $U = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 \\ 1 & \zeta^2 & \zeta \end{smallmatrix}\right)$. Then f^U has indeterminacy locus

$$I(f^U) = \big\{[1,0,0],[0,1,0],[0,0,1]\big\}.$$

To ease notation, we let $F = f^U$. Letting $\pi = \zeta - 1$, a short calculation shows that F has the form

$$\begin{aligned} F(X,Y,Z) = [&AXY + BXZ + CYZ, BXY + CXZ + AYZ, \\ &\qquad\qquad\qquad\qquad CXY + AXZ + BYZ], \end{aligned}$$

where

$$\begin{aligned} A &= \pi bg - (2\pi+3)g + (\pi+3), \\ B &= \pi bg + (\pi+3)g - (2\pi+3), \\ C &= \pi bg + \pi g + \pi. \end{aligned}$$

This system of linear equations relating (A,B,C) to $(bg,g,1)$ has determinant 27, so A,B,C are not all 0. Further, some linear algebra yields

$$\begin{aligned} A = B \quad &\Longleftrightarrow \quad g = 1, \\ A = B = 0 \quad &\Longleftrightarrow \quad b = g = 1. \end{aligned}$$

Momentarily writing $F = F_{A,B,C}$ to indicate the dependence on the coefficients, we observe that conjugation by the permutations

$$\sigma(X,Y,Z) := [Z,X,Y] \quad \text{and} \quad \tau(X,Y,Z) := [Y,X,Z]$$

has the effect

$$F^\sigma_{A,B,C} = F_{A,B,C} \quad \text{and} \quad F^\tau_{A,B,C} = F_{B,A,C}.$$

Thus $\sigma \in \operatorname{Aut}(F_{A,B,C})$ and $\tau \in \operatorname{Hom}(F_{A,B,C}, F_{B,A,C})$. In particular, if $A = B$, then $\tau \in \operatorname{Aut}(F_{A,B,C})$.

Let $\varphi \in \operatorname{Hom}(F,F')$. Then φ permutes the points in $I(F) = I(F')$. Suppose first that φ fixes the three points in $I(F)$, so $\varphi = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{smallmatrix}\right)$ is a diagonal matrix. Then

$$\begin{aligned} F^\varphi = [&\beta AXY + \gamma BXZ + \beta\gamma CYZ, BXY + \beta^{-1}\gamma CXZ + \gamma AYZ, \\ &\qquad\qquad\qquad\qquad \beta\gamma^{-1}CXY + AXZ + \beta BYZ], \end{aligned}$$

so $F' = F^\varphi$ if and only if

$$\begin{aligned} &[A',A',A',B',B',B',C',C',C'] \\ &\qquad\qquad = [A, \beta A, \gamma A, B, \beta B, \gamma B, \beta\gamma C, \beta^{-1}\gamma C, \beta\gamma^{-1} C]. \end{aligned}$$

We first observe that

$$\begin{aligned} A\neq 0 \quad &\Longrightarrow \quad [A',A',A']=[A,\beta A,\gamma A] \quad \Longrightarrow \quad \beta=\gamma=1,\\ B\neq 0 \quad &\Longrightarrow \quad [B',B',B']=[B,\beta B,\gamma B] \quad \Longrightarrow \quad \beta=\gamma=1. \end{aligned}$$

Hence if either A or B is non-zero, then $\beta=\gamma=1$ and φ is the identity matrix and $F=F'$.

On the other hand, if $A=B=0$, then we have the single map $[YZ,XZ,XY]$, and we already computed in Proposition 8.1(c) that

$$\operatorname{Aut}([YZ,XZ,XY])=\mathcal{S}_3\mathcal{G}_{2,2}\cong S_4.$$

Using the fact that $\sigma\in\operatorname{Aut}(F_{A,B,C})$ together with the fact that $\tau\in\operatorname{Aut}(F_{A,B,C})$ if and only if $A=B$, we have proven that

$$\begin{aligned}\bigl\{\varphi\in\operatorname{Aut}(F_{A,B,C}):\varphi \text{ leaves } I(F_{A,B,C}) \text{ invariant}\bigr\}&\\ \supseteq \begin{cases} \langle\sigma\rangle\cong C_3 & \text{if } A\neq B,\\ \langle\sigma,\tau\rangle=\mathcal{S}_3 & \text{if } A=B\neq 0,\\ \mathcal{S}_3\mathcal{G}_{2,2}\cong S_4. & \text{if } A=B=0.\end{cases}&\end{aligned} \tag{10.4}$$

And further, if $F_{A',B',C'}\neq F_{A,B,C}$, then

$$\begin{aligned}\bigl\{\varphi\in\operatorname{Hom}(F_{A,B,C},F_{A',B',C'}):\varphi \text{ leaves } I(F_{A,B,C}) \text{ invariant}\bigr\}&\\ \supseteq \begin{cases} \emptyset & \text{if } (A',B',C')\neq(B,A,C),\\ \{\tau,\tau\sigma,\tau\sigma^2\} & \text{if } (A',B',C')=(B,A,C).\end{cases}&\end{aligned} \tag{10.5}$$

Next suppose that $\varphi\in\operatorname{Hom}(F,F')$ induces a cyclic permutation of the three points in $I(F)$. Then $\sigma^i\varphi$ fixes $I(F)$ for some $i\in\{1,2\}$, and we also know that $\sigma\in\operatorname{Aut}(F)$, so $\sigma^i\varphi\in\operatorname{Hom}(F,F')$. It follows that we get no new elements of $\operatorname{Aut}(F)$ beyond those already described in (10.4), and we get no new possibilities for $\operatorname{Hom}(F,F')$.

Finally, suppose that φ induces a transposition on the set $I(F)$. Then for an appropriate choice of $i\in\{0,1,2\}$, the map $\tau\sigma^i\varphi$ fixes $I(F)$. Hence $\tau\sigma^i\varphi$ is one of the maps described by (10.4) (if $F'=F$) or by (10.5) (if $F'\neq F$), and in all cases we see that φ is already included in the list of maps in (10.4) or (10.5).

$adce\neq 0$ and $a\neq -bg$. Our map looks like

$$f(X,Y,Z)=[aX^2+bYZ,Z^2+XY,Y^2+gXZ].$$

It is a morphism, and its critical locus is the cubic curve

$$gX^3+bY^3+bgZ^3-(4a+bg)XYZ=0.$$

We consider various subcases depending on whether b and/or g vanishes.

$\boldsymbol{adce \neq 0}$ **and** $\boldsymbol{b = 0}$**.** Our map looks like

$$f(X,Y,Z) = [aX^2, Z^2 + XY, Y^2 + gXZ].$$

It is a morphism, and its critical locus is the reducible cubic curve

$$gX^3 - 4aXYZ = 0.$$

If $g \neq 0$, then $\mathrm{Crit}(f)$ is the union of the line $X = 0$ and a conic, while if $g = 0$, then $\mathrm{Crit}(f)$ is the union of the three lines $XYZ = 0$.

Suppose first that $\varphi \in \mathrm{Hom}(f, f')$ fixes the line $X = 0$ (which is necessary if $g \neq 0$). Then φ has the form $\varphi = \left(\begin{smallmatrix} 1 & 0 & 0 \\ \lambda & \alpha & \beta \\ \mu & \gamma & \delta \end{smallmatrix}\right)$. We note that $\varphi \circ f'$ has no YZ monomials, while

$$f \circ \varphi = [\cdots, 2\gamma\delta YZ + \cdots, 2\alpha\beta YZ + \cdots].$$

Hence $\alpha\beta = \gamma\delta = 0$. The non-singularity of φ precludes some possibilities, so either $\alpha = \delta = 0$ or $\beta = \gamma = 0$.

Suppose first that $\beta = \gamma = 0$. Then

$$\begin{aligned} \varphi \circ f' &= [\cdots, \text{no } XZ \text{ monomial, no } XY \text{ monomial}], \\ f \circ \varphi &= [\cdots, 2\mu\delta XZ + \cdots, 2\lambda\alpha XY + \cdots]. \end{aligned}$$

The non-singularity of φ forces $\alpha\delta \neq 0$, so we find that $\mu = \lambda = 0$, i.e., the matrix φ is diagonal. Then

$$f^\varphi = [aX^2, \alpha^{-1}\delta^2 Z^2 + XY, \alpha^2\delta^{-1} + gXZ],$$

so $f' = f^\varphi$ if and only if

$$[a', 1, 1, 1, g'] = [a, \alpha^{-1}\delta^2, 1, \alpha^2\delta^{-1}, g].$$

This occurs if and only if $(a', g') = (a, g)$ and $\alpha^3 = \delta^3 = 1$ and $\delta = \alpha^2$. So we find only the three elements of $\mathrm{Aut}(f)$ that we already had.

Next suppose that $\alpha = \delta = 0$. Then

$$\begin{aligned} \varphi \circ f' &= [\cdots, \text{no } XY \text{ monomial, no } XZ \text{ monomial}], \\ f \circ \varphi &= [\cdots, 2\mu\gamma XY + \cdots, 2\lambda\beta XZ + \cdots]. \end{aligned}$$

The non-singularity of φ tells us that $\beta\gamma \neq 0$, so $\lambda = \mu = 0$. Then

$$f^\varphi = [aX^2, \gamma^{-1}\beta^2 Z^2 + gXY, \beta^{-1}\gamma^2 Y^2 + XZ].$$

Hence $f' = f^\varphi$ if and only if

$$[a', 1, 1, 1, g'] = [a, \gamma^{-1}\beta^2, g, \beta^{-1}\gamma^2, 1].$$

This forces $g' = g$ and $a = a'g$ and $a' = ag'$, which combine to give $g = g' = \pm 1$. Further $\beta^3 = \gamma^3$ and $\gamma^2 = g\beta$. Hence $\beta^6 = \gamma^6 = (g\beta)^3 = g\beta^3$, so $\beta^3 = g$.

Thus if $g=1$, then $f'=f$ and $\mathrm{Aut}(f)$ contains three additional elements corresponding to taking $\beta\in\boldsymbol{\mu}_3$ and $\gamma=\beta^2$, while if $g=-1$, then we find that the maps $f_{a,-1}$ and $f_{-a,-1}$ are $\mathrm{PGL}_3(K)$-conjugate.

To recapitulate, if $g\ne 0$, then

$$\mathrm{Aut}(f)=\begin{cases}\left\langle\begin{pmatrix}1&0&0\\0&\zeta&0\\0&0&\zeta^2\end{pmatrix}\right\rangle=\mathcal{G}_3 & \text{if } g\ne 1,\\ \left\langle\begin{pmatrix}1&0&0\\0&\zeta&0\\0&0&\zeta^2\end{pmatrix},\begin{pmatrix}1&0&0\\0&0&\zeta\\0&\zeta^2&0\end{pmatrix}\right\rangle\cong C_3\rtimes C_2 & \text{if } g=1,\end{cases}$$

Further, distinct maps $f_{a,g}$ and $f_{a',g'}$ are $N(\mathcal{G}_3)$-conjugate if and only if $g=g'=-1$ and $a'=-a$.

