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On reducibility of quantum harmonic oscillator on $\mathbb{R}^d$ with quasiperiodic in time potential (*)

Benoît Grébert (1) and Eric Paturel (2)

Abstract. — We prove that a linear d-dimensional Schrödinger equation on $\mathbb{R}^d$ with harmonic potential $|x|^2$ and small t-quasiperiodic potential

$$i\partial_t u - \Delta u + |x|^2 u + \varepsilon V(t\omega, x)u = 0, \quad x \in \mathbb{R}^d$$

reduces to an autonomous system for most values of the frequency vector $\omega \in \mathbb{R}^n$. As a consequence any solution of such a linear PDE is almost periodic in time and remains bounded in all Sobolev norms.

Résumé. — On montre que l'équation de Schrödinger d-dimensionnelle avec potentiel harmonique $|x|^2$, perturbée par un petit potentiel quasipériodique en temps

$$i\partial_t u - \Delta u + |x|^2 u + \varepsilon V(t\omega, x)u = 0, \quad x \in \mathbb{R}^d$$

est réductible à un système autonome pour la plupart des valeurs du vecteur de fréquences $\omega \in \mathbb{R}^n$. En conséquence, toute solution d'une telle EDP linéaire est presque-périodique en temps et toutes ses normes de Sobolev restent bornées.

1. Introduction

We consider the following linear Schrödinger equation in $\mathbb{R}^d$

$$i\, u_t(t,x) + (-\Delta + |x|^2)u(t,x) + \varepsilon V(\omega t, x)u(t,x) = 0, \quad t \in \mathbb{R},\ x \in \mathbb{R}^d. \quad (1.1)$$

Here $\varepsilon > 0$ is a small parameter and the frequency vector ω of forced oscillations is regarded as a parameter in $\mathcal{D}$ an open bounded subset of $\mathbb{R}^n$.

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The function V is a real multiplicative potential, which is quasiperiodic in time: namely V is a continuous function of $(\varphi, x) \in \mathbb{T}^n \times \mathbb{R}^d$ and V is $\mathcal{H}^s$ (see (1.3)) with $s > d/2$ with respect to the space variable $x \in \mathbb{R}^d$ and real analytic with respect to the angle variable $\varphi \in \mathbb{T}^d$.

We consider the previous equation as a linear non-autonomous equation in the complex Hilbert space $L^2(\mathbb{R}^d)$ and we prove (see Theorem 2.3 below) that it reduces to an autonomous system for most values of the frequency vector ω.

The general problem of reducibility for linear differential systems with time quasi periodic coefficients, $\dot{x} = A(\omega t)x$, goes back to Bogolyubov [8] and Moser [21]. Then there is a large literature around reducibility of finite dimensional systems by means of the KAM tools. In particular, the basic local result states the following: Consider the non autonomous linear system

$$\dot{x} = A_0 x + \varepsilon F(\omega t)x$$

where A_0 and $F(\cdot)$ take values in $gl(k, \mathbb{R})$, $\mathbb{T}^n \ni \varphi \mapsto F(\varphi)$ admits an analytic extension to a strip in $\mathbb{C}^n$ and the imaginary part of the eigenvalues of A satisfy certain non resonance conditions, then for ε small enough and for ω in a Cantor set of asymptotically full measure, this linear system is reducible to a constant coefficients system. This result was then extended in many different directions (see in particular [10], [17] and [19]).

Essentially our Theorem 2.3 is an infinite dimensional (i.e. $k = +\infty$) version of this basic result.

Such kind of reducibility result for PDE using KAM machinery was first obtained by Bambusi & Graffi (see [5]) for Schrödinger equation on $\mathbb{R}$ with a x^β potential, β being strictly larger than 2. Here we follow the more recent approach developed by Eliasson & Kuksin (see [11]) for the Schrödinger equation on the multidimensional torus. The one dimensional case ($d = 1$) was considered in [15] as a consequence of a nonlinear KAM theorem. In the present paper we extend [15] to the multidimensional linear Schrödinger equation (1.1) by adapting the linear algebra tools.

All the previous mentioned articles as well as this present work concern bounded linear perturbations. Recently several results have been obtained for unbounded linear perturbations. In this case, the Hamiltonian vector field of the perturbation is an unbounded operator. In [1], the authors use pseudo-differential calculus to build a symplectic change of variable that conjugates the original Hamiltonian system to a new one where the vector field of the perturbation is bounded. This allows to apply a standard KAM procedure. This technics was used in [12] but also in [3] and [4] where the author considers unbounded perturbations of the 1d quantum harmonic oscillator. Actually this pseudo-differential approach seems to be restricted to the one dimensional case. We also mention the very recent[(1)] result [6] concerning polynomial perturbations of the quantum harmonic oscillator in d-dimensions:

$$i\, u_t(t,x) + (-\Delta + |x|^2)u(t,x) + \varepsilon W(\omega t, x, -i\nabla)u(t,x) = 0, \quad t \in \mathbb{R},\ x \in \mathbb{R}^d.$$

where W is a polynomial in (x,ξ) of degree at most two.

To state precisely our result we need some notations. Let

$$T = -\Delta + |x|^2 = -\Delta + x_1^2 + x_2^2 + \cdots + x_d^2$$

be the d-dimensional quantum harmonic oscillator. Its spectrum is the sum of d copies of the odd integers set, i.e. the spectrum of T equals

$$\widehat{\mathcal{E}} := \{d, d+2, d+4\cdots\}.$$

For $j \in \widehat{\mathcal{E}}$ we denote the associated eigenspace E_j whose dimension is

$$\text{card } \{(i_1, i_2, \cdots, i_d) \in (2\mathbb{N}-1)^d \,|\, i_1 + i_2 + \cdots + i_d = j\} := d_j \leqslant j^{d-1}.$$

We denote $\{\Phi_{j,l},\, l = 1, \cdots, d_j\}$, the basis of E_j obtained by d-tensor product of Hermite functions: $\Phi_{j,l} = \varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_d}$ for some choice of $i_1 + i_2 + \cdots + i_d = j$. Then setting

$$\mathcal{E} := \{(j,\ell) \in \widehat{\mathcal{E}} \times \mathbb{N} \,|\, \ell = 1, \cdots, d_j\}$$

$(\Phi_a)_{a\in\mathcal{E}}$ is a basis of $L^2(\mathbb{R}^d)$ and denoting

$$w_{j,\ell} = j \quad \text{for}(j,\ell) \in \mathcal{E}$$

we have

$$T\Phi_a = w_a \Phi_a, \quad a \in \mathcal{E}.$$

We define on $\mathcal{E}$ an equivalence relation:

$$a \sim b \iff w_a = w_b$$

and denote by $[a]$ the equivalence class associated with $a \in \mathcal{E}$. We notice that

$$\text{card } [a] \leqslant w_a^{d-1}. \tag{1.2}$$

For $s \geqslant 0$ an integer we define

$$\mathcal{H}^s = \left\{ f \in H^s(\mathbb{R}^d, \mathbb{C}) \,\middle|\, \begin{array}{l} x \mapsto x^\alpha \partial^\beta f \in L^2(\mathbb{R}^d), \\ \text{for any } \alpha, \beta \in \mathbb{N}^d \text{ satisfying } 0 \leqslant |\alpha| + |\beta| \leqslant s \end{array} \right\}. \tag{1.3}$$

We note that, for any $s \geqslant 0$, $\mathcal{H}^s$ is the form domain of T^s and the domain of $T^{s/2}$ (see for instance [16, Proposition 1.6.6]) and that this allows to extend the definition of $\mathcal{H}^s$ to real values of $s \geqslant 0$. Furthermore for $s > d/2$, $\mathcal{H}^s$ is an algebra.

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(1) Actually it was obtained after we finished the present work.

To a function $u \in \mathcal{H}^s$ we associate the sequence ξ of its Hermite coefficients by the formula $u(x) = \sum_{a\in\mathcal{E}} \xi_a \Phi_a(x)$. Then defining[(2)]

$$\ell_s^2 := \left\{ (\xi)_{a\in\mathcal{E}} \,\middle|\, \sum_{a\in\mathcal{E}} w_a^s |\xi_a|^2 < +\infty \right\},$$

we have for $s \geqslant 0$

$$u \in \mathcal{H}^s \iff \xi \in \ell_s^2. \tag{1.4}$$

Then we endow both spaces with the norm

$$\|u\|_s = \|\xi\|_s = \left(\sum_{a\in\mathcal{E}} w_a^s |\xi_a|^2 \right)^{1/2}.$$

If s is a positive integer, we will use the fact that the norms on $\mathcal{H}^s$ are equivalently defined as $\|T^{s/2}\varphi\|_{L^2(\mathbb{R}^d)}$ and $\sum_{0\leqslant|\alpha|+|\beta|\leqslant s} \|x^\alpha \partial^\beta \varphi\|_{L^2(\mathbb{R}^d)}$.

We finally introduce a regularity assumption on the potential V:

Definition 1.1. — *A potential* $V:\ \mathbb{T}^n \times \mathbb{R}^d \ni (\varphi, x) \mapsto V(\varphi, x) \in \mathbb{R}$ *is* s-admissible *if* $\mathbb{T}^n \ni \varphi \mapsto V(\varphi, \cdot\,)$ *is real analytic with value in* $\mathcal{H}^s$ *with*

$$\begin{cases} s \geqslant 0 & \text{if } d = 1 \\ s > 2(d-2) & \text{if } d \geqslant 2. \end{cases}$$

In particular if V is admissible then the map $\mathbb{T}^n \ni \varphi \mapsto V(\varphi, \cdot\,) \in \mathcal{H}^s$ analytically extends to

$$\mathbb{T}^n_\sigma = \{(a+ib) \in \mathbb{C}^n/2\pi\mathbb{Z}^n \mid |b| < \sigma\}$$

for some $\sigma > 0$. Now we can state our main Theorem:

Theorem 1.2. — *Assume that the potential* $V:\ \mathbb{T}^n \times \mathbb{R}^d \ni (\varphi, x) \mapsto \mathbb{R}$ *is* s-admissible *(see Definition 1.1). Then, there exists* $\delta_0 > 0$ *(depending only on* s *and* d*) and* $\varepsilon_* > 0$ *such that for all* $0 \leqslant \varepsilon < \varepsilon_*$ *there exists* $\mathcal{D}_\varepsilon \subset [0, 2\pi)^n$ *satisfying*

$$\operatorname{meas}(\mathcal{D} \setminus \mathcal{D}_\varepsilon) \leqslant \varepsilon^{\delta_0},$$

such that for all $\omega \in \mathcal{D}_\varepsilon$*, the linear Schrödinger equation*

$$i\partial_t u + (-\Delta + |x|^2)u + \varepsilon V(t\omega, x)u = 0 \tag{1.5}$$

reduces to a linear equation with constant coefficients in the energy space $\mathcal{H}^1$*. More precisely, for all* $0 < \delta \leqslant \delta_0$*, there exists* ε_0 *such that for all* $0 < \varepsilon < \varepsilon_0$ *there exists* $\mathcal{D}_\varepsilon \subset [0, 2\pi)^n$ *satisfying*

$$\operatorname{meas}(\mathcal{D} \setminus \mathcal{D}_\varepsilon) \leqslant \varepsilon^{\delta},$$

(2) Take care that our choice of the weight $w_a^{1/2}$ instead of w_a is non standard. It is motivated by the relation (1.4).

and for $\omega \in \mathcal{D}_\varepsilon$, there exist a linear isomorphism $\Psi(\varphi) = \Psi_{\omega,\varepsilon}(\varphi) \in \mathcal{L}(\mathcal{H}^{s'})$, for $0 \leqslant s' \leqslant \max(1,s)$, unitary on $L^2(\mathbb{R}^d)$, which analytically depends on $\varphi \in \mathbb{T}_{\sigma/2}$ and a bounded Hermitian operator $W = W_{\omega,\varepsilon} \in \mathcal{L}(\mathcal{H}^s)$ such that $t \mapsto u(t,\cdot) \in \mathcal{H}^1$ satisfies (1.5) if and only if $t \mapsto v(t,\cdot) = \Psi(\omega t)u(t,\cdot)$ satisfies the linear autonomous equation

$$i\partial_t v + (-\Delta + |x|^2)v + \varepsilon W v = 0\,.$$

Furthermore, for all $0 \leqslant s' \leqslant \max(1,s)$,

$$\|\Psi(\varphi)-\mathrm{Id}\|_{\mathcal{L}(\mathcal{H}^{s'},\mathcal{H}^{s'+2\beta})},\ \left\|\Psi(\varphi)^{-1}-\mathrm{Id}\right\|_{\mathcal{L}(\mathcal{H}^{s'},\mathcal{H}^{s'+2\beta})} \leqslant \varepsilon^{1-\delta/\delta_0} \quad \forall\, \varphi \in \mathbb{T}^n_{\sigma/2}.$$

On the other hand, the infinite matrix $(W_a^b)_{a,b\in\mathcal{E}}$ of the operator W written in the Hermite basis ($W_a^b = \int_{\mathbb{R}^d} \Phi_a W(\Phi_b)\mathrm{d}x$) is block diagonal, i.e.

$$W_a^b = 0 \quad \text{if } w_a \neq w_b$$

and, denoting by $[V](x) = \int_{\mathbb{T}^d} V(\varphi,x)\mathrm{d}\varphi$ the mean value of V on the torus $\mathbb{T}^d$, and by $([V]_a^b)_{a,b\in\mathcal{E}}$ the corresponding infinite matrix, we have

$$\left\|(W_a^b)_{a,b\in\mathcal{E}} - \Pi\big(([V]_a^b)_{a,b\in\mathcal{E}}\big)\right\|_{\mathcal{L}(\mathcal{H}^s)} \leqslant \varepsilon^{1/2}, \tag{1.6}$$

where Π is the projection on the diagonal blocks.

As a consequence of our reducibility result, we prove the following corollary concerning the solutions of (1.1).

Corollary 1.3. — *Assume that $(\varphi,x) \mapsto V(\varphi,x)$ is s-*admissible *(see Definition 1.1). Let $1 \leqslant s' \leqslant \max(1,s)$ and let $u_0 \in \mathcal{H}^{s'}$. Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and $\omega \in \mathcal{D}_\varepsilon$, there exists a unique solution $u \in \mathcal{C}\big(\mathbb{R}\,;\,\mathcal{H}^s\big)$ of (1.5) such that $u(0) = u_0$. Moreover, u is almost-periodic in time and satisfies*

$$(1-\varepsilon C)\|u_0\|_{\mathcal{H}^{s'}} \leqslant \|u(t)\|_{\mathcal{H}^{s'}} \leqslant (1+\varepsilon C)\|u_0\|_{\mathcal{H}^{s'}}, \quad \forall\ t \in \mathbb{R}, \tag{1.7}$$

for some $C = C(s',s,d)$.

Another way to understand the result of Theorem 1.2 is in term of Floquet operator (see [10] or [22]). Consider on $L^2(\mathbb{T}^n) \otimes L^2(\mathbb{R}^d)$ the Floquet Hamiltonian operator

$$K := i\sum_{k=1}^{n} \omega_k \frac{\partial}{\partial \varphi_k} - \Delta + |x|^2 + \varepsilon V(\varphi,x), \tag{1.8}$$

then we have

Corollary 1.4. — *Assume that $(\varphi,x) \mapsto V(\varphi,x)$ is s-*admissible *(see Definition 1.1). There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and $\omega \in \mathcal{D}_\varepsilon$, the spectrum of the Floquet operator K is pure point.*

Let us explain our general strategy of proof of Theorem 1.2.

