ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

MURRAY MARSHALL

Representations of non-negative polynomials having finitely many zeros

Tome XV, nº 3 (2006), p. 599-609.

<http://afst.cedram.org/item?id=AFST_2006_6_15_3_599_0>

Annales de la Faculté des Sciences de Toulouse Vol. XV, n° 3, 2006
pp. 599–609

Representations of non-negative polynomials having finitely many zeros[(*)]

Murray Marshall [(1)]

Abstract. — Consider a compact subset K of real n-space defined by polynomial inequalities $g_1 \geqslant 0, \ldots, g_s \geqslant 0$. For a polynomial f non-negative on K, natural sufficient conditions are given (in terms of first and second derivatives at the zeros of f in K) for f to have a presentation of the form $f = t_0 + t_1 g_1 + \ldots + t_s g_s$, t_i a sum of squares of polynomials. The conditions are much less restrictive than the conditions given by Scheiderer in [11, Cor. 2.6]. The proof uses Scheiderer's main theorem in [11] as well as arguments from quadratic form theory and valuation theory. We also explain how the basic lemma of Kuhlmann, Marshall and Schwartz in [3] can be used to simplify the proof of Scheiderer's main theorem, and compare the two approaches.

Résumé. — Soit K une partie compacte de $\mathbf{R}^n$ définie par les inégalités polynomiales $g_1 \geqslant 0, \ldots, g_s \geqslant 0$. Pour un polynôme positif f sur K, des conditions suffisantes naturelles sont dégagées (en termes des dérivées premières et secondes en les zéros de f dans K) pour que f puisse se représenter sous la forme $f = t_0 + t_1 g_1 + \cdots + t_s g_s$, où les t_i sont des sommes de carrés de polynômes. Les conditions sont bien plus générales que celles mises en évidence par Scheiderer dans [11, Cor. 2.6]. La démonstration utilise le théorème principal de Scheiderer [11] ainsi que des arguments de la théorie des formes quadratiques et de celle de la valuation. L'article explique également comment le lemme fondamental de Kuhlmann, Marshall et Schwartz [3] peut être mis à profit pour simplifier le théorème principal de Scheiderer, et compare les deux approches.

Let K be a basic closed semialgebraic set in $\mathbb{R}^n$ defined by polynomial inequalities $g_1 \geqslant 0, \ldots, g_s \geqslant 0$, where $g_1, \ldots, g_s \in \mathbb{R}[x_1, \ldots, x_n]$, and let $f \in \mathbb{R}[x_1, \ldots, x_n]$. In [12], Schmüdgen proves that, if K is compact and f is strictly positive on K, then f belongs to the quadratic preordering

(*) Reçu le 15 novembre 2004, accepté le 25 janvier 2005
(1) Department of Computer Science, University of Saskatchewan, Saskatoon, SK Canada, S7N 5E6
E-mail: marshall@snoopy.usask.ca
This research was supported in part by NSERC of Canada.

generated by $g_1, \dots, g_s$. Denote by M the (smaller) quadratic module generated by $g_1, \dots, g_s$. Results of Putinar [8] and Jacobi [1] show that, if M is archimedean, then $f > 0$ on K implies $f \in M$. The question of exactly when M is archimedean is studied in detail in [2]. In [11], extending earlier results in the preordering case in [10], Scheiderer gives sufficient conditions for $f \geqslant 0$ on K to imply $f \in M$. In [11, Cor. 2.6], as an application of his methods, Scheiderer extends the Putinar-Jacobi result to include the case where $f \geqslant 0$ on K and, at each zero of f in K, the partial derivatives of f vanish and the hessian of f is positive definite.

The Putinar-Jacobi result serves as the theoretical underpinning for an optimization algorithm based on semidefinite programming due to Lasserre; see [4] or [5]. According to the Putinar-Jacobi result, if M is archimedean, the minimum value of any polynomial f on K is equal to $\sup\{c \in \mathbb{R} \mid f - c \in M\}$. This latter number can be approximated by Lasserre's algorithm. One is naturally interested in knowing when $f - c \in M$ holds when c is the exact minimum of f on K, e.g., see [4, Th. 2.1 and Remark 2.2]. Although [11, Cor. 2.6] sheds light on this question, its usefulness is limited by the unrealistic constraints on the boundary zeros.

In Section 1 we review basic terminology and results and, at the same time, we use the Basic Lemma in [3] to give a short proof of the main result in [11]. In Section 2 we prove that the constraints on the boundary zeros in [11, Cor. 2.6] can be replaced by constraints which are much less restrictive and much more natural; see Theorem 2.3. In the Appendix, we examine the Basic Lemma in [3], and we compare this result to Lemma 2.6 in [10], which is the key result in the approach taken by Scheiderer in [10] and [11].