If $g=0$, then $\mathrm{Aut}(f)$ contains the above maps, but we must also consider the possibility that $\varphi\in\mathrm{Aut}(f)$ non-trivially permutes the three lines $XYZ=0$ in $\mathrm{Crit}(f)$, which is the following case.

***adce* $\ne$ 0 and *b* = *g* = 0.** Our map looks like

$$f(X,Y,Z)=[aX^2,Z^2+XY,Y^2],$$

and we are looking for maps $\varphi\in\mathrm{Hom}(f,f')$ that induce a non-trivial permutation of the lines $XYZ=0$. Such a φ has the form $\varphi=\psi\pi$ with $\pi\in\mathcal{S}_3$ and ψ diagonal. The map f^ψ has the form $F_{A,B,C}:=[AX^2,BZ^2+CXY,DY^2]$ for some non-zero A,B,C,D. We claim that a non-trivial permutation $\pi\in\mathcal{S}_3$ cannot take a map of the form $F_{A,B,C}$ to another map of the same form. Lacking a clever argument, we simply compute the effect of each permutation:

$$\begin{aligned}
\pi&=[Y,X,Z] & F^\pi_{A,B,C}&=[BZ^2+CXY,AY^2,DX^2],\\
\pi&=[Z,Y,X] & F^\pi_{A,B,C}&=[DY^2,BX^2+CZY,AZ^2],\\
\pi&=[X,Z,Y] & F^\pi_{A,B,C}&=[AX^2,DZ^2,BY^2+CXZ],\\
\pi&=[Y,Z,X] & F^\pi_{A,B,C}&=[DZ^2,AY^2,BX^2+CYZ],\\
\pi&=[Z,X,Y] & F^\pi_{A,B,C}&=[BY^2+CXZ,DX^2,AZ^2].
\end{aligned}$$

This completes the proof that we obtain no new maps if $b=g=0$.

***abdce* $\ne$ 0 and *g* = 0.** Our map looks like

$$f(X,Y,Z)=[aX^2+bYZ,Z^2+XY,Y^2].$$

Its critical locus is the singular irreducible cubic curve

$$bY^3+bgZ^3-4aXYZ=0$$

having a node at $[1,0,0]$, and the two tangent lines at the node are $YZ = 0$. Hence any $\varphi \in \mathrm{Hom}(f,f')$ must either fix or swap the two lines $YZ = 0$ (which will also force it to fix their intersection point $[1,0,0]$).

Suppose first that it fixes the nodal tangent lines. Then φ has the form $\varphi = \left(\begin{smallmatrix} 1 & \lambda & \nu \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{smallmatrix}\right)$, and we find that

$$\begin{aligned} \varphi \circ f' &= [\cdots, \beta Z^2 + \beta XY, \cdots], \\ f \circ \varphi &= [\cdots, \gamma^2 Z^2 + \beta XY + \beta\lambda Y^2 + \beta\nu YZ, \cdots]. \end{aligned}$$

The absence of Y^2 and YZ terms, together with the invertibility of φ (which implies that $\beta \neq 0$) forces $\lambda = \nu = 0$, i.e., φ is diagonal. Then

$$f^\varphi = [aX^2 + \beta\gamma bYZ, \beta^{-1}\gamma^2 Z^2 + XY, \beta^2\gamma^{-1}Y^2],$$

so $f' = f^\varphi$ if and only if

$$[a', b', 1, 1, 1] = [a, \beta\gamma b, \beta^{-1}\gamma^2, 1, \beta^2\gamma^{-1}],$$

so if and only if $a' = a$ and $b' = \beta\gamma b$ and $\beta^{-1}\gamma^2 = \beta^2\gamma^{-1} = 1$. The last condition is equivalent to $\beta^3 = 1$ and $\gamma = \beta^2$, so in particular $\beta\gamma = 1$. Hence $f' = f^\varphi$ if and only if $f' = f$ and φ is one of the three maps that we already knew was in $\mathrm{Aut}(f)$.

Next suppose that φ swaps the nodal tangent lines. Then φ has the form $\varphi = \left(\begin{smallmatrix} 1 & \lambda & \nu \\ 0 & 0 & \beta \\ 0 & \gamma & 0 \end{smallmatrix}\right)$, and we find that

$$\begin{aligned} \varphi \circ f' &= [\cdots, \cdots, \gamma Z^2 + \gamma XY], \\ f \circ \varphi &= [\cdots, \cdots, \beta^2 Z^2]. \end{aligned}$$

The lack of an XY monomial forces $\gamma = 0$, contradicting the invertibility of φ. Hence there are no $\varphi \in \mathrm{Hom}(f,f')$ that swap the lines $XY = 0$.

***$abcdeg \neq 0$* and *$ace \neq -bdg$*.** We have resumed using the general form

$$f(X,Y,Z) = [aX^2 + bYZ, cZ^2 + dXY, eY^2 + gXZ],$$

so the earlier dehomogenized condition $a \neq -bg$ for f to be a morphism becomes $ace \neq -bdg$.

It is clear that every diagonal map $\varphi \in \mathrm{PGL}_3$ preserves the form of f, as does the transposition $[X,Z,Y]$. We are going to prove that the full set of elements of PGL_3 that preserves this general form is the group generated by diagonal maps and this transposition, except for one exceptional case.

The map f is a morphism, and its critical locus is the cubic curve

$$\mathrm{Crit}(f) : adgX^3 + bdeY^3 + bcgZ^3 - (bdg + 4ace)XYZ = 0. \tag{10.6}$$

Conjugating by $\psi := [uX, vY, wZ]$ gives

$$f^\psi = [uaX^2 + u^{-1}vwbYZ, v^{-1}w^2cZ^2 + udXY, v^2w^{-1}eY^2 + ugXZ]$$

with critical locus

$$\mathrm{Crit}(f^\psi) : u^3adgX^3 + v^3bdeY^3 + w^3bcgZ^3 - uvw(bdg + 4ace)XYZ = 0.$$

So taking $u^3 = (adg)^{-1}$, $v^3 = (bde)^{-1}$, and $w^3 = (bcg)^{-1}$, the critical locus becomes

$$\mathrm{Crit}(f^\psi) = \{X^3 + Y^3 + Z^3 - \Delta XYZ = 0\},$$

where $\Delta = (ace)^{-1/3}(bdg)^{-2/3}(bdg+4ace)$. A short computation shows that the cubic curve $\mathrm{Crit}(f^\psi)$ is non-singular if and only if $\Delta^3 \neq 27$. (As we will see later, if $\Delta^3 = 27$, then $\mathrm{Crit}(f^\psi)$ is a union of three lines.) Using the formula for Δ, we observe that

$$\begin{aligned} \Delta^3 - 27 &= (bdg + 4ace)^3 - 27(ace)(bdg)^2 \\ &= (bdg + ace)(bdg - 8ace)^2. \end{aligned}$$

We have ruled out $bdg = -ace$, so we are reduced to two cases, which we consider in turn.

***abcdeg* $\neq$ 0 and *bdg* $\neq$ −*ace* and *bdg* $\neq$ 8*ace*.** The Hessian of $X^3 + Y^3 + Z^3 - \Delta XYZ$ is

$$\det\begin{pmatrix} 6X & -\Delta Z & -\Delta Y \\ -\Delta Z & 6Y & -\Delta X \\ -\Delta Y & -\Delta X & 6Z \end{pmatrix} = -6\Delta^2(X^3 + Y^3 + Z^3) + 2(108 - \Delta^3)XYZ.$$

The flex points of the smooth cubic curve $\mathrm{Crit}(f^\psi)$ are thus the roots of $(27 - \Delta^3)XYZ = 0$. Our assumptions imply that $\Delta^3 \neq 27$, so the flex points are the nine points with $XYZ = 0$, i.e., the points

$$P_i := \begin{cases} [1, -\zeta^i, 0], & \text{for } i = 0,1,2, \\ [0, 1, -\zeta^i], & \text{for } i = 3,4,5, \\ [-\zeta^i, 0, 1], & \text{for } i = 6,7,8, \end{cases}$$

where we recall that ζ is a primitive cube root of unity.

To ease notation, we let $F = f^\psi$, and we write F as

$$F(X, Y, Z) = [AX^2 + BYZ, CZ^2 + DXY, EY^2 + GXZ], \tag{10.7}$$

where $A, \dots, G$ are monomials in fractional powers of $a, \dots, g$. More precisely, tracking through their dependence on $a, \dots, g$, they satisfy the multiplicative relations

$$AD = BC, \quad AG = BE, \quad BCG = 1,$$

and if $a,\dots,g$ are generic, these are the only such relations that they satisfy.[8] We also let

$$\Gamma = \{\text{flex points of } \mathrm{Crit}(F)\} = \{P_0, P_1, \dots, P_8\}.$$

Suppose that $\varphi \in \mathrm{Hom}(F,F')$. Then φ permutes the nine points in Γ, but not entirely independently. The line through any two points in Γ contains a unique third point of Γ, so the φ-images of two points in Γ determines the φ-image of a third point. So if we choose three non-colinear points, for example P_0, P_1, P_3, then the map φ is uniquely determined by the images of these three points, where those images must be chosen from among the non-colinear triples in Γ. Unfortunately, there are a large number of possibilities.

For each triple of indices (i,j,k) such that P_i, P_j, P_k are not co-linear, we let $\varphi \in \mathrm{PGL}_3$ be a general map satisfying

$$\varphi(P_0) = P_i, \quad \varphi(P_1) = P_j, \quad \varphi(P_3) = P_k.$$

Thus φ has the form

$$\varphi = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ -1 & -\zeta & 1 \\ 0 & 0 & -1 \end{pmatrix}^{-1},$$

where the matrix with $*$ entries is the matrix whose columns are the point P_i, P_j, P_k, and where $\beta,\gamma \in K^*$ are arbitrary. (If necessary, we may write $\varphi_{i,j,k,\beta,\gamma}$ to indicate the dependence of φ on the various parameters.)