In the phase space $\mathcal{H}^s \times \mathcal{H}^s$ endowed with the symplectic 2-form $idu \wedge d\bar{u}$ equation (1.1) reads as the Hamiltonian system associated with the Hamiltonian function

$$H(u,\bar{u}) = h(u,\bar{u}) + \varepsilon q(\omega t, u, \bar{u}) \tag{1.9}$$

where

$$h(u,\bar{u}) := \int_{\mathbb{R}^d} \big(|\nabla u|^2 + |x|^2|u|^2\big)\mathrm{d}x,$$
$$q(\omega t, u, \bar{u}) := \int_{\mathbb{R}^d} V(\omega t, x)|u|^2\mathrm{d}x.$$

Decomposing u and $\bar{u}$ on the basis $(\Phi_{j,l})_{(j,l)\in\mathcal{E}}$ of real valued functions,

$$u = \sum_{a\in\mathcal{E}} \xi_a \Phi_a, \quad \bar{u} = \sum_{a\in\mathcal{E}} \eta_a \Phi_a$$

the phase space $(u,\bar{u}) \in \mathcal{H}^s \times \mathcal{H}^s$ becomes the phase space $(\xi,\eta) \in Y_s$

$$Y_s = \{\zeta = (\zeta_a \in \mathbb{C}^2,\ a \in \mathcal{E}) \mid \|\zeta\|_s < \infty\}$$

where

$$\|\zeta\|_s^2 = \sum_{a\in\mathcal{E}} |\zeta_a|^2 w_a^s.$$

We endow Y_s with the symplectic structure $i\mathrm{d}\xi \wedge \mathrm{d}\eta$. In this setting the Hamiltonians read

$$h = \sum_{a\in\mathcal{E}} w_a \xi_a \eta_a,$$
$$q = \langle \xi, Q(\omega t)\eta\rangle$$

where Q is the infinite matrix whose entries are

$$Q_a^b(\omega t) = \int_{\mathbb{R}^d} V(\omega t, x)\Phi_a(x)\Phi_b(x)\mathrm{d}x \tag{1.10}$$

defining a linear operator on $\ell^2(\mathcal{E},\mathbb{C})$ and $\langle\,\cdot\,,\cdot\,\rangle$ is the natural pairing on $\ell^2(\mathcal{E},\mathbb{C})$: $\langle \xi,\eta\rangle = \sum_{a\in\mathcal{E}} \xi_a\eta_a$ (no complex conjugation). Therefore Theorem 1.2 is equivalent to the reducibility problem for the Hamiltonian system associated with the quadratic non autonomous Hamiltonian

$$\sum_{a\in\mathcal{E}} w_a\xi_a\eta_a + \varepsilon\langle \xi, Q(\omega t)\eta\rangle. \tag{1.11}$$

This reducibility is obtained by constructing a canonical change of variables close to identity that conjugates the Hamiltonian system associated with (1.11) to the Hamiltonian equation associated with an autonomous Hamiltonian

$$\sum_{a\in\mathcal{E}} w_a\xi_a\eta_a + \varepsilon\langle\xi, Q_\infty\eta\rangle$$

where Q_∞ is block diagonal: $(Q_\infty)_a^b = 0$ for $w_a \neq w_b$. This last condition means that, in the new variables, there is no interaction between modes of different energies, and this leads to Corollary 1.3.

The proof of the reducibility theorem is based on the following analysis already used in [5], [11], [15]: the non homogeneous Hamiltonian system

$$\begin{cases} \dot\xi_a = -iw_a\xi_a - i\varepsilon\left({}^tQ(\omega t)\xi\right)_a \\ \dot\eta_a = iw_a\eta_a + i\varepsilon\left(Q(\omega t)\eta\right)_a \end{cases} \quad a\in\mathcal{E} \tag{1.12}$$

is equivalent to the homogeneous system

$$\begin{cases} \dot\xi_a = -iw_a\xi_a - i\varepsilon\left({}^tQ(\varphi)\xi\right)_a \\ \dot\eta_a = iw_a\eta_a + i\varepsilon\left(Q(\varphi)\eta\right)_a \qquad a\in\mathcal{E}, \\ \dot\varphi = \omega. \end{cases} \tag{1.13}$$

Consequently the canonical change of variables is constructed applying a KAM strategy to the Hamiltonian

$$H(y,\varphi,\xi,\eta) = \omega\cdot y + \sum_{a\in\mathcal{E}} w_a\xi_a\eta_a + \varepsilon\langle\xi, Q(\varphi)\eta\rangle$$

in the extended phase space $\mathcal{P}_s = \mathbb{R}^n\times\mathbb{T}^n\times Y_s$.

Remark 1.5. — We can also prove a similar reducibility result for the Klein Gordon equation on the sphere $\mathbb{S}^d$, or for the beam equation on $\mathbb{T}^d$, by adapting the matrix space $\mathcal{M}_{s,\beta}$ defined in Section 2 (see [14]). Nevertheless, since we need a regularizing effect of the perturbation ($\beta > 0$ in (2.2)), in order to apply our method we cannot use it for NLS on compact domains.

Remark 1.6. — The resolution of the reducibility problem for a linear Hamiltonian PDE leads naturally to a KAM result for the corresponding nonlinear PDE. Actually the KAM procedure for nonlinear perturbations consists, roughly speaking, in an iterative procedure where at each step one linearizes the nonlinear equation around an approximate solution and one reduces this linearized equation to a PDE with constant coefficients. This approach is possible in the case of the Klein Gordon equation on the sphere $\mathbb{S}^d$ (see [14]) or in the one dimensional case (see [15]) with analytic regularity in the space direction x: the extension to the d-dimensional quantum harmonic oscillator, following the realms of this paper and [14], is the goal of a forthcoming paper.

Remark 1.7. — As a difference with [11] and [15], we work here in spaces of *finite* regularity in the space variable x. This allows us to get a better control of the inverse of block diagonal matrices, especially when the dimensions of the blocks are unbounded. In return, working with finite regularity in x forbids any loss of regularity during the KAM step which is applied infinitely many times (this is classically bypassed in the analytic case with a reduction of the analyticity strip).

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2. Reducibility theorem.

In this section we state an abstract reducibility theorem for quadratic quasiperiodic in time Hamiltonians of the form

$$\sum_{a\in\mathcal{E}} \lambda_a \xi_a \eta_a + \varepsilon\langle \xi, Q(\omega t)\eta\rangle.$$

2.1. Setting

First we need to introduce some notations.

Linear space. Let $s \geqslant 0$, we consider the complex weighted ℓ^2-space

$$\ell^2_s = \{\xi = (\xi_a \in \mathbb{C},\ a \in \mathcal{E}) \mid \|\xi\|_s < \infty\}$$

where

$$\|\xi\|_s^2 = \sum_{a\in\mathcal{E}} |\xi_a|^2 w_a^s.$$

Then we define

$$Y_s = \ell^2_s \times \ell^2_s = \{\zeta = (\zeta_a \in \mathbb{C}^2,\ a \in \mathcal{E}) \mid \|\zeta\|_s < \infty\}$$

where(3)

$$\|\zeta\|_s^2 = \sum_{a\in\mathcal{E}} |\zeta_a|^2 w_a^s.$$

(3) We provide $\mathbb{C}^2$ with the euclidian norm, $|\zeta_a| = |(\xi_a, \eta_a)| = \sqrt{|\xi_a|^2 + |\eta_a|^2}$.

We provide the spaces Y_s, $s \geqslant 0$, with the symplectic structure $i\mathrm{d}\xi \wedge \mathrm{d}\eta$. To any C^1-smooth function defined on a domain $\mathcal{O} \subset Y_s$, we associate the Hamiltonian equation

$$\begin{cases} \dot{\xi} = -i\nabla_\eta f(\xi, \eta) \\ \dot{\eta} = i\nabla_\xi f(\xi, \eta) \end{cases}$$

where $\nabla f = {}^t(\nabla_\xi f, \nabla_\eta f)$ is the gradient with respect to the scalar product in Y_0. For any C^1-smooth functions, F, G, defined on a domain $\mathcal{O} \subset Y_s$, we define the Poisson bracket

$$\{F, G\} = i \sum_{a \in \mathcal{E}} \frac{\partial F}{\partial \xi_a}\frac{\partial G}{\partial \eta_a} - \frac{\partial G}{\partial \xi_a}\frac{\partial F}{\partial \eta_a}.$$

We will also consider the extended phase space

$$\mathcal{P}_s = \mathbb{R}^n \times \mathbb{T}^n \times Y_s \ni (y, \varphi, (\xi, \eta)).$$

For any C^1-smooth functions, F, G, defined on a domain $\mathcal{O} \subset \mathcal{P}_s$, we define the extended Poisson bracket (denoted by the same symbol)

$$\{F, G\} = \nabla_y F \nabla_\varphi G - \nabla_y G \nabla_\varphi F + i \sum_{a \in \mathcal{E}} \frac{\partial F}{\partial \xi_a}\frac{\partial G}{\partial \eta_a} - \frac{\partial G}{\partial \xi_a}\frac{\partial F}{\partial \eta_a}. \tag{2.1}$$

Infinite matrices. We denote by $\mathcal{M}_{s,\beta}$ the set of infinite matrices $A : \mathcal{E} \times \mathcal{E} \to \mathbb{C}$ that satisfy

$$|A|_{s,\beta} := \sup_{a,b \in \mathcal{E}} (w_a w_b)^\beta \left\| A_{[a]}^{[b]} \right\| \left(\frac{\sqrt{\min(w_a, w_b)} + |w_a - w_b|}{\sqrt{\min(w_a, w_b)}} \right)^{s/2} < \infty \tag{2.2}$$

where $A_{[a]}^{[b]}$ denotes the restriction of A to the block $[a] \times [b]$ and $\|\cdot\|$ denotes the operator norm. Further we denote $\mathcal{M} = \mathcal{M}_{0,0}$. We will also need the space $\mathcal{M}_{s,\beta}^+$ the following subspace of $\mathcal{M}_{s,\beta}$: an infinite matrix $A \in \mathcal{M}$ is in $\mathcal{M}_{s,\beta}^+$ if

$$|A|_{s,\beta+} := \sup_{a,b \in \mathcal{E}} (w_a w_b)^\beta (1 + |w_a - w_b|) \left\| A_{[a]}^{[b]} \right\| \left(\frac{\sqrt{\min(w_a, w_b)} + |w_a - w_b|}{\sqrt{\min(w_a, w_b)}} \right)^{s/2} < \infty.$$

The following structural lemma is proved in Appendix:

Lemma 2.1. — *Let $0 < \beta \leqslant 1$ and $s \geqslant 0$ there exists a constant $C \equiv C(\beta, s) > 0$ such that*

(i) *Let $A \in \mathcal{M}_{s,\beta}$ and $B \in \mathcal{M}_{s,\beta}^+$. Then AB and BA belong to $\mathcal{M}_{s,\beta}$ and*

$$|AB|_{s,\beta},\ |BA|_{s,\beta} \leqslant C |A|_{s,\beta} |B|_{s,\beta+}.$$

(ii) *Let* $A, B \in \mathcal{M}^+_{s,\beta}$. *Then* AB *and* BA *belong to* $\mathcal{M}^+_{s,\beta}$ *and*

$$|AB|_{s,\beta+},\ |BA|_{s,\beta+} \leqslant C|A|_{s,\beta+}|B|_{s,\beta+}.$$

(iii) *Let* $A \in \mathcal{M}_{s,\beta}$. *Then for any* $t \geqslant 1$, $A \in \mathcal{L}(\ell^2_t, \ell^2_{-t})$ *and*

$$\|A\xi\|_{-t} \leqslant C|A|_{s,\beta}\|\xi\|_t.$$

(iv) *Let* $A \in \mathcal{M}^+_{s,\beta}$. *Then* $A \in \mathcal{L}(\ell^2_{s'}, \ell^2_{s'+2\beta})$ *for all* $0 \leqslant s' \leqslant s$ *and*

$$\|A\xi\|_{s'+2\beta} \leqslant C|A|_{s,\beta+}\|\xi\|_{s'}.$$

Moreover $A \in \mathcal{L}(\ell^2_1, \ell^2_1)$ *and*

$$\|A\xi\|_1 \leqslant C|A|_{s,\beta+}\|\xi\|_1.$$

Notice that in particular, for all $\beta > 0$, matrices in $\mathcal{M}^+_{0,\beta}$ define bounded operator on ℓ^2_1 but, even for s large, we cannot insure that $\mathcal{M}_{s,\beta} \subset \mathcal{L}(\ell^2)$.

Normal form.

Definition 2.2. — *A matrix* $Q : \mathcal{E} \times \mathcal{E} \to \mathbb{C}$ *is in normal form, and we denote* $Q \in \mathcal{NF}$, *if*

(i) Q *is Hermitian, i.e.* $Q^a_b = \overline{Q^b_a}$,
(ii) Q *is block diagonal, i.e.* $Q^a_b = 0$ *for all* $w_a \neq w_b$.

Notice that a block diagonal matrix with bounded blocks in operator norm defines a bounded operator on ℓ^2 and thus we have $\mathcal{M}_{s,\beta} \cap \mathcal{NF} \subset \mathcal{L}(\ell^2_s)$. To a matrix $Q = (Q^b_a) \in \mathcal{L}(\ell^2_t, \ell^2_{-t})$ we associate in a unique way a quadratic form on $Y_s \ni (\zeta_a)_{a\in\mathcal{E}} = (\xi_a, \eta_a)_{a\in\mathcal{E}}$ by the formula

$$q(\xi, \eta) = \langle \xi, Q\eta\rangle = \sum_{a,b\in\mathcal{E}} Q^b_a \xi_a \eta_b.$$

We notice for later use that

$$\{q_1, q_2\}(\xi, \eta) = -i\langle \xi, [Q_1, Q_2]\eta\rangle \tag{2.3}$$

where

$$[Q_1, Q_2] = Q_1Q_2 - Q_2Q_1$$

is the commutator of the two matrices Q_1 and Q_2. If $Q \in \mathcal{M}_{s,\beta}$ then

$$\sup_{a,b\in\mathcal{E}} \left\|(\nabla_\xi\nabla_\eta q)^{[b]}_{[a]}\right\| \leqslant \frac{|Q|_{s,\beta}}{(w_a w_b)^\beta}\left(\frac{\sqrt{\min(w_a, w_b)}}{\sqrt{\min(w_a, w_b)} + |w_a - w_b|}\right)^{s/2}. \tag{2.4}$$

Parameter. In all the paper ω will play the role of a parameter belonging to $\mathcal{D}_0 = [0, 2\pi)^n$. All the constructed functions will depend on ω with C^1 regularity. When a function is only defined on a Cantor subset of $\mathcal{D}_0$ the regularity has to be understood in the Whitney sense.

A class of quadratic Hamiltonians. Let $s \geqslant 0$, $\beta > 0$, $\mathcal{D} \subset \mathcal{D}_0$ and $\sigma > 0$. We denote by $\mathcal{M}_{s,\beta}(\mathcal{D},\sigma)$ the set of C^1 mappings

$$\mathcal{D} \times \mathbb{T}_\sigma \ni (\omega,\varphi) \to Q(\omega,\varphi) \in \mathcal{M}_{s,\beta}$$

which is real analytic in $\varphi \in \mathbb{T}_\sigma := \{\varphi \in \mathbb{C}^n \mid |\Im\varphi| < \sigma\}$. This space is equipped with the norm

$$[Q]^{\mathcal{D},\sigma}_{s,\beta} = \sup_{\substack{\omega\in\mathcal{D},\ j=0,1\\ |\Im\varphi|<\sigma}} |\partial^j_\omega Q(\omega,\varphi)|_{s,\beta}\,. \tag{2.5}$$

In view of Lemma 2.1 (iii), to a matrix $Q \in \mathcal{M}_{s,\beta}(\mathcal{D},\sigma)$ we can associate the quadratic form on Y_1

$$q(\xi,\eta;\omega,\varphi) = \langle \xi, Q(\omega,\varphi)\eta\rangle$$

and we have

$$|q(\xi,\eta;\omega,\varphi)| \leqslant [Q]^{\mathcal{D};\sigma}_{s,\beta}\, \|(\xi,\eta)\|_1^2 \quad \text{for}(\xi,\eta)\in Y_1,\ \omega\in\mathcal{D},\ \varphi\in\mathbb{T}_\sigma\,. \tag{2.6}$$

The subspace of $\mathcal{M}_{s,\beta}(\mathcal{D},\sigma)$ formed by Hamiltonians S such that $S(\omega,\varphi) \in \mathcal{M}^+_{s,\beta}$ is denoted by $\mathcal{M}^+_{s,\beta}(\mathcal{D},\sigma)$ and is equipped with the norm

$$[S]^{\mathcal{D},\sigma}_{s,\beta+} = \sup_{\substack{\omega\in\mathcal{D},\ j=0,1\\ |\Im\varphi|<\sigma}} |\partial^j_\omega S(\omega,\varphi)|_{s,\beta+}\,.$$

The space of Hamiltonians $N \in \mathcal{M}_{s,\beta}(\mathcal{D},\sigma)$ that are independent of φ will be denoted by $\mathcal{M}_{s,\beta}(\mathcal{D})$ and is equipped with the norm

$$[N]^{\mathcal{D}}_{s,\beta} = \sup_{\omega\in\mathcal{D},\ j=0,1} |\partial^j_\omega N(\omega)|_{s,\beta}\,.$$

Hamiltonian flow. To any $S \in \mathcal{M}^+_{s,\beta}$ with $s \geqslant 0$ and $\beta > 0$ we associate the symplectic linear change of variable on Y_s:

$$(\xi,\eta) \mapsto (e^{-i\,{}^tS}\xi, e^{iS}\eta).$$

It is well defined and invertible in $\mathcal{L}(Y_{s'})$ for all $0 \leqslant s' \leqslant \max(1,s)$ as a consequence of Lemma 2.1 (iv). We note that it corresponds to the flow at time 1 generated by the quadratic Hamiltonian $(\xi,\eta) \mapsto \langle\xi, S\eta\rangle$. Notice that a necessary and sufficient condition for this flow to preserve the symmetry $\eta = \overline{\xi}$ (verified by any initial condition considered in this paper) is

$${}^tS = \overline{S}\,, \tag{2.7}$$

that is, S is a hermitian matrix.