1. The main result in [11]

Let A be a commutative ring with 1. For simplicity, assume $\frac{1}{2} \in A$. A *quadratic module* in A is a subset M of A satisfying $M+M \subseteq M$, $1 \in M$, and $a^2 M \subseteq M$ for all $a \in A$. We say M is *archimedean* if for each $a \in A$ there exists a natural number n such that $n + a \in M$. A *quadratic preordering* in A is a quadratic module in A which is also closed under multiplication. $\sum A^2$ denotes the set of sums of squares in A.

For any subset S of A, denote by $K = K_S$ the set of all ring homomorphisms $\alpha : A \to \mathbb{R}$ satisfying $\alpha(S) \geqslant 0$. For $a \in A$, define $\hat{a} : K \to \mathbb{R}$ by $\hat{a}(\alpha) := \alpha(a)$. Give K the weakest topology such that each $\hat{a}$, $a \in A$, is continuous. The map $a \mapsto \hat{a}$ defines a ring homomorphism from A to

$\mathrm{Cont}(K, \mathbb{R})$, the ring of all continuous functions from K to $\mathbb{R}$. For simplicity of notation, we usually suppress the 'hat' in the notation, denoting the function $\hat{a}$ simply by a, i.e., $a(\alpha) := \alpha(a)$.

According to the Basic Lemma in [3], if K is compact and $a, b \in A$ are such that $(a, b) = (1)$ and a and b are $\geqslant 0$ when viewed as functions on K, then there exist $c, d \in A$ such that c and d are > 0 on K and $1 = ca + db$. See the Appendix for further discussion of this result.

Denote by $M = M_S$ the quadratic module in A generated by S and by $T = T_S$ the quadratic preordering in A generated by S. Clearly $M \subseteq T$ and $K_S = K_M = K_T$. If M (or T) is archimedean, then K is compact. The converse is false in general. According to the Kadison-Dubois representation theorem, e.g., see [6], if T is archimedean, then for each $f \in A$, $f > 0$ on K $\Rightarrow f \in T$. According to Jacobi's representation theorem in [1] (also see [6] or [7]), if M is archimedean, then for each $f \in A$, $f > 0$ on $K \Rightarrow f \in M$.

If we only assume that $f \geqslant 0$ on K, then it is no longer true in general that $f \in M$ (or T). Combining the Basic Lemma in [3] with Jacobi's representation theorem, yields the following key result[1], which is due to Scheiderer [11].[2] The preordering version of this result is found already in [10].

Lemma 1.1. — *Suppose M is archimedean and $f \in A$ is such that $f \geqslant 0$ on K. Then $f \in M$ iff $f \in M + (f^2)$.*

Note: $M + (f^2) = M - A^2 f^2 = M - \sum A^2 f^2$. The inclusions $M - A^2 f^2 \subseteq M - \sum A^2 f^2 \subseteq M + (f^2)$ are clear. For the inclusion $M + (f^2) \subseteq M - A^2 f^2$, use the identify $a = (\frac{a+1}{2})^2 - (\frac{a-1}{2})^2$ to obtain $af^2 = (\frac{a+1}{2})^2 f^2 - (\frac{a-1}{2})^2 f^2$.

Proof. — One implication is clear. For the other, suppose $f \in M + (f^2)$. By the Note, $f = s - tf^2$, i.e., $f(1 + tf) = s$, for some $s \in M$, $t \in \sum A^2$. It is clear that $(f, 1 + tf) = (1)$ and also that $f, 1 + tf$ are $\geqslant 0$ on K. According to the Basic Lemma in [3], there exist $a, b \in A$ such that $1 = af + b(1 + tf)$ with $a, b > 0$ on K. By Jacobi's representation theorem,

---

(1) For clarity (although we never use this fact later) we note that Lemma 1.1 implies [11, Prop. 1.4]: Suppose $f \geqslant 0$ on K and $f = s + tb$ where $s \in M$, $tM \subseteq M$, and $b > 0$ on the zero set of f in K. By Jacobi's representation theorem, $b \in M + (f^2)$, so $tb \in M + (f^2)$. Since $s \in M$, this implies $f = s + tb \in M + (f^2)$, so $f \in M$ by Lemma 1.1.

(2) The proof given in [11] is based on [11, Prop. 1.4] which, in turn, is based on [10, Lemma 2.6].