We compute F^φ and pick out the coefficients of the 12 monomials that do not appear in F'. Each of those coefficients is a linear combination of $A,\dots,G$, with coefficients that are polynomials in β and γ. We accumulate this data in the form

$$\begin{pmatrix} \text{coeff of } Y^2 \text{ in } X\text{-coord of } F^\varphi \\ \text{coeff of } Z^2 \text{ in } X\text{-coord of } F^\varphi \\ \text{coeff of } XY \text{ in } X\text{-coord of } F^\varphi \\ \text{coeff of } XZ \text{ in } X\text{-coord of } F^\varphi \\ \text{coeff of } X^2 \text{ in } Y\text{-coord of } F^\varphi \\ \text{coeff of } Y^2 \text{ in } Y\text{-coord of } F^\varphi \\ \text{coeff of } XZ \text{ in } Y\text{-coord of } F^\varphi \\ \text{coeff of } YZ \text{ in } Y\text{-coord of } F^\varphi \\ \text{coeff of } X^2 \text{ in } Z\text{-coord of } F^\varphi \\ \text{coeff of } Z^2 \text{ in } Z\text{-coord of } F^\varphi \\ \text{coeff of } XY \text{ in } Z\text{-coord of } F^\varphi \\ \text{coeff of } YZ \text{ in } Z\text{-coord of } F^\varphi \end{pmatrix} = M_{i,j,k} \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ G \end{pmatrix}, \tag{10.8}$$

where $M_{i,j,k}$ is a 12-by-6 matrix whose entries are polynomials in the ring $\mathbb{Q}(\zeta)[\beta,\gamma]$. (At times we may write $M_{i,j,k}(\beta,\gamma)$ to indicate the dependence of $M_{i,j,k}$ on β and γ.) The fact that F^φ is not allowed to have any of the indicated monomials implies that (10.8) is the zero vector, and then our

[8] These formulas will explain why, when we compute the maps $\varphi \in \mathrm{PGL}_3$ that preserve the form (10.7), we don't get all diagonal matrices. More precisely, the effect of the diagonal matrix with entries α,β,γ is to multiply $AB^{-1}C^{-1}D$ by $(\alpha/\gamma)^3$ and to multiply BCG by γ^3, so α and γ must be cube roots of 1.

assumption that $A, \ldots, G$ are all non-zero implies that the matrix $M_{i,j,k}$ has rank at most 5. (Indeed, it implies the far stronger statement that the column null space of $M_{i,j,k}$ contains a vector whose coordinates are all non-zero.)

Unfortunately, there are 432 valid i, j, k triples, and even exploiting various symmetries, there are too many cases to check by hand. So we give a computer assisted proof via the following algorithm.

Step 1. For each valid choice of i, j, k, we computed the determinants of various of the 6-by-6 minors of $M_{i,j,k}$ and set them equal to 0. This gave many simultaneous equations for the two unknowns β and γ. We used resultants on pairs of equations to eliminate γ, and then took the gcd of pairs of equations with respect to β. Taking the square-free part, we obtained a separable polynomial $\Pi_{i,j,k}(\beta)$ satisfying:

$$\operatorname{rank} M_{i,j,k}(\beta,\gamma) \leqslant 5 \text{ for some } \beta,\gamma \in K^* \quad \Longrightarrow \quad \Pi_{i,j,k}(\beta) = 0.$$

The output from our program showed that

$$\Pi_{i,j,k}(\beta) \;\Big|\; (\beta^6 - 1)\left(\beta + \frac{1}{2}\right).$$

Indeed, the roots of $\Pi_{i,j,k}(\beta)$ are 6'th roots of unity except in the six cases $M_{6,8,0}$, $M_{7,8,0}$, $M_{6,8,1}$, $M_{7,8,1}$, $M_{6,8,2}$, $M_{7,8,2}$, for which $\Pi_{i,j,k}(\beta)$ also had $\beta = -\frac{1}{2}$ as a root.

Step 2. Loop through all valid i, j, k and all $\beta_0 \in \boldsymbol{\mu}_6 \cup \{-\frac{1}{2}\}$. We let

$$r = r_{i,j,k}(\beta_0,\gamma) := \operatorname{rank} M_{i,j,k}(\beta_0,\gamma)$$

denote the rank of $M_{i,j,k}(\beta_0,\gamma)$ over the function field $K(\gamma)$, i.e., where γ is an indeterminate. We also write $r_{i,j,k}(\beta_0,\gamma_0)$ for the rank of the matrix when we set $\gamma = \gamma_0$.

Step 2.1. If $r_{i,j,k}(\beta_0,\gamma) \leqslant 5$, compute the null space of $M_{i,j,k}(\beta_0,\gamma)$ over the function field $\mathbb{C}(\gamma)$. We found that there are 144 choices of (i,j,k,β_0) for which $r_{i,j,k}(\beta_0,\gamma) \leqslant 5$, and in every case, every vector in $\operatorname{Null}\bigl(M_{i,j,k}(\beta_0,\gamma)\bigr)$ has at least one coordinate equal to 0.

Example 10.3. — We illustrate with an example. Let $k \in \{3,4,5\}$. Then $r_{0,1,k}(1,\gamma) = 3$, i.e., the matrix $M_{0,1,k}(1,\gamma)$ has rank 3 over the function field $\mathbb{C}(\gamma)$. One then checks that every vector

$$[A,\ldots,G] \in \operatorname{Null}_{\mathbb{C}(\gamma)}\bigl(M_{0,1,k}(1,\gamma)\bigr)$$

has $B = E = 0$. However, if we further specialize by setting $\gamma = 1$, then $M_{0,1,k}(1,1)$ is the 0-matrix. The associated elements of PGL_3 are the diagonal matrices

$$\varphi_{0,1,3,1,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varphi_{0,1,4,1,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta \end{pmatrix}, \quad \varphi_{0,1,5,1,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}.$$

Step 2.2. Let $M'_{i,j,k}(\beta_0,\gamma)$ be a 12-by-r matrix whose column span over the function field $\mathbb{C}(\gamma)$ is the same as the column span of $M_{i,j,k}(\beta_0,\gamma)$. (In most cases we have $r = 6$ and $M' = M$, but as noted in Step 2.1, there are 144 cases with $r \leqslant 5$.)

We computed the determinants of various r-by-r minors of $M'_{i,j,k}(\beta_0,\gamma)$ and then computed their gcd. We found exactly 72 values of (i,j,k,β_0) such that there exists some γ_0 with

$$r_{i,j,k}(\beta_0,\gamma_0) < r_{i,j,k}(\beta_0,\gamma),$$

i.e., such that the generic rank over $\mathbb{C}(\gamma)$ is strictly larger than the specialized rank over $\mathbb{C}$ for some $\gamma = \gamma_0 \in \mathbb{C}$. More precisely, there are 36 cases with generic rank $r_{i,j,k}(\beta_0,\gamma) = 6$ and 18 cases each with $r_{i,j,k}(\beta_0,\gamma) = 5$ and 3. Further, and most importantly, we found in all 72 cases that $\beta_0^3 = 1$ and $\gamma_0^6 = 1$. The complete set of (i,j,k,β_0) is given in Table 10.3.

Step 3. It remains to compute the null space of $M_{i,j,k}(\beta_0,\gamma_0)$ as β_0 and γ_0 range over $\beta_0 \in \boldsymbol{\mu}_3$ and $\gamma_0 \in \boldsymbol{\mu}_6$. For each (i,j,k,β_0,γ_0) such that $\varphi_{i,j,k}$ is invertible, we check whether the null space of $M_{i,j,k}(\beta_0,\gamma_0)$ contains a vector (A,B,C,D,E,G) whose coordinates are all non-zero. It turns out that in every such case the matrix $M_{i,j,k}(\beta_0,\gamma_0)$ is identically 0. Table 10.4 lists the values of (i,j,k,β_0,γ_0), together with the associated map $\varphi \in \mathrm{PGL}_3$. We observe that the φ in Table 10.4 consist of the 9 diagonal maps satisfying $\varphi^3 = 1$, together with the conjugation of these 9 maps by the transposition $[X,Z,Y]$.

This long calculation completes the proof that under our assumptions that $abcdeg \neq 0$ and $ace \neq -bdg$, maps of the form

$$f(X,Y,Z) = [aX^2 + bYZ, cZ^2 + dXY, eY^2 + gXZ] \tag{10.9}$$

satisfy

$$\mathrm{Hom}(f,f') \subseteq \mathcal{D} \cup \pi\mathcal{D},$$

where $\mathcal{D} \subset \mathrm{PGL}_3$ is the group of diagonal matrices and π is the transposition $\pi = [X,Z,Y]$. We normalize maps of the form (10.9) by conjugating by $[uX,vY,wZ]$ with $u = a^{-1}$, $v^3 = c^{-1}e^{-2}$, and $w^3 = c^{-2}e^{-1}$. This puts f into the normalized form

$$f_{b,d,g}(X,Y,Z) = [X^2 + bYZ, Z^2 + dXY, Y^2 + gXZ]$$

M	r	β_0	M	r	β_0	M	r	β_0	M	r	β_0
$M_{0,1,3}$	3	1	$M_{1,0,3}$	5	ζ	$M_{0,1,6}$	6	ζ	$M_{8,7,1}$	6	1
$M_{0,1,4}$	3	1	$M_{1,0,4}$	5	ζ	$M_{0,1,7}$	6	ζ	$M_{7,8,1}$	6	1
$M_{0,1,5}$	3	1	$M_{1,0,5}$	5	ζ	$M_{0,1,8}$	6	ζ	$M_{9,7,1}$	6	1
$M_{2,0,3}$	3	1	$M_{0,2,3}$	5	ζ	$M_{2,0,6}$	6	ζ	$M_{7,9,1}$	6	1
$M_{2,0,4}$	3	1	$M_{0,2,4}$	5	ζ	$M_{2,0,7}$	6	ζ	$M_{9,8,1}$	6	1
$M_{2,0,5}$	3	1	$M_{0,2,5}$	5	ζ	$M_{2,0,8}$	6	ζ	$M_{8,9,1}$	6	1
$M_{1,2,3}$	3	1	$M_{2,1,3}$	5	ζ	$M_{1,2,6}$	6	ζ	$M_{8,7,2}$	6	1
$M_{1,2,4}$	3	1	$M_{2,1,4}$	5	ζ	$M_{1,2,7}$	6	ζ	$M_{7,8,2}$	6	1
$M_{1,2,5}$	3	1	$M_{2,1,5}$	5	ζ	$M_{1,2,8}$	6	ζ	$M_{9,7,2}$	6	1
$M_{7,6,3}$	3	ζ^2	$M_{6,7,3}$	5	ζ^2	$M_{1,0,6}$	6	ζ^2	$M_{7,9,2}$	6	1
$M_{6,8,3}$	3	ζ^2	$M_{8,6,3}$	5	ζ^2	$M_{1,0,7}$	6	ζ^2	$M_{9,8,2}$	6	1
$M_{8,7,3}$	3	ζ^2	$M_{7,8,3}$	5	ζ^2	$M_{1,0,8}$	6	ζ^2	$M_{8,9,2}$	6	1
$M_{7,6,4}$	3	ζ^2	$M_{6,7,4}$	5	ζ^2	$M_{0,2,6}$	6	ζ^2	$M_{8,7,2}$	6	1
$M_{6,8,4}$	3	ζ^2	$M_{8,6,4}$	5	ζ^2	$M_{0,2,7}$	6	ζ^2	$M_{7,8,2}$	6	1
$M_{8,7,4}$	3	ζ^2	$M_{7,8,4}$	5	ζ^2	$M_{0,2,8}$	6	ζ^2	$M_{9,7,2}$	6	1
$M_{7,6,5}$	3	ζ^2	$M_{6,7,5}$	5	ζ^2	$M_{2,1,6}$	6	ζ^2	$M_{7,9,2}$	6	1
$M_{6,8,5}$	3	ζ^2	$M_{8,6,5}$	5	ζ^2	$M_{2,1,7}$	6	ζ^2	$M_{9,8,2}$	6	1
$M_{8,7,5}$	3	ζ^2	$M_{7,8,5}$	5	ζ^2	$M_{2,1,8}$	6	ζ^2	$M_{8,9,2}$	6	1

Table 10.3. Matrices $M_{i,j,k}(\beta_0,\gamma)$ with generic rank r such that there exists a γ_0 such that $M_{i,j,k}(\beta_0,\gamma_0)$ has rank strictly smaller than r

that depends on three non-zero parameters b,d,g satisfying $bdg\neq -1$.

