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Non-explosion criteria for rough differential equations driven by unbounded vector fields (*)

Ismael Bailleul (1) and Remi Catellier (2)

Abstract. — We give in this note a simple treatment of the non-explosion problem for rough differential equations driven by unbounded vector fields and weak geometric rough paths of arbitrary roughness.

Résumé. — On traite de façon simple dans cette note du problème de la non explosion des solutions d'équations différentielles rugueuses conduites par des champs de vecteurs non bornés et un p-rough path faiblement géométrique, pour p quelconque.

1. Introduction

Although rough paths theory has now been explored for twenty years, a few elementary questions are still begging for a definite answer. We recommend [1, 18, 24] for gentle introductions to rough differential equations, from different points of view. We consider the existence problem for the local time and occupation time of solutions to rough differential equations as the main open problem, in relation with reflection problems. At a more fundamental level, the question of global in time existence of solutions of a rough differential equation

$$\mathrm{d}z_t = \mathrm{F}(z_t)\,\mathrm{d}\mathbf{X}_t, \tag{1.1}$$

under relaxed boundedness assumptions on the vector fields $\mathrm{F} = (V_1, \ldots, V_\ell)$ has not been clarified so far. We recall in Appendix A what needs to be known

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about weakly geometric rough paths. Given a weak geometric p-rough path $\mathbf{X}$ defined on some time interval $[0,T]$, the preceding equation is known to have a solution defined on the whole of $[0,T]$ if the driving vector fields V_i are C_b^γ, for some regularity exponent $\gamma > p$; see for instance T. Lyons' seminal paper [23] or the lecture notes [24]. One would ideally like to relax these boundedness assumptions to some linear growth assumption, but the following elementary counter-examples of Gubinelli and Lejay [22] shows that this is not sufficient. Consider the dynamics (1.1) on $\mathbb{R}^2$, with $\mathrm{F} = (V_1, V_2)$, and vector fields $V_1(x,y) = (x\sin(y), x)$ and $V_2(x,y) = 0$, driven by the non-geometric pure area rough path $\mathbf{X}_t = 1 + t(1\otimes 1)$. Writing $z_t = (x_t, y_t)$, one sees that z is actually the solution of the ordinatry differential equation

$$\dot{z}_t = (\dot{x}_t, \dot{y}_t) = \big(x_t \sin(y_t)^2 + x_t^2\cos(y_t),\, x_t\sin(y_t)\big).$$

The solution started from an initial condition of the form $(a,0)$, with a positive, has constant null y-component and has an exploding x-component since $\dot{x}_t = x_t^2$.

The non-explosion problem was explored in a number of works for differential equations driven by p-rough paths, for $2 \leqslant p < 3$, especially in the works of Davie [14] and Lejay [21, 22]. Davie provides essentially the sharpest result in the regime $2 \leqslant p < 3$.

- To make it simple, assume F is C^3 and has linear growth: $\big|\mathrm{F}(x)\big| \lesssim |x|$. Theorem 6.1 (a) in [14] provides a non-explosion criterion in terms of the growth rate of $D^2\mathrm{F}$
$$\big|D^2\mathrm{F}(x)\big| \leqslant h(|x|).$$
There is no explosion if $h(r) \lesssim \frac{1}{r}$, and
$$\int^\infty \left(\frac{r^{\gamma-2}}{h(r)}\right)^{\frac{p-1}{\gamma-1}} \frac{\mathrm{d}r}{r^p} = \infty.$$
Davie's criterion is shown to be sharp in the class of all p-rough paths, $2 \leqslant p < 3$, with an example of a rough differential equation where explosion can happen for some appropriate choice of a non-weak geometric rough path in case the criterion is not satisfied (see [14, Section 6]). The limit case for Davie's criterion is $h(r) = \frac{O(1)}{r}$. We essentially recover that bound.
- Lejay [21] works with Banach space valued weak geometric p-rough paths, with $2 \leqslant p < 3$. In the setting where the vector fields V_i are C^3 with bounded derivates and are required to have growth rate $\big|V_i(x)\big| \lesssim g(|x|)$, he shows non-explosion of solutions to equation (1.1) under the condition that $\sum_k \frac{1}{g(k)^p}$ diverges. The limit case is $g(r) \simeq r^{\frac{1}{p}}$.

- The analysis of Friz and Victoir [20, Exercice 10.56], gives a criterion comparable to ours, with an erronous proof. They use a pattern of proof that is implemented in a linear setting and cannot work in a nonlinear framework as it bears heavily on a scaling argument (see the proof of Theorem 10.53). One can see part of the present work as a correct or alternative proof of their statement.

We identify in the sequel a vector field V on $\mathbb{R}^d$ with the first order differential operator $f \mapsto (Df)(V)$. For a tuple $I = (i_1, \dots, i_k) \in \{1, \dots, \ell\}^k$, and vector fields $V_1, \dots, V_\ell$, we define the differential operators

$$V_I := V_{i_1} \cdots V_{i_k}, \quad \text{and} \quad V_{[I]} := \big[V_{i_1}, \dots, [V_{i_{k-1}}, V_{i_k}]\big],$$

under proper regularity assumptions on the V_i. (Note that the operator $V_{[I]}$ is actually of order one, so $V_{[I]}$ is a vector field.) The local increment $z_t - z_s$ of a solution z to the rough differential equation (1.1) is known to be well-approximated by the time 1 value of the ordinary differential equation

$$y'_r = \sum_{k=1}^{[p]} \sum_{I \in \{1,\dots,\ell\}^k} \Lambda_{ts}^{k,I} V_{[I]}\big(s, y_r(x)\big), \tag{1.2}$$

where $\boldsymbol{\Lambda}_{ts} := \log \mathbf{X}_{ts}$, and $0 \leqslant r \leqslant 1$ (see [3] or [6] for instance). The following simplified version of our main result, Theorem 2.3, actually gives a non-explosion result in terms of growth assumptions on the vector fields $V_{[I]}$ that appear in the approximate dynamics (1.2). Pick an arbitrary $p > 1$ and a weak geometric p-rough path $\mathbf{X}$.

Theorem 1.1. — *Take* $\mathrm{F} = (V_1, \dots, V_\ell)$*. There is no explosion for the solutions of the rough differential equation* (1.1) *if the functions* $V_{i_1} \cdots V_{i_n}\,\mathrm{Id}$ *have linear growth and are* C^2 *with bounded derivatives, for any* $1 \leqslant n \leqslant [p]$ *and any tuple* $(i_1, \dots, i_n) \in [\![1, \ell]\!]^n$.

In the case where $2 \leqslant p < 3$, the assumption of the previous theorem is implied be the weaker but more understandable criterion

$$\big|D_x^2 \mathrm{F}\big| \vee \big|D_x^3 \mathrm{F}\big| \lesssim \frac{1}{1 + |x|},$$

for a multiplicative implicit constant independent of $x \in \mathbb{R}^d$. Note that Theorem 2.3 is sharper than that statement as it involves the vector fields $V_{[I]}$ (recall [22, Example 3]). We mention here that we have been careful on the growth rate of the different quantities but that one can optimize the regularity assumptions that are made on the vector fields V_i to get slightly sharper results. This explains the discrepancy between Davie's optimal criterion in the case $2 \leqslant p < 3$ and our result. We leave the proof of these refinements to the reader. Note also here that one can replace $\mathbb{R}^d$ by a Banach space and give versions of the statements involving infinite dimensional rough paths, to the price of using slightly different notations, such as in [2]. There is no difference between the finite and the infinite dimensional settings for the explosion problem.

Our main result, Theorem 2.3, holds for dynamics (1.1) with a drift and time-dependent vector fields. It is proved in Section 2 on the basis of some intermediate technical estimates whose proof is given in Section 3. Theorem 2.3 holds for Hölder p-rough paths. A similar statement holds for more general continuous rough paths, with finite p-variation, such as proved in Section 4 with other corollaries and extensions. We note that B. Driver has proven similar results to what is proved here in his very recent works [16, 17], independently of the present work.

Notation. — We gather here a number of notations that are used throughout the paper.

- Given a positive finite time horizon T, we denote by Δ_T the simplex $\{(t,s) \in [0,T]^2 \,:\, 0 \leqslant s \leqslant t \leqslant T\}$.
- We give in the Appendix A a quick introduction to the notion of weak geometric rough paths. We refer to the reader to Lyons' seminal article [23] or any textbook or lectures notes on rough paths [1, 5, 20, 24, 25] for more than the basics on rough paths theory and mention here that we work throughout with finite dimensional weak geometric Hölder p-rough paths $\mathbf{X} = 1 \oplus X^1 \oplus \cdots \oplus X^{[p]}$, with values in $\bigoplus_{i=0}^{[p]} (\mathsf{R}^\ell)^{\otimes i}$ say, and norm
$$\|\mathbf{X}\| := \max_{1 \leqslant i \leqslant [p]} \sup_{0 \leqslant s < t \leqslant T} \frac{\left|X^i_{ts}\right|^{\frac{1}{i}}}{|t-s|^{\frac{1}{p}}}.$$
Note that if $\mathbf{\Lambda} = \left(0 \oplus \Lambda^1 \oplus \cdots \oplus \Lambda^{[p]}\right)$ is the logarithm of the rough path $\mathbf{X}$, we have for all $0 \leqslant s \leqslant t \leqslant T$, all $i \in \{1, \dots, [p]\}$,
$$\left|\Lambda^i_{ts}\right| \lesssim_i \|\mathbf{X}\|^i |t-s|^{\frac{i}{p}}$$
- Last, we use the notation $a \lesssim b$ to mean that a is smaller than a constant times b, for some universal numerical constant.

2. Solution flows to rough differential equations

Pick $\alpha \in [0,1]$. A finite dimensional-valued function f defined on $\mathbb{R}^d$ is said to have α-growth if
$$\sup_{x \in \mathbb{R}^d} \frac{\left|f(x)\right|}{\left(1+|x|\right)^\alpha} < +\infty.$$

Furthermore for $\alpha \in [0,1]$ and $i \geqslant 1$ we write

$$f \in C^i_{\alpha,b}(\mathbb{R}^d;\mathbb{R}^d)$$

if f has α-growth and f is C^i with bounded derivatives. Note that $C^i_{0,b}(\mathbb{R}^d;\mathbb{R}^d) = C^i_b$.

Let V_0 and $V_1,\dots,V_d : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ be time-dependent vector fields on $\mathbb{R}^d$.

Assumption 1 (Space regularity and growth). — *For any $1 \leqslant n \leqslant [p]$ and for any tuple $I \in \{1,\dots,\ell\}^n$,*

- *the vector fields $V_0(s,\cdot\,)$ and $V_{[I]}(s,\cdot\,)$ lie in $C^2_{\alpha,b}$, uniformly in time,*
- *for all indices $1 \leqslant k_1,\dots,k_n \leqslant [p]$ with $\sum k_i \leqslant [p]$, and all tuples $I_{k_i} \in \{1,\dots,\ell\}^{k_i}$, the functions*

$$V_0(s,\cdot\,)V_{[I_{n-1}]}(s,\cdot\,)\cdots V_{[I_1]}(s,\cdot\,)\,\mathrm{Id} \quad \textit{and} \quad V_{[I_n]}(s,\cdot\,)\cdots V_{[I_1]}(s,\cdot\,)\,\mathrm{Id}$$

are $C^2_{\alpha,b}$, uniformly in time.

One can trade in the above assumption some growth condition on the V_i against some growth condition on its derivatives; this is the rationale for introducing the notion of α-growth.

Assumption 2 (Time regularity and growth). — *There exists some regularity exponents $\kappa_1 \geqslant \frac{1+[p]-p}{p}$ and $\kappa_2 \geqslant \frac{[p]}{p}$ with the following properties.*

- *One has*

$$\sup_{x\in B(0,R)}\ \sup_{0\leqslant s<t\leqslant T} \frac{\big|V_0(t,x)-V_0(s,x)\big|}{|t-s|^{\kappa_1}} \lesssim (1+R)^\alpha,$$

- *For all $1 \leqslant n \leqslant [p]$ and $1 \leqslant k_1,\dots,k_n \leqslant [p]$, with $\sum_{i=1}^n k_i \leqslant [p]$, for all tuples $I_i \in \{1,\dots,d\}^{k_i}$, we have*

$$\sup_{x\in B(0,R)}\ \sup_{0\leqslant s<t\leqslant T} \frac{\big|V_{[I_n]}(t,\cdot\,)\cdots V_{[I_1]}(t,\cdot\,)(x) - V_{[I_n]}(s,\cdot\,)\cdots V_{[I_1]}(s,\cdot\,)(x)\big|}{|t-s|^{\kappa_2}} \lesssim (1+R)^\alpha.$$

We assume that the derivative in x of the $V_{[I_n]}(t,\cdot\,)\cdots V_{[I_1]}(t,\cdot\,)$ also satisfies the previous estimate.

Let $\mathbf{X}$ be an $\mathbb{R}^\ell$-valued weak geometric Hölder p-rough path. Set $\mathbf{\Lambda}_{ts} := \log \mathbf{X}_{ts}$, for all $0 \leqslant s \leqslant t \leqslant T$, and denote by μ_{ts} the time 1 map of the ordinary differential equation

$$y'_r = (t-s)V_0\big(s,y_r(x)\big) + \sum_{k=1}^{[p]} \sum_{I\in\{1,\dots,\ell\}^k} \Lambda^{k,I}_{t,s} V_{[I]}\big(s,y_r(x)\big) \tag{2.1}$$

that associates to x the value at time 1 of the solution path to that equation with initial condition x. Note that Assumption 1 ensures that (2.1) is well-defined up to time 1. Following [3], we define a solution flow to the rough differential equation

$$\mathrm{d}\varphi_t = V_0(t,\varphi_t)\mathrm{d}t + \mathrm{F}(t,\varphi_t)\mathrm{d}\mathbf{X}_t, \tag{2.2}$$

where $\mathrm{F} := (V_1,\ldots,V_\ell)$, as a flow locally well-approximated by μ. Here, we take advantage in this definition of some variant of the definition of [3] introduced by Cass and Weidner in [9]. For a parameter a, the notation C_a stands for a constant depending only on a.

Definition 2.1. — *A flow $\varphi : \Delta_T \times \mathbb{R}^d \mapsto \mathbb{R}^d$ is said to be a* solution flow to the rough differential equation (2.2) *if there exists an exponent $\eta > 1$ independent of $\mathbf{X}$, such that one can associate to any positive radius R two positive constants $C_{R,\mathbf{X}}$ and $\varepsilon_{\mathbf{X}}$ such that one has*

$$\sup_{x\in B(0,R)} \big|\varphi_{ts}(x) - \mu_{ts}(x)\big| \leqslant C_{R,\mathbf{X}}\,|t-s|^\eta, \tag{2.3}$$

whenever $|t-s| \leqslant \varepsilon_{\mathbf{X}}$.

Note that we require the flow to be *globally defined in time and space*, unlike local flows of possibly exploding ordinary, or rough, differential equations. The latter are only defined on an open set of $\mathbb{R}_+\times\mathbb{R}^d$ depending on $\mathbf{X}$. This definition differs from the corresponding definition in [3] in the fact that ε_R is required to be independent of $\mathbf{X}$. We first state a local in time existence result for the flow, in the spirit of [3].

