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Semilinear wave equation on manifolds ^(*)

F. D. ARARUNA, G. O. ANTUNES AND L. A. MEDEIROS ⁽¹⁾

Dedicated to M. Milla Miranda in the occasion of his 60th. anniversary.

RÉSUMÉ. — Dans ce travail nous étudions un problème pour les équations des ondes non linéaire définies dans une variété. Ce problème a été motivé par J.L.Lions [8], p. 134. Pour l'existence de solutions nous avons appliqué la méthode de Galerkin. Le comportement asymptotique des solutions a été examiné aussi.

ABSTRACT. — In this paper, we study a type of second order evolution equation on the lateral boundary Σ of the cylinder $Q = \Omega \times]0, T[$, with Ω an open bounded set of \mathbb{R}^n . In this problem is fundamental that the unknown function solves an elliptic problem on Ω . This results are motivated by Lions [8], pg. 134 where he works with another type of nonlinearity.

1. Introduction

Let Ω be a bounded open set of \mathbb{R}^n ($n \geq 1$) with smooth boundary Γ . Let ν be the outward normal unit vector to Γ and $T > 0$ a real number. We consider the cylinder $Q = \Omega \times]0, T[$ with lateral boundary $\Sigma = \Gamma \times]0, T[$.

We investigate existence and asymptotic behaviour of weak solution for the problem

$$\begin{cases} w'' + \frac{\partial w}{\partial \nu} + F(w) + \beta(x) w' = 0 & \text{on } \Sigma, \\ w(0) = w_0, \quad w'(0) = w_1 & \text{on } \Gamma, \end{cases} \quad (1.1)$$

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where the prime means the derivative with respect to t , $\frac{\partial w}{\partial \nu}$ normal derivative of w and $F : \mathbb{R} \rightarrow \mathbb{R}$ is a function that satisfies

$$F \text{ continuous and } sF(s) \geq 0, \forall s \in \mathbb{R}. \quad (1.2)$$

It is important to call the attention to the reader that the idea employed in this work comes from Lions [8], pg. 134. The main point consists in adding to (1.1) an elliptic equation in Ω to reduce the problem to a canonical model of Mathematical Physics, but in this case on a manifold which is the lateral boundary Σ of the cylinder Q . A Similar type of problem, also motivated by Lions [8], can be seen in Cavalvanti and Domingos Cavalcanti [2].

The plan of this article is the following: In the section 2, we give notations, terminology and we treat the linear case associated to (1.1). In the section 3, we prove existence for weak solution when F satisfies the condition (1.2), approximating F by Lipschitz functions. In this Lipschitz case, we employ Picard's successive approximations and then we apply the Strauss' method [9]. Finally in the section 4, we obtain the asymptotic behaviour by the method of pertubation of energy as in Zuazua [10].

2. Notations, Assumptions and Results

Denote by $|\cdot|$, (\cdot, \cdot) and $\|\cdot\|$, $((\cdot, \cdot))$ the inner product and norm, respectively, of $L^2(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$.

For

$$G(s) = \int_0^s F(\sigma) d\sigma$$

we will denote a primitive of F .

We consider the following assumption on β in (1.1) :

$$\beta \in L^\infty(\Gamma) \text{ such that } \beta(x) \geq \beta_0 > 0, \text{ a.e. on } \Gamma. \quad (2.1)$$

As was said in the introduction, for $\lambda > 0$, let us consider the problem

$$\left\{ \begin{array}{l} -\Delta w + \lambda w = 0 \quad \text{in } Q, \\ w'' + \frac{\partial w}{\partial \nu} + F(w) + \beta(x) w' = 0 \quad \text{on } \Sigma, \\ w(0) = w_0, \quad w'(0) = w_1 \quad \text{on } \Gamma. \end{array} \right. \quad (2.2)$$

From elliptic theory, we know that for $\varphi \in H^{\frac{1}{2}}(\Gamma)$, the solution Φ of the boundary value problem

$$\left\{ \begin{array}{l} -\Delta \Phi + \lambda \Phi = 0 \quad \text{in } \Omega, \\ \Phi = \varphi \quad \text{on } \Gamma, \end{array} \right. \quad (2.3)$$

belongs to $H^1(\Omega, \Delta) = \{u \in H^1(\Omega); \Delta u \in L^2(\Omega)\}$. By the trace theorem, it follows that $\frac{\partial \Phi}{\partial \nu} \in H^{-\frac{1}{2}}(\Gamma)$.