We consider first conjugation by a diagonal map $\varphi=[X,\beta Y,\gamma Z]$. Then

$$f_{b,d,g}^{\varphi}(X,Y,Z)=[X^2+\beta\gamma bYZ,\beta^{-1}\gamma^2Z^2+dXY,\beta^2\gamma^{-1}Y^2+gXZ].$$

The map $f_{b,d,g}^{\varphi}$ is thus in normalized form if and only if

$$[\beta\gamma,\beta^{-1}\gamma^2,\beta^2\gamma^{-1},1,d,g]=[1,1,1,1,d,g].$$

This is true if and only if $\beta\in\boldsymbol{\mu}_3$ and $\gamma=\beta^2$, i.e., if the map φ is in the subgroup of order 3 that we already know is contained in $\operatorname{Aut}(f)$.

Next we conjugate by the composition of a diagonal map and the transposition $[X,Z,Y]$, i.e., by a map of the form $\varphi=[X,\beta Z,\gamma Y]$. Then

$$f_{b,d,g}^{\varphi}(X,Y,Z)=[X^2+\beta\gamma bYZ,\beta^2\gamma^{-1}Z^2+gXY,\beta^{-1}\gamma^2Y^2+dXZ].$$

The map $f_{b,d,g}^{\varphi}$ is thus in normalized form if and only if

$$[\beta\gamma,\beta^2\gamma^{-1},\beta^{-1}\gamma^2,1,g,d]=[1,1,1,1,d,g].$$

The first four coordinates force $\beta\in\boldsymbol{\mu}_3$ and $\gamma=\beta^2$, as expected. The last three coordinates force $d=g$, so we find that $f_{b,d,g}$ and $f_{b,g,d}$ are

M	β_0	γ_0	ϕ	M	β_0	γ_0	ϕ
$M_{1,2,4} = 0$	1	1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$M_{8,7,4} = 0$	ζ^2	1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \zeta \\ 0 & \zeta & 0 \end{pmatrix}$
$M_{1,2,5} = 0$	1	1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}$	$M_{7,9,4} = 0$	ζ^2	1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$M_{1,2,6} = 0$	1	1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta \end{pmatrix}$	$M_{9,8,4} = 0$	ζ^2	1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \zeta^2 \\ 0 & \zeta^2 & 0 \end{pmatrix}$
$M_{3,1,4} = 0$	1	ζ	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta \end{pmatrix}$	$M_{8,7,5} = 0$	ζ^2	ζ	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \zeta^2 \\ 0 & \zeta & 0 \end{pmatrix}$
$M_{3,1,5} = 0$	1	ζ	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$M_{7,9,5} = 0$	ζ^2	ζ	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \zeta \\ 0 & 1 & 0 \end{pmatrix}$
$M_{3,1,6} = 0$	1	ζ	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}$	$M_{9,8,5} = 0$	ζ^2	ζ	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \zeta^2 & 0 \end{pmatrix}$
$M_{2,3,4} = 0$	1	ζ^2	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}$	$M_{8,7,6} = 0$	ζ^2	ζ^2	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \zeta & 0 \end{pmatrix}$
$M_{2,3,5} = 0$	1	ζ^2	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta \end{pmatrix}$	$M_{7,9,6} = 0$	ζ^2	ζ^2	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \zeta^2 \\ 0 & 1 & 0 \end{pmatrix}$
$M_{2,3,6} = 0$	1	ζ^2	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$M_{9,8,6} = 0$	ζ^2	ζ^2	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \zeta \\ 0 & \zeta^2 & 0 \end{pmatrix}$

Table 10.4. Matrices $M_{i,j,k}(\beta_0, \gamma_0)$ with $\beta_0, \gamma_0 \in \mathbb{C}^*$ such that $\mathrm{Null}\big(M_{i,j,k}(\beta_0,\gamma_0)\big)$ contains a vector with all non-zero coordinates, with the associated $\varphi \in \mathrm{PGL}_3$

PGL_3-conjugates, and if $d = g$, then $\mathrm{Aut}(f_{b,d,g})$ contains the transposition $[X, Z, Y]$, so is isomorphic to $C_3 \rtimes C_2$.

***abcdeg* $\neq$ 0 and *bdg* = 8*ace*.** We want to make a change of variables so that $b = 2a$, $d = 2c$ and $g = 2e$. Conjugating by $[uX, vY, wZ]$ with

$$u = (adg)^{-1/3}, \quad v = (bde)^{-1/3}, \quad w = (bcg)^{-1/3}$$

puts f into this form, i.e.,

$$f(X,Y,Z) = \big[a(X^2 + 2YZ), c(Z^2 + 2XY), e(Y^2 + 2XZ)\big]. \qquad (10.10)$$

N.B. This only works because of our assumption that $bdg = 8ace$. So we are reduced to studying $f = f_{a,c,e}$ in this normalized form. The critical locus is

$$\mathrm{Crit}(f_{a,c,e}) = \{X^3 + Y^3 + Z^3 - 3XYZ = 0\}.$$

The critical locus decomposes as a union of three lines via the factorization

$$X^3 + Y^3 + Z^3 - 3XYZ = (X + Y + Z)(X + \zeta Y + \zeta^2 Z)(X + \zeta^2 Y + \zeta Z).$$

The pairwise intersections of these three lines in $\mathrm{Crit}(f_{a,c,e})$ gives the following set of three points,

$$\big\{[1,1,1], [1,\zeta,\zeta^2], [1,\zeta^2,\zeta]\big\}.$$

Every $\varphi \in \operatorname{Hom}(f, f')$ stabilizes this set, so has the form

$$\varphi = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 \\ 1 & \zeta^2 & \zeta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \pi \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta^2 & \zeta \\ 1 & \zeta & \zeta^2 \end{pmatrix} \in \mathrm{PGL}_3$$

for some $\beta, \gamma \in K^*$ and some permutation $\pi \in \mathcal{S}_3$.

Just as in the previous case, for each $\pi \in \mathcal{S}_3$ we compute f^φ and pick out the coefficients of the 12 monomials that do not appear in f'. Each of those coefficients is a linear combination of a, c, e, with coefficients that are polynomials in β and γ. Just as in (10.8), we accumulate this data in the form

$$\begin{pmatrix} \text{coeff of } Y^2 \text{ in } X\text{-coord of } f^\varphi \\ \text{coeff of } Z^2 \text{ in } X\text{-coord of } f^\varphi \\ \vdots \\ \text{coeff of } XY \text{ in } Z\text{-coord of } f^\varphi \\ \text{coeff of } YZ \text{ in } Z\text{-coord of } f^\varphi \end{pmatrix} = M_\pi \begin{pmatrix} a \\ c \\ e \end{pmatrix}, \tag{10.11}$$

where M_π is a 12-by-3 matrix whose entries are polynomials in the ring $\mathbb{Q}(\zeta)[\beta, \gamma]$. (At times we may write $M_\pi(\beta, \gamma)$ to indicate the dependence of M_π on β and γ.) The fact that f^φ is not allowed to have any of the indicated monomials implies that (10.11) is the zero vector, and then our assumption that a, c, e do not vanish implies that the matrix M_π has rank at most 2. (Indeed, it implies the far stronger statement that the column null space of M_π contains a vector whose coordinates are all non-zero.)

There are only 6 choices for $\pi \in \mathcal{S}$, which we compute in turn. For each π we computed the determinants of the 3-by-3 minors of M_π. Of these 220 minors, exactly 160 have non-zero determinant, and aside from factors of the form $c\beta^i\gamma^j$ with $c \in \mathbb{Z}$, these 160 non-zero determinants yield exactly 8 distinct polynomials in $\mathbb{Q}(\zeta_3)[\beta, \gamma]$. Taking pairwise resultants to eliminate β (respectively γ) and then pairwise gcds, we find that a necessary condition for $\operatorname{rank} M_\pi(\beta, \gamma) \leqslant 2$ is that β and γ satisfy

$$\beta^7 = \gamma^7 = 1.$$

For each such pair (β, γ), we compute the null space of $M_\pi(\beta, \gamma)$ and use it to find all maps f of the form (10.10) and all φ such that f^φ has the same form.

For example, taking $\beta = \gamma = 1$ gives $M_\pi(1, 1) = 0$, so we obtain maps φ that are allowed for every f. More precisely, the elements $\pi \in \mathcal{S}_3$ of order dividing 3 give the three maps

$$[X, Y, Z], [X, \zeta Y, \zeta^2 Z], [X, \zeta^2 Y, \zeta Z] \in \operatorname{Aut}(f)$$

that we already know are in $\operatorname{Aut}(f)$, while the three elements $\pi \in \mathcal{S}_3$ of order 2 give maps φ satisfying $f^\varphi_{a,c,e} = f_{a,e,c}$.

