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Weak analytic hyperbolicity of generic hypersurfaces of high degree in $\mathbb{P}^4$ (*)

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Abstract. — In this article we prove that every entire curve in a generic hypersurface of degree $d \geqslant 593$ in $\mathbb{P}^4_{\mathbb{C}}$ is algebraically degenerated i.e there exists a proper subvariety which contains the entire curve.

Résumé. — Dans cet article nous démontrons que toute courbe entière dans une hypersurface générique de degré $d \geqslant 593$ dans $\mathbb{P}^4_{\mathbb{C}}$ est algébriquement dégénérée i.e il existe une sous-variété propre qui contient la courbe entière.

1. Introduction

In 1970, S. Kobayashi conjectured in [8] that a generic hypersurface X in $\mathbb{P}^n_{\mathbb{C}}$ is hyperbolic provided that $d = \deg(X) \geqslant 2n-1$, for $n \geqslant 3$. For $n = 3$, it was obtained by Demailly and El Goul in [4] that $d \geqslant 21$ implies the hyperbolicity of very generic hypersurfaces X in $\mathbb{P}^3_{\mathbb{C}}$.

In this paper, we would like to prove the following:

Theorem 1.1. — *Let $X \subset \mathbb{P}^4_{\mathbb{C}}$ be a generic hypersurface such that $d = deg(X) \geqslant 593$. Then every entire curve $f : \mathbb{C} \to X$ is algebraically degenerated, i.e there exists a proper subvariety $Y \subset X$ such that $f(\mathbb{C}) \subset Y$.*

This result is a weaker version of the conjecture in dimension 3, because to obtain the full conjecture one needs to prove that the entire curve is constant.

The proof of the theorem is based on two techniques. Consider $\mathcal{X} \subset \mathbb{P}^4 \times \mathbb{P}^{N_d}$ the universal hypersurface of degree d in $\mathbb{P}^4$.

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In the first section we construct meromorphic vector fields on the space $J_3^v(\mathcal{X})$ of vertical 3-jets of $\mathcal{X}$. This technique, initiated by Clemens [2], Ein [5], Voisin [14], was generalized by Y.-T. Siu [13] and detailed by M. Paun in dimension 2 [10]. Here we generalize M. Paun's computations in dimension 3 and obtain that a pole order equal to 12 is enough to obtain a "large" space of global sections of the twisted tangent bundle.

In the second section, we summarize the main facts about the bundle of jet differentials of order k and degree m, $E_{k,m}T_X^*$. The idea, in hyperbolicity questions, is that global sections of this bundle vanishing on ample divisors provide algebraic differential equations for any entire curve $f : \mathbb{C} \to X$. Therefore, the main point is to produce enough algebraically independent global holomorphic jet differentials. In the case of surfaces in $\mathbb{P}^3$ of degree $d \geqslant 15$, one can produce global jet differentials of order 2 vanishing on ample divisors using a Riemann-Roch computation and a Bogomolov vanishing theorem (see [3]). Y.-T. Siu, in [13], described a way to produce global jet differentials vanishing on ample divisors for hypersurfaces of sufficiently large degree d in $\mathbb{P}^n$, for any n. One problem is that the bound obtained for d is quite high. If we are interested in the degree d for smooth hypersurfaces of $\mathbb{P}^4$, an interesting result obtained in [12], is the existence of global jet differentials of order 3 vanishing on ample divisors for $d \geqslant 97$.

In the last section we complete the proof of the theorem using Siu's approach [13], by taking the derivative of the jet differential in the direction of the vector fields constructed in the first part, and an observation made by Mihai Paun in [10] to avoid the use of McQuillan results (cf. [9]).

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2. Vector fields

In this first section, we generalize to dimension 3 the approach of Mihai Paun [10] which gives some precisions to Siu's ideas [13] in dimension 2. Consider $\mathcal{X} \subset \mathbb{P}^4 \times \mathbb{P}^{N_d}$ the universal hypersurface given by the equation

$$\sum_{|\alpha|=d} a_\alpha Z^\alpha = 0, \text{where}[a] \in \mathbb{P}^{N_d}\text{and}[Z] \in \mathbb{P}^4.$$

We use the notations: for $\alpha = (\alpha_0, ..., \alpha_4) \in \mathbb{N}^5$, $|\alpha| = \sum_i \alpha_i$ and if $Z = (Z_0, Z_1, ..., Z_4)$ are homogeneous coordinates on $\mathbb{P}^4$, then $Z^\alpha = \prod Z_j^{\alpha_j}$. $\mathcal{X}$ is a smooth hypersurface of degree $(d, 1)$ in $\mathbb{P}^4 \times \mathbb{P}^{N_d}$. We denote by

$J_3(\mathcal{X})$ the manifold of the 3-jets in $\mathcal{X}$, and $J_3^v(\mathcal{X})$ the submanifold of $J_3(\mathcal{X})$ consisting of 3-jets in $\mathcal{X}$ tangent to the fibers of the projection $\pi : \mathcal{X} \to \mathbb{P}^{N_d}$.