When S also depends smoothly on φ, $\mathbb{T}^n \ni \varphi \mapsto S(\varphi) \in \mathcal{M}^+_{s,\beta}$ we associate to S the symplectic linear change of variable on the extended phase space $\mathcal{P}_s$:

$$\Phi_S(y,\varphi,\xi,\eta) \mapsto (\widetilde{y},\varphi,e^{-i\,{}^tS}\xi, e^{iS}\eta) \tag{2.8}$$

where $\widetilde{y}$ is the solution at time $t=1$ of the equation $\dot{\widetilde{y}} = \langle e^{-i\,{}^tS}\xi, \nabla_\varphi S e^{iS}\eta\rangle$ with $\widetilde{y}(0)=y$. We note that it corresponds to the flow at time 1 generated by the Hamiltonian $(y,\varphi,\xi,\eta)\mapsto\langle\xi,S(\varphi)\eta\rangle$. Concretely we will never calculate $\widetilde{y}$ explicitly since the non homogeneous Hamiltonian system (1.12) is equivalent to the system (1.13) where the variable conjugated to φ is not required.

2.2. Hypothesis on the spectrum

Now we formulate our hypothesis on λ_a, $a\in\mathcal{E}$:

Hypothesis H1 (Asymptotics). — *We assume that there exists an absolute constant $c_0>0$ such that*

$$\lambda_a \geqslant c_0\, w_a \quad a\in\mathcal{E} \tag{2.9}$$

and

$$|\lambda_a-\lambda_b| \geqslant c_0|w_a-w_b| \quad a,b\in\mathcal{E} \tag{2.10}$$

Hypothesis H2 (second Melnikov condition in measure). — *There exist absolute constants $\alpha_1>0$, $\alpha_2>0$ and $C>0$ such that the following holds: for each $\kappa>0$ and $K\geqslant 1$ there exists a closed subset $\mathcal{D}'=\mathcal{D}'(\kappa,K)\subset\mathcal{D}$ (where $\mathcal{D}$ is the initial set of vector frequencies) satisfying*

$$\operatorname{meas}(\mathcal{D}\setminus\mathcal{D}') \leqslant CK^{\alpha_1}\kappa^{\alpha_2} \tag{2.11}$$

such that for all $\omega\in\mathcal{D}'$, all $k\in\mathbb{Z}^n$ with $0<|k|\leqslant K$ and all $a,b\in\mathcal{E}$ we have

$$|k\cdot\omega+\lambda_a-\lambda_b| \geqslant \kappa(1+|w_a-w_b|). \tag{2.12}$$

2.3. The reducibility Theorem

Let us consider the non autonomous Hamiltonian

$$H_\omega(t,\xi,\eta)=\sum_{a\in\mathcal{E}}\lambda_a\xi_a\eta_a+\varepsilon\langle\xi,Q(\omega t)\eta\rangle \tag{2.13}$$

and the associated Hamiltonian system on Y_s

$$\begin{cases}\dot{\xi}=-iN_0\xi-i\varepsilon\,{}^tQ(\omega t)\xi\\ \dot{\eta}=iN_0\eta+i\varepsilon Q(\omega t)\eta\end{cases} \tag{2.14}$$

where $N_0=\operatorname{diag}(\lambda_a\,|\,a\in\mathcal{E})$.

Theorem 2.3. — *Fix $s\geqslant 0$, $\sigma>0$, $\beta>0$. Assume that $(\lambda_a)_{a\in\mathcal{E}}$ satisfies Hypotheses A1, A2, and that $Q\in\mathcal{M}_{s,\beta}(\mathcal{D},\sigma)$. Fix $0<\delta\leqslant\delta_0:=\frac{\beta^2\alpha_2}{16(2+d+2\beta\alpha_2)(d+2\beta)}$. Then there exists $\varepsilon_*>0$ and if $0<\varepsilon<\varepsilon_*$, there exist*

(i) *a Cantor set* $\mathcal{D}_\varepsilon \subset \mathcal{D}$ *with* $Meas(\mathcal{D} \setminus \mathcal{D}_\varepsilon) \leqslant \varepsilon^\delta$*;*

(ii) *a* C^1 *family (in* $\omega \in \mathcal{D}_\varepsilon$*) of real analytic (in* $\varphi \in \mathbb{T}_{\sigma/2}$*) linear, unitary and symplectic coordinate transformation on* Y_0*:*

$$\begin{cases} \quad Y_0 \to Y_0 \\ (\xi,\eta) \mapsto \Psi_\omega(\varphi)(\xi,\eta) = \langle \overline{M_\omega(\varphi)}\xi, M_\omega(\varphi)\eta\rangle, \quad \omega \in \mathcal{D}_\varepsilon,\ \varphi \in \mathbb{T}_{\sigma/2}\,; \end{cases}$$

(iii) *a* C^1 *family of quadratic autonomous Hamiltonians in normal form*

$$\mathcal{H}_\omega = \langle \xi, N(\omega)\eta\rangle, \quad \omega \in \mathcal{D}_\varepsilon\,,$$

where $N(\omega) \in \mathcal{NF}$*, in particular block diagonal (i.e.* $N_a^b = 0$ *for* $w_a \neq w_b$*), and is close to* $N_0 = \mathrm{diag}(\lambda_a \,|\, a \in \mathcal{E})$*:* $N(\omega) - N_0 \in \mathcal{M}_{s,\beta}$ *and*

$$\|N(\omega) - N_0\|_{s,\beta} \leqslant 2\varepsilon \quad \omega \in \mathcal{D}_\varepsilon\,; \tag{2.15}$$

such that $t \mapsto (\xi(t),\eta(t))$ *is a solution of* (2.14) *in* Y_1 *if and only if* $t \mapsto \Psi_\omega(\omega t)((\xi(t),\eta(t)))$ *is a solution of the autonomous Hamiltonian system associated with* $\mathcal{H}_\omega$*:*

$$\begin{cases} \dot\xi = -iN(\omega)\xi \\ \dot\eta = iN(\omega)\eta\,. \end{cases}$$

Furthermore $\Psi_\omega(\varphi)$ *and* $\Psi_\omega(\varphi)^{-1}$ *are bounded operators from* $Y_{s'}$ *into itself for all* $0 \leqslant s' \leqslant \max(1,s)$ *and they are close to identity:*

$$\|M_\omega(\varphi) - \mathrm{Id}\|_{\mathcal{L}(\ell^2_{s'},\ell^2_{s'+2\beta})}\,,\ \left\|M_\omega(\varphi)^{-1} - \mathrm{Id}\right\|_{\mathcal{L}(\ell^2_{s'},\ell^2_{s'+2\beta})} \leqslant \varepsilon^{1-\delta/\delta_0}\,. \tag{2.16}$$

Remark 2.4. — Although $\Psi_\omega(\varphi)$ is defined on Y_0, the normal form N (in particular N_0) defines a quadratic form on Y_s only when $s \geqslant 1$. Nevertheless its flow is well defined and continuous from Y_0 into itself (cf. (3.7)). Fortunately our change of variable $\Psi_\omega(\varphi)$ is always well defined on Y_1 even when $Q \in \mathcal{M}_{0,\beta}(\mathcal{D},\sigma)$ (i.e. when $s = 0$). This is essentially a consequence of the second part of Lemma 2.1 (iv).

Remark 2.5. — Notice that $\Psi_\omega(\varphi) - \mathrm{Id} \in \mathcal{L}(Y_s, Y_{s+2\beta})$, i.e. it is a regularizing operator.

Theorem 2.3 is proved in Section 4.

3. Applications to the quantum harmonic oscillator on $\mathbb{R}^d$

In this section we prove Theorem 1.2 as a corollary of Theorem 2.3. We use notations introduced in the introduction.

3.1. Verification of the hypothesis

We first verify the hypothesis of Theorem 1.2:

Lemma 3.1. — *When $\lambda_a = w_a$, $a \in \mathcal{E}$, Hypothesis H1 and H2 hold true with $c_0 = 1/2$ and $\mathcal{D} = [0,1]^n$.*

Proof. — The asymptotics A1 are trivially verified with $c_0 = 1$. For $\tau > n$ we define the diophantine set

$$G_\tau(\kappa) := \left\{ \omega \in [0,2\pi)^n \,\middle|\, |\langle \omega, k\rangle + j| \geqslant \frac{\kappa}{|k|^\tau}, \text{ for all } j \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^n \setminus \{0\} \right\}.$$

A classical argument leads to

$$\text{meas}\left([0,2\pi)^n \setminus G_\tau(\kappa)\right) \leqslant C\kappa \sum \frac{1}{|k|^\tau} \leqslant C(\tau)\kappa.$$

Since $w_a - w_b \in \mathbb{Z}$, Hypothesis A2 is satisfied choosing

$$\mathcal{D} = [0,1]^n, \quad \mathcal{D}' = G_{n+1}(\kappa K^{n+1}), \quad \alpha_1 = n+1 \text{and} \alpha_2 = 1.$$

□

Lemma 3.2. — *Let $d \geqslant 1$. Suppose that*

$$\begin{cases} s \geqslant 0 & \text{if } d = 1 \\ s > 2(d-2) & \text{if } d \geqslant 2 \end{cases}$$

and $V \in \mathcal{H}^s$. Then there exists $\beta(d,s) > 0$ such that the matrix Q defined by

$$Q_a^b = \int_{\mathbb{R}^d} V(x)\Phi_a(x)\Phi_b(x)\mathrm{d}x$$

belongs to $\mathcal{M}_{s,\beta(d,s)}$. Moreover, there exists $C(d,s) > 0$ such that

$$|Q|_{s,\beta} \leqslant C(d,s)\,\|V\|_s\,.$$

As a consequence if V is admissible (see Definition 1.1) then, defining

$$Q_a^b(\varphi) = \int_{\mathbb{R}^d} V(\varphi, x)\Phi_a(x)\Phi_b(x)\mathrm{d}x,$$

the mapping $\varphi \mapsto Q(\varphi)$ belongs to $\mathcal{M}_{s,\beta}(\mathcal{D}_0, \sigma)$ for some $\sigma > 0$.

Proof. — First we notice that

$$\left\| Q_{[a]}^{[b]} \right\| = \sup_{\|u\|,\|v\|=1} |\langle Q_{[a]}^{[b]} u, v\rangle| = \sup_{\substack{\Psi_a \in E_{[a]},\ \|\Psi_a\|=1 \\ \Psi_b \in E_{[b]},\ \|\Psi_b\|=1}} \left| \int_{\mathbb{R}^d} V(x)\Psi_a\Psi_b\,\mathrm{d}x \right|,$$

where $E_{[a]}$ (resp. $E_{[b]}$) is the eigenspace of T associated with the cluster $[a]$ (resp. $[b]$). Then we follow arguments developed in [2, Proposition 2] and already used in the context of the harmonic oscillator in [13]. The basic idea lies in the following commutator lemma: Let A be a linear operator which maps $\mathcal{H}^s$ into itself and define the sequence of operators

$$A_N := [T, A_{N-1}], \quad A_0 := A$$

then by [2, Lemma 7], we have for any $a, b \in \mathcal{E}$ with $w_a \neq w_b$, for any $\Psi_a \in E_{[a]}$, $\Psi_b \in E_{[b]}$ and any $N \geqslant 0$

$$|\langle A\Psi_a, \Psi_b\rangle| \leqslant \frac{1}{|w_a - w_b|^N} |\langle A_N \Psi_a, \Psi_b\rangle| = \frac{1}{|w_a - w_b|^N} \|\Psi_b\|_{L^\infty} \|A_N \Psi_a\|_{L^1}\,.$$

Let A be the operator given by the multiplication by the function $V(x)$. Then, by an induction argument,

$$A_N = \sum_{0\leqslant|\alpha|\leqslant N} C_{\alpha,N} D^\alpha \quad \text{with} C_{\alpha,N} = \sum_{0\leqslant|\beta|\leqslant 2N-|\alpha|} P_{\alpha,\beta,N}(x) D^\beta V$$

and $P_{\alpha,\beta,N}$ are polynomials of degree less than $2N - |\alpha| - |\beta|$.

We first address the case $d = 1$, that we treat in the same way as in [15]. In this case, we have in [18] the following estimate on L^∞ norm of Hermite eigenfunctions with $\|\Psi_b\|_{L^2} = 1$,

$$\|\Psi_b\|_{L^\infty} \leqslant w_b^{-1/12}\,. \tag{3.1}$$

On the other hand, for $N \geqslant 0$, we have

$$\begin{aligned}
&\|A_N \Psi_a\|_{L^1}\\
&\leqslant \sum_{0\leqslant|\alpha|\leqslant N} \sum_{0\leqslant|\beta|\leqslant 2N-|\alpha|} \|P_{\alpha,\beta,N}(x) D^\beta V D^\alpha \Psi_a\|_{L^1}\\
&\leqslant C \sum_{0\leqslant|\alpha|\leqslant N} \sum_{0\leqslant|\beta|\leqslant 2N-|\alpha|} \sum_{|\gamma|\leqslant 2N-|\alpha|-|\beta|} \|\langle x\rangle^\gamma D^\beta V D^\alpha \Psi_a\|_{L^1}\\
&\leqslant C \sum_{0\leqslant|\alpha|\leqslant N} \sum_{0\leqslant|\beta|\leqslant 2N-|\alpha|} \sum_{|\gamma|\leqslant 2N-|\beta|} \|\langle x\rangle^\gamma D^\beta V\|_{L^2} \sum_{|\gamma'|\leqslant \alpha} \|\langle x\rangle^{-\gamma'} D^\alpha \Psi_a\|_{L^2}\\
&\leqslant C\|V\|_{2N} \|\Psi_a\|_N\,,
\end{aligned}$$

where $\langle x\rangle^\alpha = \Pi_{i=1}^d (1 + |x_i|^2)^{\alpha_i/2}$ for $\alpha \in \mathbb{N}^d$. Moreover, since $T\Psi_a = w_a \Psi_a$ and $\|\Psi_a\|_{L^2} = 1$,

$$\|\Psi_a\|_N \leqslant C w_a^{N/2}. \tag{3.2}$$

Therefore choosing $N = s/2$, we obtain

$$\left|\int_{\mathbb{R}^d} \Psi_a \Psi_b V \mathrm{d}x\right| \leqslant \frac{C}{w_b^{1/12}} \left(\frac{\sqrt{w_a}}{|w_a - w_b|}\right)^{s/2} \|V\|_s.$$