More interesting are the cases for which $\beta = \zeta_7$ is a primitive 7'th root of unity. Writing ζ_3 instead of ζ for our chosen primitive cube root of unity, we find that $\beta = \zeta_7$ is possible in exactly the following situations:

$$\gamma = \zeta_7^3 \quad \text{and} \quad [a, c, e] = [1, \zeta_3, \zeta_3^2],$$
$$\gamma = \zeta_7^5 \quad \text{and} \quad [a, c, e] = [1, \zeta_3^2, \zeta_3].$$

Since we know how to swap c and e, it suffices to consider the first case, for which we find that $\mathrm{Aut}(f)$ contains the following element of order 7:

$$\begin{pmatrix} \zeta_7^3+\zeta_7+1 & \zeta_3\zeta_7^3-\zeta_3^2\zeta_7+1 & \zeta_3^2\zeta_7^3+\zeta_3\zeta_7+1 \\ \zeta_3^2\zeta_7^3+\zeta_3\zeta_7+1 & \zeta_7^3+\zeta_7+1 & \zeta_3\zeta_7^3-\zeta_3^2\zeta_7+1 \\ \zeta_3\zeta_7^3-\zeta_3^2\zeta_7+1 & \zeta_3^2\zeta_7^3+\zeta_3\zeta_7+1 & \zeta_7^3+\zeta_7+1 \end{pmatrix} \in \mathrm{Aut}\big(f_{1,\zeta_3,\zeta_3^2}\big).$$

Of course, $\mathrm{Aut}(f)$ also contains the powers of this map, and composing with one of the transpositions in $\mathcal{S}_3$ gives a map in $\mathrm{Hom}\big(f_{1,\zeta_3,\zeta_3^2}, f_{1,\zeta_3^2,\zeta_3}\big)$.

We note that these two maps f for which $\mathrm{Aut}(f)$ contains an element of order 7 are both conjugate to the map $[Z^2, X^2, Y^2]$ that we studied in Propositions 6.1 and 6.2. Indeed, one finds that $\varphi = \begin{pmatrix} 1 & \zeta_3^2 & \zeta_3 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & 1 & 1 \end{pmatrix}$ gives

$$[Z^2, X^2, Y^2]^\varphi = [X^2 + 2YZ, \zeta_3(Z^2 + 2XY), \zeta_3^2(Y^2 + 2XZ)].$$

So we can refer to Proposition 6.2 for the fact that these maps have automorphism group exactly equal to $C_7 \rtimes C_3$.

It remains to check that maps of Types $C_3(n)$ and $C_3(n')$ are not $N(\mathcal{G}_3)$-conjugate for $n \neq n'$, nor indeed are they PGL_3-conjugate. To do this, we note that the only case in which the indeterminacy and critical loci for $C_3(n)$ and $C_3(n')$ have the same geometry is

$$C_3(2) \stackrel{?}{=} C_3(3), \quad I(f) = 1 \text{ point}, \quad \mathrm{Crit}(f) = \text{triple line}.$$

Any $\varphi \in \mathrm{Hom}(f, f')$ preserves the geometry of these loci, so we ask if the maps

$$f = [aX^2 + YZ, XY, Y^2] \quad \text{and} \quad f' = [bYZ, Z^2 + XY, Y^2]$$

can be conjugate to one another. Since $\mathrm{Crit}(f) = \mathrm{Crit}(f')$ is the line $Y = 0$ (with multiplicity 3), we must have $\varphi\big(\{Y = 0\}\big) = \{Y = 0\}$. But we observe that

$$f\big(\mathrm{Crit}(f)\big) = f\big(\{Y = 0\}\big) = \big\{[1, 0, 0]\big\} \in \mathrm{Crit}(f),$$
$$f'\big(\mathrm{Crit}(f')\big) = f'\big(\{Y = 0\}\big) = \big\{[0, 1, 0]\big\} \notin \mathrm{Crit}(f').$$

Hence if there were a map $\varphi \in \mathrm{Hom}(f, f')$, then we would find that

$$\begin{aligned} f'\big(\mathrm{Crit}(f')\big) &= f^\varphi\big(\mathrm{Crit}(f^\varphi)\big) = f^\varphi\big(\mathrm{Crit}(f)^\varphi\big) \\ &= \Big(f\big(\mathrm{Crit}(f)\big)\Big)^\varphi \in \mathrm{Crit}(f)^\varphi = \mathrm{Crit}(f^\varphi) = \mathrm{Crit}(f'). \end{aligned}$$

This contradiction shows that f and f' are not $\mathrm{PGL}_3(K)$-conjugate. $\square$

Proof of Proposition 9.2. — The map $[X, \zeta Y, Z]$ is the map τ_0 defined in Section 5. Setting $m = 0$ in Table 5.1 and reducing the entries modulo 3 yields

	X^2	Y^2	Z^2	XY	XZ	YZ
X-coord	0	2	0	1	0	1
Y-coord	2	1	2	0	2	0
Z-coord	0	2	0	1	0	1

Hence assuming that $\tau_0 \in \mathrm{Aut}(f)$ leads to the following three families of maps:

$$\begin{aligned}
f_{0,0} &:= [aX^2 + bZ^2 + cXZ, dXY + eYZ, gX^2 + hZ^2 + iXZ],\\
f_{0,1} &:= [aXY + bYZ, cY^2, dXY + eYZ],\\
f_{0,2} &:= [aY^2, bX^2 + cZ^2 + dXZ, eY^2].
\end{aligned}$$

For the first two maps we use Table 5.2 to compute

$$\begin{aligned}
\mu^{\mathcal{O}(1)}(f_{0,0}, L_{k,\ell}) &\leqslant \{-k, 3k + 2\ell, k + \ell, -3k - \ell\} \xrightarrow{(k,\ell)=(1,-2)} -1,\\
\mu^{\mathcal{O}(1)}(f_{0,1}, L_{k,\ell}) &\leqslant \{-\ell, -2k, -2k - 2\ell\} \xrightarrow{(k,\ell)=(1,1)} -1.
\end{aligned}$$

Hence $f_{0,0}$ and $f_{0,1}$ are $\mathcal{D}$-unstable. (The latter also has degree 1, of course.) And finally, we see that the map $f_{0,2}$ is not dominant, since its image is contained in the line $\{eX = aZ\}$. (Or, if $a = e = 0$, then $f_{0,2}(\mathbb{P}^2) = [0, 1, 0]$.) $\square$

11. Proof of Theorems 1.2 and 1.3 and of Corollary 1.4

In this and the next section, we use our accumulated results to prove Theorems 1.2 and 1.3 and Corollary 1.4. We recall that the assumptions for both the theorem and corollary are that $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a dominant rational map of degree 2 lying in the semi-stable locus of Rat_2^2 and such that $\infty > \#\,\mathrm{Aut}(f) \geqslant 3$.

We remark that the computation of the indeterminacy and critical loci of maps in the various families as described in Table 1.1 is an elementary, albeit tedious, calculation, so in some cases we have omitted the details.

We start with the assumption that $\mathrm{Aut}(f)$ is finite and of order at least 3. Let G be a finite group of order at least 3. Then either $p \mid \#G$ for some odd prime p, or else G is a 2-group of order at least 4. In the former case, Cauchy's theorem says that G contains an element of order p, while in the latter case, the strong form of the first Sylow theorem [8, Theroem 2.12.1] says that G contains a subgroup of order 4, hence contains a copy of either C_2^2 or C_4.

Suppose that there is an odd prime p with $p \mid \mathrm{Aut}(f)$, and let $G \subset \mathrm{Aut}(f)$ be a subgroup of order p. Lemma 4.1(a) tells us that after $\mathrm{PGL}_3(K)$-conjugation, we may assume that

$$G = \left\langle \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^m \end{smallmatrix} \right) \right\rangle$$

for some primitive p'th root of unity ζ and some integer m. For future reference, we also remark that when $m = 1$, we may instead use the map $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right)$, which is PGL_3-conjugate to $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta \end{smallmatrix} \right)$.

We consider the case that $p \geqslant 5$. Then Proposition 6.1 and the semistability assumption tell us that f has one of the following three forms:

$$[aX^2 + YZ, bXY, cXZ], \quad [YZ, X^2, Y^2], \quad [Z^2, X^2, Y^2].$$

Proposition 6.2 says that maps f in the first family have a copy of $\mathbb{G}_m$ in $\mathrm{Aut}(f)$, while the second map satisfies $\mathrm{Aut}(f) \cong C_5$, and the third map satisfies $\mathrm{Aut}(f) \cong C_7 \rtimes C_3$. (A conjugate of this last case also appears in the compendium of maps for which $\mathrm{Aut}(f)$ contains an element of order 3.) This completes the proof of the automorphism parts of Theorems 1.2 and 1.3 and of Corollary 1.4 in the case that $\#\mathrm{Aut}(f)$ is divisible by a prime $p \geqslant 5$.

Similarly, if $p = 3$, i.e., if $3 \mid \#\mathrm{Aut}(f)$, then the automorphism parts of Theorems 1.2 and 1.3 and of Corollary 1.4 can be deduced from Proposition 10.1, although we need to do some work to put the maps into the indicated forms. (Using the above remark and Proposition 10.1(b), we need only consider the case $m = 2$.)

Thus let $f = [aX^2 + bYZ, cZ^2 + dXY, eY^2 + gXZ]$. If $ace \neq 0$, then the transformation

$$\psi(X, Y, Z) = \left[(ce)^{2/3}X, ac^{1/3}Y, ae^{1/3}Z\right]$$

yields

$$f^\psi(X, Y, Z) = [X^2 + ab(ce)^{-1}YZ, Z^2 + a^{-1}dXY, Y^2 + a^{-1}gXZ].$$

In other words, if $ace \neq 0$, we can find a normal form for f with $a = c = e = 1$. We apply this transformation to the maps of Type $C_3(1)$, $C_3(5)$, $C_3(6)$, $C_3(7)$, and $C_3(8)$ in Table 10.1, which serves to give them a more uniform description. Types $C_3(1)$ and $C_3(7)$ already have this form.

For $C_3(5)$ we find that

$$f^\psi(X, Y, Z) = [X^2 - a^2b^{-1}YZ, Z^2 - a^{-1}XY, Y^2 - a^{-1}bXZ].$$

Thus

$$f^{\psi}(X,Y,Z) = [X^2 + BYZ, Z^2 + DXY, Y^2 + GXZ] \quad \text{with } BDG = -1.$$

Further, the conditions that a and/or b equal 1 become

$$\begin{aligned} a = b = 1 &\iff D = G = -1, & a = 1,\ b \neq 1 &\iff D = -1 \neq G,\\ a \neq 1,\ b = 1 &\iff D = G \neq -1, & a \neq 1,\ b \neq 1 &\iff D \neq -1,\ D \neq G. \end{aligned}$$

For $C_3(6)$ we find that

$$f^{\psi}(X,Y,Z) = [X^2 + abYZ, Z^2 + a^{-1}XY, Y^2].$$

So with a slight relabeling, Type $C_3(6)$ becomes simply

$$f^{\psi}(X,Y,Z) = [X^2 + bYZ, Z^2 + dXY, Y^2],$$

i.e., it's $C_3(7)$ with $g = 0$.