Let us consider the set $\Omega_0 := (Z_0 \neq 0) \times (a_{0d000} \neq 0) \subset \mathbb{P}^4 \times \mathbb{P}^{N_d}$. We assume that global coordinates are given on $\mathbb{C}^4$ and $\mathbb{C}^{N_d}$. The equation of $\mathcal{X}$ becomes

$$\mathcal{X}_0 := (z_1^d + \sum_{|\alpha|\leqslant d, \alpha_1<d} a_\alpha z^\alpha = 0)$$

Then the equations of $J_3^v(\mathcal{X}_0)$ in $\mathbb{C}^4 \times \mathbb{C}^{N_d} \times \mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4$ can be written:

$$\sum_{|\alpha|\leqslant d, a_{d000}=1} a_\alpha z^\alpha = 0 \tag{2.1}$$

$$\sum_{j=1}^{4} \sum_{|\alpha|\leqslant d, a_{d000}=1} a_\alpha \frac{\partial z^\alpha}{\partial z_j} \xi_j^{(1)} = 0 \tag{2.2}$$

$$\sum_{j=1}^{4} \sum_{|\alpha|\leqslant d, a_{d000}=1} a_\alpha \frac{\partial z^\alpha}{\partial z_j} \xi_j^{(2)} + \sum_{j,k=1}^{4} \sum_{|\alpha|\leqslant d, a_{d000}=1} a_\alpha \frac{\partial^2 z^\alpha}{\partial z_j \partial z_k} \xi_j^{(1)} \xi_k^{(1)} = 0 \tag{2.3}$$

$$\begin{aligned} \sum_{j=1}^{4} \sum_{|\alpha|\leqslant d, a_{d000}=1} & a_\alpha \frac{\partial z^\alpha}{\partial z_j} \xi_j^{(3)} + \sum_{j,k=1}^{4} 3 \sum_{|\alpha|\leqslant d, a_{d000}=1} a_\alpha \frac{\partial^2 z^\alpha}{\partial z_j \partial z_k} \xi_j^{(2)} \xi_k^{(1)} \\ & + \sum_{j,k,l=1}^{4} \sum_{|\alpha|\leqslant d, a_{d000}=1} a_\alpha \frac{\partial^3 z^\alpha}{\partial z_j \partial z_k \partial z_l} \xi_j^{(1)} \xi_k^{(1)} \xi_l^{(1)} = 0 \end{aligned} \tag{2.4}$$

Consider now a vector field

$$V = \sum_{|\alpha|\leqslant d, \alpha_1<d} v_\alpha \frac{\partial}{\partial a_\alpha} + \sum_j v_j \frac{\partial}{\partial z_j} + \sum_{j,k} w_{j,k} \frac{\partial}{\partial \xi_j^{(k)}}$$

on the vector space $\mathbb{C}^4 \times \mathbb{C}^{N_d} \times \mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4$. The conditions to be satisfied by V to be tangent to $J_3^v(\mathcal{X}_0)$ are

$$\sum_{|\alpha|\leqslant d, \alpha_1<d} v_\alpha z^\alpha + \sum_{j=1}^{4} \sum_{|\alpha|\leqslant d, a_{d000}=1} a_\alpha \frac{\partial z^\alpha}{\partial z_j} v_j = 0$$

$$\begin{aligned} \sum_{j=1}^{4} \sum_{|\alpha|\leqslant d, \alpha_1<d} v_\alpha \frac{\partial z^\alpha}{\partial z_j} \xi_j^{(1)} & + \sum_{j,k=1}^{4} \sum_{|\alpha|\leqslant d, a_{d000}=1} a_\alpha \frac{\partial^2 z^\alpha}{\partial z_j \partial z_k} v_j \xi_k^{(1)} \\ & + \sum_{j=1}^{4} \sum_{|\alpha|\leqslant d, a_{d000}=1} a_\alpha \frac{\partial z^\alpha}{\partial z_j} w_j^{(1)} = 0 \end{aligned}$$

$$\sum_{|\alpha|\leqslant d,\alpha_1<d}(\sum_{j=1}^{4}\frac{\partial z^\alpha}{\partial z_j}\xi_j^{(2)}+\sum_{j,k=1}^{4}\frac{\partial^2 z^\alpha}{\partial z_j\partial z_k}\xi_j^{(1)}\xi_k^{(1)})v_\alpha$$
$$+\sum_{j=1}^{4}\sum_{|\alpha|\leqslant d,a_{d000}=1}a_\alpha(\sum_{k=1}^{4}\frac{\partial^2 z^\alpha}{\partial z_j\partial z_k}\xi_k^{(2)}+\sum_{k,l=1}^{4}\frac{\partial^3 z^\alpha}{\partial z_j\partial z_k\partial z_l}\xi_k^{(1)}\xi_l^{(1)})v_j$$
$$+\sum_{|\alpha|\leqslant d,a_{d000}=1}(\sum_{j,k=1}^{4}a_\alpha\frac{\partial^2 z^\alpha}{\partial z_j\partial z_k}(w_j^{(1)}\xi_k^{(1)}+w_k^{(1)}\xi_j^{(1)})+\sum_{j=1}^{4}a_\alpha\frac{\partial z^\alpha}{\partial z_j}w_j^{(2)})=0$$