Theorem 2.2. — *Let the vector fields V_0 and $(V_1,\ldots,V_\ell)$ satisfy Assumption 1 and Assumption 2.*

- *There exists a positive constant a_1 such that for all $R > 0$, and all $(t,s)\in\Delta_T$ with*
$$|t-s|^{\frac{1}{p}}(1+R)^{\frac{\alpha}{[p]+1}}\big(1+\|\mathbf{X}\|\big) < a_1, \tag{2.4}$$
there is a unique flow $\varphi : [s,t]^2\times B(0,R)\to\mathbb{R}^d$ satisfying the estimate (2.3) with
$$\eta = \frac{[p]+1}{p},\ \text{ and } C_{R,\mathbf{X}} = a_2(1+R)^\alpha\big(1+\|\mathbf{X}\|\big)^{[p]+1},$$
for some universal positive constant a_2. One writes $\varphi(\mathbf{X})$ to emphasize the dependence of φ on $\mathbf{X}$.
- *Given a weak geometric rough path $\mathbf{X}$ and $(s,t)\in\Delta_T$ and R such that condition (2.4) holds, then $\varphi(\mathbf{X}')$ is well-defined on $[s,t]\times B(0,R)$ for $\mathbf{X}'$ sufficiently close to $\mathbf{X}$, and $\varphi(\mathbf{X}')$ converges to $\varphi(\mathbf{X})$ in $L^\infty\big([s,t]\times B(0,R)\big)$ as $\mathbf{X}'$ tends to $\mathbf{X}$.*

One says that φ depends continuously on $\mathbf{X}$ in the topology of uniform convergence on bounded sets. As you can see from the statement of Theorem 2.2, the quantity $|t-s|$ is only required in that case to be smaller than a constant depending on $\mathbf{X}$ and R, unlike what is required from a solution defined globally in time. The proof of Theorem 2.2 mimics the proof of the analogue local in time result proved in [3]. As the proof of latter contains typos that makes reading it hard, we give in Section 3 a self-contained proof of this result.

THEOREM 2.3. — *Let V_0 and $(V_1, \ldots, V_\ell)$ satisfy Assumption 1 and Assumption 2. There exists a unique global in time solution flow φ to the rough differential equation* (2.2).

- *One can choose in the defining relation* (2.3) *for a solution flow*

$$\eta = \frac{1+[p]}{p}, \quad \varepsilon_{\mathbf{X}} = c_1\big(1+\|\mathbf{X}\|\big)^{-p}, \quad C_{R,\mathbf{X}} = c_2(1+R)^\alpha\big(1+\|\mathbf{X}\|\big)^{[p]+1},$$

  for some universal positive constants c_1, c_2.
- *One has for all $f \in C_b^{[p]+1}$ and all $|t-s| \leqslant \varepsilon_{\mathbf{X}}$ the estimate*

$$\sup_{x\in B(0,R)} \left| f\circ\varphi_{t,s}(x) - \left\{ f(x) + (t-s)V_0(s,\cdot)f + \sum_{k=1}^{[p]} \sum_{I\in\{0,\ldots,\ell\}^k} X_{t,s}^{k,I} V_I(s,\cdot)f \right\}(x) \right| \\ \lesssim \|f\|_{C_b^{[p]+1}} (1+R)^{\alpha([p]+1)} \big(1+\|\mathbf{X}\|\big)^{[p]+1} |t-s|^{\frac{[p]+1}{p}}.$$

  When $f = \mathrm{Id}$, one can replace $(1+R)^{\alpha([p]+1)}$ by $(1+R)^\alpha$ and $\|f\|_{C_b^n}$ by 1 in the previous bound.
- *The map that associates φ to $\mathbf{X}$ is continuous from the set of weak geometric Hölder p-rough paths into the set of continuous flows endowed with the topology of uniform convergence on bounded sets.*
- *Finally, there exists two positive universal constants c_3, c_4 such that setting*

$$N := \Big[c_3\big(1+\|\mathbf{X}\|\big)^p\Big],$$

  one has for all $(t,s) \in \Delta_T$,

$$\sup_{x\in B(0,R)} \big|\varphi_{s,t}(x) - x\big| \lesssim \begin{cases} (1+R)\left(\left(1 + c_4\frac{|t-s|^{\frac{1}{p}}N}{(1+R)^{1-\alpha}}\right)^{\frac{1}{1-\alpha}} - 1\right), & \text{if } \alpha < 1 \\ (1+R)|t-s|^{\frac{1}{p}} e^{c_4 N|t-s|^{\frac{1}{p}}}, & \text{if } \alpha = 1. \end{cases}$$

The non-trivial part of the proof consists in proving that one can patch together the local flows contructed in Theorem 2.2 and define a globally well-defined flow. As this requires a careful track of a number of quantities, we provide a proof of the technical results in Section 3. Since it is the main contribution of this work, we also give a proof of this theorem using some results of lemmas and propositions of Section 3.

Note that in a probabilistic context, the bound of Theorem 2.3 is not optimal. Indeed, in a Gaussian-process framework, the homogeneous Hölder-norm of the rough path enjoys Gaussian tails. Hence, when $p > 2$ one can not integrate the bound of Theorem 2.3. Following Cass, Litterer and Lyons [7], one could instead work with p-variation norms and local accumulated variation. This is what we are doing in Subsection 4.3 and Theorem 4.4. Nevertheless, the main difficulty of the problem of non-explosion comes from the growth of the vector fields, regardless of which norm is used to measure the size of the rough path. If one seeks for a almost sure bound, Hölder-rough path bounds are good enough. We present here a full proof in this setting and refer to Section 4.3 for the general case using p-variation bounds.

Proof of Theorem 2.3. — Fix $(s,t) \in \Delta_T$. For $n \geqslant 0$ and $0 \leqslant k \leqslant 2^n$ set $t_k^n := k2^{-n}(t-s) + s$ and $\mu_{t,s}^n := \mu_{t_{2^n}^n, t_{2^n-1}^n} \circ \cdots \circ \mu_{t_1^n, t_0^n}$. Here is the major input for the proof of the statement. Proposition 3.8 below states the existence of universal positive constants $c_1 < 1$ and c_2 such that for

$$|t-s|^{\frac{1}{p}} \big(1 + \|\mathbf{X}\|\big) \leqslant c_1^p$$

we have for all $n \geqslant 0$ the estimate

$$\sup_{x \in B(0,R)} \big|\mu_{ts}^n(x) - \mu_{ts}(x)\big| \leqslant c_2 \, |t-s|^{\frac{1+[p]}{p}} \, \big(1 + \|\mathbf{X}\|\big)^{[p]+1} \, (1+R)^\alpha. \quad (2.5)$$

An elementary Gronwall type bound proved in Lemma 3.1 also gives the estimate

$$|\mu_{t,s}| \leqslant R + c_2(1+R)^\alpha.$$

Putting those two bounds together, one gets the existence of a positive constant c such that one has

$$\big\|\mu_{t_k^n t_{k-1}^n} \circ \cdots \circ \mu_{t_1^n t_0^n}\big\|_{L^\infty(B(0,R))} \leqslant R + c\,(1+R)^\alpha,$$

for all $0 \leqslant k \leqslant 2^n - 1$. Let $n = n(R)$ be the least integer such that

$$2^{-n\,\frac{1}{p}} \, (1+R)^{\frac{\alpha}{[p]+1}} \, |t-s|^{\frac{1}{p}} \big(1 + \|\mathbf{X}\|\big) \leqslant \frac{a_1}{(1+2c)^{\frac{\alpha}{[p]+1}}}.$$

This is the smallest integer such that for all the intervals (t_k^n, t_{k+1}^n) satisfy the assumption of Theorem 2.2, with starting point $\mu_{t_k^n t_{k-1}^n} \circ \cdots \circ \mu_{t_1^n t_0^n}(x)$ and $x \in B(0,R)$. Then, we have for all $m_0, \ldots, m_{2^n-1} \in \mathbb{N}$,

$$\Big\|\mu^{m_{2^n-1}}_{t_{2^n}^n t_{2^n-1}^n} \circ \cdots \circ \mu^{m_0}_{t_1^n t_0^n} - \mu_{ts}\Big\|_{L^\infty(B(0,R))} \leqslant c_1 \big|t-s\big|^{\frac{1+[p]}{p}} (1+R)^\alpha \big(1 + \|\mathbf{X}\|\big)^{[p]+1}.$$

Sending successively $m_{2^n-1}, \dots, m_0$ to ∞ and using the continuity of φ with respect to its $\mathbb{R}^d$-valued argument gives

$$\left\|\varphi_{t^n_{2^n} t^n_{2^n-1}} \circ \cdots \circ \varphi_{t^n_1 t^n_0} - \mu_{ts}\right\| \leqslant c_2 \, |t-s|^{\frac{1+[p]}{p}} \, (1+R)^\alpha \left(1+\|\mathbf{X}\|\right)^{[p]+1}. \quad (2.6)$$

Set, for $x \in B(0,R)$,

$$\varphi_{ts}(x) := \varphi_{t^n_{2^n} t^n_{2^n-1}} \circ \cdots \circ \varphi_{t^n_1 t^n_0}(x).$$

Splitting the intervals (t^n_k, t^n_{k+1}) into dyadic sub-intervals, one shows that for all $u \in [s,t]$ of the form $u = k2^{-N}(t-s)+s$, one has

$$\varphi_{t,u} \circ \varphi_{u,s}(x) = \varphi_{t,s}(x).$$

Finally, since the map

$$(x,s,t) \to \varphi_{t^n_{k+1}, t^n_k}(x)$$

is a continuous for all $0 \leqslant k \leqslant 2^n - 1$, so is φ. This proves the first item of Theorem 2.3.

The second item is a byproduct of the bound of Equation (2.3) and Corollary 3.5 below. The third item of the statement is straightforward given that φ is constructed from patching together local solution flows.

Choose finally a positive constant c_3 big enough such that setting

$$N := \left[c_3\left(1+\|\mathbf{X}\|\right)^p\right],$$

one has $\frac{t-s}{N} \leqslant \varepsilon_{\mathbf{X}}$ and $\left(1+\|\mathbf{X}\|\right)N^{-\frac{1}{p}} \leqslant 1$. Define also

$$t_i := \frac{i}{N}(t-s)+s,$$

and $R_0 := 0$ and

$$R_i := \sup_{x \in B(0,R)} \left|\varphi_{t_i s}(x) - x\right|,$$

for $1 \leqslant i \leqslant N$. Note that

$$C_{|t_{i+1}-t_i|, \|\mathbf{X}\|} = \sum_{i=1}^{[p]} \left(\frac{1+\|\mathbf{X}\|}{N^{\frac{1}{p}}}\right)^i |t-s|^{\frac{i}{p}} \lesssim |t-s|^{\frac{1}{p}},$$

for a universal positive multiplicative factor. We thus have

$$\begin{aligned}
\varphi_{t_i s}(x) - x = {} & \varphi_{t_i t_{i-1}}\big(\varphi_{t_{i-1} s}(x)\big) - \mu_{t_i t_{i-1}}\big(\varphi_{t_{i-1} s}(x)\big) \\
& + \mu_{t_i t_{i-1}}\big(\varphi_{t_{i-1} s}(x)\big) - \varphi_{t_{i-1} s}(x) \\
& + \varphi_{t_{i-1} s}(x) - x,
\end{aligned}$$

and there is an absolute positive constant K such that

$$R_i \leqslant R_{i-1} + K(1+R+R_{i-1})^\alpha |t-s|^{\frac{1}{p}};$$

the bounds on $\varphi_{ts}(x) - x$ given in the statement follows from that relation. $\square$

As a corollary of Theorem 2.3, one proves in Theorem 4.5 the differentiability of the solution flow with respect to some parameters. This theorem will be of crucial importance in the forthcoming work [4]; we state it here in a readily usable form.

Assumption 3. — *Let A be a Banach, parameter space and let U be a bounded open subset of A. Let $(V_i)_{0\leqslant i\leqslant \ell}$ be time and parameter-dependent vector fields on $\mathbb{R}^d$ with the following regularity properties.*

- *There exists some exponents $\kappa_1 > \frac{1+[p]-p}{p}$ and $\kappa_2 > \frac{[p]}{p}$, such that we have for all integers β_1, β_2 with $0 \leqslant \beta_1 + \beta_2 \leqslant [p]+1$,*

$$\sup_{0\leqslant s\leqslant t\leqslant T} \frac{\left\| D_a^{\beta_1} D_x^{\beta_2} V_0(t,\cdot,\cdot) - D_a^{\beta_1} D_x^{\beta_2} V_0(s,\cdot,\cdot)\right\|_{L^\infty(\mathbb{R}^d\times U)}}{|t-s|^{\kappa_1}} < +\infty,$$

- *For all $1 \leqslant i \leqslant \ell$, and all integers β_1, β_2 with $0 \leqslant \beta_1 + \beta_2 \leqslant [p]+2$, we have*

$$\sup_{0\leqslant s\leqslant t\leqslant T} \frac{\left\| D_a^{\beta_1} D_x^{\beta_2} V_i(t,\cdot,\cdot) - D_a^{\beta_1} D_x^{\beta_2} V_i(s,\cdot,\cdot)\right\|_{L^\infty(\mathbb{R}^d\times U)}}{|t-s|^{\kappa_2}} < +\infty$$

Refer to Definition 4.3 in Section 4.3 for the definition of the *local accumulation* N_β of $\mathbf{X}$ (which is a quantity linked to the p-variation norm of $\mathbf{X}$, defined in the same section).

Theorem 2.4. — *Let $\mathbf{X}$ be a $\mathbb{R}^\ell$ valued weak geometric Hölder p-rough path and suppose that $V_0, V_1, \ldots, V_\ell$ satisfy Assumptions 3. Let $\varphi(a,\cdot)$ stand for all $a \in U$ for the solution flow to the equation*

$$\mathrm{d}\varphi(a,\cdot) = V_0\big(t,a,\varphi(a,\cdot)\big)\mathrm{d}t + \sigma\big(t,a,\varphi(a,\cdot)\big)\mathrm{d}\mathbf{X}_t. \tag{2.7}$$

Then for all $1 \leqslant s \leqslant t \leqslant T$, the function $(a,x) \mapsto \varphi_{ts}(a,x)$ is differentiable and

- *for $|t-s|^{\frac{1}{p}}\big(1+\|\mathbf{X}\|\big) \lesssim 1$, and $a \in U$,*

$$\sup_{x\in\mathbb{R}^d} \big|D_a\varphi_{ts}(a,x)\big| \lesssim |t-s|^{\frac{1}{p}}\big(1+\|\mathbf{X}\|\big)^{[p]}$$

- *there exists positive constants β and c such that one has*

$$\sup_{x\in\mathbb{R}^d} \big|D_a\varphi_{ts}(a,x)\big| \lesssim |t-s|^{\frac{1}{p}}\big(1+\|\mathbf{X}\|\big)e^{cN_\beta}$$

for all $0 \leqslant s \leqslant t \leqslant T$.

3. Complete proof of Theorem 2.2

The structure of the proof is simple. One first proves C^2 estimates on the time r map of the ordinary differential equation (2.1), this is the content of Lemma 3.1. Building on a Taylor formula given in Lemma 3.3, and quantified in Lemma 3.4 and Corollary 3.5, one shows in Proposition 3.7 that the μ's defined what could be called a "*local approximate flow*", after [3]. We then follow the construction recipe of a flow from an approximate flow given in [3], by patching together the local flows. The crucial global in time existence result is obtained as a consequence of a Grönwall type argument, as can be expected from the fact that, in their simplest form, the growth assumptions of Theorem 2.3 mean that all the vector fields appearing in the approximate dynamics have α-growth. Readers familiar with [3] can go directly to Section 4.

Recall the definition of y_r as the solution of the ordinary differential equation (2.1) defining μ_{ts}. The first step in the analysis consists in getting some local in space C^2 estimate on $y_r(\,\cdot\,) - \mathrm{Id}$, with $y_r(\,\cdot\,)$ seen as a function of the initial condition x in (2.1). Set

$$C_{|t-s|,\|\mathbf{X}\|} := |t-s| + \sum_{i=1}^{[p]} |t-s|^{\frac{i}{p}} \|\mathbf{X}\|^i.$$

Lemma 3.1. — *Assume V_0 and $(V_1, \dots, V_\ell)$ satisfy the space regularity Assumption 1, and pick $(s,t) \in \Delta_T$ with $|t-s|^{\frac{1}{p}}\big(1 + \|\mathbf{X}\|\big) \leqslant 1$. Then*

- $\big|y_r(x) - x\big| \lesssim \big(1 + |x|\big)^\alpha C_{|t-s|,\|\mathbf{X}\|}$,
- $\big|Dy_r(x) - \mathrm{Id}\big| \lesssim C_{|t-s|,\|\mathbf{X}\|}$,
- $\big|D^2 y_r(x)\big| \lesssim C_{|t-s|,\|\mathbf{X}\|}$.