If $\sqrt{w_a} \leqslant |w_a - w_b|$ this leads to

$$\left|\int_{\mathbb{R}^d} \Psi_a \Psi_b V \mathrm{d}x\right| \leqslant C \frac{2^{s/2}}{w_b^{1/12}} \left(\frac{\sqrt{w_a}}{\sqrt{w_a} + |w_a - w_b|}\right)^{s/2} \|V\|_s. \tag{3.3}$$

On the other hand, if $\sqrt{w_a} \geqslant |w_a - w_b|$ then $\frac{\sqrt{w_a}}{\sqrt{w_a}+|w_a-w_b|} \geqslant \frac{1}{2}$ and since, using (3.1),

$$\left|\int_{\mathbb{R}^d} \Psi_a \Psi_b V \mathrm{d}x\right| \leqslant \|\Psi_b\|_{L^\infty} \|\psi_a\|_{L^2} \|V\|_{L^2} \leqslant w_b^{-\frac{1}{12}} \|V\|_{L^2}$$

(3.3) is still true providing that C is large enough. Exchanging a and b gives

$$\begin{aligned}\left|\int_{\mathbb{R}^d} \Psi_a \Psi_b V \mathrm{d}x\right| &\leqslant \frac{2^{s/2}C}{\max(w_a,w_b)^{1/12}} \left(\frac{\sqrt{\min(w_a,w_b)}}{\sqrt{\min(w_a,w_b)}+|w_a-w_b|}\right)^{s/2} \|V\|_s \\ &\leqslant \frac{2^{s/2}C}{(w_a w_b)^{1/24}} \left(\frac{\sqrt{\min(w_a,w_b)}}{\sqrt{\min(w_a,w_b)}+|w_a-w_b|}\right)^{s/2} \|V\|_s\,, \quad (3.4)\end{aligned}$$

hence $Q \in \mathcal{M}_{s,1/24}$ and $|Q|_{s,1/24} \leqslant C(d,s)\|V\|_s$. The case $s \notin 2\mathbb{N}$ comes after a standard interpolation argument, the Stein–Weiss theorem (see e.g. [7, Corollary 5.5.4]): indeed, fixing a, b and $s_0 = 2N$, we may estimate the norm of the linear form $V \mapsto \int_{\mathbb{R}^d} \Psi_a \Psi_b V \mathrm{d}x$ acting on $\mathcal{H}^s$ for $s = \theta s_0$, $\theta \in [0,1]$, using the direct estimate

$$\left|\int_{\mathbb{R}^d} \Psi_a \Psi_b V \mathrm{d}x\right| \leqslant \frac{C'}{(w_a w_b)^{1/24}} \|V\|_{L^2}$$

and (3.4), and we get

$$\left|\int_{\mathbb{R}^d} \Psi_a \Psi_b V \mathrm{d}x\right| \leqslant \frac{C'}{(w_a w_b)^{1/24}} \left(\frac{\sqrt{\min(w_a,w_b)}}{\sqrt{\min(w_a,w_b)}+|w_a-w_b|}\right)^{\theta s_0/2} \|V\|_{\theta s_0}\,.$$

We now treat the case $d \geqslant 2$. Take $p > 2$ if $d = 2$ and $2 < p < \frac{2d}{d-2}$ if $d \geqslant 3$. Using the Hölder inequality, we get, for $\frac{1}{p} + \frac{1}{q} = 1$,

$$|\langle A\Psi_a, \Psi_b\rangle| \leqslant \frac{1}{|w_a - w_b|^N} \|\Psi_b\|_{L^p} \|A_N \Psi_a\|_{L^q}\,.$$

In [18], the L^p estimate on Hermite eigenfunctions (with $\|\Psi_b\|_{L^2} = 1$) gives

$$\|\Psi_b\|_{L^p} \leqslant w_b^{-\widetilde{\beta}(p)}\,,$$

with $\widetilde{\beta}(p) = \frac{1}{3p}$ if $d = 2$ (and $p \geqslant 10/3$) and $\widetilde{\beta}(p) = \frac{1}{2}\left(\frac{d}{3p} - \frac{d-2}{6}\right) > 0$ if $d > 2$ and $\frac{2(d+3)}{d+1} \leqslant p < \frac{2d}{d-2}$. Moreover, we may estimate $\|A_N \Psi_a\|_{L^q}$, using

Young inequality (with $\frac{1}{2}+\frac{1}{r}=\frac{1}{q}$)

$$\begin{aligned}\|A_N\Psi_a\|_{L^q} &\leqslant \sum_{0\leqslant|\alpha|\leqslant N}\ \sum_{0\leqslant|\beta|\leqslant 2N-|\alpha|} \|P_{\alpha,\beta,N}(x)D^\beta V D^\alpha\Psi_a\|_{L^q}\\ &\leqslant C\Bigg(\sum_{\substack{0\leqslant|\alpha|\leqslant N/2\\ 0\leqslant|\beta|\leqslant 2N-|\alpha|}}\ \sum_{|\gamma|\leqslant 2N-\beta}\|\langle x\rangle^\gamma D^\beta V\|_{L^2}\sum_{|\gamma'|\leqslant\alpha}\|\langle x\rangle^{-\gamma'}\Psi_a\|_{L^r}\\ &\qquad + \sum_{\substack{N/2<|\alpha|\leqslant N\\ 0\leqslant|\beta|\leqslant 2N-|\alpha|}}\ \sum_{|\gamma|\leqslant 2N-|\alpha|-|\beta|}\|\langle x\rangle^\gamma D^\beta V\|_{L^r}\|D^\alpha\Psi_a\|_{L^2}\Bigg)\\ &\leqslant C\Big(\|V\|_{2N}\|\Psi_a\|_{N/2+\nu}+\|V\|_{3N/2+\nu}\|\Psi_a\|_N\Big),\end{aligned}$$

using the embedding $\mathcal{H}^\nu(\mathbb{R}^d)\hookrightarrow H^\nu(\mathbb{R}^d)$ composed with the Sobolev embedding $H^\nu(\mathbb{R}^d)\hookrightarrow L^r(\mathbb{R}^d)$, valid for $\nu\geqslant d\left(\frac{1}{2}-\frac{1}{r}\right)=\frac{d}{p}>\frac{d-2}{2}$. Hence, for $s=2N$ and $\nu\leqslant\frac{N}{2}=\frac{s}{4}$, i.e. $s>2(d-2)$, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^d}\Psi_a\Psi_b V\mathrm{d}x\right| &\leqslant \frac{C_N}{w_b^{\tilde\beta(p)}}\frac{1}{|w_a-w_b|^{s/2}}\|\Psi_a\|_{s/2}\|V\|_s\\ &\leqslant \frac{C_N}{w_b^{\tilde\beta(p)}}\frac{w_a^{s/4}}{|w_a-w_b|^{s/2}}\|V\|_s,\end{aligned}$$

and thus

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}\Psi_a\Psi_b V\mathrm{d}x\right|\\ &\qquad\leqslant \frac{C'_N}{(w_aw_b)^{\tilde\beta(p)/2}}\left(\frac{\min(w_a,w_b)^{1/2}}{\min(w_a,w_b)^{1/2}+|w_a-w_b|}\right)^{s/2}\|V\|_s, \quad (3.5)\end{aligned}$$

using the same trick as in the case $d=1$. Now fixing $p(d,s)$ satisfying all the constraints $2<p<\frac{2d}{d-2}$ and $p\geqslant\frac{4d}{s}$ (which is always possible since $\frac{4d}{s}<\frac{2d}{d-2}$) and defining $\beta(d,s)=\widetilde{\beta}(p(d,s))$ gives the result for an even integer s satisfying $s>2(d-2)$. In order to get the estimate for any real number $s>2(d-2)$, we interpolate: we take any even integer s_0 larger than s, and define $s_1=0$ and $p=+\infty$ in the case $d=2$, and $s_1=2(d-2)$, $p=\frac{2d}{d-2}$ if $d>2$. There exists $\theta\in]0,1]$ such that $s=\theta s_0+(1-\theta)s_1$. Moreover, following the last computations, we easily find

$$\left|\int_{\mathbb{R}^d}\Psi_a\Psi_b V\mathrm{d}x\right|\leqslant C\left(\frac{\min(w_a,w_b)^{1/2}}{\min(w_a,w_b)^{1/2}+|w_a-w_b|}\right)^{s_1/2}\|V\|_{s_1}. \quad (3.6)$$

Hence, using [7, Corollary 5.5.4], (3.5) and (3.6), interpolation gives the desired estimate for $s_1<s\leqslant s_0$. □

3.2. Proof of Theorem 1.2 and Corollaries 1.3, 1.4

The Schrödinger equation (1.5) is a Hamiltonian system on $\mathcal{H}^s \times \mathcal{H}^s$ ($s \geqslant 1$) governed by the Hamiltonian function (1.9). Expanding it on the orthonormal basis $(\Phi_a)_{a\in\mathcal{E}}$, it is equivalent to the Hamiltonian system on Y_s governed by (1.11) which reads as (2.14) with $\lambda_a = w_a$ and Q given by (1.10). By Lemmas 3.1, 3.2, if V is s-admissible, we can apply Theorem 2.3 to (1.11) and this leads to Theorem 1.2. More precisely, in the new coordinates given by Theorem 2.3, $(\xi'(t), \eta'(t)) = (\overline{M_\omega(\omega t)}\xi, M_\omega(\omega t)\eta)$, the system (1.12) becomes autonomous and decomposes in blocks as follows (remark that since N is in normal form we have ${}^tN = \overline{N}$):

$$\begin{cases} \dot{\xi}'_{[a]} = -i\overline{N}_{[a]}\xi'_{[a]} & a \in \widehat{\mathcal{E}} \\ \dot{\eta}'_{[a]} = iN_{[a]}\eta'_{[a]} & a \in \widehat{\mathcal{E}}. \end{cases} \tag{3.7}$$

In particular, the solution $u(t,x)$ of (1.5) corresponding to the initial datum $u_0(x) = \sum_{a\in\mathcal{E}} \xi(0)_a\Phi_a(x) \in \mathcal{H}^1$ reads $u(t,x) = \sum_{a\in\mathcal{E}} \xi(t)_a\Phi_a(x)$ with

$$\xi(t) = {}^tM_\omega(\omega t)e^{-i\overline{N}t}\ \overline{M_\omega}(0)\xi(0)\,. \tag{3.8}$$

In other words, let us define the transformation $\Psi(\varphi) \in \mathcal{L}(\mathcal{H}^s)$ by

$$\Psi(\varphi)\left(\sum_{a\in\mathcal{E}} \xi_a\Phi_a(x)\right) = \sum_{a\in\mathcal{E}} \left(\overline{M_\omega(\varphi)}\xi\right)_a \Phi_a(x)\,.$$

Then $u(t,x)$ satisfies (1.5) if and only if $v(t,\cdot\,) = \Psi(\omega t)u(t,\cdot\,)$ satisfies

$$i\partial_t v + (-\Delta + |x|^2)v + \varepsilon W(v) = 0\,,$$

where W is defined as follows:

$$W\left(\sum_{a\in\mathcal{E}} \xi_a\Phi_a\right) = \sum_{a\in\mathcal{E}} (N_\omega\xi)_a\,\Phi_a\,.$$

Furthermore, remembering the construction of N_ω (see (4.36) and (4.25)) we get that

$$\|N_\omega - (N_0 + \widetilde{N}_1)\| \leqslant 2\varepsilon_1 = 2\varepsilon^{3/2}$$

which leads to (1.6). This achieves the proof of Theorem 1.2.

To prove Corollary 1.3 let us explicit the formula (3.8). The exponential map $e^{-i\overline{N}t}$ decomposes on the finite dimensional blocks:

$$(e^{-i\overline{N}t})_{[a]} = e^{-i\overline{N}_{[a]}t}$$

and $\overline{N}_{[a]}$ diagonalizes in orthonormal basis:

$$P_{[a]}\overline{N}_{[a]}{}^tP_{[a]} = \mathrm{diag}(\mu_c), \quad P_{[a]}{}^tP_{[a]} = I_{d_a}$$

where $P_{[a]}$ is some block matrix and μ_c are real numbers that, in view of (2.15), satisfy

$$|\mu_a - \lambda_a| \leqslant C\frac{\varepsilon}{w_a^{2\beta}}, \quad a \in \mathcal{E}.$$

Thus

$$u(t,x) = \sum_{a\in\mathcal{E}} \xi_a(t)\Phi_a(x)$$

where

$$\xi(t) = {}^tM_\omega(\omega t)PD(t)\,{}^tP\overline{M_\omega}(0)\xi(0) \tag{3.9}$$

with

$$D(t) = \mathrm{diag}(e^{i\mu_c t},\ c\in\mathcal{E})$$

and P is the ℓ^2 unitary block diagonal map whose diagonal blocks are $P_{[a]}$. In particular the solutions are all almost periodic in time with frequencies vector (ω,μ). Furthermore, since $\|P\xi\|_s = \|\xi\|_s$ and $M_\omega(\varphi)$ is close to identity (see estimate (2.16)) we deduce (1.7).

Now it remains to prove Corollary 1.4. Defining, for any $c\in\mathcal{E}$ the sequence $\delta^c\in\ell^2$ as $\delta^c_c = 1$ and $\delta^c_a = 0$ if $a\neq c$, then the function $u(t,x)$ defined as

$$u(t,x) = e^{i\mu_c t}\sum_{a\in[c]} \left({}^tM_\omega(\omega t)P\delta^c\right)_a \Phi_a(x)$$

solves (1.5) if and only if $\mu_c + k\cdot\omega$ is an eigenvalue of K defined in (1.8), with associated eigenfunction

$$(\theta,x)\mapsto e^{ik\cdot\theta}\sum_{a\in[c]} \left({}^tM_\omega(\theta)P\delta^c\right)_a \Phi_a(x)\,.$$

This shows that the spectrum of the Floquet operator (1.8) equals $\{\mu_c + k\cdot\omega \mid k\in\mathbb{Z}^n,\ c\in\mathcal{E}\}$ and thus Corollary 1.4 is proved.

4. Proof of Theorem 2.3

4.1. General strategy

Let h be a Hamiltonian in normal form:

$$h(y,\varphi,\xi,\eta) = \omega\cdot y + \langle\xi, N(\omega)\eta\rangle \tag{4.1}$$

with N in normal form (see Definition 2.2). Notice that at the beginning of the procedure N is diagonal,

$$N = N_0 = \mathrm{diag}(w_a,\ a\in\mathcal{E})$$

and is independent of ω. Let q be a quadratic Hamiltonian of the form

$$q(\xi,\eta)=\langle \xi, Q(\varphi)\eta\rangle$$

and of size $\mathcal{O}(\varepsilon)$.

We search for a quadratic hamiltonian $\chi(\varphi,\xi,\eta)=\langle\xi,S(\varphi)\eta\rangle$ with $S=\mathcal{O}(\varepsilon)$ such that its time-one flow $\Phi_S\equiv\Phi_S^{t=1}$ transforms the Hamiltonian $h+q$ into

$$(h+q(\varphi))\circ\Phi_S=h_++q_+(\varphi),$$

where h_+ is a new normal form, ε-close to h, and the new perturbation q_+ is of size $\mathcal{O}(\varepsilon^2)$.

As a consequence of the Hamiltonian structure we have (at least formally) that

$$(h+q(\varphi))\circ\Phi_S=h+\{h,\chi\}+q(\varphi)+\mathcal{O}(\varepsilon^2).$$

So to achieve the goal above we should solve the *homological equation*:

$$\{h,\chi\}=h_+-h-q(\varphi)+\mathcal{O}(\varepsilon^2). \tag{4.2}$$

or equivalently (see (2.1) and (2.3))

$$\omega\cdot\nabla_\varphi S-i[N,S]=N_+-N-Q+\mathcal{O}(\varepsilon^2). \tag{4.3}$$

Repeating iteratively the same procedure with h_+ instead of h, we will construct a change of variable Φ such that

$$(h+q(\varphi))\circ\Phi=h_\infty\,,$$

with $h_\infty=\omega\cdot y+\langle\xi,N_\infty(\omega)\eta\rangle$ in normal form. Note that we will be forced to solve the homological equation, not only for the diagonal normal form N_0, but for more general normal form Hamiltonians (4.1) with N close to N_0.