Formally, we have by (2.3) that

$$0 = \int_{\Omega} \nabla \Phi \nabla \Psi dx + \lambda \int_{\Omega} \Phi \Psi dx - \int_{\Gamma} \frac{\partial \Phi}{\partial \nu} \Psi d\Gamma.$$

We take $\Psi \in H^1(\Omega, \Delta)$ and we define

$$a(\Phi, \Psi) = \int_{\Omega} \nabla \Phi \nabla \Psi dx + \lambda \int_{\Omega} \Phi \Psi dx \quad (2.4)$$

Thus, by (2.4)

$$a(\Phi, \Psi) = \langle \gamma_1 \Phi, \gamma_0 \Psi \rangle,$$

where γ_0 and γ_1 are the traces of order zero and one, respectively, and $\langle \cdot, \cdot \rangle$ represents the duality pairing between $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$.

We consider the scheme

$$\begin{array}{ccc} \varphi \in H^{\frac{1}{2}}(\Gamma) & \xrightarrow{\gamma_0^{-1}} & \Phi \in H^1(\Omega, \Delta) \\ & \searrow A & \swarrow \gamma_1 \\ & \frac{\partial \Phi}{\partial \nu} \in H^{-\frac{1}{2}}(\Gamma) & \end{array}$$

Thus

$$A = \gamma_1 \circ \gamma_0^{-1} : H^{\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma), \quad A\varphi = \frac{\partial \Phi}{\partial \nu}.$$

Therefore A is self-adjoint and $A \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma), H^{-\frac{1}{2}}(\Gamma))$.

Moreover, we have

$$\langle A\varphi, \varphi \rangle = a(\Phi, \Phi) \quad (2.5)$$

and so by (2.4) we get

$$\begin{aligned} \langle A\varphi, \varphi \rangle &= \int_{\Omega} |\nabla \Phi|^2 dx + \lambda \int_{\Omega} |\Phi|^2 dx \geq \min\{1, \lambda\} \|\Phi\|_{H^1(\Omega)}^2 \geq \\ &\geq \alpha \|\gamma_0 \Phi\|^2 = \alpha \|\varphi\|^2, \end{aligned}$$

proving that A is positive.

We formulate now the problem on Σ . For this, we define

$$w(t)|_{\Gamma} = u(t) \quad \text{and} \quad \frac{\partial w(t)}{\partial \nu}|_{\Gamma} = Au(t).$$

In this way, the problem (1.2) is reduced to find a function $u : \Sigma \rightarrow \mathbb{R}$ such that

$$\begin{cases} u'' + Au + F(u) + \beta(x)u' = 0 & \text{on } \Sigma, \\ u(0) = u_0, \quad u'(0) = u_1 & \text{on } \Gamma, \end{cases}$$

which will be investigated in the section 3.

Firstly we will state a result that guarantees the existence and uniqueness of solution for the linear problem associated the (1.1).

THEOREM 2.1. — *Given $(u_0, u_1, f) \in H^{\frac{1}{2}}(\Gamma) \times L^2(\Gamma) \times L^2(\Sigma)$, there exists a unique function $u : \Sigma \rightarrow \mathbb{R}$ such that*

$$u \in C^0(0, T; H^{\frac{1}{2}}(\Gamma)) \cap C^1(0, T; L^2(\Gamma)), \quad (2.6)$$

$$u'' + Au + \beta u' = f \quad \text{in } L^2(0, T; H^{-\frac{1}{2}}(\Gamma)), \quad (2.7)$$

$$u(0) = u_0, \quad u'(0) = u_1 \quad \text{on } \Gamma. \quad (2.8)$$

Moreover we have the energy inequality

$$\frac{1}{2} |u'(t)|^2 + \frac{\alpha}{2} \|u(t)\|^2 \leq \frac{1}{2} |u_1|^2 + \frac{\alpha}{2} \|u_0\|^2 + \int_0^T (f(s), u'(s)) ds, \quad \text{a.e in } [0, T]. \quad (2.9)$$

Proof. — In the proof of this linear case, we employ the Faedo-Galerkin's method. \square

3. Existence of Solution

The goal of this section is to obtain existence of solutions for the problem (1.1).