$a, b, ab \in M$. Multiplying the equation $1 = af + b(1+tf)$ by bf, yields $bf = abf^2 + b^2(1+tf)f = abf^2 + b^2 s \in M$. Multiplying this same equation by f yields $f = af^2 + b(1+tf)f = af^2 + bf + btf^2 \in M$. □

To exploit Lemma 1.1 we also use another lemma[3], which we will be applying to the quadratic module $M + (f^2)$.

Lemma 1.2. — *Let M be a quadratic module, $J := M \cap -M$. Then*

(1) J is an ideal.

(2) For each minimal prime I over J, $(M+I) \cap -(M+I) = I$. Equivalently, for all $s_1, s_2 \in M$, $s_1 + s_2 \in I \Rightarrow s_1, s_2 \in I$.

(3) $(M + \sqrt{J}) \cap -(M + \sqrt{J}) = \sqrt{J}$.

Proof. — (1) This is well-known. See [7, Prop. 5.1.3].

(2) Let I be a minimal prime ideal over J. Suppose $s_1, s_2 \in M$ and $s_1 + s_2 \in I$. Since the maximal ideal of the localization of A/J at I/J is nilpotent, $u(s_1+s_2)^n \in J$ for some integer $n \geqslant 1$ and some $u \notin I$. Thus $u^2(s_1+s_2)^n \in J$. We can choose n to be odd. Note that $s_1^i s_2^{n-i} \in M$, e.g., if i is even, then $n-i$ is odd and $s_1^i \in A^2$, $s_2^{n-i} \in A^2 s_2$, so $s_1^i s_2^{n-i} \in A^2 s_2 \subseteq M$. Thus expanding and transposing terms yields $-u^2 s_1^n \in M$. Since we also have $u^2 s_1^n \in M$, this yields $u^2 s_1^n \in J$. Since $J \subseteq I$ and I is prime and $u \notin I$, this implies $s_1 \in I$. The proof that $s_2 \in I$ is the same.

(3) Since $\sqrt{J}$ is the intersection of the minimal primes lying over J, this is clear from (2). □

For each $\alpha \in K$, denote by $\hat{A}_\alpha$ the completion of A at the ideal $I_\alpha := \ker(\alpha)$ and by $\hat{I}_\alpha$ the extension of I_α to $\hat{A}_\alpha$.

Theorem 1.3 (Scheiderer, [11, Th. 1.11]). — *Suppose A is noetherian, M is archimedean, $f \geqslant 0$ on K and A/J' is has dimension $\leqslant 0$, where $J' := (M + (f)) \cap -(M + (f))$. Then the following are equivalent:*

(1) $f \in M$.

(2) $f \in M + (f^2)$.

[3] Lemma 1.2 (2) has other uses as well, e.g., it can also be used to derive the abstract Stellensätze for quadratic modules.

(3) For each zero α of f in K, f belongs to the quadratic module in $\hat{A}_\alpha$ generated by the image of M.

Proof. — (1) $\Rightarrow$ (3) is clear. (2) $\Rightarrow$ (1) is just Lemma 1.1. It remains to show (3) $\Rightarrow$ (2). Since $A/I_\alpha^k \cong \hat{A}_\alpha/\hat{I}_\alpha^k$, (3) implies that $f \in M + I_\alpha^k$ for each $k \geqslant 1$. Define $J := (M + (f^2)) \cap -(M + (f^2))$. Clearly $M + J = M + (f^2)$. Note that J and J' have the same nilradical. This follows from Lemma 1.2 (3), using the fact that $f^2 \in J$, so $f \in \sqrt{J}$. By the Chinese Remainder Theorem, A/J is the direct product of rings of the form $A/(I^k + J)$, where I a prime ideal of A containing J and $k \geqslant 1$ is sufficiently large. Thus $f \in M + J$ iff $f \in M + I^k + J$ for each such I and each k sufficiently large. It remains to show that $\{I_\alpha \mid \alpha \in K, f(\alpha) = 0\}$ is the complete set of prime ideals lying over J. Let I be any minimal (= maximal) prime ideal lying over J. Since M is archimedean, $M+I$ is also archimedean and $-1 \notin M+I$ by Lemma 1.2 (2), so there exists a ring homomorphism $\alpha : A \to \mathbb{R}$ with $\alpha(M + I) \geqslant 0$. Since I is maximal, $I = I_\alpha$. $\square$

Note: The preordering version of Theorem 1.3 is found already in [10].