4.2. Homological equation

In this section we will consider a homological equation of the form

$$\omega\cdot\nabla_\varphi S-i[N,S]+Q=\text{remainder} \tag{4.4}$$

with N in normal form close to N_0 and $Q\in\mathcal{M}_{s,\beta}$. We will construct a solution $S\in\mathcal{M}^+_{s,\beta}$.

Proposition 4.1. — *Let $\mathcal{D}\subset\mathcal{D}_0$. Let $\mathcal{D}\ni\rho\mapsto N(\omega)\in\mathcal{NF}$ be a $\mathcal{C}^1$ mapping that verifies*

$$\left\|\partial_\omega^j(N(\omega)-N_0)_{[a]}\right\|\leqslant\frac{c_0}{4w_a^{2\beta}} \tag{4.5}$$

for $j=0,1$, $a\in\mathcal{E}$ *and* $\omega\in\mathcal{D}$. *Let* $Q\in\mathcal{M}_{s,\beta}$, $0<\kappa\leqslant c_0/2$ *and* $K\geqslant 1$. *Then there exists a subset* $\mathcal{D}'=\mathcal{D}'(\kappa,K)\subset\mathcal{D}$, *satisfying*

$$\operatorname{meas}(\mathcal{D}\setminus\mathcal{D}')\leqslant CK^{\gamma_1}\kappa^{\gamma_2}, \tag{4.6}$$

and there exist $\mathcal{C}^1$*-functions* $\widetilde{N}:\mathcal{D}'\to\mathcal{M}_{s,\beta}\cap\mathcal{NF}$, $S:\mathbb{T}^n_\sigma\times\mathcal{D}'\to\mathcal{M}^+_{s,\beta}$ *hermitian and* $R:\mathbb{T}^n_\sigma\times\mathcal{D}'\to\mathcal{M}_{s,\beta}$, *analytic in* φ, *such that*

$$\omega\cdot\nabla_\varphi S-i[N,S]=\widetilde{N}-Q+R \tag{4.7}$$

and for all $(\varphi,\omega)\in\mathbb{T}^n_{\sigma'}\times\mathcal{D}'$, $\sigma'<\sigma$, *and* $j=0,1$

$$\left|\partial^j_\omega R(\varphi,\omega)\right|_{s,\beta}\leqslant C\,\frac{K^{1+\frac{d}{2}}e^{-\frac12(\sigma-\sigma')K}}{\kappa^{1+\frac{d}{2\beta}}(\sigma-\sigma')^n}\sup_{\substack{|\Im\varphi|<\sigma\\ j=0,1}}|\partial^j_\omega Q(\varphi)|_{s,\beta}\,, \tag{4.8}$$

$$\left|\partial^j_\omega S(\varphi,\omega)\right|_{s,\beta+}\leqslant C\frac{K^{d+1}}{\kappa^{\frac{d}{\beta}+2}(\sigma-\sigma')^n}\sup_{\substack{|\Im\varphi|<\sigma\\ j=0,1}}|\partial^j_\omega Q(\varphi)|_{s,\beta}\,, \tag{4.9}$$

$$\left|\partial^j_\omega \widetilde{N}(\omega)\right|_{s,\beta}\leqslant\sup_{\substack{|\Im\varphi|<\sigma\\ j=0,1}}|\partial^j_\omega Q(\varphi)|_{s,\beta}\,. \tag{4.10}$$

The constant C *depends on* n, d, s, β *and* $|\omega|$, $\gamma_2=\frac{\beta\alpha_2}{d+1+\beta\alpha_2}$ *and* $\gamma_1=\max(\alpha_1,2+d+n)$.

Proof. — Written in Fourier variables (w.r.t. φ), (4.7) reads

$$i\omega\cdot k\ \widehat{S}(k)-i[N,\widehat{S}(k)]=\delta_{k,0}\widetilde{N}-\widehat{Q}(k)+\widehat{R}(k) \tag{4.11}$$

where $\delta_{k,j}$ denotes the Kronecker symbol.

We decompose the equation into “components” on each product block $[a]\times[b]$:

$$L\,\widehat{S}^{[b]}_{[a]}(k)=-i\delta_{k,0}\widetilde{N}^{[b]}_{[a]}+i\widehat{Q}^{[b]}_{[a]}(k)-i\widehat{R}^{[b]}_{[a]}(k) \tag{4.12}$$

where the operator $L:=L(k,[a],[b],\omega)$ is the linear operator, acting in the space of complex $[a]\times[b]$-matrices defined by

$$L\,M=\big(k\cdot\omega\ I-N_{[a]}(\omega)\big)M+MN_{[b]}(\omega)$$

with $N_{[a]}=N^{[a]}_{[a]}$.

First we solve this equation when $k=0$ and $w_a=w_b$ by defining

$$\widehat{S}^{[a]}_{[a]}(0)=0,\quad \widehat{R}^{[a]}_{[a]}(0)=0\text{and}\widetilde{N}^{[a]}_{[a]}=\widehat{Q}^{[a]}_{[a]}(0).$$

Then we set $\widetilde{N}^{[b]}_{[a]}=0$ for $w_a\neq w_b$ in such a way $\widetilde{N}\in\mathcal{M}_{s,\beta}\cap\mathcal{NF}$ and satisfies

$$|\widetilde{N}|_{s,\beta}\leqslant|\widehat{Q}(0)|_{s,\beta}.$$

The estimates of the derivatives with respect to ω are obtained by differentiating the expressions for $\widetilde{N}$.

It remains to consider the case when $k \neq 0$ or $w_a \neq w_b$. The matrix $N_{[a]}$ can be diagonalized in an orthonormal basis:

$$ {}^tP_{[a]}N_{[a]}P_{[a]} = D_{[a]}. $$

Then we denote $\widehat{S'}^{[b]}_{[a]} = {}^tP_{[a]}\widehat{S}^{[b]}_{[a]}P_{[b]}$, $\widehat{Q'}^{[b]}_{[a]} = {}^tP_{[a]}\widehat{Q}^{[b]}_{[a]}P_{[b]}$ and $\widehat{R'}^{[b]}_{[a]} = {}^tP_{[a]}\widehat{R}^{[b]}_{[a]}P_{[b]}$ and we notice for later use that $\|\widehat{M'}^{[b]}_{[a]}\| = \|M^{[b]}_{[a]}\|$ for $M = S, Q, R$. In this new variables the homological equation (4.12) reads

$$ (k\cdot\omega - D_{[a]})\widehat{S'}^{[b]}_{[a]}(k) + S'^{[b]}_{[a]}(k)D_{[b]} = i\widehat{Q'}^{[b]}_{[a]}(k) - i\widehat{R'}^{[b]}_{[a]}(k). \tag{4.13} $$

This equation can be solved term by term: let $a, b \in \mathcal{E}$, we set

$$ \begin{aligned} &\widehat{R'}^{[b]}_{[a]}(k) = 0 && \text{for } |k| \leqslant K, \\ &\widehat{R'}_{j\ell}(k) = \widehat{Q'}_{j\ell}(k), && j\in[a],\ \ell\in[b],\ |k| > K, \end{aligned} \tag{4.14} $$

and

$$ \begin{aligned} \widehat{S'}^{[b]}_{[a]}(k) &= 0 \qquad \text{for } |k| > K \text{ or for } k=0 \text{ and } w_a = w_b, \\ \big(\widehat{S'}^{[b]}_{[a]}(k)\big)_{j\ell} &= \frac{i}{k\cdot\omega\ -\alpha_j + \beta_\ell}\big(\widehat{Q'}^{[b]}_{[a]}(k)\big)_{j\ell} \quad \text{in the other cases.} \end{aligned} \tag{4.15} $$

Here $\alpha_j(\omega)$ and $\beta_\ell(\omega)$ denote eigenvalues of $N_{[a]}(\omega)$ and $N_{[b]}(\omega)$, respectively. Before the estimations of such matrices, first remark that with this resolution, we ensure that

$$ \overline{\big(\widehat{Q'}^{[b]}_{[a]}(k)\big)_{j\ell}} = \big(\widehat{Q'}^{[a]}_{[b]}(-k)\big)_{\ell j} \ \Rightarrow\ \overline{\big(\widehat{S'}^{[b]}_{[a]}(k)\big)_{j\ell}} = \big(\widehat{S'}^{[a]}_{[b]}(-k)\big)_{\ell j} $$

hence, if Q' verifies condition (2.7), then this is also the case for S', hence the flow induced by S preserves the symmetry $\eta = \bar{\xi}$.

First notice that (4.14) classically leads to (see for instance [20])

$$ |R(\varphi)|_{s,\beta} = |R'(\varphi)|_{s,\beta} \leqslant C\frac{e^{-\frac{1}{2}(\sigma-\sigma')K}}{(\sigma-\sigma')^n}\sup_{|\Im\theta|<\sigma}|Q(\theta)|_{s,\beta}, \quad \text{for} |\Im\varphi| < \sigma'. $$

In order to estimate S, we will use Lemma 4.3 stated at the end of this section and proved in the appendix. We face the small divisors

$$ k\cdot\omega - \alpha_j(\omega) + \beta_\ell(\omega), \quad j\in[a],\ \ell\in[b]. \tag{4.16} $$

To estimate them, we have to distinguish two cases, depending on whether $k = 0$ or not.

The case $k=0$. — In that case, we know that $w_a \neq w_b$ and we use (4.5)[4] and (2.10) to get

$$|\alpha_j(\omega)-\beta_\ell(\omega)| \geqslant c_0|w_a-w_b| - \frac{c_0}{4w_a^{2\beta}} - \frac{c_0}{4w_b^{2\beta}} \geqslant \kappa(1+|w_a-w_b|).$$

This last estimate allows us to use Lemma 4.3 to conclude that

$$|\widehat{S}(0)|_{\beta+} \leqslant C\frac{1}{\kappa^{1+\frac{d}{2\beta}}}|\widehat{Q}(0)|_\beta\,. \tag{4.17}$$

The case $k \neq 0$. — Using Hypothesis A2, for any $\eta>0$, there is a set $\mathcal{D}_1=\mathcal{D}(2\eta,K)$,

$$\operatorname{meas}(\mathcal{D}\setminus\mathcal{D}_1) \leqslant CK^{\alpha_1}\eta^{\alpha_2},$$

such that for all $\omega\in\mathcal{D}_1$ and $0<|k|\leqslant K$

$$|k\cdot\omega\ -\lambda_a(\omega)+\lambda_b(\omega)| \geqslant 2\eta(1+|w_a-w_b|).$$

By (4.5) this implies

$$\begin{aligned}|k\cdot\omega\ -\alpha_j(\omega)+\beta_\ell(\omega)| &\geqslant 2\eta(1+|w_a-w_b|) - \frac{c_0}{4w_a^{2\beta}} - \frac{c_0}{4w_b^{2\beta}}\\ &\geqslant \eta(1+|w_a-w_b|)\end{aligned}$$

if

$$w_b \geqslant w_a \geqslant \left(\frac{c_0}{2\eta}\right)^{\frac{1}{2\beta}}.$$

Let now $w_a \leqslant (\frac{c_0}{2\eta})^{\frac{1}{2\beta}}$. We note that $|k\cdot\omega\ -\lambda_a(\omega)+\lambda_b(\omega)|\leqslant 1$ implies that $w_b \leqslant 1+(\frac{c_0}{2\eta})^{\frac{1}{2\beta}}+C|k| \leqslant C((\frac{c_0}{2\eta})^{\frac{1}{2\beta}}+K)$. Since $|\partial_\omega(k\cdot\omega)(\frac{k}{|k|}))|=|k|\geqslant 1$, we get, using condition (4.5),

$$\left|\partial_\omega\left(k\cdot\omega-\alpha_j(\omega)+\beta_\ell(\omega)\left(\frac{k}{|k|}\right)\right)\right| \geqslant 1/2\,. \tag{4.18}$$

Then we recall the following classical lemma:

Lemma 4.2. — *Let $f:[0,1]\mapsto\mathbb{R}$ a C^1-map satisfying $|f'(x)|\geqslant\delta$ for all $x\in[0,1]$ and let $\kappa>0$ then*

$$\operatorname{meas}\{x\in[0,1]\,|\,|f(x)|\leqslant\kappa\} \leqslant \frac{\kappa}{\delta}.$$

Using (4.18) and the Lemma 4.2, we conclude that

$$|k\cdot\omega-\alpha_j(\omega)+\beta_\ell(\omega)| \geqslant \kappa(1+|w_a-w_b|) \quad \forall\, j\in[a],\ \forall\,\ell\in[b] \tag{4.19}$$

holds outside a set $F_{[a],[b],k}$ of measure $\leqslant Cw_a^d w_b^d(1+|w_a-w_b|)\kappa$.

[4] We use that the modulus of the eigenvalues are controlled by the operator norm of the matrix.

If F is the union of $F_{[a],[b],k}$ for $|k|\leqslant K$, $[a],[b]\in\widehat{\mathcal{E}}$ such that $w_a\leqslant(\frac{c_0}{2\eta})^{\frac{1}{2\beta}}$, $w_b\leqslant C((\frac{c_0}{2\eta})^{\frac{1}{2\beta}}+K)$ and $|w_a-w_b|\leqslant CK$, we have

$$\begin{aligned}\operatorname{meas}(F)&\leqslant C\Big(\frac{c_0}{2\eta}\Big)^{\frac{1}{2\beta}}\Big(\Big(\frac{c_0}{2\eta}\Big)^{\frac{1}{2\beta}}+K\Big)K^n\Big(\Big(\frac{c_0}{2\eta}\Big)^{\frac{1}{2\beta}}+K\Big)^d\Big(\frac{c_0}{2\eta}\Big)^{\frac{d}{2\beta}}K\kappa\\&\leqslant CK^{n+d+2}\eta^{-\frac{1+d}{\beta}}\kappa\,.\end{aligned}$$

Now we choose η such that

$$\eta^{\alpha_2}=\eta^{-\frac{d+1}{\beta}}\kappa\quad\text{i.e.}\eta=\kappa^{\frac{\beta}{d+1+\beta\alpha_2}}\,.$$

Then, as $\beta\leqslant 1$, $\eta\geqslant\kappa$ and we have

$$\operatorname{meas}(F)\leqslant CK^{n+d+2}\kappa^{\frac{\beta\alpha_2}{d+1+\beta\alpha_2}}\,.$$

Let $\mathcal{D}_2=\mathcal{D}_1\cup F$, we have

$$\operatorname{meas}(\mathcal{D}\setminus\mathcal{D}_2)\leqslant CK^{\alpha_1}\eta^{\alpha_2}+CK^{n+d+2}\kappa^{\frac{\beta\alpha_2}{d+1+\beta\alpha_2}}\leqslant CK^{\gamma_1}\kappa^{\gamma_2}$$

with $\gamma_1=\max(\alpha_1,2+d+n)$, $\gamma_2=\frac{\beta\alpha_2}{d+1+\beta\alpha_2}$. Further, by construction, for all $\rho\in\mathcal{D}_3$, $0<|k|\leqslant K$, $a,b\in\mathcal{E}$ and $j\in[a]$, $\ell\in[b]$ we have

$$|\langle k,\omega\rangle\ -\alpha_j(\omega)+\beta_\ell(\omega)|\geqslant\kappa(1+|w_a-w_b|).$$

Hence using Lemma 4.3 and in view of (4.15), we get that $\widehat{S}'(k)\in\mathcal{M}^+_{s,\beta}$ and

$$|\widehat{S}'(k)|_{s,\beta+}\leqslant C\frac{|\widehat{Q}(k)|_{s,\beta}K^{\frac{d}{2}}}{\kappa^{1+\frac{d}{2\delta}}},\quad 0<|k|\leqslant K\,.$$

Combining this last estimate with (4.17) we obtain a solution S satisfying for any $|\Im\varphi|<\sigma'$

$$|S(\varphi)|_{s,\beta+}\leqslant C\frac{K^{\frac{d}{2}}}{(\sigma-\sigma')^n\kappa^{1+\frac{d}{2\delta}}}\sup_{|\Im\varphi|<\sigma}|Q(\varphi)|_{s,\beta}$$