$$\sum_{|\alpha|\leqslant d,\alpha_1<d}(\sum_{j=1}^{4}\frac{\partial z^\alpha}{\partial z_j}\xi_j^{(3)}+3\sum_{j,k=1}^{4}\frac{\partial^2 z^\alpha}{\partial z_j\partial z_k}\xi_j^{(2)}\xi_k^{(1)}$$
$$+\sum_{j,k,l=1}^{4}\frac{\partial^3 z^\alpha}{\partial z_j\partial z_k\partial z_l}\xi_j^{(1)}\xi_k^{(1)}\xi_l^{(1)})v_\alpha$$
$$+\sum_{j=1}^{4}\sum_{|\alpha|\leqslant d,a_{d000}=1}a_\alpha(\sum_{k=1}^{4}\frac{\partial^2 z^\alpha}{\partial z_j\partial z_k}\xi_k^{(3)}+3\sum_{k,l=1}^{4}\frac{\partial^3 z^\alpha}{\partial z_j\partial z_k\partial z_l}\xi_k^{(2)}\xi_l^{(1)}$$
$$+\sum_{k,l,m=1}^{4}\frac{\partial^4 z^\alpha}{\partial z_j\partial z_k\partial z_l\partial z_m}\xi_k^{(1)}\xi_l^{(1)}\xi_m^{(1)})v_j$$
$$+\sum_{|\alpha|\leqslant d,a_{d000}=1}(\sum_{j,k,l=1}^{4}a_\alpha\frac{\partial^3 z^\alpha}{\partial z_j\partial z_k\partial z_l}$$
$$(w_j^{(1)}\xi_k^{(1)}\xi_l^{(1)}+\xi_j^{(1)}w_k^{(1)}\xi_l^{(1)}+\xi_j^{(1)}\xi_k^{(1)}w_l^{(1)})$$
$$+3\sum_{j,k=1}^{4}a_\alpha\frac{\partial^2 z^\alpha}{\partial z_j\partial z_k}(w_j^{(2)}\xi_k^{(1)}+\xi_j^{(2)}w_k^{(1)})+\sum_{j=1}^{4}a_\alpha\frac{\partial z^\alpha}{\partial z_j}w_j^{(3)})=0$$

Now we can introduce the first package of vector fields tangent to $J_3^v(\mathcal{X}_0)$. We denote by $\delta_j\in\mathbb{N}^4$ the multi-index whose j-component is equal to 1 and the other are zero.

For $\alpha_1\geqslant 4$:

$$V_\alpha^{4000}:=\frac{\partial}{\partial a_\alpha}-4z_1\frac{\partial}{\partial a_{\alpha-\delta_1}}+6z_1^2\frac{\partial}{\partial a_{\alpha-2\delta_1}}-4z_1^3\frac{\partial}{\partial a_{\alpha-3\delta_1}}+z_1^4\frac{\partial}{\partial a_{\alpha-4\delta_1}}.$$

For $\alpha_1 \geqslant 3, \alpha_2 \geqslant 1$:

$$
\begin{aligned}
V_\alpha^{3100} \quad : \quad = & \frac{\partial}{\partial a_\alpha} - 3z_1 \frac{\partial}{\partial a_{\alpha-\delta_1}} - z_2 \frac{\partial}{\partial a_{\alpha-\delta_2}} + 3z_1 z_2 \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_2}} \\
& + 3z_1^2 \frac{\partial}{\partial a_{\alpha-2\delta_1}} - 3z_1^2 z_2 \frac{\partial}{\partial a_{\alpha-2\delta_1-\delta_2}} - z_1^3 \frac{\partial}{\partial a_{\alpha-3\delta_1}} + z_1^3 z_2 \frac{\partial}{\partial a_{\alpha-3\delta_1-\delta_2}}.
\end{aligned}
$$

For $\alpha_1 \geqslant 2, \alpha_2 \geqslant 2$:

$$
\begin{aligned}
V_\alpha^{2200} \quad : \quad = & \frac{\partial}{\partial a_\alpha} - z_2 \frac{\partial}{\partial a_{\alpha-\delta_2}} - z_1 \frac{\partial}{\partial a_{\alpha-\delta_1}} + z_1 z_2^2 \frac{\partial}{\partial a_{\alpha-\delta_1-2\delta_2}} \\
& + z_1^2 z_2 \frac{\partial}{\partial a_{\alpha-2\delta_1-\delta_2}} - z_1^2 z_2^2 \frac{\partial}{\partial a_{\alpha-2\delta_1-2\delta_2}}.
\end{aligned}
$$

For $\alpha_1 \geqslant 2, \alpha_2 \geqslant 1, \alpha_3 \geqslant 1$:

$$
\begin{aligned}
V_\alpha^{2110} \quad : \quad = & \frac{\partial}{\partial a_\alpha} - z_3 \frac{\partial}{\partial a_{\alpha-\delta_3}} - z_2 \frac{\partial}{\partial a_{\alpha-\delta_2}} - 2z_1 \frac{\partial}{\partial a_{\alpha-\delta_1}} + z_2 z_3 \frac{\partial}{\partial a_{\alpha-\delta_2-\delta_3}} \\
& + 2z_1 z_3 \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_3}} + 2z_1 z_2 \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_2}} + z_1^2 \frac{\partial}{\partial a_{\alpha-2\delta_1}} \\
& - 2z_1 z_2 z_3 \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_2-\delta_3}} - z_1^2 z_3 \frac{\partial}{\partial a_{\alpha-2\delta_1-\delta_3}} - z_1^2 z_2 \frac{\partial}{\partial a_{\alpha-2\delta_1-\delta_2}} \\
& + z_1^2 z_2 z_3 \frac{\partial}{\partial a_{\alpha-2\delta_1-\delta_2-\delta_3}}.
\end{aligned}
$$

For $\alpha_1 \geqslant 1, \alpha_2 \geqslant 1, \alpha_3 \geqslant 1, \alpha_4 \geqslant 1$:

$$
\begin{aligned}
V_\alpha^{1111} \quad : \quad = & \frac{\partial}{\partial a_\alpha} - z_1 \frac{\partial}{\partial a_{\alpha-\delta_1}} - z_2 \frac{\partial}{\partial a_{\alpha-\delta_2}} - z_3 \frac{\partial}{\partial a_{\alpha-\delta_3}} \\
& - z_4 \frac{\partial}{\partial a_{\alpha-\delta_4}} + z_2 z_3 z_4 \frac{\partial}{\partial a_{\alpha-\delta_2-\delta_3-\delta_4}} + z_1 z_3 z_4 \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_3-\delta_4}} \\
& + z_1 z_2 z_3 \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_2-\delta_3}} + z_1 z_2 z_4 \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_2-\delta_4}} \\
& - z_1 z_2 z_3 z_4 \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_2-\delta_3-\delta_4}}.
\end{aligned}
$$

Similar vector fields are constructed by permuting the z-variables, and changing the index α as indicated by the permutation. The pole order of the previous vector fields is equal to 4.