2. Application to semialgebraic sets

We specialize to the case where A is the coordinate ring of an algebraic set V in $\mathbb{R}^n$, i.e., V is the set of common zeros of some finite set of polynomials in $\mathbb{R}[x_1, \dots, x_n]$, and A is the ring of all polynomial functions on V, equivalently, $A = \frac{\mathbb{R}[x_1,\dots,x_n]}{\mathcal{I}(\mathcal{V})}$ where $\mathcal{I}(\mathcal{V})$ denotes the ideal of all polynomials vanishing on V. For example, if $V = \mathbb{R}^n$, then $A = \mathbb{R}[x_1, \dots, x_n]$. Ring homomorphisms $\alpha : A \to \mathbb{R}$ are naturally identified with points of V. For a subset S of A, K_S is identified with the set of points $\{p \in V \mid \forall g \in S, g(p) \geqslant 0\}$.

By Schmüdgen's theorem [12] (also see [6] or [7]), if S is finite then K_S compact implies T_S is archimedean. In general, K_S compact does not imply M_S is archimedean; see [2]. According to Putinar's criterion [1] [8] (also see [6] or [7]), M is archimedean iff $N - \sum_{i=1}^n x_i^2 \in M$ for some number N. Thus, if K is compact, we can always 'force' M to be archimedean simply by adding $N - \sum_{i=1}^n x_i^2$ to the set S, for some large N.

Suppose now that V is irreducible, $\dim(V) = d$, and $p \in V$ is a non-singular point of V. Let $t_1, \dots, t_d \in A$ be a system of uniformizing parameters at p. The completion of A at p is $\mathbb{R}[[t_1, \dots, t_d]]$, the ring of formal power series. Each $f \in A$ decomposes as $f = f_0 + f_1 + f_2 + \dots$ where f_i is a form of degree i in the variables $t_1, \dots, t_d$. A necessary condition for f to have a local minimum at p is that $f_1 = 0$ and f_2 is PSD. A sufficent condition for

f to have a local minimum at p is that $f_1 = 0$ and f_2 is PD. This is just the second derivative test for local minima.

Lemma 2.1 [10, Example 3.18]. — *Suppose $f = f_1+f_2+\dots$ is an element of the power series ring $\mathbb{R}[[t_1,\dots,t_d]]$ such that $f_1 = 0$ and f_2 is PD. Then f is a sum of squares in $\mathbb{R}[[t_1,\dots,t_d]]$.*

Proof. — For any element c in the maximal ideal of $\mathbb{R}[[t_1,\dots,t_d]]$, $1+c$ is a unit and also a square. Making a linear change in variables, we can assume $f_2 = t_1^2+\dots+t_d^2$. Thus

$$f = t_1^2+\dots+t_d^2+a+bt_1+ct_1^2 = \frac{1}{2}t_1^2(1+2c)+\frac{1}{2}(t_1+b)^2+t_2^2+\dots+t_d^2+a-\frac{b^2}{2},$$

where a is a sum of terms of degree $\geqslant 3$ in $t_2,\dots,t_d$, b is a sum of terms of degree $\geqslant 2$ in $t_2,\dots,t_d$, and c is a sum of terms of degree $\geqslant 1$ in $t_1,\dots,t_d$. By induction on d, $t_2^2+\dots+t_d^2+a-\frac{b^2}{2}$ is a sum of squares in $\mathbb{R}[[t_2,\dots,t_d]]$. $\square$

Note: This is not the proof given in [10, Example 3.18].

We refer to the two conditions $f_1 = 0$, f_2 is PD, as the *hessian conditions* at p. We also want to consider boundary hessian conditions. Fix k, $0 \leqslant k \leqslant d$, and consider the region $\mathcal{R}$ in V defined by the inequalities $t_i \geqslant 0$, $i = 1,\dots,k$, i.e., $\mathcal{R} = K_{\{t_1,\dots,t_k\}}$. Suppose $f \in A$, $f = f_0+f_1+f_2+\dots$. A necessary condition for $f|_{\mathcal{R}}$ to have a local minimum at p is that $f_1 = a_1t_1+\dots+a_kt_k$ with $a_i \geqslant 0$, $i = 1,\dots,k$, and the quadratic form $f_2(0,\dots,0,t_{k+1},\dots,t_d)$ is PSD. A sufficient condition for $f|_{\mathcal{R}}$ to have a local minimum at p is that $f_1 = a_1t_1+\dots+a_kt_k$ with $a_i > 0$, $i = 1,\dots,k$, and the quadratic form $f_2(0,\dots,0,t_{k+1},\dots,t_d)$ is PD. These facts are well-known. In any case, they are easy to check. We refer to these (last) conditions as the *boundary hessian conditions* with respect to $t_1,\dots,t_k$ at p. If $k = 0$ these are precisely the hessian conditions mentioned earlier.