The estimates for the derivatives with respect to ρ are obtained by differentiating (4.12) which leads to

$$L(\partial_\omega\widehat{S}_{[a]}^{[b]}(k,\omega))=-(\partial_\omega L)\widehat{S}_{[a]}^{[b]}(k,\omega)+i\partial_\omega\widehat{Q}_{[a]}^{[b]}(k,\omega)-i\partial_\omega\widehat{R}_{[a]}^{[b]}(k,\omega)$$

which is an equation of the same type as (4.12) for $\partial_\omega\widehat{S}_{[a]}^{[b]}(k,\omega)$ and $\partial_\omega\widehat{R}_{[a]}^{[b]}(k,\omega)$ where $i\widehat{Q}_{[a]}^{[b]}(k,\omega)$ is replaced by $B_{[a]}^{[b]}(k,\omega)=-(\partial_\omega L)\widehat{S}_{[a]}^{[b]}(k,\omega)+i\partial_\omega\widehat{Q}_{[a]}^{[b]}(k,\omega)$. This equation is solved by defining

$$\begin{aligned}\partial_\omega\widehat{S}_{[a]}^{[b]}(k,\omega)&=\chi_{|k|\leqslant K}(k)L(k,[a],[b],\omega)^{-1}B_{[a]}^{[b]}(k,\omega),\\\partial_\omega\widehat{R}_{[a]}^{[b]}(k,\omega)&=-i\chi_{|k|>K}(k)B_{[a]}^{[b]}(k,\omega)=\chi_{|k|>K}(k)\partial_\rho\widehat{Q}_{[a]}^{[b]}(k,\omega)\end{aligned}$$

Since

$$|(\partial_\omega L)\widehat{S}(k,\omega)|_{s,\beta} \leqslant C(K+2(\|\partial_\omega A_0\|+\delta_0))|\widehat{S}(k,\omega)|_{s,\beta} \leqslant CK|\widehat{S}(k,\omega)|_{s,\beta}$$

we obtain

$$|B(k,\omega)|_{s,\beta} \leqslant CK\kappa^{-\frac{d}{2\beta}-1}K^{d/2}|\big(|\widehat{Q}(k)|_{s,\beta}+|\partial_\omega\widehat{Q}(k)|_{s,\beta}\big)$$

and thus following the same strategy as in the resolution of (4.12) we get for $|\Im\varphi|<\sigma'$

$$|\partial_\omega S(\varphi)|_{s,\beta+} \lesssim \frac{K^{d+1}}{\kappa^{\frac{d}{\beta}+2}(\sigma-\sigma')^n}\left(\sup_{|\Im\varphi|<\sigma}|Q(\varphi)|_{s,\beta}+\sup_{|\Im\varphi|<\sigma}|\partial_\omega Q(\varphi)|_{s,\beta}\right),$$

$$|\partial_\omega R(\varphi)|_{s,\beta} \lesssim \frac{K^{1+\frac{d}{2}}e^{-\frac{1}{2}(\sigma-\sigma')K}}{\kappa^{1+\frac{d}{2\beta}}(\sigma-\sigma')^n}\left(\sup_{|\Im\varphi|<\sigma}|Q(\varphi)|_{s,\beta}+\sup_{|\Im\varphi|<\sigma}|\partial_\omega Q(\varphi)|_{s,\beta}\right).$$

□

We end this section with the key Lemma which is an adaptation of Proposition 2.2.4 in [9] (a similar Lemma is also proved in [14]):

Lemma 4.3. — *Let $A\in\mathcal{M}$ and let $B(k)$ defined for $k\in\mathbb{Z}^n$ by*

$$B(k)_j^l = \frac{1}{k\cdot\omega\ -\mu_j+\mu_l}A_j^l, \quad j\in[a],\ \ell\in[b] \tag{4.20}$$

where $\omega\in\mathbb{R}^n$ and $(\mu_a)_{a\in\mathcal{E}}$ is a sequence of real numbers satisfying

$$|\mu_a-w_a| \leqslant \min\left(\frac{C_\mu}{w_a^\delta},\frac{1}{4}\right), \quad \text{for all } a\in\mathcal{E} \tag{4.21}$$

for a given $C_\mu>0$ and $\delta>0$, and such that for all $a,b\in\mathcal{E}$ and all $|k|\leqslant K$

$$|k\cdot\omega\ -\mu_a+\mu_b| \geqslant \kappa(1+|w_a-w_b|). \tag{4.22}$$

Then $B\in\mathcal{M}$ and there exists a constant $C>0$ depending only on C_μ, $|\omega|$ and δ such that

$$\left\|B(k)_{[a]}^{[b]}\right\| \leqslant C\frac{K^{\frac{d}{2}}}{\kappa^{1+\frac{d}{2\delta}}(1+|w_a-w_b|)}\left\|A_{[a]}^{[b]}\right\| \quad \text{for all} a,b\in\mathcal{E},\ |k|\leqslant K.$$

The proof is based on the fact that the lemma is trivially true when $\mu_a=w_a$ is constant on each block. It is given in Appendix B.

4.3. The KAM step

Theorem 2.3 is proved by an iterative KAM procedure. We begin with the initial Hamiltonian $H_\omega=h_0+q_0$ where

$$h_0(y,\varphi,\xi,\eta)=\omega\cdot y+\langle\xi,N_0\eta\rangle\,, \tag{4.23}$$

$N_0 = \mathrm{diag}(w_a,\ a \in \mathcal{E})$, $\omega \in \mathcal{D}_0$ and the quadratic perturbation $q_0(\varphi, \xi, \eta) = \langle \xi, Q_0(\omega, \varphi)\eta\rangle$ with $Q_0 = \varepsilon Q \in \mathcal{M}_{s,\beta}(\sigma_0, \mathcal{D}_0)$ where $\sigma_0 = \sigma$. Then we construct iteratively the change of variables Φ_{S_m}, the normal form $h_m = \omega \cdot y + \langle \xi, N_m \eta\rangle$ and the perturbation $q_m(\varphi, \xi, \eta; \omega) = \langle \xi, Q_m(\omega, \varphi)\eta\rangle$ with $Q_m \in \mathcal{M}_{s,\beta}(\sigma_m, \mathcal{D}_m)$ as follows: assume that the construction is done up to step $m \geqslant 0$ then

(1) using Proposition 4.1 we construct $S_{m+1}(\omega, \varphi)$ solution of the homological equation for $\omega \in \mathcal{D}_{m+1}$ and $\varphi \in \mathbb{T}^n_{\sigma_{m+1}}$

$$\omega \cdot \nabla_\varphi S_{m+1} - i[N_m, S_{m+1}] + Q_m = \widetilde{N}_m + R_m \tag{4.24}$$

with $\widetilde{N}_m(\omega)$, $R_m(\omega, \varphi)$ defined for $\omega \in \mathcal{D}_{m+1}$ and $\varphi \in \mathbb{T}_{\sigma_{m+1}}$ by

$$\widetilde{N}_m(\omega) = ((\delta_{[j]=[\ell]} \widehat{Q}_m(0))_{j\ell})_{j,\ell \in \mathcal{E}} \tag{4.25}$$

$$R_m(\omega, \varphi) = \sum_{|k| > K_m} \widehat{Q}_m(\omega, k) e^{ik\cdot\varphi}\,; \tag{4.26}$$

(2) we define $Q_{m+1},\ N_{m+1}$ for $\omega \in \mathcal{D}_{m+1}$ and $\varphi \in \mathbb{T}_{\sigma_{m+1}}$ by

$$N_{m+1} = N_m + \widetilde{N}_m\,, \tag{4.27}$$

and

$$Q_{m+1} = R_m + \int_0^1 e^{itS_{m+1}} \Big[(1-t)(N_{m+1} - N_m + R_{m+1}) + tQ_m, S_{m+1}\Big] e^{-it\bar{S}_{m+1}} \mathrm{d}t\,. \tag{4.28}$$

By construction, if Q_m and N_m are hermitian, so are R_m, S_{m+1}, by the resolution of the homological equation, and also N_{m+1} and Q_{m+1}. Then we define

$$\begin{aligned} h_{m+1}(y, \varphi, \xi, \eta; \omega) &= \omega \cdot y + \langle \xi, N_{m+1}(\omega)\eta\rangle\,,\\ s_{m+1}(y, \varphi, \xi, \eta; \omega) &= \langle \xi, S_{m+1}(\omega, \varphi)\eta\rangle\,,\\ q_{m+1}(y, \varphi, \xi, \eta; \omega) &= \langle \xi, Q_{m+1}(\omega, \varphi)\eta\rangle\,. \end{aligned} \tag{4.29}$$

Recall that Φ^t_S denotes the time t flow associated with S (see (2.8)) and $\Phi_S = \Phi^1_S$. For any regular Hamiltonian f we have, using the Taylor expansion of $g(t) = f \circ \Phi^t_{S_{m+1}}$ between $t = 0$ and $t = 1$

$$f \circ \Phi_{S_{m+1}} = f + \{f, s_{m+1}\} + \int_0^1 (1-t)\{\{f, s_{m+1}\}, s_{m+1}\} \circ \Phi^t_{S_{m+1}} \mathrm{d}t\,.$$

Therefore we get for $\omega \in \mathcal{D}_{m+1}$

$$\begin{aligned}(h_m + q_m) \circ \Phi_{S_{m+1}} &= h_m + \{h_m, s_{m+1}\} \\ &\quad + \int_0^1 (1-t)\{\{h_m, s_{m+1}\}, S_{m+1}\} \circ \Phi^t_{S_{m+1}} \mathrm{d}t \\ &\quad + q_m + \int_0^1 \{q_m, s_{m+1}\} \circ \Phi^t_{S_{m+1}} \mathrm{d}t \\ &= h_m + \langle \xi, (\widetilde{N}_m + R_m)\eta\rangle \\ &\quad + \int_0^1 \{(1-t)\langle \xi, (\widetilde{N}_m + R_m)\eta\rangle + tq_m, s_{m+1}\} \circ \Phi^t_{S_{m+1}} \mathrm{d}t \\ &= h_{m+1} + q_{m+1}\end{aligned}$$

where for the last equality we used (2.3) and (2.8).

4.4. Iterative lemma

Following the general scheme (4.24)–(4.29) we have

$$(h_0 + q_0) \circ \Phi^1_{S_1} \circ \cdots \circ \Phi^1_{S_m} = h_m + q_m$$

where $q_m(\xi, \eta) = \langle \xi, Q_m \eta\rangle$ with $Q_m \in \mathcal{M}_{s,\beta}(\mathcal{D}_m, \sigma_m)$ and $h_m = \omega \cdot y + \langle \xi, N_m \eta\rangle$ with N_m in normal form. At step m the Fourier series are truncated at order K_m and the small divisors are controlled by κ_m. Now we specify the choice of all the parameters for $m \geqslant 0$ in term of ε_m which will control(5) $[Q_m]^{\mathcal{D}_m,\sigma_m}_{s,\beta}$.

First we define $\varepsilon_0 = \varepsilon$, $\sigma_0 = \sigma$ and for $m \geqslant 1$ we choose

$$\begin{aligned}\sigma_{m-1} - \sigma_m &= C_* \sigma_0 m^{-2}, \\ K_m &= 2(\sigma_{m-1} - \sigma_m)^{-1} \ln \varepsilon_m^{-1}, \\ \kappa_m &= \varepsilon^\delta_{m-1}\end{aligned}$$

where $(C_*)^{-1} = 2 \sum_{j \geqslant 1} \frac{1}{j^2}$ and $\delta > 0$.

Lemma 4.4. — *Let $0 < \delta' \leqslant \delta'_0 := \frac{\beta}{8(d+2\beta)}$. There exists ε_* depending on δ', d, n, s, β, γ, α_1, α_2 and h_0 such that, for $0 < \varepsilon \leqslant \varepsilon_*$ and*

$$\varepsilon_m = \varepsilon_0^{(3/2)^m} \quad m \geqslant 0\,,$$

we have the following: For all $m \geqslant 1$ there exist $\mathcal{D}_m \subset \mathcal{D}_{m-1}$, $S_m \in \mathcal{M}_{s,\beta+}(\mathcal{D}_m, \sigma_m)$, $h_m = \langle \omega, y\rangle + \langle \xi, N_m \eta\rangle$ where $N_m \in \mathcal{M}_{s,\beta}(\mathcal{D}_m)$ is in normal form and there exists $Q_m \in \mathcal{M}_{s,\beta}(\mathcal{D}_m, \sigma_m)$ such that for $m \geqslant 1$

(5) The norm $[\,\cdot\,]^{\mathcal{D}_m,\sigma_m}_{s,\beta}$ is defined in (2.5).

(i) *The mapping*

$$\Phi_m(\,\cdot\,,\omega,\varphi)=\Phi^1_{S_m}\ :\ Y_s\to Y_s,\quad \omega\in\mathcal{D}_m,\ \varphi\in\mathbb{T}_{\sigma_m} \tag{4.30}$$

is a linear isomorphism linking the Hamiltonian at step $m-1$ and the Hamiltonian at step m, i.e.

$$(h_{m-1}+q_{m-1})\circ\Phi_m=h_m+q_m.$$

(ii) *we have the estimates*

$$\mathrm{meas}(\mathcal{D}_{m-1}\setminus\mathcal{D}_m)\leqslant\varepsilon_{m-1}^{\alpha\delta'}, \tag{4.31}$$

$$[\widetilde{N}_{m-1}]_{s,\beta}^{\mathcal{D}_m}\leqslant\varepsilon_{m-1}, \tag{4.32}$$

$$[Q_m]_{s,\beta}^{\mathcal{D}_m,\sigma_m}\leqslant\varepsilon_m, \tag{4.33}$$

$$\|\Phi_m(\,\cdot\,,\omega,\varphi)-\mathrm{Id}\|_{\mathcal{L}(Y_s,Y_{s+2\beta})}\leqslant\varepsilon_{m-1}^{1-\nu\delta'},\ for\varphi\in\mathbb{T}_{\sigma_m},\ \omega\in\mathcal{D}_m\,. \tag{4.34}$$

The exponent α and ν are given by the formulas $\nu=4(\frac{d}{\beta}+2)$ and $\alpha=\frac{\beta\alpha_2}{2+d+2\beta\alpha_2}$.