The maps $y_r(\,\cdot\,)$ are thus C^1_b, uniformly in $r \in [0,1]$.

Proof. — Apply repeatedly Grönwall lemma. We only prove the estimate for $y_r(x) - x$ and leave the remaining details to the reader. It suffices to write

$$\begin{aligned} |y_r(x) - x| \leqslant (t-s)\big|V_0(s,x)\big| &+ \sum_{k=1}^{[p]} \sum_{I \in \{1,\dots,\ell\}^k} \big|\Lambda^{k,I}_{t,s}\big|\,\big|V_{[I]}(s,x)\big| \\ &+ (t-s)\int_0^r \Big|V_0\big(s, y_u(x)\big) - V_0(s,x)\Big| \mathrm{d}u \\ &+ \sum_{k=1}^{[p]} \sum_{I \in \{1,\dots,\ell\}^k} \big|\Lambda^{k,I}_{t,s}\big| \int_0^r \Big|V_{[I]}\big(s, y_u(x)\big) - V_{[I]}(s,x)\Big| \mathrm{d}u \end{aligned}$$

$$\lesssim C\big(|t-s|, \|\mathbf{X}\|\big)\left(\big(1+|x|\big)^{\alpha} + \int_0^r |y_u(x)-x|\mathrm{d}u\right)$$

to get the conclusion from Grönwall lemma, using the fact that $C_{|t-s|,\|\mathbf{X}\|} \lesssim 1$, for $|t-s|^{\frac{1}{p}}\big(1+\|\mathbf{X}\|\big) \leqslant 1$. The derivative equations satisfied by Dy_r and D^2y_r are used to get the estimates of the statement on these quantities, using once again that the condition of the statement imposes to C to be of order 1. □

Remark 3.2. — Would Assumption 1 require in addition that the vector fields $V_0(s,\cdot)$ and $V_{[I]}(s,\cdot)$ were $C^{n+2}_{\alpha,b}$, uniformly in $0 \leqslant s \leqslant T$, we would then have the estimate

$$\sup_{2\leqslant k\leqslant n+2} \left|D^k y_r(x)\right| \lesssim C_{|t-s|,\|\mathbf{X}\|},$$

under the assumption that $|t-s|^{\frac{1}{p}}\big(1+\|\mathbf{X}\|\big) \leqslant 1$.

The second step of the analysis is an elementary explicit Taylor expansion; see [3] for the model situation. Given $1 \leqslant n \leqslant [p]$, set

$$\Delta_1^n := \Big\{\big(r_n,\dots,r_1\big) \in [0,1]^n \ : \ r_n \leqslant r_{n-1} \leqslant \cdots \leqslant r_1\Big\}$$

and

$$\mathfrak{I}_{n,[p]} := \left\{(I_1,\dots,I_n) \in \{1,\dots,d\}^{k_1} \times \cdots \times \{1,\dots,d\}^{k_n} \ ; \sum_{m=1}^n k_m \leqslant [p]\right\};$$

indices k_m above are non-null.

Lemma 3.3. — *Assume V_0 and $(V_1,\dots,V_\ell)$ satisfy the space regularity Assumption 1. For any $1 \leqslant n \leqslant [p]$ and any vector space valued function f on $\mathbb{R}^d$ of class C^n we have the Taylor formula*

$$\begin{aligned}
f\big(\mu_{ts}(x)\big) = f(x) &+ (t-s)\big(V_0(s,\cdot)f\big)(x) \\
&+ \sum_{i=1}^n \frac{1}{i!} \sum_{\mathfrak{I}_{i,[p]}} \prod_{m=1}^i \Lambda_{ts}^{k_m,I_m} \big(V_{[I_i]}(s,\cdot)\cdots V_{[I_1]}(s,\cdot)f\big)(x) \\
&+ \sum_{\mathfrak{I}_{n,[p]}} \prod_{m=1}^n \Lambda_{ts}^{k_m,I_m} \int_{\Delta_1^n} \Big\{\big(V_{[I_n]}(s,\cdot)\cdots V_{[I_1]}(s,\cdot)f\big)\big(y_{r_n}(x)\big) \\
&\qquad\qquad - \big(V_{[I_n]}(s,\cdot)\cdots V_{[I_1]}(s,\cdot)f\big)(x)\Big\}\,\mathrm{d}r \\
&+ \varepsilon_{ts}^{n,f}(x)
\end{aligned}$$

where

$$\begin{aligned}\varepsilon_{ts}^{n,f}(x) := (t-s)\int_0^1 \Big\{\big(V_0(s,\cdot)f\big)\big(y_r(x)\big) - \big(V_0(s,\cdot)f\big)(x)\Big\}\,\mathrm{d}r \\ + \sum_{i=1}^{n-1}\frac{1}{i!}\sum_{\mathfrak{I}_{i,[p]}}(t-s)\prod_{m=1}^{i}\Lambda_{ts}^{k_m,I_m} \\ \times\int_{\Delta_1^{i+1}}\big(V_0(s,\cdot)V_{[I_i]}(s,\cdot)\cdots V_{[I_1]}(s,\cdot)f\big)\big(y_{r_{i+1}}(x)\big)\mathrm{d}r \\ + \sum_{i=2}^{n}\sum_{\substack{\mathfrak{I}_{i-1,[p]}\\ k_1+\cdots+k_i\geqslant[p]+1}}\prod_{m=1}^{i}\Lambda_{ts}^{k_m,I_m}\int_{\Delta_1^i}\big(V_{[I_i]}(s,\cdot)\cdots V_{[I_1]}(s,\cdot)f\big)\big(y_{r_i}(x)\big)\,\mathrm{d}r.\end{aligned}$$

Proof. — The proof is done by induction, and relies on the following fact. For all $u\in[0,1]$ and all $g\in C^1(\mathbb{R}^d;\mathbb{R}^d)$, we have

$$\begin{aligned}g(y_r)-g(x) = (t-s)\int_0^r\big(V_0(s,\cdot)g\big)(y_u)\,\mathrm{d}u \\ + \sum_{\substack{1\leqslant k\leqslant[p]\\ I\in\{1,\dots,\ell\}^k}}\Lambda_{t,s}^{k,I}\int_0^r\big(V_{[I]}(s,\cdot)g\big)(y_u)\,\mathrm{d}u;\end{aligned}$$

this is step 1 of the induction. For step 2, apply step 1 successively to $g=f$ and $u=1$, then $g=\big(V_{[I]}(s,\cdot)f\big)$ and $u=r$. This gives

$$\begin{aligned}&f\big(\mu_{ts}(x)\big)-f(x)\\ &\quad= (t-s)\big(V_0(s,\cdot)f\big)(x) + (t-s)\int_0^1\Big\{\big(V_0(s,\cdot)f\big)(y_r) - \big(V_0(s,\cdot)f\big)(x)\Big\}\,\mathrm{d}r\\ &\qquad+\sum_{\substack{1\leqslant k\leqslant[p]\\ I\in\{1,\dots,\ell\}^k}}\Lambda_{ts}^{k,I}\big(V_{[I]}(s,\cdot)f\big)(x)\\ &\qquad+\sum_{\substack{1\leqslant k\leqslant[p]\\ I\in\{1,\dots,\ell\}^k}}(t-s)\Lambda_{ts}^{k,I}\int_0^1\int_0^{r_1}\big(V_0(s,\cdot)V_{[I]}(s,\cdot)f\big)(y_{r_2})\,\mathrm{d}r_2\mathrm{d}r_1\\ &\qquad+\sum_{\substack{1\leqslant k_1,k_2\leqslant[p]\\ I_1\in\{1,\dots,\ell\}^{k_1}\\ I_2\in\{1,\dots,\ell\}^{k_2}}}\prod_{m=1}^{2}\Lambda_{ts}^{k_m,I_m}\int_0^1\int_0^{r_1}\big(V_{[I_2]}(s,\cdot)V_{[I_1]}(s,\cdot)f\big)(y_{r_2})\,\mathrm{d}r_2\mathrm{d}r_1.\end{aligned}$$

The last term of the right hand side can be decomposed into

$$\begin{aligned}
&\frac{1}{2}\sum_{\mathfrak{I}_{2,[p]}}\prod_{m=1}^{2}\Lambda_{ts}^{k_m,I_m}\big(V_{[I_2]}(s,\cdot\,)V_{[I_1]}(s,\cdot\,)f\big)(x)\\
&\quad+\sum_{\mathfrak{I}_{2,[p]}}\prod_{m=1}^{2}\Lambda_{ts}^{k_m,I_m}\int_{\Delta_1^2}\Big\{\big(V_{[I_2]}(s,\cdot\,)V_{[I_1]}(s,\cdot\,)f\big)(y_{r_2})\\
&\qquad\qquad\qquad\qquad\qquad\qquad-\big(V_{[I_2]}(s,\cdot\,)V_{[I_1]}(s,\cdot\,)f\big)(x)\Big\}\mathrm{d}r_2\mathrm{d}r_1\\
&\quad+\sum_{\substack{k_1+k_2\geqslant[p]+1\\ \mathfrak{I}_{1,[p]}}}\prod_{m=1}^{2}\Lambda_{ts}^{k_m,I_m}\int_{\Delta_1^2}\big(V_{[I_2]}(s,\cdot\,)V_{[I_1]}(s,\cdot\,)f\big)(y_{r_2})\,\mathrm{d}r_2\mathrm{d}r_1;
\end{aligned}$$

this proves step 2 of the induction. The n to $(n+1)$ induction step is done similarly, and left to the reader. □

Given $f\in C_b^n\big(\mathbb{R}^d,\mathbb{R}^d\big)$, set

$$\|f\|_n:=\big|f(0)\big|+\sup_{k\in\{1,\dots,n\}}\big\|D^kf\big\|_\infty.$$

Assumption H. — *A function $g\in C^n\big(\mathbb{R}^d,\mathbb{R}^d\big)$ is said to satisfy this assumption if for all $1\leqslant k_1,\dots,k_n\leqslant[p]$ with $\sum_{i=1}^p k_i\leqslant[p]$, and all tuples $I_{k_i}\in\{1,\dots,\ell\}^{k_i}$, the functions*

$$V_0(s,\cdot\,)V_{[I_{n-1}]}(s,\cdot\,)\cdot V_{[I_1]}(s,\cdot\,)g\quad\textit{and}\quad V_{[I_n]}(s,\cdot\,)\cdot V_{[I_1]}(s,\cdot\,)g$$

are $C^2_{\alpha,b}$, uniformly in $s\in[0,T]$.

Lemma 3.4. — *Assume Assumption 1 holds, and pick a function $f\in C_b^n\big(\mathbb{R}^d,\mathbb{R}^d\big)$, for some $2\leqslant n\leqslant[p]$. Given $(s,t)\in\Delta_T$ with $|t-s|^{\frac{1}{p}}\big(1+\|\mathbf{X}\|\big)\leqslant 1$, we have*

$$\sup_{x\in B(0,R)}\big|\varepsilon_{ts}^{n,f}(x)\big|\lesssim\|f\|_n(1+R)^{n\alpha}\big(1+\|\mathbf{X}\|\big)^{[p]+1}(t-s)^{\frac{1+[p]}{p}},$$

for all positive radius R.

- *If furthermore $D^{n+1}f$ exists and is a bounded function, then*

  $$\sup_{x\in B(0,R)}\big|D_x\varepsilon_{ts}^{n,f}\big|\lesssim\|f\|_{n+1}(1+R)^{n\alpha}\big(1+\|\mathbf{X}\|\big)^{[p]+1}(t-s)^{\frac{1+[p]}{p}}.$$

- *If finally f satisfies Assumption H, then the previous bound on $D_x\varepsilon_{ts}^{n,f}$ holds with $(1+R)^\alpha$ in place of $(1+R)^{n\alpha}$.*

Proof. — Write $\mathrm{d}r$ for $\mathrm{d}r_i \dots \mathrm{d}r_1$ on Δ_1^i, and recall that

$$\begin{aligned}
\varepsilon_{ts}^{n,f}(x) &= (t-s)\int_0^1 \Big\{\big(V_0(s,\cdot)f\big)(y_r) - \big(V_0(s,\cdot)f\big)(x)\Big\}\,\mathrm{d}r \\
&\quad + \sum_{i=1}^{n-1}\frac{1}{i!}\sum_{\mathfrak{I}_{i,[p]}}(t-s)\prod_{m=1}^{i}\Lambda_{ts}^{k_m,I_m} \\
&\qquad\qquad \times \int_{\Delta_1^{i+1}}\Big\{V_0(s,\cdot)V_{[I_i]}(s,\cdot)\cdots V_{[I_1]}(s,\cdot)f\Big\}(y_{r_{i+1}})\,\mathrm{d}r \\
&\quad + \sum_{i=2}^{n}\sum_{\substack{\mathfrak{I}_{i-1,[p]} \\ k_1+\dots+k_i\geqslant[p]+1}}\prod_{m=1}^{i}\Lambda_{ts}^{k_m,I_m}\int_{\delta_1^i}\Big\{V_{[I_i]}(s,\cdot)\cdots V_{[I_1]}(s,\cdot)f\Big\}(y_{r_i})\,\mathrm{d}r.
\end{aligned} \tag{3.1}$$

Recall also that $C_{|t-s|,\|\mathbf{X}\|} \lesssim 1$ under the assumption of the statement.

As we have for all positive radius R, and all points $x, y \in B(0,R)$, the estimate

$$\big|V_0(s,\cdot)f(x) - V_0(s,\cdot)f(y)\big| \leqslant \|f\|_n\big(1+R\big)^{\alpha}|x-y|$$

uniformly in $0 \leqslant s \leqslant T$, it follows from Lemma 3.1 that

$$\begin{aligned}
&\left|(t-s)\int_0^1 \Big\{\big(V_0(s,\cdot)f\big)(y_r) - \big(V_0(s,\cdot)f\big)(x)\Big\}\mathrm{d}r\right| \\
&\qquad\qquad\qquad \lesssim \|f\|_n (t-s)\, C_{|t-s|,\|\mathbf{X}\|}\,(1+R)^{2\alpha}.
\end{aligned}$$

Note that if $V_0(s,\cdot)f$ is globally Lipschitz continuous one can replace $(1+R)^{2\alpha}$ above by $(1+R)^{\alpha}$.

We estimate the size of the spatial derivative of the first term in the above decomposition of $\varepsilon_{ts}^{n,f}$ writing

$$\begin{aligned}
&\left|(t-s)\int_0^1 \Big(D\big(V_0(s,\cdot)f\big)\big(y_r(x)\big)Dy_r(x) - D\big(V_0(s,\cdot)f\big)\big(x\big)\Big)\right|\mathrm{d}r \\
&\quad\lesssim \left|(t-s)\int_0^1 \mathrm{d}r\Big(D\big(V_0(s,\cdot)f\big)\big(x\big)\Big)\Big(Dy_r(x) - \mathrm{Id}\Big)\right| \\
&\qquad\quad + \left|(t-s)\int_0^1 \mathrm{d}r\Big(D\big(V_0(s,\cdot)f\big)\big(y_r(x)\big) - D\big(V_0(s,\cdot)f\big)\big(x\big)\Big)Dy_r(x)\right| \\
&\quad\lesssim \|f\|_n\,|t-s|^{1+\frac{1}{p}}\,(1+R)^{2\alpha}\,\big(1+\|\mathbf{X}\|\big)^{[p]}.
\end{aligned}$$

Once again, one can replace $(1+R)^{2\alpha}$ by $(1+R)^{\alpha}$ if f satisfies Assumption H.