Lemma 2.2. — *Suppose $0 \leqslant k \leqslant d$ and that $f = f_1+f_2+\dots \in \mathbb{R}[[t_1,\dots,t_d]]$ satisfies the boundary hessian conditions with respect to $t_1,\dots,t_k$. Then f lies in the quadratic module in $\mathbb{R}[[t_1,\dots,t_d]]$ generated by $t_1,\dots,t_k$.*

Proof. — Write $f = \sum_{i=1}^k a_it_i+h+\sum_{i=1}^k t_ih_i = h+\sum_{i=1}^k t_i(a_i+h_i)$ where the a_i are positive reals, $h = f_2(0,\dots,0,t_{k+1},\dots,t_d)+$ terms of degree $\geqslant 3$

in $t_{k+1}, \dots, t_d$, and each h_i is in the maximal ideal. Since h is a sum of squares by Lemma 2.1, and each $a_i + h_i$ is a square, the conclusion is clear. $\square$

We come to the main result of this paper.

Theorem 2.3. — *Suppose V is an irreducible algebraic set, M is archimedean and $f \geqslant 0$ on K. Suppose, for each zero p of f in K, p is a non-singular point of V, and there exist $g^{(1)}, \dots, g^{(k)} \in M$, $0 \leqslant k \leqslant d$, where $d := \dim(V)$, such that:*

(1) $g^{(1)}, \dots, g^{(k)}$ are part of a system of uniformizing parameters at p, and

(2) f satisfies the boundary hessian conditions with respect to $g^{(1)}, \dots, g^{(k)}$ at p.

Then $f \in M$.

Note: There is no assumption here that the quadratic module M is finitely generated.

Theorem 2.3 extends [11, Cor. 2.6] substantially in that Theorem 2.3 allows for a variety of commonly occuring sorts of boundary minima, whereas [11, Cor. 2.6] does not. Of course, Theorem 2.3 and [11, Cor. 2.6] both extend the Putinar-Jacobi result.

Proof. — In view of Theorem 1.3 and Lemma 2.2, it suffices to show that the ring A/J has dimension $\leqslant 0$, where $J := (M+(f)) \cap -(M+(f))$. Let I be a minimal prime over J and let L be the field of fractions of A/I. By Lemma 1.2, $(M+I) \cap -(M+I) = I$, so the image of M generates a proper quadratic module in L. By Zorn's lemma,, we have a semiordering Q of L containing the image of M. Let B be the associated valuation ring of L (see [6, Prop. 3.3.6] or [7, Prop. 5.3.2]). Since M is archimedean, $A/I \subseteq B$. The mapping from B to its residue field $\mathbb{R}$ defines a point p of K with $f(p) = 0$. By our hypothesis, p is a non-singular point of V, and we have some $0 \leqslant k \leqslant d$, and $g^{(1)}, \dots, g^{(k)} \in M$ satisfying conditions (1) and (2) in the statement of the theorem. Fix local generators $h^{(d+1)}, \dots, h^{(n)}$ at p for the ideal $\mathcal{I}(V)$. Changing coordinates, we may assume that $p = 0$, $h^{(i)} = x_i + h_2^{(i)} + h_3^{(i)} + \dots$, where $h_j^{(i)}$ is a form of degree j in $x_1, \dots, x_n$, $j \geqslant 2$, $i = d+1, \dots, n$, $x_1, \dots, x_d$ are uniformizing parameters at 0, $f = a_1 x_1 + \dots + a_k x_k + f_2 + \dots$ with $a_i > 0$, $i = 1, \dots, k$, and $g^{(i)} = x_i + g_2^{(i)} + \dots$, $i = 1, \dots, k$, and the quadratic form

$$h := f_2(0,\dots,0,x_{k+1},\dots,x_d) - \sum_{i=1}^{k} a_i g_2^{(i)}(0,\dots,0,x_{k+1},\dots,x_d) \qquad (2.1)$$

is PD. We are tacitly assuming here that f and the $g^{(i)}$ have been suitably modified by polynomials in the ideal $(h^{(d+1)},\dots,h^{(n)})$ so that the quadratic forms f_2, $g_2^{(i)}$ involve only the variables $x_1,\dots,x_d$. We want to show $x_1,\dots,x_n$ all belong to I, i.e., are zero in L. Suppose this is not the case. We compute in the field L. As in the proof of Lemma 2.2, we have an equation