Lemma 2.1. — *For any* $(v_i)_{1\leqslant i\leqslant 4}\in\mathbb{C}^4$, *there exist* $v_\alpha(a)$, *with degree at most 1 in the variables* (a_γ), *such that* $V:=\sum_\alpha v_\alpha(a)\frac{\partial}{\partial a_\alpha}+\sum_j v_j\frac{\partial}{\partial z_j}$ *is tangent to* $J_3^v(\mathcal{X}_0)$ *at each point.*

Proof. — We impose the additional conditions of vanishing for the coefficients of $\xi_j^{(1)}$ in the second equation (respectively of $\xi_j^{(1)}\xi_k^{(1)}$ in the third equation and $\xi_j^{(1)}\xi_k^{(1)}\xi_l^{(1)}$ in the fourth equation) for any $1\leqslant j\leqslant k\leqslant l\leqslant 4$. Then the coefficients of $\xi_j^{(2)}$ (respectively $\xi_j^{(2)}\xi_k^{(1)}$ and $\xi_j^{(3)}$) are automatically zero in the third (respectively fourth) equation. The resulting 35 equations are

$$\sum_{|\alpha|\leqslant d,\alpha_1<d} v_\alpha z^\alpha+\sum_{j=1}^4\sum_{|\alpha|\leqslant d,a_{d000}=1} a_\alpha\frac{\partial z^\alpha}{\partial z_j}v_j=0$$

$$\sum_{|\alpha|\leqslant d,\alpha_1<d} v_\alpha\frac{\partial z^\alpha}{\partial z_j}+\sum_{k=1}^4\sum_{|\alpha|\leqslant d,a_{d000}=1} a_\alpha\frac{\partial^2 z^\alpha}{\partial z_j\partial z_k}v_k=0$$

$$\sum_{\alpha}\frac{\partial^2 z^\alpha}{\partial z_j\partial z_k}v_\alpha+\sum_{l=1}^4\sum_{|\alpha|\leqslant d,a_{d000}=1} a_\alpha\frac{\partial^3 z^\alpha}{\partial z_j\partial z_k\partial z_l}v_l=0$$

$$\sum_{\alpha}\frac{\partial^3 z^\alpha}{\partial z_j\partial z_k\partial z_l}v_\alpha+\sum_{m=1}^4\sum_{|\alpha|\leqslant d,a_{d000}=1} a_\alpha\frac{\partial^4 z^\alpha}{\partial z_j\partial z_k\partial z_l\partial z_m}v_m=0$$

Now we can observe that if the $v_\alpha(a)$ satisfy the first equation, they automatically satisfy the other ones because the v_α are constants with respect to z. Therefore it is sufficient to find (v_α) satisfying the first equation. We identify the coefficients of $z^\rho=z_1^{\rho_1}\ z_2^{\rho_2}\ z_3^{\rho_3}\ z_4^{\rho_4}$:

$$v_\rho+\sum_{j=1}^4 a_{\rho+\delta_j}v_j(\rho_j+1)=0$$

$\square$

Another family of vector fields can be obtained thanks to the generalization to dimension 3 of a lemma (cf. [10]) given by Mihai Paun in dimension 2. Consider a 4×4-matrix $A=(A_j^k)\in\mathcal{M}_4(\mathbb{C})$ and let $\widetilde{V}:=\sum_{j,k}w_j^{(k)}\frac{\partial}{\partial\xi_j^{(k)}}$, where $w^{(k)}:=A\xi^{(k)}$, for $k=1,2,3$.

LEMMA 2.2. — *There exist polynomials* $v_\alpha(z,a) := \sum_{|\beta|\leqslant 3} v_\beta^\alpha(a) z^\beta$ *where each coefficient* v_β^α *has degree at most 1 in the variables* (a_γ) *such that*

$$V := \sum_\alpha v_\alpha(z,a) \frac{\partial}{\partial a_\alpha} + \widetilde{V}$$

is tangent to $J_3^v(\mathcal{X}_0)$ *at each point.*

Proof. — We impose the additional conditions of vanishing for the coefficients of $\xi_j^{(1)}$ in the second equation (respectively of $\xi_j^{(1)}\xi_k^{(1)}$ in the third equation and $\xi_j^{(1)}\xi_k^{(1)}\xi_l^{(1)}$ in the fourth equation) for any $1 \leqslant j \leqslant k \leqslant l \leqslant 4$. Then the coefficients of $\xi_j^{(2)}$ (respectively $\xi_j^{(2)}\xi_k^{(1)}$ and $\xi_j^{(3)}$) are automatically zero in the third (respectively fourth) equation. The resulting 35 equations are

$$\sum_{|\alpha|\leqslant d, \alpha_1 < d} v_\alpha z^\alpha = 0 \ (5)$$

$$\sum_{|\alpha|\leqslant d, \alpha_1<d} v_\alpha \frac{\partial z^\alpha}{\partial z_j} + \sum_{k=1}^{4} \sum_{|\alpha|\leqslant d, a_{d000}=1} a_\alpha \frac{\partial z^\alpha}{\partial z_k} A_k^j = 0 \quad (6_j)$$