And for $C_3(8)$, we find that

$$f^{\psi}(X,Y,Z) = [X^2 + 2(ce)^{-1}YZ, Z^2 + 2cXY, Y^2 + 2eXZ].$$

In other words, we obtain the $C_3(7)$ form with $bdg = 8$. Further, the only cases with $\mathrm{Aut}(f) \neq C_3$ are c a primitive cube root of unity and $e = c^2$, with these two cases being conjugate. In particular, $ce = 1$.

We next consider Types $C_3(2)$, $C_3(3)$, and $C_3(4)$. We claim that they may all be put into the form $f_{a,c,g} := [aX^2 + YZ, cZ^2 + XY, Y^2 + gXZ]$ with $(a,b,g) \neq (0,0,0)$ and one or more of a,c,g equal to 0. For Type $C_3(2)$, we already have $f = f_{a,0,g}$. For Types $C_3(3)$ and $C_3(4)$, which have the form $f = [bYZ, Z^2 + XY, Y^2 + gXZ]$ with $b \neq 0$, the transformation $\psi = [b^{2/3}X, b^{1/3}Y, Z]$ yields $f^{\psi} = [YZ, b^{-1}Z^2 + XY, Y^2 + gXZ]$, so we get maps $f_{0,c,g}$ with $c \neq 0$. We also observe that the PGL_3-conjugacies for Type $C_3(4)$ become the maps $\varphi = [c^{1/3}X, c^{2/3}g^{1/3}Y, g^{2/3}Z]$ which have the effect $f_{0,c,g}^{\varphi} = f_{0,c/g,1/g}$.

We next consider the maps such that $\mathrm{Aut}(f)$ contains an element of order 4. The description of these maps in Proposition 9.1 is already in the form that we want.

Finally we consider the maps such that $\mathrm{Aut}(f)$ contains a subgroup of type C_2^2. To fit these maps, which are described in Proposition 8.1, into a single family, we apply the transformation $\psi = [X/\sqrt{d}, Y, Z]$ to the map $f = [X^2+Y^2-Z^2, dXY, eXZ]$. This gives $f^{\psi} = [d^{-1}X^2+Y^2-Z^2, XY, ed^{-1}XZ]$, so these maps have the form $f_{a,e} = [aX^2 + Y^2 - Z^2, XY, eXZ]$. Then one family in Proposition 8.1 is $f_{a,e}$ with $a \neq 0$ and the other family is $f_{0,e}$.

We conclude this section with the one part of Corollary 1.4 that is not immediate from the main theorems. Corollary 1.4 (b) asserts that each group listed in (a) occurs as the full automorphism group of a semistable map of degree 2. Theorems 1.2 and 1.3, together with appropriately chosen maps from Table 1.1, take care of $G \in \{C_3, C_4, C_5, C_2^2\}$. That same table gives us maps with $\mathrm{Aut}(f) \in \{S_3, S_4, C_7 \rtimes C_3\}$. For these groups it suffices to point out that the identity component of the normalizers of the associated subgroups of PGL_3 is in all cases equal to the group of diagonal matrices, and hence semistability for (say) $\mathcal{G}_3$ or $\mathcal{G}_7$ gives semistability for the larger group. It remains to deal with the cases $G = C_1$ and $G = C_2$.

Consider the family of maps
$$f_{u,v} = [YZ, X^2, uXY + vY^2].$$
Then $\mathrm{Aut}(f_{1,0}) \cong C_4$ and $\mathrm{Aut}(f_{0,1}) \cong C_5$. Hence the generic member of this family has $\mathrm{Aut}(f_{u,v}) = C_1$.

Similarly, consider the family of maps
$$f_{u,v,w} = [uY^2 + vZ^2, XY, wY^2 + 2XZ].$$
Then $\mathrm{Aut}(f_{0,1,1}) \cong C_4$ and $\mathrm{Aut}(f_{1,-1,0}) \cong C_2^2$, so generically $\mathrm{Aut}(f_{u,v,w})$ is either C_1 or C_2. Since $\varphi = [X, -Y, Z] \in \mathrm{Aut}(f_{u,v,w})$, we conclude that a generic map in the family satisfies $\mathrm{Aut}(f_{u,v,w}) \cong C_2$.

12. Computation of Dynamical and Topological Degrees

In this section we compute the dynamical and topological degrees of the various maps in Table 1.1. The following elementary results will be useful, especially in establishing that a map is algebraically stable, i.e., satisfies $\lambda_1(f) = \deg(f)$. Results of this sort, and much more, appear in [4], but for the convenience of the reader, we give the short proofs.

Lemma 12.1. — *Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a dominant rational map.*

(a) *If f is a morphism, then*
$$\lambda_1(f) = \deg(f) \quad \textit{and} \quad \lambda_2(f) = \deg(f)^2.$$
(b) $\lambda_1(f) < \deg(f)$ *if and only if there is a curve $\Gamma \subset \mathbb{P}^2$ and an integer $n \geqslant 1$ such that $f^n(\Gamma) \subseteq I(f)$.*
(c) *Let $\Gamma \subset \mathbb{P}^2$ be a curve such that $f(\Gamma)$ consists of a single point. Then $\Gamma \subseteq \mathrm{Crit}(f)$.*

Proof. — (a) This is standard and elementary.

(b) Let
$$f(X, Y, Z) = [F_1(X, Y, Z), G_1(X, Y, Z), H_1(X, Y, Z)],$$

and define inductively

$$[F_{n+1}, G_{n+1}, H_{n+1}] = [F_n(F_1, G_1, H_1), G_n(F_1, G_1, H_1), H_n(F_1, G_1, H_1)].$$

Then

$$f^n(X, Y, Z) = [F_n(X, Y, Z), G_n(X, Y, Z), H_n(X, Y, Z)],$$

so $\lambda_1(f) < \deg(f)$ if and only if there exists an n such that F_n, G_n, H_n have a non-trivial common factor in $K[X, Y, Z]$. Taking the smallest such n, if $R(X, Y, Z)$ is the common factor, then the curve $\Gamma = \{R = 0\}$ satisfies $f^{n-1}(\Gamma) \subset I(f)$.

(c) This is just the chain rule. We view Γ as an abstract curve with an embedding $j : \Gamma \hookrightarrow \mathbb{P}^2$. Differentiating the constant function $f \circ j$ gives $(Df \circ j) \cdot \nabla j = 0$. The fact that j is non-constant, i.e., Γ is a curve, tells us that $\nabla j(t) \neq \mathbf{0}$ for all but finitely many $t \in \Gamma$, and hence that $\det Df\bigl(j(t)\bigr) = 0$ for all but finitely many $t \in \Gamma$. Taking Zariski closures gives $\Gamma = \operatorname{Image}(j) \subset \{\det Df = 0\} = \operatorname{Crit}(f)$. $\square$

The remainder of this section is devoted to computing the dynamical and topological degrees of the maps in Table 1.1.

Types 1.1, 1.3, 1.5, 1.7.1.8, 3.4, 8.1.
These maps are morphisms, so Lemma 12.1 gives $\lambda_1(f) = 2$ and $\lambda_2(f) = 4$.

Types 1.2 ($\boldsymbol{C_3}$) : $f = [X^2 - YZ, Z^2 - XY, Y^2 - XZ]$.
A short calculation shows that $f^2(X, Y, Z) = [X, Y, Z]$, so $\deg(f^n)$ alternates between 2 and 1. In particular, $\lambda_1(f) = \lambda_2(f) = 1$.

Types 1.4, 1.6 ($\boldsymbol{\mathcal{G}_3}$) : $f = [X^2 + bYZ, Z^2 + dXY, Y^2 + gXZ]$, $bdg = -1$.
The critical locus has the equation $dgX^3 + bdY^3 + bgZ^3 - 3XYZ = 0$. Using the assumption that $bdg = -1$, we find that the critical locus is the union of the three lines

$$L_k = \{(dg)^{1/3}X + \zeta_3^k(bd)^{1/3}Y + \zeta_3^{2k}(bg)^{1/3}Z = 0\}, \quad k = 0, 1, 2,$$

where ζ_3 is a primitive cube root of unity. We compute

$$f(L_k) = \bigl\{[b^{2/3}, \zeta_3^k d^{2/3}, \zeta_3^{2k} g^{2/3}]\bigr\} \quad \text{for } k = 0, 1, 2.$$

In particular, $f\bigl(\operatorname{Crit}(f)\bigr)$ consists of three points. The indeterminacy locus of f is

$$I(f) = \bigl\{[b^{1/3}, g^{1/3}, d^{1/3}], [b^{1/3}, \zeta_3 g^{1/3}, \zeta_3^2 d^{1/3}], [b^{1/3}, \zeta_3^2 g^{1/3}, \zeta_3 d^{1/3}]\bigr\},$$

where $b^{1/3}$ and $d^{1/3}$ are arbitrary fixed cube roots of b and d, and then $g^{1/3}$ is set equal to $-1/b^{1/3}d^{1/3}$. Thus for generic values of b, d, g (satisfying $bdg = -1$), the orbits of the three points $f(L_1), f(L_2), f(L_3)$ will not hit $I(f)$, so generically we have $\lambda_1(f) = 2$. However, we expect that there is a countable set of values of b, d, g such that $\lambda_1(f) < 2$. To see why, we observe that for each $n \geqslant 1$, the equation

$$f^n(L_0) = f^{n-1}(b^{2/3}, d^{2/3}, g^{2/3}) = [b^{1/3}, g^{1/3}, d^{1/3}] \in I(f)$$

yields two homogeneous polynomial equations for the point $[b^{1/3}, g^{1/3}, d^{1/3}]$. Substituting the inhomogeneous value $g^{1/3} = 1/b^{1/3}d^{1/3}$ yields two inhomogeneous polynomials equations for $(b^{1/3}, d^{1/3})$, and hence a finite number of values for (b, d, g). Varying n should then yield a countable number of exceptional values of (b, d, g) with $f^n(L_0) \in I(f)$, and thus with $\lambda_1(f) < 2$. This applies to Type 1.6, for which b, d, g satisfy the single relation $bdg = -1$. For maps of Type 1.4 with $(b, d, g) = (b, b^{-1}, -1)$, there is only one degree of freedom, so it seems plausible that for these maps we have $\lambda_1(f) = 2$.