$$0 = f = h + \sum_{i=1}^{k} g^{(i)}(a_i + h_i) + r \qquad (2.2)$$

where r is a sum of terms of degree at least 3 in $x_1,\dots,x_n$, h is the quadratic form defined by equation (2.1), and each h_i is a linear form in $x_1,\dots,x_d$. We compare values. Since h is PD, it can be written as $h = u_{k+1}^2 + \dots + u_d^2$ where each u_i is a linear combination of $x_{k+1},\dots,x_d$, with the associated matrix non-singular, so, if not all of the $x_{k+1},\dots,x_d$ are zero in L, then not all the $u_{k+1},\dots,u_d$ are zero in L. Also,

$$v(h) = \min\{2v(u_i) \mid i = k+1,\dots,d\} = \min\{2v(x_i) \mid i = k+1,\dots,d\}. \qquad (2.3)$$

Here, v denotes the valuation on L associated to the valuation ring B. Consider an index i such that $v(x_i) \leqslant v(x_j)$ for all $j = 1,\dots,n$. Then $v(h) \geqslant 2v(x_i)$, $v(h_j g^{(j)}) \geqslant 2v(x_i)$, and $v(r) \geqslant 3v(x_i)$. For $j > d$, $0 = h^{(j)} = x_j + h_2^{(j)} + \dots$, so $v(x_j) \geqslant 2v(x_i) > v(x_i)$. It follows that $i \leqslant d$.

Case 1. $v(x_j) > v(x_i)$ for all $j \in \{1,\dots,k\}$. Then $i \in \{k+1,\dots,d\}$, the $h_j g^{(j)}$ and r have value strictly greater than $2v(x_i) = v(h)$, and we obtain $0 = h(1 + \frac{\sum_{j=1}^{k} h_j g^{(j)} + r}{h}) + \sum_{j=1}^{k} a_j g^{(j)}$. Since each term in this sum is in Q, this forces each of the terms to be zero. This contradicts $x_i \neq 0$ in L.

Case 2. $i \in \{1,\dots,k\}$, say $i = 1$. Then $v(a_1 g^{(1)} + \dots + a_k g^{(k)}) > v(g^{(1)}) = v(x_1)$. If $v(g^{(1)}) = 2v(t)$ for some $t \in L^*$ then, dividing $a_1 g^{(1)} + \dots + a_k g^{(k)}$ by t^2 and passing to the residue field $\mathbb{R}$ yields a contradiction. Thus we can assume $v(g^{(1)}) \notin 2v(L^*)$. Let w be the coarsest valuation on L coarser than v subject to the condition $w(g^{(1)}) \notin 2w(L^*)$. Then $w(x_j) \geqslant w(x_1)$ for all j and $w(x_1) = w(g^{(1)}) \neq 0$, so $w(x_1) > 0$. Working with the binary form $\langle 1, g^{(1)} \rangle$, as in the proof of the Bröcker-Prestel criterion for weak isotropy (e.g., see [6, Claim 3, page 56]), we have $g^{(1)}(1+z) \in Q$ for all $z \in L$ with $w(z) > 0$. Rewriting equation (2) in the form $0 = f = a_1 g^{(1)}(1 + \frac{h + r + \sum_{j=1}^{k} h_j g^{(j)}}{a_1 g^{(1)}}) +$

$\sum_{j=2}^k a_j g^{(j)}$, and using the fact that $w(h + r + \sum_{j=1}^k h_j g^{(j)}) \geqslant 2w(x_1) > w(x_1) = w(g^{(1)})$, this implies that each term in this sum belongs to Q, so each term in this sum is zero. This contradicts $x_1 \neq 0$ in L. □

Note: Suppose V is irreducible, p is a non-singular point of V, $t_1, \dots, t_d \in A$ is a system of uniformizing parameters at p, $0 \leqslant k \leqslant d$ and $\mathcal{R} = K_{\{t_1,\dots,t_k\}}$. If $f \in A$ is arbitrary such that $f|_{\mathcal{R}}$ has a local minimum at p, then it seems clear that, with probability 1, the boundary hessian conditions at p will be satisfied. Thus one might expect the hypotheses of Theorem 2.3 to hold rather frequently. As was mentioned earlier, this would seem to have implications for polynomial optimization [4] [5].

At the same time, it is not clear how one can weaken the hypotheses of Theorem 2.3 in any substantial way, at least, for $d \geqslant 3$. The following example seems to be well-known. See [9, Prop. 6.1] for additional examples.