THEOREM 3.1. — *Consider F satisfying (1.2) and suppose*

$$(u_0, u_1) \in H^{\frac{1}{2}}(\Gamma) \times L^2(\Gamma) \quad \text{and} \quad G(u_0) \in L^1(\Gamma).$$

Then there exists a function $u : \Sigma \rightarrow \mathbb{R}$ such that

$$u \in L^\infty(0, T; H^{\frac{1}{2}}(\Gamma)), \quad (3.1)$$

$$u' \in L^\infty(0, T; L^2(\Gamma)), \quad (3.2)$$

$$u'' + Au + F(u) + \beta u' = 0 \quad \text{in } L^1\left(0, T; H^{-\frac{1}{2}}(\Gamma) + L^1(\Gamma)\right), \quad (3.3)$$

$$u(0) = u_0, \quad u'(0) = u_1 \quad \text{on } \Gamma. \quad (3.4)$$

To prove the Theorem 3.1, the following Lemma will be used:

LEMMA 3.1. — *Assume that $(u_0, u_1) \in H^{\frac{1}{2}}(\Gamma) \times L^2(\Gamma)$ and suppose that the function F satisfies*

$$F : \mathbb{R} \rightarrow \mathbb{R} \text{ be Lipschitz function such that } sF(s) \geq 0, \quad \forall s \in \mathbb{R}. \quad (3.5)$$

Then there exists only one function $u : \Sigma \rightarrow \mathbb{R}$ satisfying the conditions

$$u \in L^\infty\left(0, T; H^{\frac{1}{2}}(\Gamma)\right), \quad (3.6)$$

$$u' \in L^\infty(0, T; L^2(\Gamma)), \quad (3.7)$$

$$u'' + Au + F(u) + \beta u' = 0 \quad \text{in } L^2\left(0, T; H^{-\frac{1}{2}}(\Gamma)\right), \quad (3.8)$$

$$u(0) = u_0, \quad u'(0) = u_1 \quad \text{on } \Gamma. \quad (3.9)$$

Furthermore

$$\begin{aligned} & \frac{1}{2} |u'(t)|^2 + \frac{\alpha}{2} \|u(t)\|^2 + \int_{\Gamma} G(u(x, t)) d\Gamma \leq \frac{1}{2} |u_1|^2 + \\ & + \frac{\alpha}{2} \|u_0\|^2 + \int_{\Gamma} G(u_0(x)) d\Gamma, \quad \text{a.e in } [0, T]. \end{aligned} \quad (3.10)$$

Proof of Lemma 3.1. — The proof will be done employing the Picard successive approximations method. Let us consider the sequence of successive approximations

$$u_0, u_1, u_2, \dots, u_n, \dots \quad (3.11)$$

defined as the solutions of the linear problems

$$\begin{cases} u_n'' + Au_n + F(u_{n-1}) + \beta u_n' = 0 & \text{on } \Sigma, \\ u_n(0) = u_0, \quad u_n'(0) = u_1 & \text{on } \Gamma. \end{cases} \quad (3.12)$$

Using that F is Lipschitz and from Theorem 2.1, one can prove, using induction, that (3.12) has a solution for each $n \in \mathbb{N}$ with the regularity claimed in the Theorem 2.1. We will prove now that the sequence (3.11) converges to a function $u : \Sigma \rightarrow \mathbb{R}$ in the conditions of the Lemma 3.1.