The two other terms in the decomposition (3.1) of $\varepsilon_{ts}^{n,f}$ are estimated in the same way. Remark that

$$\sup_{x\in B(0,R)} \Big|\big(V_0(s,\cdot)V_{[I_i]}(s,\cdot)\cdots V_{[I_1]}(s,\cdot)f\big)(x)\Big| \lesssim \|f\|_n(1+R)^{(i+1)\alpha}$$

and that

$$\sup_{x\in B(0,R)} \Big|D\big(V_0(s,\cdot)V_{[I_i]}(s,\cdot)\cdots V_{[I_1]}(s,\cdot)f\big)(x)\Big| \lesssim \|f\|_{n+1}(1+R)^{(i+1)\alpha}.$$

One can replace in the previous bounds the first term $(1+R)^{(i+1)\alpha}$ by $(1+R)^{\alpha}$ and the second term $(1+R)^{(i+1)\alpha}$ by 1 if f satisfies Assumption H. So

$$\left|\sum_{i=1}^{n-1}\frac{1}{i!}\sum_{\mathfrak{I}_{i,[p]}}(t-s)\prod_{m=1}^{i}\Lambda_{ts}^{k_m,I_m}\int_{\Delta_1^{i+1}}\big(V_0(s,\cdot)V_{[I_i]}(s,\cdot)\cdots V_{[I_1]}(s,\cdot)f\big)(y_{r_{i+1}})\,\mathrm{d}r\right|$$
$$\lesssim \|f\|_n\,(t-s)^{1+\frac{1}{p}}\,(1+R)^{n\alpha}\,\big(1+\|\mathbf{X}\|\big)^{[p]}$$

and

$$\left|\sum_{i=1}^{n-1}\frac{1}{i!}\sum_{\mathfrak{I}_{i,[p]}}(t-s)\prod_{m=1}^{i}\Lambda_{ts}^{k_m,I_m}\right.$$
$$\left.\times\int_{\Delta_1^{i+1}}D\Big\{V_0(s,\cdot)V_{[I_i]}(s,\cdot)\cdots V_{[I_1]}(s,\cdot)f\big)\big(y_{r_{i+1}}(x)\Big\}Dy_{r_{i+1}}(x)\,\mathrm{d}r\right|$$
$$\lesssim \|f\|_{n+1}\,(t-s)^{1+\frac{1}{p}}\,(1+R)^{n\alpha}\,\big(1+\|\mathbf{X}\|\big)^{[p]}.$$

Once again, if the function f satisfies Assumption H, one can replace $(1+R)^{n\alpha}$ by $(1+R)^{\alpha}$ in the first bound and $(1+R)^{n\alpha}$ by 1 in the second bound.

The analysis of the last term in the right hand side of the decomposition (3.1) for $\varepsilon_{ts}^{n,f}$ is a bit trickier since greater powers of $\|\mathbf{X}\|$ can pop out. Indeed, one has

$$\left|\sum_{i=2}^{n}\sum_{\substack{\mathfrak{I}_{i-1,[p]}\\ k_1+\cdots+k_i\geqslant[p]+1}}\prod_{m=1}^{i}\Lambda_{ts}^{k_m,I_m}\int_{\Delta_1^i}\Big\{V_{[I_i]}(s,\cdot)\cdots V_{[I_1]}(s,\cdot)f\Big\}(y_{r_i})\,\mathrm{d}r\right|$$
$$\lesssim \|f\|_n\sum_{l=1}^{[p]}\big(1+\|\mathbf{X}\|\big)^{[p]+i}|t-s|^{\frac{[p]+i}{p}}\,(1+R)^{n\alpha}.$$

But recall that $|t-s|^{\frac{1}{p}}\big(1+\|\mathbf{X}\|\big)\leqslant 1$, so we have $\big(1+\|\mathbf{X}\|\big)^{i-1}|t-s|^{\frac{i-1}{p}}\lesssim 1$. Hence for all $i\in\{1,\dots,[p]\}$; this gives the expected upper bound. The same idea is used for the spatial derivatives. Once again, one can replace $(1+R)^{n\alpha}$ by $(1+R)^{\alpha}$ if the function f satisfies Assumption H. $\square$

Corollary 3.5. — *We have*

$$\sup_{x\in B(0,R)}\Bigg| f\circ\mu_{ts}(x) - \Bigg\{ f(x) + (t-s)V_0(s,\cdot)f(x) + \sum_{k=1}^{[p]}\sum_{I\in\{0,\ldots,\ell\}^k} X_{ts}^{k,I}V_I(s,\cdot)f\Bigg\}(x)\Bigg| \lesssim \|f\|_{[p]+1}\,(1+R)^{\alpha([p]+1)}\,\big(1+\|\mathbf{X}\|\big)^{[p]+1}\,|t-s|^{\frac{[p]+1}{p}}.$$

for all $f\in C^{[p]+1}_{\alpha,b}$*, and* $1\leqslant k\leqslant [p]$*. We also have*

$$\sup_{x\in B(0,R)}\Bigg|\mu_{ts}(x) - \Bigg(x + (t-s)V_0(s,x) + \sum_{k=1}^{[p]}\sum_{I\in\{0,\ldots,\ell\}^k} X_{ts}^{k,I}V_I(s,x)\Bigg)\Bigg| \lesssim (1+R)^{\alpha}\big(1+\|\mathbf{X}\|\big)^{[p]+1}|t-s|^{\frac{[p]+1}{p}}.$$

and

$$\sup_{x\in B(0,R)}\Bigg|D\mu_{ts}(x) - \Bigg(\mathrm{Id} + (t-s)DV_0(s,x) + \sum_{k=1}^{[p]}\sum_{I\in\{0,\ldots,\ell\}^k} X_{ts}^{k,I}DV_I(s,x)\Bigg)\Bigg| \lesssim (1+R)^{\alpha}\big(1+\|\mathbf{X}\|\big)^{[p]+1}|t-s|^{\frac{[p]+1}{p}}.$$

Proof. — We only have to bound the sum over $\mathfrak{I}_{n,[p]}$ of the terms

$$\prod_{m=1}^{n}\Lambda_{ts}^{k_m,I_m}\int_{\Delta_n}\mathrm{d}r\Big(\big(V_{[I_n]}(s,\cdot)\cdots V_{[I_1]}(s,\cdot)f\big)(y_{r_n}) - \big(V_{[I_n]}(s,\cdot)\cdots V_{[I_1]}(s,\cdot)f\big)(x)\Big)$$

for $n=[p]$, thanks to Lemmas 3.3 and 3.4. We have $k_1=\cdots=k_{[p]}=1$, on $\mathfrak{I}_{[p],[p]}$. As we know that we have

$$\Big|\big(V_I(s,\cdot)f\big)(x) - \big(V_I(s,\cdot)f\big)(y)\Big| \lesssim (1+R)^{\alpha[p]}\|f\|_{[p]+1}|x-y|,$$

for all $I \in \{1,\dots,d\}^{[p]}$, and all $x, y \in B(0,R)$, it follows that

$$\begin{aligned}\Bigg| \sum_{I_1,\dots,I_{[p]}\in\{1,\dots,\ell\}} \Big(\Lambda^1_{ts}\Big)^{[p]} \int_{\Delta_1^n} \Big\{ \big(V_{I_{[p]}}(s,\cdot)\cdots V_{I_1}(s,\cdot) f\big)(y_{r_n}) \\ - \big(V_{I_{[p]}}(s,\cdot)\cdots V_{I_1}(s,\cdot) f\big)(x) \Big\}\,\mathrm{d}r \Bigg| \\ \lesssim (1+R)^{\alpha([p]+1)} \sum_{k=1}^{[p]} |t-s|^{\frac{i+[p]}{p}} \|\mathbf{X}\|^{[p]+i} \\ \lesssim (1+R)^{\alpha([p]+1)} (t-s)^{\frac{1+[p]}{p}} \big(1+\|\mathbf{X}\|\big)^{[p]+1}.\end{aligned}$$

The first estimate of the corollary follows then from the fact that $\exp(\boldsymbol{\Lambda}_{ts}) = \mathbf{X}_{ts}$. The two other estimates are consequences of the fact that the identity map satisfies Assumption 1 and Assumption H. $\square$

Remark 3.6. — As in Remark 3.2, one can require that V_0 and $V_{[I]}$ are more regular, and ask

- For all $1 \leqslant k_1,\dots,k_n \leqslant [p]$, with $\sum_{i=1}^n k_i \leqslant [p]$, and all $I_{k_i} \in \{1,\dots,\ell\}^{k_i}$, the functions

  $$V_0(s,\cdot)V_{[I_{n-1}]}(s,\cdot)\cdots V_{[I_1]}(s,\cdot) \text{ and } V_{[I_n]}(s,\cdot)\cdots V_{[I_1]}(s,\cdot)$$

  are C_b^{2+n} with α-growth, uniformly in time.

Under that stronger regularity assumption, we have for all $2 \leqslant k \leqslant n+1$,

$$\begin{aligned}\sup_{x\in B(0,R)} \left| D^k\mu_{ts}(x) - \left\{ (t-s)D^kV_0(s,x) + \sum_{j=1}^{[p]} \sum_{I\in\{0,\dots,\ell\}^j} X^{j,I}_{ts} D^kV_I(s,x) \right\} \right| \\ \lesssim (1+R)^\alpha \big(1+\|\mathbf{X}\|\big)^{[p]+1} |t-s|^{\frac{[p]+1}{p}}.\end{aligned}$$

The next proposition shows that μ satisfies a localized version of an *approximate flow*; see [3].

Proposition 3.7. — *Given $0 \leqslant s \leqslant u \leqslant t \leqslant T$, with $(t-s)^{\frac{1}{p}}\big(1+\|\mathbf{X}\|\big)^{[p]} \leqslant 1$, we have*

$$\begin{aligned}\sup_{x\in B(0,R)} \big|\mu_{tu}\circ\mu_{us}(x) - \mu_{ts}(x)\big| \vee \big|D\big(\mu_{tu}\circ\mu_{us}\big)(x) - D\big(\mu_{ts}\big)(x)\big| \\ \lesssim (1+R)^\alpha \big(1+\|\mathbf{X}\|\big)^{[p]+1} |t-s|^{\frac{1+[p]}{p}}.\end{aligned}$$

Proof. — First, remark that

$$\begin{aligned}\mu_{tu}\circ\mu_{us}(x) = \mu_{us}(x) &+ (t-u)V_0\big(u,\mu_{us}(x)\big)\\ &+\sum_{i=1}^{[p]}\frac{1}{i!}\sum_{\mathfrak{I}_{i,[p]}}\prod_{m=1}^{i}\Lambda_{tu}^{k_m,I_m}\Big\{V_{[I_i]}(u,\cdot\,)\cdots V_{[I_1]}(u,\cdot\,)\Big\}\big(\mu_{us}(x)\big)\\ &+\widetilde{\varepsilon}_{tu}\big(\mu_{us}(x)\big),\end{aligned}$$

where $\widetilde{\varepsilon}_{ts}(x) := \varepsilon_{ts}^{[p],\mathrm{Id}}(x) + \varepsilon'_{ts}(x)$ and

$$\varepsilon'_{ts}(x) := \sum_{I\in\{1,\dots,d\}^{[p]}}\prod_{m=1}^{[p]}\Lambda_{t,s}^{1,i_k}\int_{\Delta_{[p]}}\Big\{(V_I\,\mathrm{Id})(s,y_{r_n}) - (V_I\,\mathrm{Id})(s,x)\Big\}\,\mathrm{d}r,$$

for any $0\leqslant a\leqslant b\leqslant T$. As we also have

$$\begin{aligned}&\mu_{us}(x)-\mu_{ts}(x)\\ &\quad= -(t-u)V_0(s,x)\\ &\qquad+\sum_{i=1}^{[p]}\frac{1}{i!}\sum_{\mathfrak{I}_{i,[p]}}\Big\{\prod_{m=1}^{i}\Lambda_{us}^{k_m,I_m}-\prod_{m=1}^{i}\Lambda_{ts}^{k_m,I_m}\Big\}\big(V_{[I_i]}(s,\cdot\,)\cdots V_{[I_1]}(s,\cdot\,)\big)(x)\\ &\qquad+\widetilde{\varepsilon}_{us}(x)+\widetilde{\varepsilon}_{ts}(x),\end{aligned}$$

this gives

$$\begin{aligned}&\mu_{tu}\circ\mu_{us}(x)-\mu_{ts}(x)\\ &\quad=(t-u)\Big(V_0\big(u,\mu_{us}(x)\big)-V_0\big(s,\mu_{us}(x)\big)\Big)\\ &\qquad+(t-u)\Big(V_0\big(s,\mu_{us}(x)\big)-V_0\big(s,x\big)\Big)\\ &\qquad+\sum_{i=1}^{[p]}\frac{1}{i!}\sum_{\mathfrak{I}_{i,[p]}}\Bigg\{\prod_{m=1}^{i}\Lambda_{tu}^{k_m,I_m}+\prod_{m=1}^{i}\Lambda_{us}^{k_m,I_m}-\prod_{m=1}^{i}\Lambda_{ts}^{k_m,I_m}\Bigg\}\\ &\qquad\qquad\qquad\big(V_{[I_i]}(s,\cdot\,)\cdots V_{[I_1]}(s,\cdot\,)\,\mathrm{Id}\big)(x)\\ &\qquad+\sum_{i=1}^{[p]}\frac{1}{i!}\sum_{\mathfrak{I}_{i,[p]}}\prod_{m=1}^{i}\Lambda_{tu}^{k_m,I_m}\Big\{\big(V_{[I_i]}(u,\cdot\,)\cdots V_{[I_1]}(u,\cdot\,)\,\mathrm{Id}\big)\big(\mu_{us}(x)\big)\\ &\qquad\qquad\qquad-\big(V_{[I_i]}(s,\cdot\,)\cdots V_{[I_1]}(s,\cdot\,)\,\mathrm{Id}\big)\big(\mu_{us}(x)\big)\Big\}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{[p]}\frac{1}{i!}\sum_{\mathfrak{I}_{i,[p]}}\prod_{m=1}^{i}\Lambda_{tu}^{k_m,I_m}\Big\{\big(V_{[I_i]}(s,\cdot\,)\cdots V_{[I_1]}(s,\cdot\,)\,\mathrm{Id}\big)\big(\mu_{u,s}(x)\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad -\big(V_{[I_i]}(s,\cdot\,)\cdots V_{[I_1]}(s,\cdot\,)\,\mathrm{Id}\big)(x)\Big\}\\
&+\widetilde{\varepsilon}_{tu}\big(\mu_{us}(x)\big)+\widetilde{\varepsilon}_{us}\big(x\big)+\widetilde{\varepsilon}_{ts}(x)\\
=&:(1)+\cdots+(6).
\end{aligned}
$$

The bounds of the statement can be read on that decomposition; we give the details for $(\mu_{tu}\circ\mu_{us})(x)$ and live the details of the estimate for its derivative to the reader.

It follows from Assumption 2 on the time regularity of V_0 and $V_{[I_i]}\dots V_{[I_1]}\,\mathrm{Id}$ that

$$
\begin{aligned}
\big|(1)\big|+\big|(4)\big| &\lesssim (1+R)^{\alpha}\Big((t-u)(u-s)^{\kappa_1}+\big(1+\|\mathbf{X}\|\big)^{[p]}(t-u)^{\frac{1}{p}}(u-s)^{\kappa_2}\Big)\\
&\lesssim (1+R)^{\alpha}(t-s)^{\frac{1+[p]}{p}}\big(1+\|\mathbf{X}\|\big)^{[p]}.
\end{aligned}
$$

Lemma 3.4 takes care of the remainder terms (6). By using Lemma 3.1 and the fact that V_0 is Lipschitz continuous in space, uniformly in time, one gets

$$
\big|(2)\big|\lesssim (t-u)(u-s)^{\frac{1}{p}}\big(1+\|\mathbf{X}\|\big)^{[p]}(1+R)^{\alpha}\lesssim (t-s)^{\frac{1+[p]}{p}}\big(1+\|\mathbf{X}\|\big)^{[p]}(1+R)^{\alpha}.
$$

To estimate the terms (3) and (5), set

$$
g(s,\cdot\,):=V_{[I_i]}(s,\cdot\,)\cdots V_{[I_1]}(s,\cdot\,)\,\mathrm{Id}\,.
$$

We start by doing a Taylor expansion of $g\big(s,\mu_{ts}(x)\big)$ using Lemma 3.3, to the order $n=[p]-\sum_{j=1}^{i}k_j$. As $g(s,\cdot\,)$ satisfies Assumption H as a consequence of Assumption 1, one can use Lemma 3.4 to get the expected bounds, using the fact that $\mathbf{X}_{u,s}\mathbf{X}_{t,u}=\mathbf{X}_{t,s}$ and $\exp(\boldsymbol{\Lambda})=\mathbf{X}$. Details of these algebraic computations can be found in the proof of the corresponding statement in [3]. □

Remark. — One has similar local bounds for higher derivatives of $\mu_{tu}\circ\mu_{us}-\mu_{ts}$ in the setting of Remark 3.6.