In order to compute the topological degree, we set $f(X, Y, Z) = [\alpha, \beta, 1]$ for generic α, β and solve for $[X, Y, Z]$. This gives two equations

$$X^2 + bYZ = \alpha(Y^2 + gXZ), \quad Z^2 + dXY = \beta(Y^2 + gXZ).$$

We dehomogenize $x = X/Z$ and $y = Y/Z$. Then we can solve the second equation for x and substitute into the first equation to find

$$(\beta^2 - d^2\alpha)b^2d^2y^4 + (d^2b^2 - \beta\alpha)bd^2y^3 + (\beta^2 - d^2\alpha)by + (d^2b^2 - \alpha\beta) = 0.$$

The discriminant of this quartic equation is a mess, but part of it looks like

$$\mathrm{Disc}(f) = -27b^8d^4(\beta^6 - b^4d^8)^2 + \alpha \cdot \big(\text{polynomial in } \mathbb{Z}[b, d, \alpha, \beta]\big).$$

In particular, since $bdg = -1$, we see that for generic α, β, the quartic has distinct roots. If those roots lead to points not in $I(f)$, which we expect to be true for most (b, d, g) triples, then $\lambda_2(f) = \#f^{-1}(\alpha, \beta) = 4$. On the other hand, the general inequality $\lambda_2 \leqslant \lambda_1^2$ shows that we should expect $\lambda_2 < 4$ for countably many (b, d, g).

We illustrate with the extreme case $b = d = g = -1$, which is the map of Type 1.2. In that case, the discriminant quartic factors (essentially) as

$$(y^3 - 1)\left(y - \frac{1 - \alpha\beta}{\beta^2 - \alpha}\right).$$

The three roots with $y \in \boldsymbol{\mu}_3$ lead to points in $I(f)$, so we find that $\#f^{-1}(\alpha, \beta) = 1$, which confirms our earlier computation.

Type 2.1 $(\mathcal{G}_3)$: $f = [YZ, XY, Y^2 + gXZ]$, $g \neq 0$.
Here $\mathrm{Crit}(f)$ is a line and a conic and $\#I(f) = 2$,

$$\mathrm{Crit}(f) = \{Y = 0\} \cup \{Y^2 = gXZ\}, \quad I(f) = \big\{[0, 0, 1], [1, 0, 0]\big\}.$$

We have $f\big(\{Y = 0\}\big) = [0, 0, 1] \in I(f)$, so $\lambda_1(f) < 2$. The degree sequence of f is $2, 3, 5, 8, 13, 21, 34, \ldots$, which suggests that $\deg(f^n)$ is the $(n + 2)$'nd Fibonacci number.

We dehomogenize $Y = 1$, so $f(x,z) = \bigl(z/x, (1+gxz)/x\bigr)$. Setting $f(x,z) = (\alpha,\beta)$ with generic α,β, we find that $z = \alpha x$ and $\alpha g x^2 - \beta x + 1 = 0$. Since $g \neq 0$, we have $\lambda_2(f) = \#f^{-1}(\alpha,\beta) = 2$.

Types 2.2, 2.6 ($\mathcal{G}_3$) : $f = [YZ, cZ^2 + XY, Y^2 + gXZ]$, $cg \neq 0$.
The critical locus is a nodal cubic that never gets mapped to a point, so $\lambda_1(f) = 2$. Dehomogenizing with respect to X and setting $\bigl((cz^2+y)/yz, (y^2+gz)/yz\bigr) = (\alpha,\beta)$ with generic α,β leads to $y = cz^2/(\alpha z - 1)$ and $c(c-\alpha\beta)z^3 + (c\beta - g\alpha^2)z^2 + 2\alpha g z - g = 0$. (Note that $z = 0$ is not a valid value.) Hence $\lambda_2(f) = 3$.

Types 2.3 ($\mathcal{G}_3$) : $f = [YZ, cZ^2 + XY, Y^2]$, $c \neq 0$.
Here $I(f) = \bigl\{[1,0,0]\bigr\}$ and $\operatorname{Crit}(f) = 3 \cdot \{Y = 0\}$. We have

$$\{Y=0\} \xrightarrow{f} [0,1,0] \xrightarrow{f} [0,0,1] \xrightarrow{f} [0,1,0],$$

so although the critical locus maps to a point, that point is part of a 2-cycle, so it never hits $I(f)$. Hence $\lambda_1(f) = 2$. Further, $f^{-1}(X,Y,Z) = [b^2YZ - X^2, b^2Z^2, bXZ]$, so f is birational and $\lambda_2(f) = 1$.

Type 2.4 ($\mathcal{G}_3$) : $f = [aX^2 + YZ, XY, Y^2]$, $a \neq 0$.
The critical locus is a triple line, $\operatorname{Crit}(f) = 3 \cdot \{Y = 0\}$, and we have $f\bigl(\{Y = 0\}\bigr) = [1,0,0] \in \operatorname{Fix}(f)$. Hence by the usual argument via Lemma 12.1, we find that $\lambda_1(f) = 2$. Further, $f^{-1}(X,Y,Z) = [YZ, Z^2, XZ - aY^2]$ shows that f is birational, so $\lambda_2(f) = 1$.

Type 3.1 ($\mathcal{G}_4$) : $f = [Z^2, XY, Y^2]$, $a = e = 0$.
The dynamical degree of a monomial map may be computed using the formula in [7]. In affine coordinates the map is $f(x,y) = (y^{-2}, xy^{-1})$ with exponent matrix $\bigl(\begin{smallmatrix} 0 & -2 \\ 1 & -1 \end{smallmatrix}\bigr)$. Then [7] says that $\lambda_1(f)$ is the spectral radius of the exponent matrix, so $\lambda_1(f) = \left|\frac{-1+\sqrt{-7}}{2}\right| = \sqrt{2}$. And setting $f(x,y) = (\alpha,\beta)$ with generic α,β, we see that $f^{-1}(\alpha,\beta)$ consists of the two points $(\beta\gamma,\gamma)$ with $\gamma^2 = \alpha^{-1}$. Hence $\lambda_2(f) = 2$.

Type 3.2 ($\mathcal{G}_4$) : $f = [Z^2, XY, Y^2 + eXZ]$, $e \neq 0$.
We have

$$I(f) = \bigl\{[1,0,0]\bigr\} \quad \text{and} \quad \operatorname{Crit}(f) = \{Z = 0\} \cup \{eXZ = 2Y^2\}.$$

Both $f\bigl(\{Z=0\}\bigr)$ and $f\bigl(\{eXZ = 2Y^2\}\bigr)$ are curves, so $\lambda_1(f) = 2$. Next we dehomogenize with $Z = 1$. Then $f(x,y) = \bigl((y^2+ex)^{-1}, xy(y^2+ex)^{-1}\bigr)$. Setting $f(x,y) = (\alpha,\beta)$ with generic α,β leads to $x = \alpha^{-1}\beta y^{-1}$ and $\alpha y^3 - y + \beta e = 0$, so $\lambda_2(f) = \#f^{-1}(\alpha,\beta) = 3$.

Type 3.3 ($\mathcal{G}_4$) : $f=[aX^2+Z^2,XY,Y^2]$, $a\neq 0$.
We have

$$I(f)=\bigl\{[1,0,\pm\sqrt{-a}]\bigr\}\quad\text{and}\quad \mathrm{Crit}(f)=\{Z=0\}\cup 2\cdot\{Y=0\}.$$

The orbit of the line $Y=0$ is

$$\{Y=0\}\xrightarrow{f}\bigl\{[1,0,0]\bigr\}\xrightarrow{f}\bigl\{[1,0,0]\bigr\},$$

so the orbit of $Y=0$ never lands in $I(f)$. The image of the line $Z=0$ is a curve,

$$\{Z=0\}\xrightarrow{f}\{XZ=aY^2\}.$$

Hence $\lambda_1(f)=2$. Next we dehomogenize with $Z=1$. Then $f(x,y)=\bigl((ax^2+1)/y^2,x/y\bigr)$. Setting $f(x,y)=(\alpha,\beta)$ with generic α,β leads to $x=\beta y$ and $(a\beta^2-a)y^2+1=0$. Thus $\lambda_2(f)=\#f^{-1}(\alpha,\beta)=2$.

Types 4.1, 4.2, 4.3, 4.4($\mathcal{G}_4$) : $f=[YZ,X^2+cZ^2,XY]$.
Proposition 9.3 says that f satisfies $\deg(f^n)\leqslant 2n+1$, so $\lambda_1(f)=1$. Since $\lambda_2\leqslant\lambda_1^2$ in general, it follows that $\lambda_2(f)=1$. (Alternatvely, it is not hard to write down the inverse map.)

Types 5.1, 5.2, 5.3 ($\mathcal{G}_{2,2}$) : $f=[Y^2-Z^2,XY,eXZ]$, $e\neq 0$.
Proposition 8.2 tells us that $\deg(f^n)\leqslant n+1$, so just as in the previous case, we have $\lambda_1(f)=\lambda_2(f)=1$.

Types 5.4, 5.5($\mathcal{G}_{2,2}$) : $f=[aX^2+Y^2-Z^2,XY,eXZ]$, $ae\neq 0$.
We have

$$I(f)=\bigl\{[0,1,1],[0,1,-1]\bigr\}$$

and

$$\mathrm{Crit}(f)=\{X=0\}\cup\{aX^2-Y^2+Z^2=0\}.$$

The line $X=0$ is sent to a fixed point of f,

$$f\bigl(\{X=0\}\bigr)=[1,0,0]\in\mathrm{Fix}(f),$$

while the conic $aX^2-Y^2+Z^2=0$ is sent to another conic,

$$\begin{aligned}
&f\bigl(\{aX^2-Y^2+Z^2=0\}\bigr)\\
&\qquad=\bigl\{[aX^2+Y^2-Z^2,XY,eXZ]:aX^2-Y^2+Z^2=0\bigr\}\\
&\qquad=\bigl\{[2aX^2,XY,eXZ]:aX^2-Y^2+Z^2=0\bigr\}\\
&\qquad=\bigl\{[2aX,Y,eZ]:aX^2-Y^2+Z^2=0\bigr\}\\
&\qquad=\bigl\{[u,v,w]:a(u/2a)^2-v^2+(w/e)^2=0\bigr\}\\
&\qquad=\bigl\{[X,Y,Z]:X^2/4a-Y^2+Z^2/e^2=0\bigr\}.
\end{aligned}$$

Lemma 12.1 (c) tells us that the only curve in $\mathbb{P}^2$ that f maps to a point is the line $\{X = 0\}$. Now suppose that $\lambda_1(f) < 2$. Then Lemma 12.1 (b) says that there is a curve $\Gamma \subset \mathbb{P}^2$ and an $n \geqslant 1$ such that $f^n(\Gamma) \subset I(f)$. In particular, $f^n(\Gamma)$ is a point, so there is some $0 \leqslant m < n$ such that $f^m(\Gamma)$ is a curve and $f^{m+1}(\Gamma)$ is a point. But we have just shown that this forces $f^m(\Gamma)$ to be the line $X = 0$ and forces $f^{m+1}(\Gamma)$ to be the point $[1, 0, 0]$, which is not in $I(f)$. This completes the proof that $\lambda_1(f) = 2$.