For this end, we define $v_n = u_n - u_{n-1}$ which is the unique solution of the problem

$$\begin{cases} v_n'' + Av_n + F(u_{n-1}) - F(u_{n-2}) + \beta v_n' = 0 & \text{on } \Sigma, \\ v_n(0) = 0, \quad v_n'(0) = 0 & \text{on } \Gamma. \end{cases} \quad (3.13)$$

By the energy inequality (2.9), we have

$$\frac{1}{2} |v_n'(t)|^2 + \frac{\alpha}{2} \|v_n(t)\|^2 \leq - \int_0^t (F(u_{n-1}) - F(u_{n-2}), v_n'(s)) ds. \quad (3.14)$$

Set

$$e_n(t) = \text{ess sup}_{s \in]0, t[} \left\{ \frac{1}{2} |v_n'(s)|^2 + \frac{\alpha}{2} \|v_n(s)\|^2 \right\}. \quad (3.15)$$

Thus, since F is Lipschitz, we have

$$- \int_0^t (F(u_{n-1}) - F(u_{n-2}), v_n'(s)) ds \leq C \int_0^t |v_{n-1}(s)|^2 ds + \frac{1}{2} e_n(t). \quad (3.16)$$

We have also

$$|v_{n-1}(s)|^2 \leq C e_{n-1}(s). \quad (3.17)$$

Combining (3.14) – (3.17), we get

$$e_n(t) \leq C \int_0^t e_{n-1}(s) ds,$$

and, by iteration, we obtain, for $n = 1, 2, \dots$, that

$$e_n(t) \leq e_0 C_T \frac{(Ct)^n}{n!},$$

hence, we conclude that the series $\sum_{n=1}^{\infty} e_n(t)$ is uniformly convergent on $]0, T[$. By the definition of $e_n(t)$, see (3.15), it follows that the series $\sum_{n=1}^{\infty} (u_n' - u_{n-1}')$ and $\sum_{n=1}^{\infty} (u_n - u_{n-1})$ are convergents in the norms of $L^\infty(0, T; L^2(\Gamma))$ and $L^\infty(0, T; H^{\frac{1}{2}}(\Gamma))$, respectively. Therefore, there exists $u : \Sigma \rightarrow \mathbb{R}$ such that

$$u_n \rightarrow u \text{ strong in } L^\infty(0, T; H^{\frac{1}{2}}(\Gamma)), \quad (3.18)$$

$$u_n' \rightarrow u' \text{ strong in } L^\infty(0, T; L^2(\Gamma)). \quad (3.19)$$

Since F is Lipschitz, we have by (3.18) that

$$F(u_n) \rightarrow F(u) \text{ strong in } L^\infty(0, T; L^2(\Gamma)). \quad (3.20)$$

Then, by the convergences (3.18) – (3.20), we can pass to the limit in (3.12) and we obtain, by standard procedure, a unique function u satisfying (3.6) – (3.10). \square

We will prove now the main result.

Proof of Theorem 3.1. — By Strauss [9], there exists a sequence of functions $(F_\nu)_{\nu \in \mathbb{N}}$, such that each $F_\nu : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz and $(F_\nu)_{\nu \in \mathbb{N}}$ approximates F uniformly on bounded sets of \mathbb{R} . Since the initial data u_0 is not necessarily bounded, we have to approximate u_0 by bounded functions of $H^{\frac{1}{2}}(\Gamma)$. We consider the functions $\xi_j : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\xi_j(s) = \begin{cases} -j, & \text{if } s < -j, \\ s, & \text{if } |s| \leq j, \\ j, & \text{if } s > j. \end{cases}$$

Considering $\xi_j(u_0) = u_{0j}$, we have by Kinderlehrer and Stampacchia [5] that the sequence $(u_{0j})_{j \in \mathbb{N}} \subset H^{\frac{1}{2}}(\Gamma)$ is bounded a.e. in Γ and

$$u_{0j} \rightarrow u_0 \text{ strong in } H^{\frac{1}{2}}(\Gamma). \quad (3.21)$$