$$\sum_\alpha \frac{\partial^2 z^\alpha}{\partial z_j \partial z_k} v_\alpha + \sum_{\alpha,p} a_\alpha \frac{\partial^2 z^\alpha}{\partial z_j \partial z_p} A_p^k + \sum_{\alpha,p} a_\alpha \frac{\partial^2 z^\alpha}{\partial z_k \partial z_p} A_p^j = 0 \ (7_{jk})$$

$$\sum_\alpha \frac{\partial^3 z^\alpha}{\partial z_j \partial z_k \partial z_l} v_\alpha + \sum_{\alpha,p} a_\alpha \frac{\partial^3 z^\alpha}{\partial z_p \partial z_k \partial z_l} A_p^j + \sum_{\alpha,p} a_\alpha \frac{\partial^3 z^\alpha}{\partial z_j \partial z_p \partial z_l} A_p^k + \sum_{\alpha,p} a_\alpha \frac{\partial^3 z^\alpha}{\partial z_j \partial z_k \partial z_p} A_p^l = 0 \ (8_{jkl})$$

The equations for the unknowns v_β^α are obtained by identifying the coefficients of the monomials z^ρ in the above equations. We can do the following reductions: using equations 6_j $v_\beta^\alpha = 0$ if $|\alpha| + |\beta| \geqslant d+1$, and using the equation of $\mathcal{X}_0$, we can assume that degree in the z_1 variable is at most $d-1$.

The monomials z^ρ in (5) are $z_1^{\rho_1}\ z_2^{\rho_2}\ z_3^{\rho_3}\ z_4^{\rho_4}$ with $\sum \rho_i \leqslant d$ and $\rho_1 \leqslant d-1$.

If all the components of ρ are greater than 3, then we obtain the following system

9. The coefficient of z^ρ in (5) impose the condition

$$\sum_{\alpha+\beta=\rho} v_\beta^\alpha = 0$$

10_j. The coefficient of the monomial $z^{\rho-\delta_j}$ in (6_j) impose the condition

$$\sum_{\alpha+\beta=\rho} \alpha_j v_\beta^\alpha = l_j(a)$$

where l_j is a linear expression in the a-variables.

11_{jj}. For $j = 1, ..., 4$ the coefficient of the monomial $z^{\rho-2\delta_j}$ in (7_{jj}) impose the condition

$$\sum_{\alpha+\beta=\rho} \alpha_j(\alpha_j - 1) v_\beta^\alpha = l_{jj}(a)$$

11_{jk}. For $1 \leqslant j < k \leqslant 4$ the coefficient of the monomial $z^{\rho-\delta_j-\delta_k}$ in (7_{jk}) impose the condition

$$\sum_{\alpha+\beta=\rho} \alpha_j(\alpha_j - 1) v_\beta^\alpha = l_{jk}(a)$$

12_{jjj}. For $j = 1, ..., 4$ the coefficient of the monomial $z^{\rho-3\delta_j}$ in (8_{jjj}) impose the condition

$$\sum_{\alpha+\beta=\rho} \alpha_j(\alpha_j - 1)(\alpha_j - 2) v_\beta^\alpha = l_{jjj}(a)$$

12_{jjk}. For $1 \leqslant j < k \leqslant 4$ the coefficient of the monomial $z^{\rho-2\delta_j-\delta_k}$ in (8_{jjk}) impose the condition

$$\sum_{\alpha+\beta=\rho} \alpha_j(\alpha_j - 1)\alpha_k v_\beta^\alpha = l_{jjk}(a)$$

12_{jkl}. For $1 \leqslant j < k < l \leqslant 4$ the coefficient of the monomial $z^{\rho-\delta_j-\delta_k-\delta_l}$ in (8_{jjk}) impose the condition

$$\sum_{\alpha+\beta=\rho} \alpha_j \alpha_k \alpha_l v_\beta^\alpha = l_{jkl}(a)$$

As in the case of dimension 2 (cf. [10]) we obtain that the determinant of the matrix associated to the system is not zero. Indeed, for each ρ the matrix whose column C_β consists of the partial derivatives of order at most 3 of the monomial $z^{\rho-\beta}$ has the same determinant, at the point $z_0 = (1,1,1,1)$, as our system. Therefore if the determinant is zero, we would have a non-identically zero polynomial

$$Q(z) = \sum_\beta a_\beta z^{\rho-\beta}$$

such that all its partial derivatives of order less or equal to 3 vanish at z_0. Thus the same is true for

$$P(z) = z^\rho Q(\frac{1}{z_1}, ..., \frac{1}{z_4}) = \sum_\beta a_\beta z^\beta.$$

But this implies $P \equiv 0$.

Finally, we conclude by Cramer's rule. The systems we have to solve are never over determined as well. The lemma is proved. $\square$

PROPOSITION 2.3. — *Let $\Sigma_0 := \{(z, a, \xi^{(1)}, \xi^{(2)}, \xi^{(3)}) \in J_3^v(\mathcal{X}) \ / \ \xi^{(1)} \wedge \xi^{(2)} \wedge \xi^{(3)} = 0\}$. Then the vector space $T_{J_3^v(\mathcal{X})} \otimes \mathcal{O}_{\mathbb{P}^4}(12) \otimes \mathcal{O}_{\mathbb{P}^{N_d}}(*)$ is generated by its global sections on $J_3^v(\mathcal{X}) \backslash \Sigma$, where Σ is the closure of Σ_0.*