Write here part of the conclusion of Proposition 3.7 under the form

$$
\sup_{x\in B(0,R)}\big|\mu_{t,u}\circ\mu_{u,s}(x)-\mu_{t,s}(x)\big|\leqslant C_0\,(1+R)^{\alpha}\,\big(1+\|\mathbf{X}\|\big)^{[p]+1}\,|t-s|^{\frac{1+[p]}{p}},
$$

for some positive constant C_0. Given $n\geqslant 1$, and $0\leqslant s\leqslant t\leqslant T$, set $t_k^n:=k2^{-n}(t-s)+s$. Pick ε_0 such that

$$
2^{-\frac{1+[p]-p}{p}}(1+2\varepsilon_0)<1
$$

and

$$L > \frac{C_0}{1 - 2^{-\frac{1+[p]-p}{p}}(1+2\varepsilon_0)}.$$

PROPOSITION 3.8. — *For all* $0 \leqslant s < t \leqslant T$ *with*

$$L\,|t-s|^{\frac{1}{p}}\left(1+\|\mathbf{X}\|\right) \leqslant \varepsilon_0,$$

and all positive radius R*, we have*

$$\sup_{|x|\leqslant R}\left|\mu_{t^n_{2^n}t^n_{2^n-1}}\circ\cdots\circ\mu_{t^n_1t^n_0}(x)-\mu_{ts}(x)\right| \leqslant L\,|t-s|^{\frac{1+[p]}{p}}\,(1+R)^\alpha\left(1+\|\mathbf{X}\|\right)^{[p]+1}.$$

Proof. — The proof is done by induction on n. Note first that we can take L enough to have $\frac{\varepsilon_0}{L} \leqslant 1$ and $C_{|t-s|,\|\mathbf{X}\|} \lesssim \varepsilon_0$. Proposition 3.7 provides the initialisation of the induction. Assume step n of the induction has been proved and set

$$u := \frac{t+s}{2} = t^{n+1}_{2^n},$$

so the statement of the proposition holds on the intervals (s,u) and (u,t). We have

$$\begin{aligned}
&\left|\left(\mu_{t^{n+1}_{2^n}t^{n+1}_{2^n-1}}\circ\cdots\circ\mu_{t^{n+1}_1t^{n+1}_0}\right)(x)\right| \\
&\qquad\leqslant \left|\left(\mu_{t^{n+1}_{2^n}t^{n+1}_{2^n-1}}\circ\cdots\circ\mu_{t^{n+1}_1t^{n+1}_0}\right)(x)-\mu_{us}(x)\right| + \left|\mu_{us}(x)\right| \\
&\qquad\leqslant L\,2^{-\frac{1+[p]}{p}}|t-s|^{\frac{1+[p]}{p}}\,(1+R)^\alpha\left(1+\|\mathbf{X}\|\right)^{[p]+1} + R \\
&\qquad\qquad + C_{|t-s|,\|\mathbf{X}\|}\,(1+R)^\alpha \\
&\qquad\leqslant R + 2(1+R)^\alpha\,\varepsilon_0.
\end{aligned}$$

and

$$\sup_{x\in B(0,R)}\left|D\mu_{tu}(x)\right| \leqslant 1+2\varepsilon_0,$$

by Lemma 3.1. Furthermore we have

$$\begin{aligned}
&\mu_{t^{n+1}_{2^{n+1}}t^{n+1}_{2^{n+1}-1}}\circ\cdots\circ\mu_{t^{n+1}_1t^{n+1}_0}(x)-\mu_{ts}(x) \\
&= \left(\mu_{t^{n+1}_{2^{n+1}}t^{n+1}_{2^{n+1}-1}}\circ\cdots\circ\mu_{t^{n+1}_{2^n+1}t^{n+1}_{2^n}}-\mu_{tu}\right)\circ\left(\mu_{t^{n+1}_{2^n}t^{n+1}_{2^n-1}}\circ\cdots\circ\mu_{t^{n+1}_1t^{n+1}_0}\right)(x) \\
&\quad + \mu_{tu}\circ\left(\mu_{t^{n+1}_{2^n}t^{n+1}_{2^n-1}}\circ\cdots\circ\mu_{t^{n+1}_1t^{n+1}_0}\right)(x) - \mu_{tu}\circ\mu_{us}(x) \\
&\quad + \mu_{tu}\circ\mu_{us}(x)-\mu_{ts}(x).
\end{aligned}$$

We thus have for all $x \in B(0,R)$, the estimate

$$
\begin{aligned}
\Big|\mu_{t^{n+1}_{2^{n+1}}t^{n+1}_{2^{n+1}-1}} \circ \cdots \circ \mu_{t^{n+1}_1 t^{n+1}_0}(x) - \mu_{ts}(x)\Big| \\
&\leqslant L \left|\frac{t-s}{2}\right|^{\frac{1+[p]}{p}} \Big(1+R+2\varepsilon_0(1+R)^\alpha\Big)^\alpha \big(1+\|\mathbf{X}\|\big)^{[p]+1} \\
&\quad + (1+2\varepsilon_0)\, L\,(1+R)^\alpha \left|\frac{t-s}{2}\right|^{\frac{1+[p]}{p}} \big(1+\|\mathbf{X}\|\big)^{[p]+1} \\
&\quad + C_0\,|t-s|^{\frac{1+[p]}{p}}(1+R)^\alpha\big(1+\|\mathbf{X}\|\big)^{[p]+1},
\end{aligned}
$$

from which the induction step follows given our choice of ε_0 and L. □

The same bound for the derivative of the approximate flow requires a bound on $|t-s|$ that *depends on* $(1+R)^\alpha$, such as described here.

Proposition 3.9. — *One can find a positive constant $\varepsilon_1 < 1$ such that for $0 \leqslant s \leqslant t \leqslant T$ with*

$$
C_0|t-s|^{\frac{1}{p}}(1+R)^{\frac{\alpha}{1+[p]}}\big(1+\|\mathbf{X}\|\big) \leqslant \varepsilon_1,
$$

we have, for all positive radius R,

$$
\begin{aligned}
\sup_{|x|\leqslant R}\Big|D\big(\mu_{t^n_{2^n}t^n_{2^n-1}} \circ \cdots \circ \mu_{t^n_1 t^n_0}\big)(x) - D\big(\mu_{ts}\big)(x)\Big| \\
&\leqslant L|t-s|^{\frac{1+[p]}{p}}(1+R)^\alpha\big(1+\|\mathbf{X}\|\big)^{[p]+1}.
\end{aligned}
$$

Proof. — The proof is a variation on the theme of the proof of Proposition 3.8. We provide the details for the reader's convenience, and keep the notation u for $\frac{s+t}{2}$. We proceed here as well by induction and loot at the "n to $n+1$" induction step of the proof.

$$
\begin{aligned}
&D\Big(\mu_{t^{n+1}_{2^{n+1}}t^{n+1}_{2^{n+1}-1}} \circ \cdots \circ \mu_{t^{n+1}_1 t^{n+1}_0}\Big)(x) - D\mu_{ts}(x) \\
&= \Big\{D\big(\mu_{t^{n+1}_{2^{n+1}}t^{n+1}_{2^{n+1}-1}} \circ \cdots \circ \mu_{t^{n+1}_{2^n+1}t^{n+1}_{2^n}}\big) - D\mu_{tu}\Big\}\big(\mu_{t^{n+1}_{2^n}t^{n+1}_{2^n-1}} \circ \cdots \circ \mu_{t^{n+1}_1 t^{n+1}_0}\big)(x) \\
&\qquad \times D\big(\mu_{t^{n+1}_{2^n}t^{n+1}_{2^n-1}} \circ \cdots \circ \mu_{t^{n+1}_1 t^{n+1}_0}\big)(x) \\
&\quad + \left\{ \big(D\mu_{tu}\big)\Big(\big(\mu_{t^{n+1}_{2^n}t^{n+1}_{2^n-1}} \circ \cdots \circ \mu_{t^{n+1}_1 t^{n+1}_0}\big)(x)\Big) - D\mu_{tu}\big(\mu_{us}(x)\big) \right\} \\
&\qquad \times D\big(\mu_{t^{n+1}_{2^n},t^{n+1}_{2^n-1}} \circ \cdots \circ \mu_{t^{n+1}_1,t^{n+1}_0}\big)(x) \\
&\quad + D\mu_{tu}\big(\mu_{us}(x)\big)\Big(D\big(\mu_{t^{n+1}_{2^n}t^{n+1}_{2^n-1}} \circ \cdots \circ \mu_{t^{n+1}_1 t^{n+1}_0}\big)(x) - D\mu_{us}(x)\Big) \\
&\quad + D\big(\mu_{tu} \circ \mu_{us}\big)(x) - D\mu_{ts}(x).
\end{aligned}
$$

We know from the induction step and the R-dependent assumption on $u-s$ that

$$\left|D\Big(\mu_{t^{n+1}_{2^n}t^{n+1}_{2^n-1}}\circ\cdots\circ\mu_{t^{n+1}_1t^{n+1}_0}\Big)(x)-D\mu_{us}(x)\right|\leqslant\varepsilon_1\,2^{-\frac{1+[p]}{p}},$$

and

$$\left|D\Big(\mu_{t^{n+1}_{2^n}t^{n+1}_{2^n-1}}\circ\cdots\circ\mu_{t^{n+1}_1t^{n+1}_0}\Big)(x)\right|\leqslant 1+2\varepsilon_1.$$

We also have from Proposition 3.8

$$\left|\Big(\mu_{t^{n+1}_{2^n}t^{n+1}_{2^n-1}}\circ\cdots\circ\mu_{t^{n+1}_1t^{n+1}_0}\Big)(x)\right|\leqslant\big(1+2\varepsilon_1\big)\,(1+R)-1,$$

and Lemma 3.1 gives us a uniform control on the Lipschitz size of the μ_{ba}. We thus have

$$\begin{aligned}
&\left|D\Big(\mu_{t^{n+1}_{2^{n+1}}t^{n+1}_{2^{n+1}-1}}\circ\cdots\circ\mu_{t^{n+1}_1t^{n+1}_0}\Big)(x)-D\mu_{ts}(x)\right|\\
&\qquad\leqslant L\,(1+2\varepsilon_1)^{1+\alpha}\,2^{-\frac{1+[p]}{p}}\,|t-s|^{\frac{1+[p]}{p}}\,\big(1+\|\mathbf{X}\|\big)^{[p]+1}\,(1+R)^{\alpha}\\
&\qquad\quad+L\,2^{-\frac{1+[p]}{p}}\,\varepsilon_1\,(1+2\varepsilon_1)\,|t-s|^{\frac{1+[p]}{p}}\,\big(1+\|\mathbf{X}\|\big)^{[p]+1}\,(1+R)^{\alpha}\\
&\qquad\quad+L\,(1+\varepsilon_1)\,2^{-\frac{1+[p]}{p}}\,|t-s|^{\frac{1+[p]}{p}}\,\big(1+\|\mathbf{X}\|\big)^{[p]+1}\,(1+R)^{\alpha}\\
&\qquad\quad+C_0\,|t-s|^{\frac{1+[p]}{p}}\,\big(1+\|\mathbf{X}\|\big)^{[p]+1}\,(1+R)^{\alpha}\\
&\qquad\leqslant\Big(C_0+b2^{-\frac{1+[p]}{p}}\big((1+2\varepsilon)^{\alpha}+\varepsilon_1(1+2\varepsilon_1)+(1+\varepsilon_1)\big)\Big)\\
&\qquad\quad\times|t-s|^{\frac{1+[p]}{p}}\,(1+R)^{\alpha}\,\big(1+\|\mathbf{X}\|\big)^{[p]+1}.
\end{aligned}$$

An adequate choice of ε_1 closes the induction step, given the definition of L. □

Remark 3.10. — In the improved regularity conditions on the vector fields stated in Remark 3.6, we have for all $2\leqslant k\leqslant n+1$ and for all

$$C_0\,|t-s|^{\frac{1}{p}}\,(1+R)^{\frac{k\alpha}{1+[p]}}\,\big(1+\|\mathbf{X}\|\big)\leqslant\varepsilon_1,$$

one have

$$\sup_{|x|\leqslant R}\left|D^k\big(\mu_{t^n_{2^n}t^n_{2^n-1}}\circ\cdots\circ\mu_{t^n_1t^n_0}\big)(x)-D^k\mu_{ts}(x)\right|\leqslant L\,|t-s|^{\frac{1+[p]}{p}}\,(1+R)^{\alpha}\,\big(1+\|\mathbf{X}\|\big)^{[p]+1}.$$

With all these preliminary results at hand, we are now in a position to give a proof of our local well-posedness result, Theorem 2.2.

Proof of Theorem 2.2. — We treat existence and uniqueness one after the other. We keep the above notations, and set, in addition,

$$\mu^n_{ts}=\mu_{t^n_{2^n}t^n_{2^n-1}}\circ\cdots\circ\mu_{t^n_1t^n_0}.$$

Local in time existence. — For all $x \in B(0,R)$,

$$\begin{aligned}
&\mu^{n+1}_{ts}(x) - \mu^{n}_{ts}(x) \\
&\quad = \sum_{k=1}^{2^n} \Big(\mu_{t^{n+1}_{2^{n+1}} t^{n+1}_{2^{n+1}-1}} \circ \cdots \circ \mu_{t^{n+1}_{2k+3} t^{n+1}_{2k+2}} \circ \big(\mu_{t^{n+1}_{2k+2} t^{n+1}_{2k+1}} \circ \mu_{t^{n+1}_{2k+1} t^{n+1}_{2k}} \big) \\
&\qquad\qquad - \mu_{t^{n+1}_{2^{n+1}} t^{n+1}_{2^{n+1}-1}} \circ \cdots \circ \mu_{t^{n+1}_{2k+3} t^{n+1}_{2k+2}} \circ \big(\mu_{t^{n+1}_{k+1} t^{n+1}_{2k}} \big) \Big) \\
&\qquad\qquad\qquad\qquad \circ \mu_{t^n_k t^n_{k-1}} \circ \cdots \circ \mu_{t^n_1 t^n_0}(x). \qquad (3.2)
\end{aligned}$$

It follows from Proposition 3.9 that the maps

$$\mu_{t^{n+1}_{2^{n+1}} t^{n+1}_{2^{n+1}-1}} \circ \cdots \circ \mu_{t^{n+1}_{2k+3} t^{n+1}_{2k+2}}$$

are Lipschitz continuous, uniformly in n, with a Lipschitz constant that depends neither on $\mathbf{X}$ nor on R. Furthermore, thanks to Proposition 3.8,

$$\Big| \mu_{t^n_k t^n_{k-1}} \circ \cdots \circ \mu_{t^n_1 t^n_0}(x) \Big| \leqslant R + 2\varepsilon_1 \,(1+R)^\alpha.$$

Finally, Proposition 3.7 tells us that

$$\big| \mu^{n+1}_{ts}(x) - \mu^{n}_{ts}(x) \big| \lesssim 2^{-n\frac{[p]+1-p}{p}} \, |t-s|^{\frac{1+[p]}{p}} \, (1+R)^\alpha \, \big(1 + \|\mathbf{X}\|\big)^{[p]+1}. \quad (3.3)$$

The sequence μ^n_{ts} is thus uniformly convergent on the ball $B(0,R)$ to a limit, continuous, function denoted by φ_{ts}; it satisfies the estimate

$$\sup_{x \in B(0,R)} \big| \varphi_{ts}(x) - \mu_{ts}(x) \big| \lesssim |t-s|^{\frac{1+[p]}{p}} \, (1+R)^\alpha \, \big(1 + \|\mathbf{X}\|\big)^{[p]+1}.$$

Finally, for all dyadic points $a \in [s,t]$ and all $x \in B(0,R)$, we have by construction

$$\varphi_{ta}(x) \circ \varphi_{as}(x) = \varphi_{ts}(x).$$

As $\mathbf{X}$ is an Hölder continuous rough path, the function $(x;s,t) \mapsto \mu_{ts}(x)$, from $B(0,R) \times \{0 \leqslant s < t \leqslant T\}$ to $\mathbb{R}^d$, is continuous. The continuity of φ as a function of $(x;s,t)$ follows in a straightforward way; its continuous dependence on $\mathbf{X}$ is a consequence of the continuous dependence of μ with respect to $\mathbf{X}$. Note however that φ_{ts} is only defined at that stage for s and t close enough.

