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Erratum: article Melles-Milman, Lemma 4.5, p. 719-720, fascicule 4, 2006

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ERRATUM

(**article Melles-Milman**, Lemma 4.5, p. 719-720, fascicule 4, 2006)

"Due to an error of the authors, a few lines were missing in the fourth paragraph of the original proof of Lemma IV.5 of our recently published paper, leaving a gap in the proof."

LEMMA 4.5. — *The same polynomials that generate $\mathcal{J}_{(0,0)}$ also generate $\mathcal{J}$ over a neighborhood of $(0,0)$ in $U \times \mathbb{C}^{n+1}$ and generate $\mathcal{I}$ over a neighborhood of $\{0\} \times \mathbb{P}^n$ in $U \times \mathbb{P}^n$.*

Proof. — Suppose that $\mathcal{J}$ is generated in a neighborhood of $(0,0)$ by $F_1(x,y), ..., F_s(x,y)$, where $F_i(x,y)$ is a homogeneous polynomial of degree d_i in y with analytic coefficients in x. We will show that $\mathcal{I}$ is generated on a neighborhood of $\{0\} \times \mathbb{P}^n$ in $U \times \mathbb{P}^n$ by the corresponding polynomials $F_i(x,\xi)$, where $[\xi] = [\xi_0 : ... : \xi_n]$ are homogeneous coordinates for $\mathbb{P}^n$. More precisely, we will show that $\mathcal{I}$ is generated on a neighborhood of any point $q \in \{0\} \times \mathbb{P}^n$ by dehomogenizations of $F_1, ..., F_s$ near q.

Choose homogeneous coordinates ξ on $\mathbb{P}^n$ such that $q = (0, [1 : 0 : ... : 0])$. Nonhomogeneous coordinates on the set $W = \{\xi_0 \neq 0\} \subset \mathbb{P}^n$ are $w_i = \frac{\xi_i}{\xi_0}$ for $1 \leqslant i \leqslant n$. We will check that $\mathcal{I}$ is generated in a neighborhood of q by the polynomials

$$\frac{F_i(x,\xi)}{\xi_0^{d_i}} = F_i\left(x, \frac{\xi}{\xi_0}\right) = F_i(x, 1, w_1, ..., w_n).$$

First we look at the maps σ_1 and σ_2 in local coordinates. We may use (x, y_0, w) as local coordinates in $\sigma_2^{-1}(U \times W) \cong U \times \mathbb{C} \times W$. Local coordinates for $U \times \mathbb{C}^{n+1}$ are $(x, y_0, y_1, ..., y_n)$, where $y_i = y_0 w_i$ for $1 \leqslant i \leqslant n$. The maps σ_1 and σ_2 are given by

$$\sigma_1(x, y_0, w) = (x, y_0, y_0 w) \qquad \text{and} \qquad \sigma_2(x, y_0, w) = (x, w).$$

Suppose that G is a holomorphic section of $\mathcal{I}$ on a neighborhood of q in $U \times \mathbb{P}^n$. Then $\sigma_2^* G$ is a holomorphic section of $\sigma_2^{-1}\mathcal{I}$ in a neighborhood of $\sigma_2^{-1}(q) = \{(0, y_0, 0) : y_0 \in \mathbb{C}\}$. The homogeneous polynomials $F_1, ..., F_s$ that generate $\mathcal{J}_{(0,0)}$ also generate $\mathcal{J} = \sigma_{1*}(\sigma_2^{-1}\mathcal{I})$ on a neighborhood of $(0,0) \in U \times C^{n+1}$ (since $\mathcal{J}$ is coherent, by the Direct Image Theorem), so their pullbacks $\sigma_1^* F_1, ..., \sigma_1^* F_s$ generate $\tilde{\mathcal{J}} := \sigma_1^{-1}\mathcal{J}$ on a neighborhood of $\sigma_1^{-1}(0,0) \in U \times \tilde{\mathbb{C}}^{n+1}$ and therefore generate $\tilde{\mathcal{I}} := \sigma_2^{-1}\mathcal{I}$ off

$H := \sigma_1^{-1}(U \times \{0\})$. Hence $\operatorname{Supp}(\tilde{\mathcal{I}}/\tilde{\mathcal{J}}) \subset H$ and therefore (by the complex analytic nullstellensatz) there exist an integer $d > 0$ and holomorphic functions $A_1, ..., A_s$ on a neighborhood of the point $(x = 0, y_0 = 0, w = 0)$ in $U \times \tilde{\mathbb{C}}^{n+1}$ such that

$$y_0^d \sigma_2^* G(x, y_0, w) = \sum_{i=1}^{s} A_i(x, y_0, w) \sigma_1^* F_i(x, y_0, w)$$

on that neighborhood. But $\sigma_2^* G(x, y_0, w) = G(x, w)$ is independent of the value of y_0 and $\sigma_1^* F_i(x, y_0, w) = F_i(x, y_0, y_0 w) = y_0^{d_i} F_i(x, 1, w)$ since F_i is homogeneous of degree d_i in y. Therefore, by comparing terms in y_0^d of both sides of the equation above, it follows that there are holomorphic functions $a_1, ..., a_s$ (on a neighborhood of $(x = 0, y_0 = 0, w = 0)$) depending only on x and w such that

$$G(x, w) = \sum_{i=1}^{s} a_i(x, w) F_i(x, 1, w).$$

Since the functions $F_i(x, 1, w)$ are the local dehomogenizations of the homogeneous polynomials $F(x, \xi)$, we are done. $\square$