Proof. — From the preceding lemmas, we are reduced to consider $V = \sum_{|\alpha| \leqslant 3} v_\alpha \frac{\partial}{\partial a_\alpha}$. The conditions for V to be tangent to $J_3^v(\mathcal{X}_0)$ are

$$\sum_{|\alpha| \leqslant 3} v_\alpha z^\alpha = 0$$

$$\sum_{j=1}^4 \sum_{|\alpha| \leqslant d, \alpha_1 < d} v_\alpha \frac{\partial z^\alpha}{\partial z_j} \xi_j^{(1)} = 0$$

$$\sum_{|\alpha| \leqslant 3} (\sum_{j=1}^4 \frac{\partial z^\alpha}{\partial z_j} \xi_j^{(2)} + \sum_{z_j \partial z_k} \xi_j^{(1)} \xi_k^{(1)}) v_\alpha = 0$$

$$\sum_{|\alpha| \leqslant 3} (\sum_{j=1}^4 \frac{\partial z^\alpha}{\partial z_j} \xi_j^{(3)} + 3 \sum_{z_j \partial z_k} \xi_j^{(2)} \xi_k^{(1)} + \sum_{j,k,l=1}^4 \frac{\partial^3 z^\alpha}{\partial z_j \partial z_k \partial z_l} \xi_j^{(1)} \xi_k^{(1)} \xi_l^{(1)}) v_\alpha = 0$$

We denote by W_{jkl} the wronskian operator corresponding to the variables z_j, z_k, z_l. We can suppose $W_{123} := \det(\xi_j^{(i)})_{1 \leqslant i,j \leqslant 3} \neq 0$. Then we can solve the previous system with $v_{0000}, v_{1000}, v_{0100}, v_{0010}$ as unknowns. By the Cramer rule, each of the previous quantity is a linear combination of the v_α, $|\alpha| \leqslant 3$, $\alpha \neq (0000)$, (1000), (0100), (0010) with coefficients rational functions in $z, \xi^{(1)}, \xi^{(2)}, \xi^{(3)}$. The denominator is W_{123} and the numerator is a polynomial whose monomials verify either:

i) degree in z at most 3 and degree in each $\xi^{(i)}$ at most 1.

ii) degree in z at most 2 and degree in $\xi^{(1)}$ at most 3, degree in $\xi^{(2)}$ at most 0, degree in $\xi^{(3)}$ at most 1.

iii) degree in z at most 2 and degree in $\xi^{(1)}$ at most 2, degree in $\xi^{(2)}$ at most 2, degree in $\xi^{(3)}$ at most 0.

iv) degree in z at most 1 and degree in $\xi^{(1)}$ at most 4, degree in $\xi^{(2)}$ at most 1, degree in $\xi^{(3)}$ at most 0.

$\xi^{(1)}$ has a pole of order 2, $\xi^{(2)}$ has a pole of order 3 and $\xi^{(3)}$ has a pole of order 4, therefore the previous vector field has order at most 12. $\square$

Remark 2.4. — If the third derivative of $f : (\mathbb{C}, 0) \to \mathcal{X}$ lies inside Σ_0 then the image of f is contained in a hyperplane section of $\mathcal{X}$.

3. Jet differentials

In this section we recall the basic facts about jet differentials following J.-P. Demailly [3].

Let X be a complex manifold. We start with the directed manifold (X, T_X). We define $X_1 := \mathbb{P}(\mathbb{T}_{\mathbb{X}})$, and $V_1 \subset T_{X_1}$:

$$V_{1,(x,[v])} := \{\xi \in T_{X_1,(x,[v])}; \pi_* \xi \in \mathbb{C}v\}$$

where $\pi : X_1 \to X$ is the natural projection. If $f : (\mathbb{C}, 0) \to (X, x)$ is a germ of holomorphic curve then it can be lifted to X_1 as $f_{[1]}$.

By induction, we obtain a tower of varieties (X_k, V_k). $\pi_k : X_k \to X$ is the natural projection. We have a tautological line bundle $\mathcal{O}_{X_k}(1)$ and we denote $u_k := c_1(\mathcal{O}_{X_k}(1))$.

Let's consider the direct image $\pi_{k*}(\mathcal{O}_{X_k}(m))$. It's a vector bundle over X which can be described with local coordinates. Let $z = (z_1, ..., z_n)$ be local coordinates centered in $x \in X$. A local section of $\pi_{k*}(\mathcal{O}_{X_k}(m))$ is a polynomial

$$P = \sum_{|\alpha_1|+2|\alpha_2|+...+k|\alpha_k|=m} R_\alpha(z) dz^{\alpha_1}...d^k z^{\alpha_k}$$

which acts naturally on the fibers of the bundle $J_k X \to X$ of k-jets of germs of curves in X, i.e the set of equivalence classes of holomorphic maps $f : (\mathbb{C}, 0) \to (X, x)$ with the equivalence relation which identifies two such maps if their derivatives agree up to order k, and which is invariant under reparametrization i.e

$$P((f \circ \phi)', ..., (f \circ \phi)^{(k)})_t = \phi'(t)^m P(f', ..., f^{(k)})_{\phi(t)}$$

for every $\phi \in \mathbb{G}_{k}$, the group of k-jets of biholomorphisms of $(\mathbb{C}, 0)$. The vector bundle $\pi_{k*}(\mathcal{O}_{X_k}(m))$ is denoted $E_{k,m}T^*_X$. This bundle of invariant jet differentials is a subbundle of the bundle of jet differentials, of order k and degree m, $E^{GG}_{k,m}T^*_X \to X$ whose fibres are complex-valued polynomials $Q(f', f'', ..., f^{(k)})$ on the fibers of $J_k X$, of weight m under the action of $\mathbb{C}^*$:

$$Q(\lambda f', \lambda^2 f'', ..., \lambda^k f^{(k)}) = \lambda^m Q(f', f'', ..., f^{(k)})$$

for any $\lambda \in \mathbb{C}^*$ and $(f', f'', ..., f^{(k)}) \in J_k X$.