Thus, for $(u_{0j}, u_1) \in H^{\frac{1}{2}}(\Gamma) \times L^2(\Gamma)$, the Lemma 3.1 says that there exists only one solution $u_{j\nu} : \Sigma \rightarrow \mathbb{R}$ satisfying (3.6) – (3.9) and the energy inequality

$$\begin{aligned} & \frac{1}{2} |u'_{j\nu}(t)|^2 + \frac{\alpha}{2} \|u_{j\nu}(t)\|^2 + \int_{\Gamma} G_\nu(u_{j\nu}(x, t)) d\Gamma \leq \frac{1}{2} |u_1|^2 + \\ & + \frac{\alpha}{2} \|u_{0j}\|^2 + \int_{\Gamma} G_\nu(u_{0j}(x)) d\Gamma. \end{aligned} \quad (3.22)$$

We need an estimate for the term $\int_{\Gamma} G_\nu(u_{0j}(x)) d\Gamma$. Since u_{0j} is bounded a.e. in Γ , $\forall j \in \mathbb{N}$, it follows that

$$F_\nu(u_{0j}) \rightarrow F(u_{0j}) \text{ uniform in } \Gamma.$$

So

$$\int_{\Gamma} G_\nu(u_{0j}(x)) d\Gamma \rightarrow \int_{\Gamma} G(u_{0j}(x)) d\Gamma \text{ uniform in } \mathbb{R}. \quad (3.23)$$

From (3.21), there exists a subsequence of $(u_{0j})_{j \in \mathbb{N}}$, which will still be denoted by $(u_{0j})_{j \in \mathbb{N}}$, such that

$$u_{0j} \rightarrow u_0 \text{ a.e. in } \Gamma.$$

Hence, by continuity of G , we have that $G(u_{0j}) \rightarrow G(u_0)$ a.e. in Γ . We also have that $G(u_{0j}) \leq G(u_0) \in L^1(\Gamma)$. Thus, by the Lebesgue's dominated convergence theorem, we get

$$G(u_{0j}) \rightarrow G(u_0) \text{ strong in } L^1(\Gamma). \quad (3.24)$$

Then, by (3.23) and (3.24), we obtain that

$$\int_{\Gamma} G_{\nu}(u_{0j}(x)) d\Gamma \leq C, \quad (3.25)$$

where C is independent of j and ν . In this way, using (3.21) and (3.25), we have from (3.22) that

$$|u'_{j\nu}|^2 + \|u_{j\nu}\|^2 + \int_{\Gamma} G(u_{j\nu}(x, t)) d\Gamma \leq C, \quad (3.26)$$

where C is independent of j , ν and t .

From (3.26), we obtain that

$$(u_{j\nu}) \text{ is bounded in } L^{\infty}\left(0, T; H^{\frac{1}{2}}(\Gamma)\right), \quad (3.27)$$

$$(u'_{j\nu}) \text{ is bounded in } L^{\infty}\left(0, T; L^2(\Gamma)\right). \quad (3.28)$$

We have that (3.27) and (3.28) are true for all pairs $(j, \nu) \in \mathbb{N}^2$, in particular, for $(i, i) \in \mathbb{N}^2$. Thus, there exists a subsequence of (u_{ii}) , which we denote by (u_i) , and a function $u : \Sigma \rightarrow \mathbb{R}$, such that

$$u_i \rightarrow u \text{ weak star in } L^{\infty}\left(0, T; H^{\frac{1}{2}}(\Gamma)\right), \quad (3.29)$$

$$u'_i \rightarrow u' \text{ weak in } L^{\infty}\left(0, T; L^2(\Gamma)\right). \quad (3.30)$$

We also have by (3.8) that

$$u''_i + Au_i + F_i(u_i) + \beta u'_i = 0 \quad \text{in } L^2\left(0, T; H^{-\frac{1}{2}}(\Gamma)\right). \quad (3.31)$$