Example 2.4. — Let f be a form of degree 6 in n variables $x_1 \dots, x_n$, $n \geqslant 3$ which is strictly positive on the unit sphere in $\mathbb{R}^n$ but is not a sum of squares of polynomials. Such an f exists by results of Hilbert, e.g., take $f = \epsilon(x_1^2 + \dots + x_n^2)^3 + g(x_1, x_2, x_3)$ where g is the (homogenized) Motzkin polynomial and $\epsilon > 0$ is sufficiently close to zero. Then f is also not a sum of squares in $\mathbb{R}[[x_1 \dots, x_n]]$. Consequently, even though f is strictly positive at every point of $\mathbb{R}^n$ different from the origin, f is not in the preordering in $\mathbb{R}[x_1, \dots, x_n]$ generated by $\delta - (x_1^2 + \dots + x_n^2)$, for any real $\delta > 0$. We remark that degree 4 examples of this sort also exist if $n \geqslant 4$ (but not if $n = 3$).

3. Appendix. The Basic Lemma in [3] and Lemma 2.6 in [10]

The Basic Lemma in [3] reads as follows:

LEMMA 3.1. — *Let X be a compact Hausdorff space, A a commutative ring with 1 with $\frac{1}{n} \in A$ for some integer $n \geqslant 2$ and $\phi : A \to Cont(X, \mathbb{R})$ a ring homomorphism. Suppose $f, g \in A$ are such that $\phi(f) \geqslant 0$, $\phi(g) \geqslant 0$ and $(f, g) = (1)$. Then there exist $s, t \in A$ such that $sf + tg = 1$ and $\phi(s), \phi(t)$ are strictly positive.*

It would seem from the proof of Lemma 1.1 (also see various proofs in [3, Section 2]) that this lemma provides an adequate substitute for Lemma 2.6 in [10].

We record here the following natural extension of Lemma 3.1. It was pointed out by Scheiderer [private communication] that this certainly must be true – and indeed it is – although the proof is not completely obvious.

Lemma 3.2. — *Let X be a compact Hausdorff space, A a commutative ring with 1 with $\frac{1}{n} \in A$ for some integer $n \geqslant 2$ and $\phi : A \to Cont(X, \mathbb{R})$ a ring homomorphism. Suppose $f_1, \ldots, f_k \in A$ are such that $\phi(f_i) \geqslant 0$, $i = 1, \ldots, n$ and $(f_1, \ldots, f_k) = (1)$. Then there exist $s_1, \ldots, s_k \in A$ such that $s_1 f_1 + \ldots + s_k f_k = 1$ and such that each $\phi(s_i)$ is strictly positive.*

Proof. — We suppress ϕ from the notation. We may assume $n \geqslant 3$. Suppose $s_1 f_1 + \ldots + s_k f_k = 1$ with $s_1, \ldots, s_k \in A$. Then $(f_i, (\sum_{j \neq i} s_j f_j)^2) = (1)$ so, by Lemma 3.1, there exist $s, t \in A$, $s, t > 0$ such that $s f_i + t(\sum_{j \neq i} s_j f_j)^2 = 1$. Expanding this yields a new presentation $t_1 f_1 + \ldots + t_k f_k = 1$ with $t_i = s$, $t_j = (\sum_{\ell < i} s_\ell f_\ell) s_j t$ for $j < i$. (One could also write down explicit formulas for the t_j for $j > i$, but we do not need these.) Thus $t_i > 0$ but also, if we know inductively that $s_j \geqslant 0$ for $j < i$, then we also have $t_j \geqslant 0$ for $j < i$. It follows by induction that there exists a presentation $s_1 f_1 + \ldots + \ldots + s_k f_k = 1$ with $s_i \geqslant 0$ for $i < n$ and $s_n > 0$. By symmetry, for each i, we have a presentation $s_{i1} f_1 + \ldots + s_{ik} f_k = 1$ with $s_{ii} > 0$ and $s_{ij} \geqslant 0$ for $j \neq i$. Choosing suitable positive integers m_j so that $\sum_{i=1}^k m_i = n^p$ for some $p \geqslant 1$, this yields $s_1 f_1 + \ldots + s_k f_k = n^p$ where $s_j := \sum_{i=1}^k m_i s_{ij}$. Dividing by n^p yields the desired presentation. □

Note: If we define $e_i = s_i f_i$, $i = 1, \ldots, k$, then $1 = e_1 + \ldots + e_k$, each e_i is $\geqslant 0$, and the zero set of e_i is equal to the zero set of f_i, i.e., Lemma 3.2 constructs a partition of unity on X corresponding to open cover $X \backslash Z(f_i))$, $i = 1, \ldots, k$ of X. Thus Lemma 3.2 can be viewed as some kind of existence theorem for partitions of unity.