Uniqueness. — Let ψ stand for another solution flow, with associated constants $\overline{\varepsilon}_{\mathbf{X}}$ and $\overline{C}_{R,\mathbf{X}}$, and exponent $\overline{\eta} > 1$. Take R and (s,t) satisfying the conditions of Proposition 3.9, with $|t-s| \leqslant \overline{C}_{R,\mathbf{X}}$. Then

$$\begin{aligned}
\Big|\varphi_{ts}(x) - \psi_{ts}(x)\Big| &\leqslant \Big|\mu^n_{ts}(x) - \varphi_{ts}(x)\Big| + \Big|\mu^n_{ts}(x) - \psi_{ts}(x)\Big| \\
&\leqslant \Big|\mu^n_{ts}(x) - \varphi_{ts}(x)\Big| \\
&\quad + \sum_{k=0}^{2^n-1} \bigg|\Big(\big(\mu_{t^n_{2^n} t^n_{2^n-1}} \circ \mu_{t^n_{k+2} t^n_{k+1}}\big) \circ \mu_{t^n_{k+1} t^n_k} \\
&\qquad\qquad - \big(\mu_{t^n_{2^n} t^n_{2^n-1}} \circ \mu_{t^n_{k+2} t^n_{k+1}}\big) \circ \psi_{t^n_{k+1} t^n_k}\Big) \circ \psi_{t^n_k t^n_{k-1}} \circ \cdots \circ \psi_{t^n_1 t^n_0}(x)\bigg| \qquad (3.4) \\
&\lesssim \Big|\mu^n_{ts}(x) - \varphi_{ts}(x)\Big| + 2^{-n(\overline{\eta}-1)}.
\end{aligned}$$

Local uniqueness follows from that estimate. We have used here the fact that the μ^n_{ts} are Lipschitz continuous, uniformly in n, and that

$$\sup_{x \in B(0,R)} \Big|\psi_{t,s} - \mu_{t,s}(x)\Big| \lesssim \overline{C}_{R,\mathbf{X}}\, |t-s|^{\overline{\eta}}. \qquad \square$$

4. Corollaries and extensions

We emphasize in the Section 4.1 and Section 4.2 two consequences on solutions to rough differential equations of the above results/computations. Young and mixed rough/Young equations are considered in Section 4.1, and differentiability of the solution flow with respect to parameters is considered in Section 4.2. The estimates on the derivative flow we get there will be used in the forthcoming work [4] on limit theorems for systems of mean field rough differential equations. We worked so far in with weak geometric Hölder p-rough paths; one can actually work with general rough paths, controlled by arbitrary controls [23]. A non-explosion criterion with quantitative estimates is provided in Section 4.3 in this more general setting.

4.1. Young and mixed rough-Young differential equations

The proofs of Theorems 2.2 and 2.3 do not use the fact the drift term is driven only by time. Instead we treat the signal $t \to t$ as a Lipschitz path, and deal with it using Young differential calculus techniques. A direct counterpart of this approach is a loss of regularity in the coefficients, either in time and space. A real reward of this approach, which does not modify the proof but requires only more notations, is an extension of the results to a mixed rough-Young differential equation.

Let V_0 and $\mathrm{F}=(V_1,\dots,V_\ell)$ be given; let another family $\mathrm{G}:=(W_1,\dots,W_m)$ of vector fields on B be given. A solution flow to the mixed rough-Young differential equation is defined as in Definition 2.1, with the 'approximate flow' μ_{ts} defined as the time 1 map of the ordinary differential equation

$$y'_r = V_0\big(s,y_r\big)(t-s) + \sum_{j=1}^{m} Y^j_{ts}\, W_j\big(s,y_r\big) + \sum_{k=}^{[p]} \sum_{I\in\{1,\dots,\ell\}^k} \Lambda^{k,I}_{ts}\, V^X_{[I]}\big(s,y_r\big).$$

The constants ε and C that appear in the defining estimate (2.3) are now allowed to depend on $R,\mathbf{X}$ and Y.

Corollary 4.1. — *Let $\mathbf{X}$ be an $\mathbb{R}^\ell$-valued weak geometric Hölder p-rough path and Y be an $\mathbb{R}^m$-valued $\frac{1}{q}$-Hölder path, with $\frac{1}{p}+\frac{1}{q}>1$ and $p\geqslant 2$. Assume (V_0,F) and (W_i,F) satisfy Assumption 1 and Assumption 2 for all $1\leqslant i\leqslant m$. Assume furthermore that there exists a positive exponent κ such that $\kappa+\frac{1}{q}>1$, and*

$$\sup_{x\in B(0,R)} \sup_{0\leqslant s<t\leqslant T} \frac{\big|W_i(t,x)-W_i(s,x)\big|}{|t-s|^\kappa} \lesssim (1+R)^\alpha.$$

Then the rough differential equation

$$\mathrm{d}\varphi_t = V_0(t,\varphi_t)\mathrm{d}t + \mathrm{G}(t,\varphi_t)\,\mathrm{d}Y_t + \mathrm{F}(t,\varphi)\,\mathrm{d}\mathbf{X}_t,$$

has a unique global in time solution flow.

On can choose the constants $\varepsilon_{\mathbf{X},Y}$ and $C_{R,\mathbf{X},Y}$ such that

$$(t-s)^{\frac{1}{q}}\,\big(1+\|Y\|\big) + (t-s)^{\frac{1}{p}}\big(1+\|\mathbf{X}\|\big) \lesssim 1,$$

and

$$C_{R,\mathbf{X},Y} \simeq (1+R)^\alpha\,\big(1+\|Y\|+\|\mathbf{X}\|\big)\,\big(1+\|\mathbf{X}\|\big)^{[p]}$$

and

$$N \simeq \max\Big\{\big[(1+\|Y\|)^q\big],\big[(1+\|\mathbf{X}\|)^p\big]\Big\}.$$

The proof is left to the reader since it is a direct modification of Section 3, with more notations.

4.2. Derivative flow

Rough differential equations

$$\mathrm{d}\varphi_t = V_0(t,\varphi_t)\mathrm{d}t + \mathrm{F}(t,\varphi_t)\,\mathrm{d}\mathbf{X}_t \tag{4.1}$$

generate flows of diffeomorphisms under appropriate regularity conditions on the driving vector fields. The pair $(\varphi, D\varphi)$, made up of φ and its differential, also satisfies an equation, with 'triangular' structure

$$\mathrm{d}(D\varphi) = DV_0(t,\varphi_t)D\varphi_t\,\mathrm{d}t + D\mathrm{F}(t,\varphi_t)D\varphi_t\,\mathrm{d}\mathbf{X}_t.$$

One can find results on derivative flows in the book [20] of Friz and Victoir, Chapter 11; see also the interesting works [12] and [11] of Coutin and Lejay. One gets another proof of the differentiability of the flow with respect to the initial point as a direct byproduct of the results of Section 2. Pick $p > 2$.

Assumption 4. — *Let V_0 and $V_1, \dots, V_\ell$ be a set of time dependent vector fields on B such that there exists two exponents with $\kappa_1 > \frac{1+[p]-p}{p}$, and $\kappa_2 + \frac{1}{p} > 1$, such that*

$$\sup_{0\leqslant s<t\leqslant T} \frac{\big\|V_0(t,\cdot) - V_0(s,\cdot)\big\|_{C_b^{2+n}}}{|t-s|^{\kappa_1}} < +\infty$$

and each V_i satisfies the estimate

$$\sup_{0\leqslant s<t\leqslant T} \frac{\big\|V_i(t,\cdot) - V_i(s,\cdot)\big\|_{C_b^{3+n}}}{|t-s|^{\kappa_2}} < +\infty.$$

Theorem 4.2. — *Let $\mathbf{X}$ be a weak geometric Hölder p-rough path and $(V_0, V_1, \dots, V_\ell)$ which satisfy Assumption 4. Let φ stand for the solution flow to the rough differential equation (4.1). Then each φ_{ts} is of class C^n, has linear growth and bounded derivatives, namely $\varphi_{ts} \in C^n_{1,b}$. Furthermore for a suitable positive constant ε_3, independent of $\mathbf{X}$, and $|t-s|^{\frac{1}{p}}\big(1+\|\mathbf{X}\|\big) \leqslant \varepsilon_3$, we have*

$$\big\|D^k\varphi_{ts} - D^k\mu_{ts}\big\|_\infty \lesssim |t-s|^{\frac{1+[p]}{p}}\big(1+\|\mathbf{X}\|\big)^{[p]+1},$$

for all $0 \leqslant k \leqslant n$. Finally there exists some positive constants $c_1, \dots, c_n$, independent of $\mathbf{X}$, such that for all $0 \leqslant s \leqslant t \leqslant T$, *and every $1 \leqslant k \leqslant n$, we have*

$$\sup_{x\in\mathbb{R}^d} \big|\varphi_{ts}(x) - x\big| \lesssim |t-s|^{\frac{1}{p}}\big(1+\|\mathbf{X}\|\big)$$

and

$$\sup_{x\in\mathbb{R}^d} \big|D^k\varphi_{ts}(x) - D^k\varphi_{s,s}(x)\big| \lesssim |t-s|^{\frac{1}{p}} e^{c_k|t-s|^{\frac{1}{p}}N},$$

where $N = \Big[c\big(1+\|\mathbf{X}\|\big)^{-p}\Big]$.

Proof. — We work here with $\alpha = 0$, so we know from the above computations that for $|t-s|^{\frac{1}{p}}\big(1+\|\mathbf{X}\|\big) \lesssim 1$, and all $0 \leqslant k \leqslant (n+1)$, we have

$$\big\|D^k\big(\mu^n_{ts} - \mu_{ts}\big)(x)\big\|_\infty \lesssim |t-s|^{\frac{1+[p]}{p}}\big(1+\|\mathbf{X}\|\big)^{[p]+1}. \tag{4.2}$$

This implies that for all $0 \leqslant k \leqslant n$ the function $D^k_{t,s}$ is Lipschitz continuous, with Lipschitz constant not greater than a constant multiple of $|t-s|^{\frac{1+[p]}{p}}\big(1+\|\mathbf{X}\|\big)^{[p]+1}$. It follows from this fact and the proof of Theorems 2.2 and 2.3 that there exists some maps $A^k_{t,s}$, such that $D^k\mu^n_{ts}$ converges uniformly to $A^k_{t,s}$ as n goes to ∞. One then needs to prove that the A^k are indeed the k-th derivative of φ and get the bounds of the statement. Note that the small time bounds are direct consequences of equation (4.2) and Remark 3.2 once we know that the $D^k\mu^n_{ts}$ converge.

We have

$$\mu^n_{ts}(x+h)-\sum_{0\leqslant j\leqslant k}\frac{1}{j!}D^k\mu_{t,s}(x)\cdot h^j = \frac{1}{k!}\int_0^1 \mathrm{d}\lambda D^{k+1}\mu^n_{ts}(\lambda h+x)\cdot((1-\lambda)h)^k h,$$

where $h^j = \underbrace{(h,\dots,h)}_{j\text{ times}}$. Hence, thanks to Remark 3.10, for all $|t-s|^{\frac{1}{p}}\big(1+\|\mathbf{X}\|\big)\lesssim 1$, the maps $D^{k+1}\mu^n_{ts}$ are bounded, *uniformly in n*, and

$$\left|\mu^n_{t,s}(x+h)-\sum_{0\leqslant j\leqslant k}\frac{1}{j!}D^k\mu^n_{t,s}(x)\cdot h^j\right| \lesssim |t-s|^{\frac{1+[p]}{p}}\big(1+\|\mathbf{X}\|\big)^{[p]+1}|h|^{k+1}.$$

The previous bound allows us to send n to ∞, and to get, as a consequence, that $A^k_{ts} = D^k\varphi_{ts}$. The construction of the global in time flow and its derivatives is done by gluing all these local flows, as above.

We now turn to the global bounds. As previously let N be the least integer such that $T^{\frac{1}{p}}N^{-\frac{1}{p}}\big(1+\|\mathbf{X}\|\big)\lesssim 1$, where the implicit multiplicative constant is chosen such that all the previous bounds hold. Setting $t_i := \frac{i}{N}(t-s)+s$, one can use the local in time bounds on some time interval of length $(t_{i+1}-t_i)$. We have

$$\begin{aligned}\varphi_{t_is}(x)-x = \varphi_{t_it_{i-1}}\big(\varphi_{t_{i-1}s}(x)\big) &- \mu_{t_it_{i-1}}\big(\varphi_{t_{i-1}s}(x)\big)\\ &+ \mu_{t_it_{i-1}}\big(\varphi_{t_{i-1}s}(x)\big) - \varphi_{t_{i-1}s}(x)\\ &+ \varphi_{t_{i-1}s}(x)-x.\end{aligned}$$

Hence, if one sets $R^0_i := \sup_x\big|\varphi_{t_i,s}(x)-x\big|$, one has

$$R^0_i \leqslant R^0_{i-1} + C|t-s|^{\frac{1}{p}} \lesssim i|t-s|^{\frac{1}{p}}.$$

Similarly, we have

$$\begin{aligned} D\varphi_{t_i s}(x) - \mathrm{Id} = \Big(D\varphi_{t_i t_{i-1}}\big(\varphi_{t_{i-1}s}(x)\big) - \mu_{t_i t_{i-1}}\big(\varphi_{t_{i-1}s}(x)\big)\Big) D\varphi_{t_{i-1}s}(x) \\ + \Big(D\mu_{t_i t_{i-1}}\big(\varphi_{t_{i-1}s}(x)\big) - \mathrm{Id}\Big) D\varphi_{t_{i-1}s}(x) \\ + D\varphi_{t_{i-1}s}(x) - \mathrm{Id}\,. \end{aligned}$$

Again,given the choice of N, one have can use all the local bounds on φ, $D\varphi$, and μ and $D\mu$, and setting $R^1_i := \sup_x \big|D\varphi_{t_i,s}(x) - \mathrm{Id}\,\big|$, one has

$$R^1_i \leqslant C|t-s|^{\frac{1}{p}} + R^1_{i-1}\big(1 + |t-s|\big)^{\frac{1}{p}},$$

and

$$\sup_x \big|D\varphi_{s,t}(x) - \mathrm{Id}\big| \lesssim |t-s|^{\frac{1}{p}} e^{c_1|t-s|^{\frac{1}{p}} N}.$$

One obtains the bounds for the higher order derivatives using Faà di Bruno formula. □

4.3. Finite p-variation rough paths

It is well-known the global bound for the differential of the flow, or the global bound for the flow for vector field with linear-growth, is not good [7], [19], [18]. Indeed, in the setting of weak geometric Hölder p-rough paths, $N \sim \big(1 + \|\mathbf{X}\|\big)^p$, and for a Gaussian rough path $\mathbf{X}$, the quantity $\|\mathbf{X}\|$ only has Gaussian tail and $\mathbb{E}\big[e^{c\|\mathbf{X}\|^p}\big] = +\infty$ for any $p > 2$ and any positive constant c. To derive some moment bounds of solutions of rough differential equations, one need more advanced tools; we recall them here for the reader's convenience.