It turns out that we have an embedding $J_k^{reg} X/\mathbb{G}_k \hookrightarrow X_k$, where $J_k^{reg} X$ denotes the space of non-constants jets.

For $k = 1$, $E_{1,m}T^*_X = S^m T^*_X$.

If X is a surface we have the following description of $E_{2,m}T^*_X$. Let W be the wronskian, $W = dz_1 d^2 z_2 - dz_2 d^2 z_1$, then every invariant differential operator of order 2 and degree m can be written

$$P = \sum_{|\alpha|+3k=m} R_{\alpha,k}(z) dz^\alpha W^k.$$

The following theorem makes clear the use of jet differentials in the study of hyperbolicity:

Theorem 3.1 ([7], [3]). — *Assume that there exist integers $k, m > 0$ and an ample line bundle L on X such that*

$$H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_k^* L^{-1}) \simeq H^0(X, E_{k,m}T^*_X \otimes L^{-1})$$

has non zero sections $\sigma_1, ..., \sigma_N$. Let $Z \subset X_k$ be the base locus of these sections. Then every entire curve $f : \mathbb{C} \to X$ is such that $f_{[k]}(\mathbb{C}) \subset Z$. In *other words, for every global $\mathbb{G}_k$−invariant polynomial differential operator P with values in L^{-1}, every entire curve $f:\mathbb{C}\to X$ must satisfy the algebraic differential equation $P(f)=0$.*

Remark 3.2. — In fact, this theorem is true for global sections of $E^{GG}_{k,m}T^*_X$ vanishing on an ample divisor.

A complex compact manifold is hyperbolic if there is no non constant entire curve $f:\mathbb{C}\to X$. Thus, the problem reduces to produce enough independant algebraic differential equations.

If $X\subset\mathbb{P}^4$ is a smooth hypersurface, we have established the next result:

THEOREM 3.3 [12]. — *Let X be a smooth hypersurface of $\mathbb{P}^4$ such that $d=\deg(X)\geqslant 97$, and A an ample line bundle, then $E_{3,m}T^*_X\otimes A^{-1}$ has global sections for m large enough and every entire curve $f:\mathbb{C}\to X$ must satisfy the corresponding algebraic differential equation.*

The proof relies on the filtration of $E_{3,m}T^*_X$ [11]:

$$Gr^{\bullet}E_{3,m}T^*_X=\bigoplus_{0\leqslant\gamma\leqslant\frac{m}{5}}\Big(\bigoplus_{\{\lambda_1+2\lambda_2+3\lambda_3=m-\gamma;\lambda_i-\lambda_j\geqslant\gamma,i<j\}}\Gamma^{(\lambda_1,\lambda_2,\lambda_3)}T^*_X\Big)$$

where Γ is the Schur functor. This filtration provides a Riemann-Roch computation of the Euler characteristic [11]:

$$\chi(X,E_{3,m}T^*_X)=\frac{m^9}{81648\times10^6}d(389d^3-20739d^2+185559d-358873)+O(m^8).$$

In dimension 3 there is no Bogomolov vanishing theorem (cf. [1]) as it is used in dimension 2 to control the cohomology group H^2, therefore we need the following proposition:

PROPOSITION 3.4 [12] . — *Let $\lambda=(\lambda_1,\lambda_2,\lambda_3)$ be a partition such that $\lambda_1>\lambda_2>\lambda_3$ and $|\lambda|=\sum\lambda_i>4(d-5)+18$. Then :*

$$h^2(X,\Gamma^{\lambda}T^*_X)\leqslant g(\lambda)d(d+13)+q(\lambda)$$

where $g(\lambda)=\frac{3|\lambda|^3}{2}\prod\limits_{\lambda_i>\lambda_j}(\lambda_i-\lambda_j)$ and q is a polynomial in λ with homogeneous components of degrees at most 5.

This proposition provides the estimate

$$h^2(X,Gr^{\bullet}E_{3,m}T^*_X)\leqslant Cd(d+13)m^9+O(m^8)$$

where C is a constant.

4. Proof of the theorem

Let us consider an entire curve $f : \mathbb{C} \to X$ in a generic hypersurface of $\mathbb{P}^4$. By Riemann-Roch and the proposition of the previous section we obtain the following lemma:

LEMMA 4.1. — *Let X be a smooth hypersurface of $\mathbb{P}^4$ of degree d, $0 < \delta < \frac{1}{18}$ then $h^0(X, E_{3,m}T_X^* \otimes K_X^{-\delta m}) \geqslant \alpha(d,\delta)m^9 + O(m^8)$, with*

$$\begin{aligned}\alpha(d,\delta) \quad = \quad & -\frac{1}{408240000000}d(-1945d^3 - 784080\delta^2 d^3 + 105030d^3\delta + \\ & 1058400\delta^3 d^3 + 103695d^2 + 7075491d + 105837083 \\ & +322256880\delta^2 d - 1260083250\delta \\ & -435002400\delta^3 d - 6819271200\delta^3 + 5051827440\delta^2 \\ & -2255850d^2\delta - 15876000\delta^3 d^2 - 81814050\delta d + 11761200\delta^2 d^2).\end{aligned}$$