From (3.29), (3.30) and observing that the injection of $H^1(\Sigma)$ in $L^2(\Sigma)$ is compact, there exists a subsequence of (u_i) , which we still denote by (u_i) , such that

$$u_i \rightarrow u \text{ a.e. in } \Sigma.$$

Since F is continuous

$$F(u_i) \rightarrow F(u) \text{ a.e. in } \Sigma.$$

Furthermore, since $u_i(x, t)$ is bounded in \mathbb{R} ,

$$F_i(u_i) - F(u_i) \rightarrow 0 \text{ a.e. in } \Sigma.$$

Therefore, we conclude

$$F_i(u_i) \rightarrow F(u) \text{ a.e. in } \Sigma. \quad (3.32)$$

Taking duality between (3.31) and u_i we obtain

$$\begin{aligned} \int_0^T (F_i(u_i), u_i(t)) dt &= \int_0^T |u_i'(t)|^2 dt - \alpha \int_0^T \|u_i(t)\|^2 dt - \\ &- (u_i'(T), u_i(T)) + (u_{0j}) - \int_0^T (\beta u_i'(t), u_i(t)) dt. \end{aligned} \quad (3.33)$$

Using (2.1), (3.6) and (3.7), we have by (3.33) that

$$\int_0^T (F_i(u_i), u_i(t)) dt \leq C, \quad (3.34)$$

where C is independent of i .

Thus, from (3.32) and (3.34), it follows by Strauss' theorem, see Strauss [9], that

$$F_i(u_i) \rightarrow F(u) \text{ strongly in } L^1(\Sigma). \quad (3.35)$$

By (3.29), (3.30) and (3.35) it is permissible to pass to the limit in (3.31) obtaining a function $u : \Sigma \rightarrow \mathbb{R}$ satisfying (3.1) – (3.4). \square

4. Asymptotic Behaviour

In this section we study the exponential decay for the energy $E(t)$ associated to the weak solution u given by the Theorem 3.1. This energy is given by

$$E(t) = \frac{1}{2} |u'(t)|^2 + \frac{\alpha}{2} \|u(t)\|^2 + \int_{\Gamma} G(u(x, t)) d\Gamma, \quad t \geq 0. \quad (4.1)$$

We consider the followings additional hypothesis:

$$0 \leq G(s) \leq sF(s), \quad \forall s \in \mathbb{R} \quad (4.2)$$

THEOREM 4.1. — *Let F satisfying (1.2) and (4.2). Then the energy (4.1) satisfies*

$$E(t) \leq 4E(0)e^{-\frac{\epsilon}{2}t}, \quad (4.3)$$

where ϵ is a positive constant.

Proof. — For an arbitrary $\epsilon > 0$, we define the perturbed energy

$$E_{\nu\epsilon}(t) = E_{\nu}(t) + \epsilon\eta(t) \quad (4.4)$$

where $E_{\nu}(t)$ is the energy similar to (4.1) associated to the solution obtained in the Lemma 3.1 and

$$\eta(t) = (u_{\nu}(t), u'_{\nu}(t)).$$

Note that

$$|\eta(t)| \leq C_2 E_{\nu}(t),$$

where $C_2 = \max\left\{C_1, \frac{1}{\alpha}\right\}$, and C_1 is the immersion constant of $H^{\frac{1}{2}}(\Gamma)$ into $L^2(\Gamma)$.

Then,

$$|E_{\nu\epsilon}(t) - E_{\nu}(t)| \leq \epsilon C_2 E_{\nu}(t),$$

or

$$(1 - \epsilon C_2) E_{\nu}(t) \leq E_{\nu\epsilon}(t) \leq (1 + \epsilon C_2) E_{\nu}(t).$$

Taking $0 < \epsilon \leq \frac{1}{2C_2}$, we get

$$\frac{E_{\nu}(t)}{2} \leq E_{\nu\epsilon}(t) \leq 2E_{\nu}(t), \quad \forall t \geq 0. \quad (4.5)$$

Multiplying the equation in (3.8) for u'_{ν} , using (2.1) and the fact of A to be positive, we obtain

$$E'_{\nu}(t) \leq -\beta_0 |u'_{\nu}(t)|^2 \leq 0. \quad (4.6)$$

Differentiating the function $\eta(t)$ and using (3.8), (4.2) and the fact of A to be positive comes that

$$\eta'(t) \leq \left(1 + \frac{\beta_1}{2\mu}\right) |u'_{\nu}(t)|^2 + \left(\frac{\beta_1\mu C_1}{2} - \alpha\right) \|u_{\nu}(t)\|^2 - \int_{\Gamma} G_{\nu}(u_{\nu})d\Gamma, \quad (4.7)$$

where $\beta_1 = \|\beta\|_{L^{\infty}(\Gamma)}$ and $\mu > 0$ to be chosen.