Definition. — *A weak geometric continuous rough path with finite p-variation is a continuous $[p]$-level weak geometric rough path such that*

$$\|\mathbf{X}\|_{[0,T],p-\mathrm{var}} := \sum_{i=1}^{[p]} \sup_{\pi \textit{ partition of } [0,T]} \left(\sum_{(t_k,t_{k+1})\in\pi} |X^i_{t,s}|^{\frac{p}{i}} \right)^{\frac{1}{p}} < +\infty$$

Set

$$w(t,s) := \|\mathbf{X}\|^p_{[s,t],p-\mathrm{var}}.$$

If $\mathbf{X}$ is a weak geometric continuous rough path with finite p-variation then w is a control; it is in particular increasing in its two variables, super-additive and continuous on the diagonal. Note also that a weak geometric Hölder p-rough path is always of finite p-variation since

$$w(t,s) \leqslant |t-s|\big(1 + \|\mathbf{X}\|\big)^p.$$

The advantage of using the p-variation norm instead of the Hölder norm is related to integrability properties for random rough paths.

Definition 4.3. — *Given $\beta > 0$ define $\tau_0^\beta = 0$ and*

$$\tau_{i+1}^\beta = \inf\{t \in [\tau_i^\beta, T] : w(\tau_i^\beta, t) \geqslant \beta\} \wedge T.$$

The quantity $N_\beta := \sup\{i \geqslant 0 : \tau_i^\beta < T\}$ is called the local accumulated variation of $\mathbf{X}$.

The following result combines results from Friz and Victoir [20] and Cass, Litterer and Lyons [7]

Theorem. — *Let $\beta > 0$, $p \geqslant 2$ and let X be a centered Gaussian process defined over some finite interval $[0, T]$. Suppose that the covariance function is of finite ρ-two dimensional variation for some $\rho \in (1, 2)$. Then for any $p \in (2\rho, 4)$, X can be lifted as a level-$[p]$ weakly geometric continuous finite p-variation rough path, and for $\beta > 0$, the process $N_\beta^{\frac{1}{\rho}}$ has a Gaussian tails, namely there exists a constant $\mu > 0$ such that*

$$\mathbb{E}\left[\exp\left(\mu N_\beta^{\frac{2}{\rho}}\right)\right] < +\infty.$$

In particular, for $p \in (2\rho, 4)$, and for any constant $C > 0$,

$$\mathbb{E}\left[\exp\left(C N_\beta\right)\right] \lesssim 1.$$

Friz and Riedel gave in [19] what is now the classical proof of this result, based on Borell's isoperimetric inequality in Gaussian spaces. Cass and Ogrodnik [8] use heat kernel estimates as a substitute to isoperimetry to prove a similar result for Markovian rough paths. Compare the following definition to definition 2.1.

Definition. — *A flow $\varphi : \Delta_T \times \mathbb{R}^d \mapsto \mathbb{R}^d$ is said to be a* solution flow to the rough differential equation (2.2) *if there exists an exponent $\eta > 1$ such that one can associate to any positive radius R two positive constants C_R and ε, independent of $\mathbf{X}$, such that one has*

$$\sup_{x \in B(0,R)} \left|\varphi_{ts}(x) - \mu_{ts}(x)\right| \leqslant C_R\, w(t,s)^\eta, \tag{4.3}$$

whenever $w(t,s) \leqslant \varepsilon$.

Theorem 4.4. — *Let $\mathbf{X}$ be a weak geometric continuous rough path with finite p-variation. Let V_0 and $(V_1, \ldots, V_\ell)$ satisfy Assumption 1 and Assumption 2. There exists a unique global in time solution flow φ to the rough differential equation (2.2).*

- *One can choose* $\eta = \frac{1+[p]}{p}$, $\varepsilon = c_1$ *and* $C_R = c_2(1+R)^\alpha$, *for some positive universal constants* c_1, c_2, *in the defining identity* (4.3).
- *One has for all* $f \in C_b^{[p]+1}$ *and all* $w(t,s) \leqslant \varepsilon$ *the estimate*

$$\sup_{x \in B(0,R)} \left| f \circ \varphi_{t,s}(x) - \left\{ f(x) + (t-s)V_0(s,\cdot)f \right.\right.$$
$$\left.\left. + \sum_{k=1}^{[p]} \sum_{I \in \{0,\dots,\ell\}^k} X_{t,s}^{k,I} V_I(s,\cdot)f \right\}(x) \right|$$
$$\lesssim \|f\|_{C_b^{[p]+1}} (1+R)^{\alpha([p]+1)} w(t,s)^{\frac{[p]+1}{p}}.$$

When $f = \mathrm{Id}$, *one can replace* $(1+R)^{\alpha([p]+1)}$ *by* $(1+R)^\alpha$ *and* $\|f\|_{C_b^n}$ *by* 1 *in the previous bound.*

- *The map that associates* φ *to* $\mathbf{X}$ *is continuous from the set of weak geometric continuous rough paths with finite p-variation into the set of continuous flows endowed with the topology of uniform convergence on bounded sets.*
- *Finally, there exists* $\beta > 0$ *and* $c_3 > 0$ *such that one has for all* $(t,s) \in \Delta_T$,

$$\sup_{x \in B(0,R)} \left| \varphi_{ts}(x) - x \right|$$
$$\lesssim \begin{cases} \left(\left((1+R)^{1-\alpha} + c_4 w(t,s)^{\frac{1}{p}} N_\beta^{1-\frac{1}{p}} \right)^{\frac{1}{1-\alpha}} - (1+R) \right), & \text{if } \alpha < 1 \\ (1+R) w(t,s)^{\frac{1}{p}} e^{c_3 N_\beta}, & \text{if } \alpha = 1. \end{cases}$$

One gets back Theorem 2.3 when $\mathbf{X}$ is an Hölder p-rough path, with N replaced by N_β.

Proof. — The proof follows exactly the same steps as the proofs of Theorems 2.2 and Theorem 2.3. We give here the main changes and leave the computations to the reader.

First, there is no loss of generality in assuming that $|t-s| \leqslant w(t,s)$; replace if necessary $w(t,s)$ by $|t-s| + w(t,s)$. Set

$$C(t,s,\mathbf{X}) := \sum_{k=1}^{[p]} w(t,s)^{\frac{k}{p}}.$$

One can replace the constant $C_{|t-s|, \|\mathbf{X}\|}$ by $C(t,s,\mathbf{X})$ in Lemma 3.1 and Remark 3.2 as soon as $w(t,s) \leqslant 1$; this ensures that $C(t,s,\mathbf{X}) \lesssim 1$. Lemma 3.3 remains the same as it relies only on algebraic manipulations. In Lemma 3.4, one has to assume that $w(t,s,\mathbf{X}) \leqslant 1$, and one can replace

$\big(1+\|\mathbf{X}\|\big)^{[p]+1}|t-s|^{\frac{1+[p]}{p}}$ in the estimates by $w(t,s)^{\frac{1+[p]}{p}}$ (recall $|t-s|\leqslant w(t,s)$). The same replacement is done in Corollary 3.5 and Remark 3.6. Finally, using the inequality $|t-s|\leqslant w(t,s)$ and the fact that the real-valued functions $u\to w(t,u)$ and $u\to w(u,s)$ are increasing, one can also replace $\big(1+\|\mathbf{X}\|\big)^{[p]+1}|t-s|^{\frac{1+[p]}{p}}$ by $w(t,s)^{\frac{1+[p]}{p}}$ in Proposition 3.7.

The proofs of Proposition 3.8 and Proposition 3.9 are a bit different, but the spirit is the same. The main difference is that one cannot say immediately that $w\big(t,\frac{t+s}{2}\big)\leqslant\frac{1}{2}w(t,s)$. But given $(t,s)\in\Delta_T$, there exists $\widetilde{u}\in(s,t)$ such that $w(t,u)=w(u,s)\leqslant\frac{1}{2}w(t,s)$. Consider any sequence of embedded partitions $(\pi^n)_{n\in\mathbb{N}}=\big((t_i^n)_{i\in\{0,\dots,n\}}\big)_{n\in\mathbb{N}}$ with mesh going to 0. One proves by induction the existence of constants $0<\beta\leqslant 1$ and $L>0$ such that for $w(t,s)\leqslant\beta$, one has for all $k\leqslant n$,

$$\sup_{x\in B(0,R)}\Big|\mu_{t_k^k,t_{k-1}^k}\circ\cdots\circ\mu_{t_1^k,t_0^k}(x)-\mu_{t,s}(x)\Big|\leqslant L\,(1+R)^{\alpha}\,w(t,s)^{\frac{[p]+1}{p}}$$

Let the integer $0\leqslant i_0\leqslant n$ be such that $t_{i_0}^{n+1}\leqslant\widetilde{u}<t_{i_0+1}^{n+1}$. One closes the induction and proves the following bound for all $n\in\mathbb{N}$ by taking $u=t_{i_0+1}^{n+1}$, using the fact that

$$w(t_{i_0+1}^{n+1},\widetilde{u})+w(\widetilde{u},t_{i_0}^{n+1})\underset{n\to\infty}{\longrightarrow}0.$$

The same trick holds for the proof of Proposition 3.9, assuming that

$$w(t,s)(1+R)^{\frac{\alpha}{1+[p]}}\leqslant\beta.$$

One can again replace in Proposition 3.8, Propositino 3.9 and Remark 3.10 $\big(1+\|\mathbf{X}\|\big)^{[p]+1}|t-s|^{\frac{1+[p]}{p}}$ by $w(t,s)^{\frac{1+[p]}{p}}$.

For the proof of the local existence, one can proceed as in Lemma 2.1 of [15], and as in the proof of Theorem 2.2. Let $\big((t_i^n)_{i\in\{0,\dots,n\}}\big)_{n\in\mathbb{N}}$ be the sequence of dyadic partitions. Remark that since w is superadditive, there exists i such that $w(t_{i+1}^n,t_{i-1}^n)\leqslant(2^n-1)^{-1}w(t,s)$. Define the partition $\widehat{\pi}=\big\{s=t_0,<\cdots<t_{i-1}^n<t_{i+1}^n<\cdots<t_{2^n}^n=t\big\}$and set $M_{t,s}^n:=\mu_{t,s}^n-\mu_{t,s}$, and

$$\widehat{M}_{t,s}^n:=\mu_{t_{2^n}^n,t_{2^n-1}^n}\circ\cdots\circ\mu_{t_{i+2}^n,t_{i+1}^n}\circ\mu_{t_{i+1}^n,t_{i-1}^n}\circ\mu_{t_{i-1}^n,t_{i-2}^n}\circ\mu_{t_1^n,t_0^n}-\mu_{t,s}.$$

We have

$$\begin{aligned}\widehat{M}_{s,t}-M_{t,s}^n=\Big\{&\mu_{t_{2^n}^n,t_{2^n-1}^n}\circ\cdots\circ\mu_{t_{i+2}^n,t_{i+1}^n}\circ\big(\mu_{t_{i+1}^n,t_i^n}\circ\mu_{t_i^n,t_{i-1}^n}\big)\\&-\mu_{t_{2^n}^n,t_{2^n-1}^n}\circ\cdots\circ\mu_{t_{i+2}^n,t_{i+1}^n}\circ\big(\mu_{t_{i+1}^n,t_{i-1}^n}\big)\Big\}\\&\circ\mu_{t_{i-1}^n,t_{i-2}^n}\circ\mu_{t_1^n,t_0^n}(x).\end{aligned}$$

The induction hypothesis and the bound $w(t,s)(1+R)^{\frac{\alpha}{1+[p]}} \leqslant \beta$, then give

$$\left|\widehat{M}^n_{s,t} - M^n_{t,s}\right| \lesssim (2^n-1)^{-\frac{[p]+1}{p}} (1+R)^\alpha w(s,t)^{\frac{[p]+1}{p}}.$$

Repeating this operation until we get the trivial partition of $[s,t]$ we see that

$$M^n_{t,s} = \sum_{k=0}^{2^n} \rho^k_{t,s},$$

with

$$\left|\rho^k_{t,s}(x)\right| \lesssim (1+R)^\alpha\, w(t,s)^{\frac{[p]+1}{p}}\, (2^n-k)^{-\frac{[p]+1}{p}}.$$

Here we crucially use the fact that the composition of the flows are globally Lipschitz continuous, uniformly in n. Hence M^n converges uniformly to a limit $\varphi_{t,s} - \mu_{t,s}$ and

$$\begin{aligned}\sup_{x\in B(0,R)} &|\varphi_{t,s}(x) - \mu_{t,s}(x)| \\ &\lesssim (1+R)^\alpha \sum_{i\geqslant 0} i^{-\frac{[p]+1}{p}} w(t,s)^{\frac{[p]+1}{p}} \lesssim (1+R)^\alpha w(t,s)^{\frac{[p]+1}{p}}.\end{aligned}$$

The remainder of the proof follows easily from the proof of Theorem 2.2 and Theorem 2.3. Indeed, by construction, φ is a flow for all dyadic points, and then by continuity for all points, and thanks to the continuity of μ with respect to $\mathbf{X}$, φ is continuous with respect to $\mathbf{X}$.

Note also that thanks to the superadditivity property of the control, one has

$$\sum_{k=0}^{2^n-1} w(t^n_{i+1}, t^n_i)^{\frac{[p]+1}{p}} \lesssim \max_{i\in\{0,\dots,2^n-1\}} w(t^n_{i+1}, t^n_i)^{\frac{[p]+1-p}{p}} w(t,s),$$

and since w is continuous on the diagonal, the above sum goes to 0 as n goes to infinity. Local uniqueness of the flow follows (see Equation (3.4)).