One would like to better understand the relationship between our approach in Section 1 and the approach in [10] and [11]. The proof of [11, Prop. 1.4] given in [11] is based on [10, Lemma 2.6], which in turn (at least in the case where X is compact) is a special case of the following general result:

Lemma 3.3. — *Let X be a compact Hausdorff space, A a commutative ring with 1 with $\frac{1}{n} \in A$ for some integer $n \geqslant 2$ and $\phi : A \to Cont(X, \mathbb{R})$ a ring homomorphism. Suppose $F = F_0 + F_1 + F_2$ is a degree two polynomial in d variables with coefficients in A, with F_i homogeneous of degree i, $i = 0, 1, 2$. Suppose, for each point $p \in X$, the quadratic form F_2 is PSD at p, and there exists $y \in \mathbb{R}^d$ such that $F(y) < 0$ holds at p. Then there exist $a \in A^d$ such that $F(a) < 0$ holds at every point of X.*

Proof. — Replacing X by the obvious quotient space, we are reduced to the case where $\phi(A)$ separates points in X. For each point $p \in X$, we have a d-tuple of continuous functions $\psi = (\psi_1, \dots, \psi_d)$ such that $F(\psi) < 0$ holds near p, e.g., we can take the ψ_i to be the constant function y_i. Use the compactness of X to construct a partition of unity $1 = \sum_{j=1^m} \tau_j$, τ_i continuous, $\tau_j \geqslant 0$, $j = 1, \dots, m$ and d-tuples of continuous functions $\psi_1, \dots, \psi_m$ such that $F(\psi_j) < 0$ holds on the set defined by the inequality $\tau_j > 0$. Define $\psi = \sum_{j=1}^m \tau_j \psi_j$. Expanding $F(\psi)$ yields $F(\psi) = \sum_i \tau_i F(\psi_i) - \sum_{i<j} \tau_i \tau_j F_2(\psi_i - \psi_j)$. Since $F(\psi_i) < 0$ holds on the set $\tau_i > 0$, the first term is $< \sum_{i=1}^m \tau_i 0 = 0$. Since F_2 is PSD, each of the terms $\tau_i \tau_j F_2(\psi_i - \psi_j)$ is non-negative. This proves $F(\psi) < 0$. Now approximate ψ using the Stone-Weierstrass theorem to get the required $a \in A^d$. $\square$

Thus, on the one hand, Lemma 3.3 can be viewed to be a consequence of the existence of continuous partitions of unity and the Stone-Weierstrass theorem. On the other hand, Lemma 3.2 can be viewed to be some sort of extension of the theorem asserting the existence of continuous partitions of unity. The existence of continuous partitions of unity and the Stone-Weierstrass theorem play no role in the proof of Lemma 3.2.

Bibliography

[1] Jacobi (T.). — A representation theorem for certain partially ordered commutative rings, Math. Zeit. 237, p. 223–235 (2001).

[2] Jacobi (T.), Prestel (A.). — Distinguished presentations of strictly positive polynomials, J. reine angew. Math. 532, p. 223–235 (2001).

[3] Kuhlmann (S.), Marshall (M.), Schwartz (N.). — Positivity, sums of squares and the multi-dimensional moment problem II, Advances in Geometry, to appear.

[4] Lasserre (J.B.). — Optimization globale et théorie des moments, C. R. Acad. Sci. Paris, 331, Série I, p. 929–934 (2000).

[5] Marshall (M.). — Optimization of polynomial functions, Canad. Math. Bull. 46, p. 575–587 (2003).

[6] Marshall (M.). — Positive polynomials and sums of squares, Dottorato de Ricerca in Matematica, Univ. Pisa, (2000).

[7] Prestel (A.), Delzell (C.). — Positive Polynomials: From Hilbert's 17th problem to real algebra, Springer Monographs in Mathematics, (2001).

[8] Putinar (M.). — Positive polynomials on compact semi-algebraic sets, Indiana Univ. Math. J. 42, p. 969–984 (1993).

[9] Scheiderer (C.). — Sums of squares of regular functions on real algebraic varieties, Trans. Amer. Math. Soc. 352, p. 1039–1069 (1999).

[10] Scheiderer (C.). — Sums of squares on real algebraic curves, Math. Zeit. 245, p. 725–760 (2003).

[11] Scheiderer (C.). — Distinguished representations of non-negative polynomials, to appear.

[12] Schmüdgen (K.). — The K-moment problem for compact semi-algebraic sets, Math. Ann. 289, p. 203–206 (1991).