Proof. — $E_{3,m}T_X^* \otimes K_X^{-\delta m}$ admits a filtration with graded pieces

$$\Gamma^{(\lambda_1,\lambda_2,\lambda_3)}T_X^* \otimes K_X^{-\delta m} = \Gamma^{(\lambda_1-\delta m,\lambda_2-\delta m,\lambda_3-\delta m)}T_X^*$$

for $\lambda_1 + 2\lambda_2 + 3\lambda_3 = m - \gamma$; $\lambda_i - \lambda_j \geqslant \gamma$, $i < j$, $0 \leqslant \gamma \leqslant \frac{m}{5}$. We compute by Riemann-Roch

$$\chi(X, E_{3,m}T_X^* \otimes K_X^{-\delta m}) = \chi(X, Gr^\bullet E_{3,m}T_X^* \otimes K_X^{-\delta m}).$$

We use the proposition of the previous section to control

$$h^2(X, E_{3,m}T_X^* \otimes K_X^{-\delta m}):$$

$$\begin{aligned}h^2(X, \Gamma^{(\lambda_1-\delta m,\lambda_2-\delta m,\lambda_3-\delta m)}T_X^*) \;\leqslant\; & g(\lambda_1-\delta m, \lambda_2-\delta m, \lambda_3-\delta m)d(d+13) \\ & +q(\lambda_1-\delta m, \lambda_2-\delta m, \lambda_3-\delta m)\end{aligned}$$

under the hypothesis $\sum \lambda_i - 3\delta m > 4(d-5) + 18$. The conditions verified by λ imply $\sum \lambda_i \geqslant \frac{m}{6}$ therefore the hypothesis will be verified if

$$m(\frac{1}{6} - 3\delta) > 4(d-5) + 18.$$

We conclude with the computation

$$\chi(X, E_{3,m}T_X^* \otimes K_X^{-\delta m}) - h^2(X, Gr^\bullet E_{3,m}T_X^* \otimes K_X^{-\delta m}) \leqslant h^0(X, E_{3,m}T_X^* \otimes K_X^{-\delta m}).$$

$\square$

Remark 4.2. — If we denote $\mathcal{X}_3^v$ the quotient of $J_3^{v,reg}(\mathcal{X})$ by the reparametrization group $\mathbb{G}_3$, one can easily verify that each vector field given at section 2 defines a section of the tangent bundle of the manifold $\mathcal{X}_3^v$.

We have a section

$$\sigma \in H^0(X, E_{3,m}T_X^* \otimes K_X^{-\delta m}) \simeq H^0(X_3, \mathcal{O}_{X_3}(m) \otimes \pi_3^* K_X^{-\delta m}).$$

with zero set Z and vanishing order $\delta m(d-5)$. Consider the family

$$\mathcal{X} \subset \mathbb{P}^4 \times \mathbb{P}^{N_d}$$

of hypersurfaces of degree d in $\mathbb{P}^4$. General semicontinuity arguments concerning the cohomology groups show the existence of a Zariski open set $U_d \subset \mathbb{P}^{N_d}$ such that for any $a \in U_d$, there exists a divisor

$$Z_a = (P_a = 0) \subset (\mathcal{X}_a)_3$$

where

$$P_a \in H^0((\mathcal{X}_a)_3, \mathcal{O}_{(\mathcal{X}_a)_3}(m) \otimes \pi_3^* K_{(\mathcal{X}_a)}^{-\delta m})$$

such that the family $(P_a)_{a \in U_d}$ varies holomorphically. We consider P as a holomorphic function on $J_3(\mathcal{X}_a)$. The vanishing order of this function as a function of dz_i, d^2z_i, d^3z_i $(1 \leqslant i \leqslant 3)$ is no more than m at a generic point of $\mathcal{X}_a$. We have $f_{[3]}(\mathbb{C}) \subset Z_a$.

Then we invoke the proposition 2.3 which gives the global generation of

$$T_{J_3^v(\mathcal{X})} \otimes \mathcal{O}_{\mathbb{P}^4}(12) \otimes \mathcal{O}_{\mathbb{P}^{N_d}}(*)$$

on $J_3^v(\mathcal{X}) \backslash \Sigma$.

If $f_{[3]}(\mathbb{C})$ lies in Σ, f is algebraically degenerated as we saw in remark 2.4. So we can suppose it is not the case.

At any point of $f_{[3]}(\mathbb{C}) \backslash \Sigma$ where the vanishing of P as a function of dz_i, d^2z_i, d^3z_i $(1 \leqslant i \leqslant 3)$ is no more than m, we can find global meromorphic vector fields $v_1, ..., v_p$ $(p \leqslant m)$ and differentiate P with these vector fields such that $v_1...v_pP$ is not zero at this point. From the above remark, we see that $v_1...v_pP$ corresponds to an invariant differential operator and its restriction to $(\mathcal{X}_a)_3$ can be seen as a section of the bundle

$$\mathcal{O}_{(\mathcal{X}_a)_3}(m) \otimes \mathcal{O}_{\mathbb{P}^4}(12p - \delta m(d-5)).$$

Assume that the vanishing order of P is larger than the sum of the pole order of the v_i in the fiber direction of $\pi : \mathcal{X} \to \mathbb{P}^{N_d}$. Then the restriction of

$v_1 ... v_p P$ to $\mathcal{X}_a$ defines a jet differential which vanishes on an ample divisor. Therefore $f_{[3]}(\mathbb{C})$ should be in its zero set. Thus $f_{[3]}(\mathbb{C})$ must be in Σ over a generic point of $\mathcal{X}_a$.

To finish the proof, we just have to see when the vanishing order of P is larger than the sum of the pole order of the v_i. This will be verified if

$$\delta(d-5) > 12.$$

So we want $\delta > \frac{12}{(d-5)}$ and $\alpha(d,\delta) > 0$. This is the case for $d \geqslant 593$.

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