It follows by (4.4), (4.6) and (4.7) that

$$E'_{\nu\epsilon}(t) \leq \left[\epsilon \left(1 + \frac{\beta_1}{2\mu} \right) - \beta_0 \right] |u'_\nu(t)|^2 - \epsilon \left(\alpha - \frac{\beta_1 \mu C_1}{2} \right) \|u_\nu(t)\|^2 - \epsilon \int_\Gamma G_\nu(u_\nu) d\Gamma. \quad (4.8)$$

Taking $\mu = \frac{\alpha}{\beta_1 C_1}$ and $0 < \epsilon \leq \frac{2\alpha\beta_0}{3\alpha + \beta_1^2 C_1}$ we get

$$E'_{\nu\epsilon}(t) \leq -E_\nu(t). \quad (4.9)$$

Choosing $\epsilon \leq \min \left\{ \frac{1}{2C_2}, \frac{2\alpha\beta_0}{3\alpha + \beta_1^2 C_1} \right\}$ then (4.5) and (4.9) occur simultaneously, therefore

$$E'_{\nu\epsilon}(t) + \frac{\epsilon}{2} E_{\nu\epsilon}(t) \leq 0,$$

that is,

$$E_\nu(t) \leq 4E_\nu(0)e^{-\frac{\epsilon}{2}t}. \quad (4.10)$$

From (3.29), (3.30) and since G_ν is continuous, we have

$$G_\nu(u_\nu(\cdot, t)) - G_\nu(u(\cdot, t)) \rightarrow 0 \text{ a.e. in } \Gamma, \forall t \geq 0. \quad (4.11)$$

But we know that $F_\nu \rightarrow F$ uniformly on bounded sets of \mathbb{R} . Then

$$G_\nu(u(\cdot, t)) \rightarrow G(u(\cdot, t)) \text{ a.e. in } \Gamma, \forall t \geq 0. \quad (4.12)$$

Thus, by (4.11) and (4.12)

$$G_\nu(u_\nu(\cdot, t)) \rightarrow G(u(\cdot, t)) \text{ a.e. in } \Gamma, \forall t \geq 0. \quad (4.13)$$

Moreover, we have, by (4.10), that

$$\int_\Gamma G_\nu(u_\nu(x, t)) d\Gamma \leq 4E_\nu(0).$$

Therefore, using (3.21), (3.23) and (3.24), we get

$$\liminf_{\nu \rightarrow \infty} \int_\Gamma G_\nu(u_\nu(x, t)) d\Gamma \leq 4E(0). \quad (4.14)$$

By (4.13), (4.14) and Fatou's lemma, we have

$$\int_{\Gamma} G(u(x, t)) d\Gamma \leq \liminf_{\nu \rightarrow \infty} \int_{\Omega} G_{\nu}(u_{\nu}(x, t)) d\Gamma.$$

Hence, passing \liminf in (4.10), we get (4.3). \square

Remark. — In the existence we can take $\lambda = 0$. For this end, we define in $H^1(\Omega)$ the norm

$$[v]^2 = \int_{\Omega} |\nabla v|^2 dx + \int_{\Gamma} |\gamma_0 v|^2 d\Gamma, \quad \forall v \in H^1(\Omega),$$

obtaining now the positivity of operator $A + \zeta I$, for $\zeta > 0$ arbitrary, like in Lions [8]. For the asymptotic behaviour, we need the additional hypothesis $\beta_0 > \zeta$.

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