The proof of global existence is similar to the proof of Theorem (2.3). Use the sequence of times $(\tau^\beta_i)_{i\in\mathbb{N}}$ from definition 4.3. We have

$$\begin{aligned}\varphi_{\tau^\beta_i,s}(x) - x = {}& \varphi_{\tau^\beta_i \tau^\beta_{i-1}}\left(\varphi_{\tau^\beta_{i-1}s}(x)\right) - \mu_{\tau^\beta_i \tau^\beta_{i-1}}\left(\varphi_{\tau^\beta_{i-1}s}(x)\right) \\ &+ \mu_{\tau^\beta_i \tau^\beta_{i-1}}\left(\varphi_{\tau^\beta_{i-1}s}(x)\right) - \varphi_{\tau^\beta_{i-1}s}(x) \\ &+ \varphi_{\tau^\beta_{i-1}s}(x) - x.\end{aligned}$$

Define $R_i := \sup_{x\in B(0,R)} \left|\varphi_{\tau^\beta_i s}(x) - x\right|$ and $R_0 = 0$. The fourth item of the statement follows then from the induction relation

$$R_i \leqslant R_{i-1} + w(\tau^\beta_i, \tau^\beta_{i-1})^{\frac{[p]+1}{p}} (1+R+R_i)^\alpha + C(\tau^\beta_i, \tau^\beta_{i-1}, \mathbf{X})(1+R+R_{i-1})^\alpha.$$

Since $w(\tau_i^\beta,\tau_{i-1}^\beta)\leqslant\beta$, one has $C(\tau_i^\beta,\tau_{i-1}^\beta,\mathbf{X})\lesssim w(\tau_i^\beta,\tau_{i-1}^\beta)^{\frac{1}{p}}$, hence

$$R_i-R_{i-1}\lesssim(1+R+R_{i-1})^\alpha\big(w(\tau_i^\beta,\tau_{i-1}^\beta)^{\frac{[p]+1}{p}}+w(\tau_i^\beta,\tau_{i-1}^\beta)^{\frac{1}{p}}\big).$$

When $\alpha=1$ one end up with the following bound:

$$R_i\lesssim R_{i-1}+(1+R)w(t,s)^{\frac{1}{p}}.$$

When $\alpha<1$, one ends up with

$$R_N\lesssim\left(\left((1+R)^{1-\alpha}+\frac{1}{1-\alpha}\sum_{i=1}^N\big(w(\tau_i,\tau_{i-1})^{\frac{[p]+1}{p}}+w(\tau_i,\tau_{i-1})^{\frac{1}{p}}\big)\right)^{\frac{1}{1-\alpha}}-(1+R)\right).$$

By using Jensen formula, one finally has the bound

$$\sum_{i=1}^N\big(w(\tau_i,\tau_{i-1})^{\frac{[p]+1}{p}}+w(\tau_i,\tau_{i-1})^{\frac{1}{p}}\big)\lesssim N^{1-\frac{1}{p}}w(t,s)^{\frac{1}{p}},$$

which ends the proof. □

Theorem 4.5. — *Let $p>2$ and $\mathbf{X}$ be a weak geometric continuous finite p-variation rough path and let $(V_0,\dots,V_\ell)$ which satisfies Assumption 4. Let φ stands for the solution flow to the rough differential equation* (4.1)*. Then each φ_{ts} is of class C^n, has linear growth and bounded derivatives, $\varphi_{ts}\in C^n_{1,b}$. Furthermore for a suitable positive constant ε_3, independent of $\mathbf{X}$, and $w(t,s)\leqslant\varepsilon_3$, we have*

$$\big\|D^k\varphi_{ts}-D^k\mu_{ts}\big\|_\infty\lesssim w(t,s)^{\frac{1+[p]}{p}},$$

for all $0\leqslant k\leqslant n$. Finally there exists $\beta>0$ and some positive constants $c_1,\dots,c_n$, independent of $\mathbf{X}$, such that for all $0\leqslant s\leqslant t\leqslant T$, *and every $1\leqslant k\leqslant n$, we have*

$$\sup_{x\in\mathbb{R}^d}\big|\varphi_{ts}(x)-x\big|\lesssim w(t,s)^{\frac{1}{p}}N_\beta$$

and

$$\sup_{x\in\mathbb{R}^d}\big|D^k\varphi_{ts}(x)-D^k\varphi_{s,s}(x)\big|\lesssim w(t,s)^{\frac{1}{p}}e^{c_kN_\beta}.$$

Proof. — We refer to the proof of Theorem 4.2. The first bound of the theorem is a direct application of Theorem 4.4 with $\alpha=0$. For the existence of derivatives and the associated bounds, one can mimic the proof of Theorem 4.2 by replacing $|t-s|^{\frac{1}{p}}\big(1+\|\mathbf{X}\|\big)^p$ by $w(t,s)$. The proof of the global bound is done in the same way as the proof of Theorem 4.4. □

Appendix A. Weakly geometric rough paths in a Nutshell

As first proved by T. Lyons in [23], in order to solve stochastic differential equation in a pathwise way, the knowledge of the path of a Brownian motion is not enough (see [24, Section 1.5] for a more precise statement). The solution proposed by Lyons to deal with this problem, which is at the heart of the rough path theory, is to consider the path together with its iterated integrals as a priori data. A weakly geometric rough path is then an idealization of a truncated sequence of iterated integrals for paths which are not differentiable.

To give a hint of the algebraic and analytical properties of iterated integrals, focus first on a smooth path x and its first iterated integral. Namely let $x : [0, T] \to \mathbb{R}^\ell$, be a smooth path, and define for $(t, s) \in \Delta_T$,

$$X^1_{ts} = \int_s^t \dot{x}_r \,\mathrm{d}r = x_t - x_s \text{ and } X^2_{ts} = \int_s^t X^1_{r,s} \otimes \dot{x}_r \,\mathrm{d}r = \int_s^t (x_r - x_s) \otimes \dot{x}_r \,\mathrm{d}r.$$

From this definition, one can extract three interesting properties.

(a) Regularity:

$$\sup_{k\in\{1,2\}} \sup_{0\leqslant s<t\leqslant T} \frac{|X^k_{t,s}|^{\frac{1}{k}}}{|t-s|} < +\infty,$$

(b) Chen's relation (see [10]): for all $0 \leqslant s \leqslant u \leqslant t \leqslant T$

$$X^2_{ts} = X^2_{us} + X^1_{us} \otimes X^1_{tu} + X^2_{tu},$$

(c) Geometric property:

$$\mathrm{Sym}(X^2_{ts}) = \frac{1}{2} X^1_{ts} \otimes X^1_{ts}.$$

The first property comes from the fact that the path x is Lipchitz continuous on $[0, T]$. The second property comes from the construction of X^2 as the iterated integral of x. The last property comes from the chain rule. It clarifies the matter to set the scene in a precise algebrai setting in order to generalized these properties to higher iterated integrals.

A.1. Truncated tensor algebra and free nilpotent groups

Let $N \geqslant 2$ to be fixed from now. Let us define $T^{(N)}$ the *truncated tensor algebra* as

$$T^{(N)} = \bigoplus_{k=0}^{N} (\mathbb{R}^\ell)^{\otimes k}.$$

The space $T^{(N)}$ is an algebra when defining

$$\mathbf{a}+\mathbf{b}=(a^0+b^0,\dots,a^N+b^N)$$

and

$$\mathbf{ab}=(c^0,c^1,\dots,c^N) \text{ with } c^k=\sum_{i=0}^{k} a^i\otimes b^{k-i}.$$

One can also define the Lie bracket between $\mathbf{a}$ and $\mathbf{b}$ by

$$[\mathbf{a},\mathbf{b}]=\mathbf{ab}-\mathbf{ba}.$$

Having this Lie bracket, it is straightforward to define the N*-step free nilpotent algebra* as

$$\mathfrak{g}^N=\bigoplus_{k=0}^{N} F^k,$$

where $F^0=\{0\}$, $F^1=\mathbb{R}^\ell$ and $F^{n+1}=[\mathbb{R}^\ell,F^n]$. Finally, one can define

$$\exp(\mathbf{a})=\sum_{j=0}^{N}\frac{\mathbf{a}^k}{k!} \text{ and } \log(\mathbf{b})=\sum_{j=1}^{N}\frac{(-1)^{j+1}}{j}(1-\mathbf{b})^j,$$

such that

$$G^N=\exp(\mathfrak{g}^N)$$

is a Lie group, called the N*-step free nilpotent group*. Note also that for all $\mathbf{a}\in G^N$, $\log(\mathbf{a})\in\mathfrak{g}^N$ is well defined. Let $e=(e_1,\dots,e_\ell)$ be the canonical basis of $\mathbb{R}^\ell$. For all $k\in\{0,\dots,N\}$ and for all $I=(i_1,\dots,i_k)\in\{1,\dots,\ell\}^k$ let us define

$$e_I=e_{i_1}\otimes\cdots\otimes e_{i_k} \text{ and } e_{[I]}=\Big[e_{i_1},\big[e_{i_2},\cdots[e_{i_{k-1}},e_{i_k}]\cdots\big]\Big].$$

Then $(e_I)_{I\in\{1,\dots,\ell\}^k}$ is a basis of $(\mathbb{R}^\ell)^{\otimes k}$, and for all $\mathbf{a}=(a^0,\dots,a^N)\in T^{(N)}$,

$$a^k=\sum_{I\in\{1,\dots,\ell\}^k} a^{k,I}e_I,$$

where $a^{k,I}\in\mathbb{R}$. Furthemore, in the same manner if $\mathbf{b}\in\mathfrak{g}^N$, $\mathbf{b}=(0,b^1,\dots,b^N)$, then for all $k\in\{1,\dots,N\}$,

$$b^k=\sum_{I\in\{1,\dots,\ell\}^k} b^{k,[I]}e_{[I]}$$

A.2. Weakly geometric rough paths

One has now all the tools to define properly weakly geometric rough paths. It is possible to transcript properties (a), (b) and (c) into the setting of truncated tensor algebra, when considering $\mathbf{X} = (1, X^1, X^2)$ as a path from Δ_T to $T^{(2)}$. Property (b) can be written as $\mathbf{X}_{ts} = \mathbf{X}_{us}\mathbf{X}_{tu}$, whereas property (c) translates itself into $\mathbf{X} \in G^2$ (which is precisely the reason G^N of the introduction of the N step free-nilpotent group). However, the paths we have in mind are only Hölder continuous with Hölder exponent $\frac{1}{p} < \frac{1}{2}$ (think to the Brownian motion). Even by using the theory of Young integrals (see [26]), it is not possible to define a generalization of the previous construction for non smooth paths. Furthermore, property (a) is less obvious is such a context. The solution proposed by Lyons is to reverse the problem and consider that the iterated integrals are given. Furthermore as seen in Section 3, it is crucial to have the first $[p]$ iterated integrals to solve rough differential equations. This leads to the following definition:

Definition A.1. — *Let $p \geqslant 2$. A* weak geometric Hölder p-rough path $\mathbf{X}$ *is a path from Δ_T to the $[p]$-step free nilpotent group $G^{[p]}$ which enjoys several propeties:*

(a) *Regularity:*

$$\|\mathbf{X}\| := \sup_{k\in\{1,\dots,[p]\}} \sup_{(t,s)\in\Delta_T} \frac{|X^k_{ts}|^{\frac{1}{k}}}{|t-s|^{\frac{1}{p}}} < +\infty.$$

(b) *Chen's relation: for all $0 \leqslant s \leqslant u \leqslant t \leqslant T$*

$$\mathbf{X}_{ts} = \mathbf{X}_{us}\mathbf{X}_{tu}.$$

Note that since $\mathbf{X} \in G^{[p]}$, there exists a unique $\mathbf{\Lambda} = (0, \Lambda^1, \dots, \Lambda^{[p]})$ such that $\exp(\mathbf{\Lambda}) = \mathbf{X}$, called the logarithm of $\mathbf{X}$. Since $\mathbf{\Lambda} = \log(\mathbf{X})$ and the logarithm in $T^{([p])}$ is polynomial, one has furthermore for all $(t,s) \in \Delta_T$,

$$|\Lambda^k_{t,s}| \leqslant \|\mathbf{X}\|^k |t-s|^{\frac{k}{p}}.$$

The fact that $\mathbf{X}$ lies in $G^{[p]}$ is the generalization of the chain rule, whereas Chen's relation is the generalization of the definition of iterated integrals. Finally, we complete this section by giving the following example of weakly geometric rough paths, proved first by Coutin and Qian in [13].

Theorem. — *Let $\frac{1}{2} \geqslant H > \frac{1}{4}$ and $\ell \geqslant 2$ and let $p > \frac{1}{H}$. Let B^H be a $\mathbb{R}^\ell$-fractional Brownian motion of Hurst parameter H. Then there exists a weak geometric Hölder p-rough path $\mathbf{B}^H \in G^{[p]}$ such that $B^{H,1}_{ts} = B^H_t - B^H_s$.*

Furthermore when $H = \frac{1}{2}$, $B = B^{\frac{1}{2}}$ is a standard ℓ-dimensional Brownian motion and $B^2_{ts} = \int_s^t (B_r - B_s) \otimes \circ \, \mathrm{d}B_r$, where the stochastic integral is in the Stratonovitch sense.

Bibliography

[1] I. Bailleul, "A flows-based approach to rough differential equations", https://perso.univ-rennes1.fr/ismael.bailleul/files/M2Course.pdf.

[2] ———, "Flows driven by Banach space-valued rough paths", in *Séminaire de Probabilités XLVI*, Lecture Notes in Mathematics, vol. 2123, Springer, 2014, p. 195-205.

[3] ———, "Flows driven by rough paths", *Rev. Mat. Iberoam.* **31** (2015), no. 3, p. 901-934.

[4] I. Bailleul, R. Catellier & F. Delarue, "Solving mean field rough differential equations", *Electron. J. Probab.* **25** (2020), article ID 21 (51 pages).

[5] F. Baudoin, *Diffusion processes and stochastic calculus*, EMS Textbooks in Mathematics, European Mathematical Society, 2014, xii+276 pages.

[6] Y. Boutaib, L. G. Gyurkó, T. J. Lyons & D. Yang, "Dimension-free Euler estimates of rough differential equations", *Rev. Roum. Math. Pures Appl.* **59** (2014), no. 1, p. 25-53.

[7] T. Cass, C. Litterer & T. J. Lyons, "Integrability and tail estimates for Gaussian rough differential equations", *Ann. Probab.* **41** (2013), no. 4, p. 3026-3050.

[8] T. Cass & M. Ogrodnik, "Tail estimates for Markovian rough paths", *Ann. Probab.* **45** (2017), no. 4, p. 2477-2504.

[9] T. Cass & M. P. Weidner, "Tree algebras over topological vector spaces in rough path theory", https://arxiv.org/abs/1604.07352v2, 2017.

[10] K.-T. Chen, "Integration of paths, geometric invariants and a generalized Baker–Hausdorff formula", *Ann. Math.* **65** (1957), p. 163-178.

[11] L. Coutin & A. Lejay, "Perturbed linear rough differential equations", *Ann. Math. Blaise Pascal* **21** (2014), no. 1, p. 103-150.

[12] ———, "Sensitivity of rough differential equations: an approach through the Omega lemma", *J. Differ. Equations* **264** (2018), no. 6, p. 3899-3917.

[13] L. Coutin & Z. Qian, "Stochastic analysis, rough path analysis and fractional Brownian motions", *Probab. Theory Relat. Fields* **122** (2002), no. 1, p. 108-140.

[14] A. M. Davie, "Differential equations driven by rough paths: an approach via discrete approximation", *AMRX, Appl. Math. Res. Express* **2** (2007), p. 9-40.

[15] A. Deya, M. Gubinelli, M. Hofmanová & S. Tindel, "A priori estimates for rough PDEs with application to rough conservation laws", https://arxiv.org/abs/1604.00437, 2016.

[16] B. Driver, "Global existence of geometric rough flows", https://arxiv.org/abs/1810.03708, 2018.

[17] ———, "On truncated logarithms of flows on a Riemannian manifold", https://arxiv.org/abs/1810.02414, 2018.

[18] P. K. Friz & M. Hairer, *A course on rough paths. With an introduction to regularity structures*, Universitext, Springer, 2014, xiv+251 pages.

[19] P. K. Friz & S. Riedel, "Integrability of (non-)linear rough differential equations and integrals", *Stochastic Anal. Appl.* **31** (2013), no. 2, p. 336-358.

[20] P. K. Friz & N. B. Victoir, *Multidimensional stochastic processes as rough paths*, Cambridge Studies in Advanced Mathematics, vol. 120, Cambridge University Press, 2010.

[21] A. Lejay, "On rough differential equations", *Electron. J. Probab.* **14** (2009), no. 12, p. 341-364.

[22] ———, "Global solutions to rough differential equations with unbounded vector fields", in *Séminaire de probabilités XLIV*, Lecture Notes in Mathematics, vol. 2046, Springer, 2012, p. 215-246.

[23] T. J. Lyons, "Differential equations driven by rough signals", *Rev. Mat. Iberoam.* **14** (1998), no. 2, p. 215-310.

[24] T. J. Lyons, M. Caruana & T. Lévy, *Differential equations driven by rough paths*, Lecture Notes in Mathematics, vol. 1908, Springer, 2007, Lectures from the 34th Summer School on Probability Theory held in Saint-Flour, July 6–24, 2004, With an introduction concerning the Summer School by Jean Picard, xviii+109 pages.

[25] T. J. Lyons & Z. Qian, *System control and rough paths*, Oxford Mathematical Monographs, Clarendon Press, 2002.

[26] L. C. Young, "An inequality of the Hölder type, connected with Stieltjes integration", *Acta Math.* **67** (1936), no. 1, p. 251-282.