To compute the topological degree $\lambda_2(f)$, we compute the inverse image of a generic point. To simplify the calculation, we compute $[x, y, z] \in f^{-1}(\alpha, \beta, e\gamma)$ with α, β, γ generic. From the last two coordinates we have $e\gamma xy = \beta exz$, so $x = 0$ or $\gamma y = \beta z$. We cannot have $x = 0$, since $f(0, Y, Z) = [1, 0, 0]$. Hence our point has the form $[x, y, z] = [\beta x, \beta y, \beta z] = [\beta x, \beta y, \gamma y]$, so the homogeneous point $[x, y] \in \mathbb{P}^1$ determines $[x, y, z]$. We next use the first two coordinates to deduce that $\beta(ax^2 + y^2 - z^2) = \alpha dxy$. Multiplying this by β and using $\beta z = \gamma y$ yields $(\beta^2 ax^2 + \beta^2 y^2 - \gamma^2 y^2) = \alpha\beta dxy$. Since α, β, γ are generic, this equation has two solutions in $\mathbb{P}^1$. This shows that $\#f^{-1}(\alpha, d\beta, e\gamma) = 2$, so $\lambda_2(f) = 2$.

Types 2.5 ($\mathcal{G}_3$) : $f = [aX^2 + YZ, XY, Y^2 + gXZ]$, $ag \neq 0$.
Here $\mathrm{Crit}(f)$ is a nodal cubic curve and the image of $\mathrm{Crit}(f)$ is also a curve. It follows from Lemma 12.1 that $\lambda_1(f) = 2$, since no iterate of f maps a curve to a point, much less to a point in $I(f)$. To compute the topological degree, we dehomogenize by setting $x = X/Y$ and $z = Z/Y$ and solving $\bigl((ax^2 + z)/x, (1 + gxz)/x\bigr) = (\alpha, \beta)$ for generic α, β. The first coordinate gives $z = \alpha x - ax^2$, and substituting into the second coordinate gives $1 + gx(\alpha x - ax^2) = \beta x$. Hence $\lambda_2(f) = 3$.

Type 7.1 ($\mathcal{G}_5$) : $f = [YZ, X^2, Y^2]$.
The map f of Type 7.1 is also monomial, but there's an easier way to calculate $\lambda_1(f)$. We note that $f^8 = [X^{16}, Y^{16}, Z^{16}]$, so in particular f^8 is a morphism. Hence

$$\lambda_1(f) = \lambda_1(f^8)^{1/8} = \deg(f^8)^{1/8} = 16^{1/8} = \sqrt{2}.$$

Similarly $\lambda_2(f) = \lambda_2(f^8)^{1/8} = 256^{1/8} = 2$.

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Appendix A. Maps with Infinite Automorphism Group

As indicated earlier, we have restricted attention in this paper to maps whose automorphism group is finite. The reason is because maps with infinite automorphism group decompose as fiber product maps, as described by the following general construction.

Definition. — *Let X and Y be varieties, and let $f : X \to X$ be a dominant rational map. We say that f* descends to Y *if there are dominant rational maps $\pi : X \to Y$ and $g : Y \to Y$ such we have a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{g} & Y \end{array}$$

We note that if $f : X \to X$ descends to Y, then analyzing the dynamics of f may be reduced, in some sense, to analyzing the dynamics of $g : Y \to Y$ and the "twisted" dynamics on the fibers. In particular, if $1 \leqslant \dim(Y) < \dim(X)$, then one is reduced to lower dimensional problems.

We let

$$\mathrm{Aut}(X, f) := \{\varphi \in \mathrm{Aut}(X) : \varphi \circ f = f \circ \varphi\}$$

and note that if the quotient of X by a subgroup $\mathcal{G} \subseteq \mathrm{Aut}(X, f)$ is well-defined, then f descends to the quotient $X/\mathcal{G}$. We give two examples.

Example A.1. — Let

$$f = [YZ, X^2 - Z^2, XY]$$

be the map of Type 4.1 with $c = -1$ in Table 1.1. Then $\mathrm{Aut}(f)$ contains a copy of $\mathbb{G}_m$,

$$\left\{ \left(\begin{smallmatrix} s & 0 & t \\ 0 & 1 & 0 \\ t & 0 & s \end{smallmatrix} \right) : s^2 - t^2 = 1 \right\} \subset \mathrm{Aut}(f).$$

The quotient of f by this subgroup yields the commutative diagram

$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{[YZ, X^2 - Z^2, XY]} & \mathbb{P}^2 \\ \downarrow{\scriptstyle [X^2 - Z^2, Y^2]} & & \downarrow{\scriptstyle [X^2 - Z^2, Y^2]} \\ \mathbb{P}^1 & \xrightarrow{[U,V] \to [-V,U]} & \mathbb{P}^1 \end{array}$$

Of course, the map f is not very interesting dynamically, since $f^2 = [X, -Y, Z]$.

Example A.2. — Here is a more interesting example. The automorphism group of the map

$$f : \mathbb{P}^2 \to \mathbb{P}^2, \quad f(X,Y,Z) = [X^2 + YZ, XY, XZ]$$

contains a copy of $\mathbb{G}_m$ via

$$\left\{ \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-1} \end{smallmatrix}\right) : t \neq 0 \right\} \subset \mathrm{Aut}(f).$$

The quotient of f gives the diagram

$$\begin{tikzcd} \mathbb{P}^2 \arrow[r, "{[X^2+YZ,XY,XZ]}"] \arrow[d, "{[X^2,YZ]}"] & \mathbb{P}^2 \arrow[d, "{[X^2,YZ]}"] \\ \mathbb{P}^1 \arrow[r, "{[U,V]\to[(U+V)^2,UV]}"] & \mathbb{P}^1 \end{tikzcd}$$

Appendix B. Semistability for Subgroups

In this section we give a general result for GIT semistability relative to subgroups. We thank Friedrich Knop for explaining how the following result is a consequence of a general theorem of Luna [12].

Proposition B.1. — *We work over an algebraically closed field K of characteristic* 0*. Let*

$$\begin{aligned} X &= \text{a smooth projective variety}, \\ G &= \text{a reductive group acting linearly on } X, \\ H &= \text{a subgroup of } G \text{ that is also reductive}, \\ N(H) &= \text{the normalizer of } H \text{ in } G. \end{aligned}$$

For $x \in X$, let

$$\begin{aligned} \mathrm{Stab}(x) &= \{g \in G : g \cdot x = x\}, \\ X^H &= \{x \in X : \mathrm{Stab}(x) \text{ contains } H\}. \end{aligned}$$

(a) *Let $x \in X^H$. Then*

$$x \text{ is } N(H)\text{-semistable} \iff x \text{ is } G\text{-semistable}.$$

(b) *The map*

$$(X^H)^{\mathrm{ss}}/N(H) \longrightarrow X^{\mathrm{ss}}/G$$

is a finite map.

(c) *Let $\tilde{X}$ be the affine cone over X, and for $x \in X^H$, let $\tilde{x} \in \tilde{X}$ be a lift. Thus each $x \in X^H$ determines a character χ_x on H via*

$$\chi_x : H \longrightarrow \mathbb{G}_m, \quad h \cdot \tilde{x} = \chi_x(h)\tilde{x}.$$

For each character $\chi \in \hat{H} := \mathrm{Hom}(H, \mathbb{G}_m)$, let $X^H_\chi := \{x \in X^H : \chi_x = \chi\}$. Then $(X^H)^{\mathrm{ss}}$ decomposes as a disjoint union

$$(X^H)^{\mathrm{ss}} = \bigcup_{\chi \in \hat{H}} (X^H_\chi)^{\mathrm{ss}}.$$

(d) *For $\chi \in \hat{H}$ and $\pi \in N(H)$, define $\chi^\pi \in \hat{H}$ by $\chi^\pi(h) = \chi(\pi^{-1}h\pi)$. Then there is an isomorphism*

$$X^H_\chi \longrightarrow X^H_{\chi^\pi}, \quad x \longmapsto \pi \cdot x.$$

Proof. — (a) (Knop [9]) Let $\tilde{X}$ be the affine cone over X, and for $x \in X^H$, let $\tilde{x} \in \tilde{X}$ be a lift. By definition [16], the point x is G-semistable if the closure of the orbit $G\tilde{x}$ does not contain the vertex 0, and similarly for $N := N(H)$. From this it is obvious that if x is G-semistable, then it is also N-semistable.

Conversely, suppose that x is N-semistable. After possibly replacing $\tilde{x}$ by a point in the (unique) closed N-orbit $\overline{N\tilde{x}}$, we may assume that $N\tilde{x}$ is closed and not equal to $\{0\}$.

The group H acts on the line $K\tilde{x}$ by a character χ, and the assumption that x is N-semistable implies that χ has finite order. In particular, $\chi(H)$ is a finite subgroup of $\mathbb{G}_m$. Let

$$\tilde{G} := G \times \chi(H) \quad \text{and} \quad \tilde{H} := \big\{(h, \chi(h)^{-1}) : h \in H\big\}.$$

Then $\tilde{G}$ acts on $\tilde{X}$ and $\tilde{x}$ is fixed by $\tilde{H}$. Also note that the normalizer $\tilde{N}$ of $\tilde{H}$ in $\tilde{G}$ is of finite index in $N \times \chi(H)$. In particular, the orbit $\tilde{N}\tilde{x}$ is closed and not equal to $\{0\}$. Now apply [12, Corollary 1], which says that

$$\tilde{N}\tilde{x} \text{ closed} \quad \Longrightarrow \quad \tilde{G}\tilde{x} \text{ closed}.$$

Hence 0 is not in the closure of $\tilde{G}\tilde{x}$, and therefore x is G-semistable. This proves (a).

(b) This is an immediate consequence of the main result of [12], which in our terminology says that $\tilde{X}^{\tilde{H}}/\tilde{N} \to \tilde{X}/\tilde{G}$ is finite.

(c) The character χ_x is well-defined, since if we choose some other lift $\tilde{x}'$ of x, then $\tilde{x}' = c\tilde{x}$ for some $c \neq 0$. It remains to show that the union is disjoint. So suppose that $x \in X^H_\chi \cap X^H_{\chi'}$. It follows that $\chi(h)\tilde{x} = \chi'(h)\tilde{x}$ for all $h \in H$, and since $\tilde{x} \neq \mathbf{0}$, we find that $\chi(h) = \chi'(h)$. Hence $\chi = \chi'$.

(d) This follows from the calculation

$$h \cdot \pi \cdot \tilde{x} = \pi \cdot (\pi^{-1} \cdot h \cdot \pi) \cdot \tilde{x} = \pi \cdot \chi_x(\pi^{-1} \cdot h \cdot \pi)\tilde{x} = \chi_x(\pi^{-1} \cdot h \cdot \pi)\pi \cdot \tilde{x},$$

which shows that $\chi_{\pi \cdot x} = \chi_x^{\pi}$. □

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