Proof. — At step 1, $h_0=\omega\cdot y+\langle\xi,N_0\eta\rangle$ and thus hypothesis (4.5) is trivially satisfied and we can apply Proposition 4.1 to construct S_1, N_1, R_1 and $\mathcal{D}_1$ such that for $\omega\in\mathcal{D}_1$

$$\omega\cdot\nabla_\varphi S_1-i[N_0,S_1]=N_1-N_0-Q_0+R_1.$$

Then, using (4.6), we have

$$\mathrm{meas}(\mathcal{D}\setminus\mathcal{D}_1)\leqslant CK_1^\gamma\kappa_1^{2\alpha}\leqslant\varepsilon_0^{\alpha\delta'}$$

for $\varepsilon=\varepsilon_0$ small enough. Using (4.9) we have for ε_0 small enough

$$[S_1]_{s,\beta+}^{\mathcal{D}_1,\sigma_1}\leqslant C\frac{K_1^{d+1}}{\kappa_1^{\frac{d}{\beta}+2}(\sigma_0-\sigma_1)^n}\varepsilon_0\leqslant\varepsilon_0^{1-\frac{1}{2}\nu\delta'}$$

with $\nu=4(\frac{d}{\beta}+2)$ and thus in view of (2.8) and assertion (iv) of Lemma 2.1 we get

$$\|\Phi_1(\,\cdot\,,\omega,\varphi)-\mathrm{Id}\|_{\mathcal{L}(Y_s,Y_{s+2\beta})}\leqslant\varepsilon_0^{1-\nu\delta'}.$$

Similarly using (4.8), (4.10) we have

$$[N_1-N_0]_{s,\beta}^{\mathcal{D}_1}\leqslant\varepsilon_0,$$

and

$$[R_1]_{s,\beta}^{\mathcal{D}_1,\sigma_1}\leqslant\varepsilon_0^{2-\nu\delta'}$$

for $\varepsilon=\varepsilon_0$ small enough. Thus using (4.28) we get

$$\begin{aligned}[Q_1]_{s,\beta}^{\mathcal{D}_1,\sigma_1}&\leqslant C[R_1]_{s,\beta}^{\mathcal{D}_1,\sigma_1}+C([N_1-N_0]_{s,\beta}^{\mathcal{D}_1}+[R_1]_{s,\beta}^{\mathcal{D}_1,\sigma_1}+[Q_0]_{s,\beta}^{\mathcal{D}_1,\sigma_1})[S_1]_{s,\beta+}^{\mathcal{D}_1,\sigma_1}\\&\leqslant C\varepsilon_0^{2-\nu\delta'}.\end{aligned}$$

Thus for $\delta' \leqslant \delta'_0$ and ε_0 small enough

$$[Q_1]_{s,\beta}^{\mathcal{D}_1,\sigma_1} \leqslant \varepsilon_0^{3/2} = \varepsilon_1.$$

Now assume that we have verified Lemma 4.4 up to step m. We want to perform the step $m+1$. We have $h_m = \omega \cdot y + \langle \xi, N_m \eta \rangle$ and since

$$[N_m - N_0]_{s,\beta}^{\mathcal{D}_m} \leqslant [N_m - N_0]_{s,\beta}^{\mathcal{D}_m} + \cdots + [N_1 - N_0]_{s,\beta}^{\mathcal{D}_1} \leqslant \sum_{j=0}^{m-1} \varepsilon_j \leqslant 2\varepsilon_0,$$

hypothesis (4.5) is satisfied and we can apply Proposition 4.1 to construct S_{m+1}, N_{m+1}, R_{m+1} and $\mathcal{D}_{m+1}$ such that for $\omega \in \mathcal{D}_{m+1}$

$$\omega \cdot \nabla_\varphi S_{m+1} - i[N_m, S_{m+1}] = N_{m+1} - N_m - Q_m + R_{m+1}.$$

Then, using (4.6), we have

$$\text{meas}(\mathcal{D}_m \setminus \mathcal{D}_{m+1}) \leqslant C K_{m+1}^{\gamma} \kappa_{m+1}^{2\alpha} \leqslant \varepsilon_m^{\alpha\delta'}$$

for ε_0 small enough. Using (4.9) we have for ε_0 small enough

$$[S_{m+1}]_{s,\beta+}^{\mathcal{D}_{m+1},\sigma_{m+1}} \leqslant C \frac{K_{m+1}^{d+1}}{\kappa_{m+1}^{\frac{d}{\beta}+2} (\sigma_m - \sigma_{m+1})^n} \varepsilon_m \leqslant \varepsilon_m^{1-\frac{1}{2}\nu\delta'}.$$

Thus in view of (2.8) and assertion (iv) of Lemma 2.1 we get

$$\|\Phi_{m+1}(\,\cdot\,,\omega,\varphi) - \text{Id}\|_{\mathcal{L}(Y_s, Y_{s+2\beta})} \leqslant \varepsilon_m^{1-\nu\delta'}.$$

Similarly using (4.8), (4.10) we have

$$[N_{m+1} - N_m]_{s,\beta}^{\mathcal{D}_{m+1}} \leqslant \varepsilon_m,$$

and

$$[R_{m+1}]_{s,\beta}^{\mathcal{D}_{m+1},\sigma_{m+1}} \leqslant \varepsilon_m^{2-\nu\delta'}$$

for ε_0 small enough. Thus using (4.28) we get

$$\begin{aligned} &[Q_{m+1}]_{s,\beta}^{\mathcal{D}_{m+1},\sigma_{m+1}} \\ &\quad \leqslant C[R_{m+1}]_{s,\beta}^{\mathcal{D}_{m+1},\sigma_{m+1}} + C\Big([N_{m+1} - N_m]_{s,\beta}^{\mathcal{D}_{m+1}} + [R_{m+1}]_{s,\beta}^{\mathcal{D}_{m+1},\sigma_{m+1}} \\ &\qquad\qquad + [Q_m]_{s,\beta}^{\mathcal{D}_{m+1},\sigma_{m+1}}\Big)[S_{m+1}]_{s,\beta+}^{\mathcal{D}_{m+1},\sigma_{m+1}} \\ &\quad \leqslant C\varepsilon_m^{2-\nu\delta'}. \end{aligned}$$

Thus for $\delta' \leqslant \delta'_0$ and ε_0 small enough

$$[Q_{m+1}]_{s,\beta}^{\mathcal{D}_{m+1},\sigma_{m+1}} \leqslant \varepsilon_m^{3/2} = \varepsilon_{m+1}. \qquad \square$$

4.5. Transition to the limit and proof of Theorem 2.3

Let

$$\mathcal{D}' = \cap_{m\geqslant 0}\mathcal{D}_m.$$

In view of (4.31), this is a Borel set satisfying

$$\mathrm{meas}(\mathcal{D}\setminus\mathcal{D}') \leqslant \sum_{m\geqslant 0}\varepsilon_m^{\alpha\delta'} \leqslant 2\varepsilon_0^{\alpha\delta'}.$$

Let us denote $\Phi_N^1(\,\cdot\,,\omega,\varphi) = \Phi_1(\,\cdot\,,\omega,\varphi)\circ\cdots\circ\Phi_N(\,\cdot\,,\omega,\varphi)$. Due to (4.30), it maps Y_s to Y_s and due to (4.34) it satisfies for $M\leqslant N$ and for $\omega\in\mathcal{D}'$, $\varphi\in\mathbb{T}_{\sigma/2}$

$$\|\Phi_N^1(\,\cdot\,,\omega,\varphi)-\Phi_M^1(\,\cdot\,,\omega,\varphi)\|_{\mathcal{L}(Y_s,Y_{s+2\beta})} \leqslant \sum_{m=M}^{N}\varepsilon_m^{1-\nu\delta'} \leqslant 2\varepsilon_M^{1-\nu\delta'}.$$

Therefore $(\Phi_N^1(\,\cdot\,,\omega,\varphi))_N$ is a Cauchy sequence in $\mathcal{L}(Y_s,Y_{s+2\beta})$. Thus when $N\to\infty$ the maps $\Phi_N^1(\,\cdot\,,\omega,\varphi)$ converge to a limit mapping $\Phi_\infty^1(\,\cdot\,,\omega,\varphi)\in\mathcal{L}(Y_s)$. Furthermore since the convergence is uniform on $\omega\in\mathcal{D}'$ and $\varphi\in\mathbb{T}_{\sigma/2}$, $(\omega,\varphi)\to\Phi_\infty^1(\,\cdot\,,\omega,\varphi)$ is analytic in φ and C^1 in ω. Moreover, defining $\delta=\alpha\delta'/2$ and taking $\delta_0=\alpha/(4\nu)$, we get

$$\|\Phi_\infty^1(\,\cdot\,,\omega,\varphi)-\mathrm{Id}\|_{\mathcal{L}(Y_s,Y_{s+2\beta})} \leqslant 2\varepsilon_0^{1-\nu\delta'} < \varepsilon_0^{1-\delta/\delta_0}. \tag{4.35}$$

By construction, the map $\Phi_m^1(\,\cdot\,,\omega,\omega t)$ conjugates the original Hamiltonian system associated with

$$H_0 = H_\omega(t,\xi,\eta) = \langle\xi,N_0\eta\rangle + \varepsilon\langle\xi,Q(\omega,\omega t)\eta\rangle$$

into the Hamiltonian system associated with

$$H_m(t,\xi,\eta) = \langle\xi,N_m\eta\rangle + \langle\xi,Q_m(\omega,\omega t)\eta\rangle.$$

By (4.33), $Q_m\to 0$ when $m\to\infty$ and by (4.32) $N_m\to N$ when $m\to\infty$ where the operator

$$N\equiv N(\omega) = N_0 + \sum_{k=1}^{+\infty}\widetilde{N}_k \tag{4.36}$$

is C^1 with respect to ω and is in normal form, since this is the case for all the $N_k(\omega)$. Further for all $\omega\in\mathcal{D}'$ we have using (4.32)

$$\|N(\omega)-N_0\|_{s,\beta} \leqslant \sum_{m=0}^{\infty}\varepsilon^m \leqslant 2\varepsilon.$$

Let us denote $\Psi_\omega(\varphi)=\Phi_\infty^1(\,\cdot\,,\omega,\varphi)$. By construction,

$$\Psi_\omega(\varphi) = \langle\overline{M_\omega(\varphi)}\xi, M_\omega(\varphi)\eta\rangle\,,$$

where

$$M_\omega(\varphi) = \lim_{j\to+\infty} e^{iS_1(\omega,\varphi)} \dots e^{iS_j(\omega,\varphi)} .$$

Further, denoting the limiting Hamiltonian $\mathcal{H}_\omega(\xi,\eta) = \langle \xi, N\eta\rangle$, the symplectic change of variables $\Psi_\omega(\omega t)$ conjugates the original Hamiltonian system associated with H_ω into the Hamiltonian system associated with $\mathcal{H}_\omega$.

This concludes the proof of Theorem 2.3.

Appendix A. Proof of Lemma 2.1

We start with two auxiliary lemmas

LEMMA A.1. — *Let $j,k,\ell \in \mathbb{N}\setminus\{0\}$ then*

$$\frac{\sqrt{\min(j,k)}}{\sqrt{\min(j,k)}+|j-k|}\,\frac{\sqrt{\min(\ell,k)}}{\sqrt{\min(\ell,k)}+|\ell-k|} \leqslant \frac{\sqrt{\min(j,\ell)}}{\sqrt{\min(j,\ell)}+|j-\ell|}. \quad \text{(A.1)}$$

Proof. — Without loss of generality we can assume $j \leqslant \ell$.

If $k \leqslant j$ then $|k-\ell| \geqslant |j-\ell|$ and thus

$$\begin{aligned}\frac{\sqrt{\min(j,\ell)}}{\sqrt{\min(j,\ell)}+|j-\ell|} &= \frac{\sqrt{j}}{\sqrt{j}+|j-\ell|} \geqslant \frac{\sqrt{j}}{\sqrt{j}+|k-\ell|}\\ &\geqslant \frac{\sqrt{k}}{\sqrt{k}+|k-\ell|} = \frac{\sqrt{\min(k,\ell)}}{\sqrt{\min(k,\ell)}+|k-\ell|}\end{aligned}$$

which leads to (A.1). The case $\ell \leqslant k$ is similar.

In the case $j<k<\ell$ inequality (A.1) is equivalent to

$$\sqrt{k}(\sqrt{j}+\ell-j) \leqslant (\sqrt{j}+k-j)(\sqrt{k}+\ell-k)$$

and splitting $\ell - j = (\ell-k)+(k-j)$ leads to

$$\sqrt{k}(\ell-k) \leqslant \sqrt{j}(\ell-k)+(k-j)(\ell-k)$$

which is true since $\sqrt{k}-\sqrt{j} \leqslant (\sqrt{k}-\sqrt{j})(\sqrt{k}+\sqrt{j}) = k-j$. □

LEMMA A.2. — *Let $j \in \mathbb{N}$ then*

$$\sum_{k\in\mathbb{N}} \frac{1}{k^\beta(1+|k-j|)} \leqslant C(\beta)$$

for a constant $C(\beta)>0$ depending only on $\beta>0$.

Proof. — We note that

$$\sum_{k\in\mathbb{N}} \frac{1}{k^\beta(1+|k-j|)} = a\star b(j)$$

where $a_k = \frac{1}{k^\beta}$ for $k\geqslant 1$, $a_k=0$ for $k\leqslant 0$ and $b_k=\frac{1}{1+|k|}$, $k\in\mathbb{Z}$. We have that $b\in\ell^p$ for any $1<p\leqslant+\infty$ and that $a\in\ell^q$ for any $\frac{1}{\beta}<q\leqslant+\infty$. Thus by Young inequality $a\star b\in\ell_r$ for r such that $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$. In particular choosing $q=\frac{2}{\beta}$ and $p=\frac{2}{2-\beta}$ we conclude that $a\star b\in\ell_\infty$. □

Proof of Lemma 2.1. — In this proof we extend the definition of the weight w_a, $a\in\mathcal{E}$, as follows: when $j\in\widehat{\mathcal{E}}$ we denote $w_j=j$.

(i). — Let $a,b\in\mathcal{E}$

$$\begin{aligned}\left\|(AB)_{[a]}^{[b]}\right\| &\leqslant \sum_{c\in\hat{\mathcal{E}}}\left\|A_{[a]}^{[c]}\right\|\left\|B_{[c]}^{[b]}\right\|\\ &\leqslant \frac{|A|_{s,\beta+}|B|_{s,\beta}}{(w_aw_b)^\beta}\left(\frac{\sqrt{\min(w_a,w_b)}}{\sqrt{\min(w_a,w_b)}+|w_a-w_b|}\right)^{s/2}\sum_{c\in\hat{\mathcal{E}}}\frac{1}{w_c^{2\beta}(1+|w_a-w_c|)}\\ &\leqslant C\frac{|A|_{s,\beta+}|B|_{s,\beta}}{(w_aw_b)^\beta}\left(\frac{\sqrt{\min(w_a,w_b)}}{\sqrt{\min(w_a,w_b)}+|w_a-w_b|}\right)^{s/2}\end{aligned}$$

where we used that by Lemma A.1

$$\frac{\sqrt{\min(w_a,w_b)}}{\sqrt{\min(w_a,w_b)}+|w_a-w_b|} \geqslant \frac{\sqrt{\min(w_a,w_c)}}{\sqrt{\min(w_a,w_c)}+|w_a-w_c|}\,\frac{\sqrt{\min(w_c,w_b)}}{\sqrt{\min(w_c,w_b)}+|w_c-w_b|}$$

and that by Lemma A.2, $\sum_{c\in\hat{\mathcal{E}}}\frac{1}{w_c^{2\beta}(1+|w_a-w_c|)}\leqslant C$ where C only depends on β.

(ii). — Similarly let $a,b\in\mathcal{L}$ and assume without loss of generality that $w_a\leqslant w_b$

$$\begin{aligned}\left\|(AB)_{[a]}^{[b]}\right\| &\leqslant \sum_{c\in\hat{\mathcal{E}}}\left\|A_{[a]}^{[c]}\right\|\left\|B_{[c]}^{[b]}\right\|\\ &\leqslant \frac{|A|_{s,\beta+}|B|_{s,\beta+}}{(w_aw_b)^\beta}\left(\frac{\sqrt{\min(w_a,w_b)}}{\sqrt{\min(w_a,w_b)}+|w_a-w_b|}\right)^{s/2}\\ &\quad \sum_{c\in\hat{\mathcal{E}}}\frac{1}{w_c^{2\beta}(1+|w_a-w_c|)(1+|w_b-w_c|)}\end{aligned}$$

$$\leqslant \frac{2|A|_{s,\beta+}|B|_{s,\beta+}}{(w_a w_b)^\beta(1+|w_a-w_b|)}\left(\frac{\sqrt{\min(w_a,w_b)}}{\sqrt{\min(w_a,w_b)}+|w_a-w_b|}\right)^{s/2}$$
$$\left(\sum_{\substack{c\in\hat{\mathcal{E}}\\ w_c\leqslant\frac12(w_a+w_b)}}\frac{1}{w_c^{2\beta}(1+|w_a-w_c|)}+\sum_{\substack{c\in\hat{\mathcal{E}}\\ w_c\geqslant\frac12(w_a+w_b)}}\frac{1}{w_c^{2\beta}(1+|w_b-w_c|)}\right)$$
$$\leqslant C\frac{|A|_{s,\beta+}|B|_{s,\beta+}}{(w_a w_b)^\beta(1+|w_a-w_b|)}\left(\frac{\sqrt{\min(w_a,w_b)}}{\sqrt{\min(w_a,w_b)}+|w_a-w_b|}\right)^{s/2}.$$

(iii). — Let $\xi\in\ell_t^2$, with $t\geqslant 1$. We have

$$\begin{aligned}\|A\xi\|_{-t}^2 &\leqslant \sum_{a\in\hat{\mathcal{E}}} w_a^{-t}\left(\sum_{b\in\hat{\mathcal{E}}}\|A_{[a]}^{[b]}\|\|\xi_{[b]}\|\right)^2\\
&\leqslant |A|_{s,\beta}^2\sum_{a\in\hat{\mathcal{E}}}\left(\sum_{b\in\hat{\mathcal{E}}}\frac{\|w_b^{t/2}\xi_{[b]}\|}{w_a^{t/2+\beta}w_b^{t/2+\beta}}\left(\frac{\sqrt{\min(w_a,w_b)}}{\sqrt{\min(w_a,w_b)}+|w_a-w_b|}\right)^{s/2}\right)^2\\
&\leqslant \sum_{a\in\hat{\mathcal{E}}}\frac{1}{w_a^{t+2\beta}}\sum_{b\in\hat{\mathcal{E}}}\frac{1}{w_b^{t+2\beta}}|A|_{s,\beta}^2\,\|\xi\|_t^2\end{aligned}$$

where we have used the Cauchy Schwarz inequality to get the last line.

(iv). — Let $\xi\in\ell_s^2$. We have

$$\begin{aligned}\|A\xi\|_{s+2\beta}^2 &\leqslant \sum_{a\in\hat{\mathcal{E}}} w_a^{s+2\beta}\left(\sum_{b\in\hat{\mathcal{E}}}\|A_{[a]}^{[b]}\|\|\xi_{[b]}\|\right)^2\\
&\leqslant |A|_{s,\beta+}^2\sum_{a\in\hat{\mathcal{E}}}\left(\sum_{b\in\hat{\mathcal{E}}}\frac{w_a^{s/2}\|w_b^{s/2}\xi_{[b]}\|}{w_b^{s/2+\beta}(1+|w_a-w_b|)}\right.\\
&\qquad\left.\left(\frac{\sqrt{\min(w_a,w_b)}}{\sqrt{\min(w_a,w_b)}+|w_a-w_b|}\right)^{s/2}\right)^2\\
&\leqslant 2^{s+1}|A|_{s,\beta+}^2\sum_{a\in\hat{\mathcal{E}}}\left(\sum_{\substack{b\in\hat{\mathcal{E}}\\ w_a\leqslant 2w_b}}\frac{\|w_b^{s/2}\xi_{[b]}\|}{w_b^{\beta}(1+|w_a-w_b|)}\right.\\
&\qquad\left.+\sum_{\substack{b\in\hat{\mathcal{E}}\\ w_a\geqslant 2w_b}}\frac{\|w_b^{s/2}\xi_{[b]}\|\min(w_a,w_b)^{\frac{s}{2}}}{w_b^{s/2+\beta}(1+|w_a-w_b|)}\right)^2\end{aligned}$$

$$\leqslant 2^{s+1}|A|^2_{s,\beta+}\sum_{a\in\hat{\mathcal{E}}}\left(\sum_{b\in\hat{\mathcal{E}}}\frac{\|w_b^{s/2}\xi_{[b]}\|}{w_b^{\beta}(1+|w_a-w_b|)}\right)^2$$

where we used that $\frac{w_a}{\sqrt{\min w_a,w_b)}+|w_a-w_b|}\leqslant\sqrt{\min w_a,w_b)}$. Then we note that

$$\sum_{b\in\hat{\mathcal{E}}}\frac{\|w_b^{s/2}\xi_{[b]}\|}{w_b^{\beta}(1+|w_a-w_b|)}=u\star v(a)$$

with $u_b=\|w_b^{s/2-\beta}\xi_{[b]}\|$ and $v_b=\frac{1}{(1+|w_b|)}$. Writing $u_b^p=\|w_b^{s/2}\xi_{[b]}\|^p\,w_b^{-\beta p}$ and using the Hölder inequality we get for $\frac{\ }{2}+\frac{1}{r}=1$

$$\sum_{b\in\hat{\mathcal{E}}}u_b^p\leqslant\left(\sum_{b\in\hat{\mathcal{E}}}\|w_b^{s/2}\xi_{[b]}\|^2\right)^{p/2}\left(\sum_{b\in\hat{\mathcal{E}}}w_b^{-\beta pr}\right)^{1/r}.$$

Choosing $p=\frac{2}{1+\beta}<2$ we have $r=\frac{1+\beta}{\beta}$ and thus $\beta pr=2>1$. Therefore $u\in\ell^p$. On the other hand, choosing $q=\frac{2}{2-\beta}>1$, we have $v\in\ell^q$. Since $1/p+1/q=3/2$ we conclude by Young inequality that $u\star v\in\ell^2$ and

$$\|u\star v\|_{\ell^2}\leqslant C\,\|u\|_{\ell^p}\,\|v\|_{\ell^q}\,.$$

This leads to the first part of (iv) since $\|u\|_{\ell^p}\leqslant C\|\xi\|_s$. Now we prove the second assertion of (iv) in a similar way: let $\xi\in\ell^2_1$, we have

$$\begin{aligned}\|A\xi\|_1^2&\leqslant\sum_{a\in\hat{\mathcal{E}}}w_a\left(\sum_{b\in\hat{\mathcal{E}}}\|A_{[a]}^{[b]}\|\|\xi_{[b]}\|\right)^2\\&\leqslant|A|^2_{s,\beta+}\sum_{a\in\hat{\mathcal{E}}}\left(\sum_{b\in\hat{\mathcal{E}}}\frac{w_a^{1/2}\|w_b^{1/2}\xi_{[b]}\|}{(w_aw_b)^{\beta}w_b^{1/2}(1+|w_a-w_b|)}\right.\\&\qquad\left.\left(\frac{\sqrt{\min(w_a,w_b)}}{\sqrt{\min(w_a,w_b)}+|w_a-w_b|}\right)^{s/2}\right)^2\\&\leqslant 2^{s+1}|A|^2_{s,\beta+}\sum_{a\in\hat{\mathcal{E}}}\left(\sum_{\substack{b\in\hat{\mathcal{E}}\\w_a\leqslant 2w_b}}\frac{\|w_b^{1/2}\xi_{[b]}\|}{(w_aw_b)^{\beta}(1+|w_a-w_b|)}\right.\\&\qquad\left.+\sum_{\substack{b\in\hat{\mathcal{E}}\\w_a\geqslant 2w_b}}\frac{\|w_b^{1/2}\xi_{[b]}\|w_a^{(1-s)/2}}{(w_aw_b)^{\beta}w_b^{1/2-s/4}(1+|w_a-w_b|)}\right)^2\end{aligned}$$

The last sum may be bounded above by (notice that $|w_a - w_b| \geqslant w_b$)

$$\sum_{\substack{b\in\hat{\mathcal{E}}\\ w_a\geqslant 2w_b}} \frac{\|w_b^{1/2}\xi_{[b]}\| w_a^{(1-s)/2}}{(w_a w_b)^\beta w_b^{1/2-s/4}(1+|w_a-w_b|)}$$

$$\leqslant \sum_{\substack{b\in\hat{\mathcal{E}}\\ w_a\geqslant 2w_b}} \frac{\|w_b^{1/2}\xi_{[b]}\|}{(w_a w_b)^\beta w_b^{1/2-s/4}(1+|w_a-w_b|)^{1/2+s/2}}$$

$$\leqslant \frac{1}{w_a^\beta}\sum_{b\in\hat{\mathcal{E}}} \frac{\|w_b^{1/2}\xi_{[b]}\|}{w_b^{1/2+\beta/2}(1+|w_a-w_b|)^{1/2+\beta/2}}\,,$$

and this last sum is the convolution product $u'\star v'(a)$, with $u'_b = \frac{\|w_b^{1/2}\xi_{[b]}\|}{w_b^{1/2+\beta}}$, which defines a ℓ^1 sequence thanks to Cauchy Schwarz inequality, and $v'_b = \frac{1}{(1+w_b)^{1/2+\beta/2}}$, which defines a ℓ^2 sequence. Therefore, it is a ℓ^2 sequence with index a. We treat the first sum in the same way as before, and we obtain

$$\|A\xi\|_1^2 \leqslant C|A|_{s,\beta+}^2\|\xi\|_1^2\,. \qquad \square$$

Appendix B. Proof of Lemma 4.3

Since we estimate the operator norm of $B_{[a]}^{[b]}$, we need to rewrite the definition (4.20) in a operator way: denoting by $D_{[a]}$ the diagonal (square) matrix with entries μ_j, for $j\in[a]$ and $D'_{[a]}$ the diagonal (square) matrix with entries $k\cdot\omega+\mu_j$, for $j\in[a]$, equation (4.20) reads

$$D'_{[a]}B_{[a]}^{[b]} - B_{[a]}^{[b]}D_{[b]} = A_{[a]}^{[b]}\,. \tag{B.1}$$

Then we distinguish 3 cases:

Case 1. — Suppose that a,b satisfy

$$\max(w_a,w_b) > K_1\min(w_a,w_b)$$

take for instance $w_a > K_1 w_b$. For $j\in[a]$ we have

$$|k\cdot\omega\ +\mu_j| \geqslant w_a - \frac{1}{4} - K|\omega| \geqslant \frac{w_a}{2} \tag{B.2}$$

if $K|\omega|\leqslant 1/4w_a$. In particular this holds true assuming

$$K_1 \geqslant 4K|\omega|. \tag{B.3}$$

As a consequence if (B.3) is satisfied then $D'_{[a]}$ is invertible and $\frac{w_a}{2}$ is an upper bound for the operator norm of its inverse. Then (B.1) is equivalent to

$$B_{[a]}^{[b]} - {D'_{[a]}}^{-1} B_{[a]}^{[b]} D_{[b]} = {D'_{[a]}}^{-1} A_{[a]}^{[b]} . \tag{B.4}$$

Next consider the operator $\mathscr{L}^1_{[a]\times[b]}$ acting on matrices of size $[a]\times[b]$ such that

$$\mathscr{L}^1_{[a]\times[b]}\left(B_{[a]}^{[b]}\right) := {D'_{[a]}}^{-1} B_{[a]}^{[b]} D_{[b]} . \tag{B.5}$$

We have

$$\left\|\mathscr{L}^1_{[a]\times[b]}\left(B_{[a]}^{[b]}\right)\right\| \leqslant \frac{2w_b}{w_a}\left\|B_{[a]}^{[b]}\right\| \leqslant \frac{2}{K_1}\left\|B_{[a]}^{[b]}\right\| , \tag{B.6}$$

hence, in operator norm, $\|\mathscr{L}^1_{[a]\times[b]}\| \leqslant \frac{1}{2}$ if $K_1 \geqslant 4$. Then the operator $\mathrm{Id} - \mathscr{L}^1_{[a]\times[b]}$ is invertible and

$$\begin{aligned}\left\|B_{[a]}^{[b]}\right\| &\leqslant \left\|\left(\mathrm{Id} - \mathscr{L}_{[a]\times[b]}\right)^{-1}\right\|\left\|{D'_{[a]}}^{-1} A_{[a]}^{[b]}\right\| \\ &\leqslant \frac{4}{w_a}\left\|A_{[a]}^{[b]}\right\| .\end{aligned}$$

But in case 1, $1 + |w_a - w_b| \leqslant 1 + w_a \leqslant 2w_a$, therefore

$$\left\|B_{[a]}^{[b]}\right\| \leqslant 8\frac{1}{1+|w_a - w_b|}\left\|A_{[a]}^{[b]}\right\|. \tag{B.7}$$

Case 2. — Suppose that a, b satisfy

$$\max(w_a, w_b) \leqslant K_1 \min(w_a, w_b) \text{ and } \max(w_a, w_b) > K_2 .$$

Notice that these two conditions imply that

$$\min(w_a, w_b) \geqslant \frac{K_2}{K_1}.$$

We define the square matrix $\widetilde{D}_{[a]} = w_a \mathbb{1}_{[a]}$, where $\mathbb{1}_{[a]}$ is the identity matrix. Then

$$\left\|D_{[a]} - \widetilde{D}_{[a]}\right\| \leqslant \frac{C_\mu}{w_a^\delta} , \tag{B.8}$$

and equation (4.20) may be rewritten as

$$\mathscr{L}^2_{[a]\times[b]}\left(B_{[a]}^{[b]}\right) - (\widetilde{D}_{[a]} - D_{[a]})B_{[a]}^{[b]} + B_{[a]}^{[b]}(\widetilde{D}_{[b]} - D_{[b]}) = A_{[a]}^{[b]} , \tag{B.9}$$

where we denote by $\mathscr{L}^2_{[a]\times[b]}$ the operator acting on matrices of size $[a]\times[b]$ such that

$$\mathscr{L}^2_{[a]\times[b]}\left(B_{[a]}^{[b]}\right) := (k\cdot\omega + w_a - w_b)\, B_{[a]}^{[b]} . \tag{B.10}$$

This dilation is invertible and (4.22) then gives, in operator norm,

$$\left\|\left(\mathscr{L}^2_{[a]\times[b]}\right)^{-1}\right\| \leqslant \frac{1}{\kappa(1+|w_a - w_b|)} . \tag{B.11}$$

This allows to write (B.9) as

$$B_{[a]}^{[b]} - \left(\mathscr{L}^2_{[a]\times[b]}\right)^{-1} \mathscr{K}_{[a]\times[b]}\left(B_{[a]}^{[b]}\right) = \left(\mathscr{L}^2_{[a]\times[b]}\right)^{-1}\left(A_{[a]}^{[b]}\right), \tag{B.12}$$

where $\mathscr{K}_{[a]\times[b]}\big(B_{[a]}^{[b]}\big) = (\widetilde{D}_{[a]} - D_{[a]})B_{[a]}^{[b]} - B_{[a]}^{[b]}(\widetilde{D}_{[b]} - D_{[b]})$. We have, thanks to (4.21), in operator norm,

$$\|\mathscr{K}_{[a]\times[b]}\| \leqslant C_\mu \left(\frac{1}{w_a^\delta} + \frac{1}{w_b^\delta}\right) \leqslant C_\mu \left(\frac{K_1}{K_2}\right)^\delta. \tag{B.13}$$

Then for

$$K_2 \geqslant K_1 \left(\frac{2C_\mu}{\kappa}\right)^{1/\delta}, \tag{B.14}$$

the operator $\mathrm{Id} - (\mathscr{L}^2_{[a]\times[b]})^{-1}\mathscr{K}_{[a]\times[b]}$ is invertible and from (B.12) we get

$$\begin{aligned}\left\|B_{[a]}^{[b]}\right\| &= \left\|\left(\mathrm{Id} - (\mathscr{L}^2_{[a]\times[b]})^{-1}\mathscr{K}_{[a]\times[b]}\right)^{-1}\right\| \left\|\left(\mathscr{L}^2_{[a]\times[b]}\right)^{-1}\left(A_{[a]}^{[b]}\right)\right\| \\ &\leqslant 2\left\|\left(\mathscr{L}^2_{[a]\times[b]}\right)^{-1}\left(A_{[a]}^{[b]}\right)\right\|.\end{aligned}$$

Hence in this case

$$\left\|B_{[a]}^{[b]}\right\| \leqslant \frac{2}{\kappa(1+|w_a - w_b|)}\left\|A_{[a]}^{[b]}\right\|. \tag{B.15}$$

Case 3. — Suppose that $a, b \in \mathcal{L}$ satisfy

$$\max(w_a, w_b) \leqslant K_1 \min(w_a, w_b) \text{ and } \max(w_a, w_b) \leqslant K_2\,.$$

In that case the size of the blocks are less than K_2^d and we have

$$|B_j^l| = \left|\frac{1}{\langle k, \omega\rangle \; + \mu_j - \mu_l}\right| |A_j^l| \leqslant \frac{1}{\kappa(1+|w_a - w_b|)}|A_j^l| \tag{B.16}$$

A majoration of the coefficients gives a poor majoration of the operator norm of a matrix, but it is sufficient here:

$$\left\|B_{[a]}^{[b]}\right\| \leqslant \frac{K_2^{d/2}}{\kappa(1+|w_a - w_b|)}\left\|A_{[a]}^{[b]}\right\|. \tag{B.17}$$

Collecting (B.7), (B.15) and (B.17) and taking into account (B.3), (B.14) leads to the result. □

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