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JAMEL BEN AMEUR, RIDHA SELMI

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Study of Anisotropic MHD system in Anisotropic Sobolev spaces^(*)

JAMEL BEN AMEUR⁽¹⁾, RIDHA SELMI⁽²⁾

ABSTRACT. — Three-dimensional anisotropic magneto-hydrodynamical system is investigated in the whole space \mathbb{R}^3 . Existence and uniqueness results are proved in the anisotropic Sobolev space $H^{0,s}$ for $s > 1/2$. Asymptotic behavior of the solution when the Rossby number goes to zero is studied. The proofs, where the incompressibility condition is crucial, use the energy method, an appropriate dyadic decomposition of the frequency space, product laws in anisotropic Sobolev spaces and Strichartz-type estimates.

RÉSUMÉ. — On étudie un système magneto-hydro-dynamique tridimensionnel dans le cas de l'espace entier \mathbb{R}^3 . On démontre l'existence et l'unicité de la solution pour des données initiales dans les espaces de Sobolev anisotropes ; $H^{0,s}$, $s > 1/2$. On étudie le comportement asymptotique de la solution lorsque le nombre de Rossby tends vers zéro. Les preuves se basent essentiellement sur des méthodes d'énergie, une décomposition adéquate de l'espace de fréquences, les lois produit dans les espaces de Sobolev anisotropes et une estimation de type Strichartz. La condition d'incompressibilité joue un rôle crucial dans les démonstrations.

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(1) Faculté des Sciences de Bizerte, Tunisie.

jamel.BenAmeur@fsb.rnu.tn

(2) Institut Supérieur d'Informatique, l'Ariana 2080, Tunisie.

Ridha.Selmi@isi.rnu.tn

1. Introduction

The purpose of this paper is to study the incompressible *MHD* model with anisotropic diffusion in the limit of small Rossby number, namely the following system

$$(MHD_{\nu_h}^\varepsilon) \left\{ \begin{array}{l} \partial_t u - \nu_h \Delta_h u + u \cdot \nabla u - b \cdot \nabla b + \frac{1}{\varepsilon} \operatorname{curl} b \times e_3 \\ \quad + \frac{1}{\varepsilon} u \times e_3 = -\nabla p \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \\ \partial_t b - \eta_h \Delta_h b + u \cdot \nabla b - b \cdot \nabla u \\ \quad + \frac{1}{\varepsilon} \operatorname{curl} (u \times e_3) = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \\ \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \\ \operatorname{div} b = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \\ (u, b)|_{t=0} = (u_0, b_0), \end{array} \right.$$

where the velocity field u , the induced magnetic perturbation b and the pressure p are unknown functions of time t and space variable $x = (x_1, x_2, x_3)$, e_3 is the third vector of the cartesian coordinate system, ν_h is the anisotropic dynamic viscosity, η_h is the anisotropic magnetic diffusivity, Δ_h denotes the horizontal Laplace operator defined by $\Delta_h = \partial_1^2 + \partial_2^2$ and ε is the Rossby number which is the ratio between the fluid's typical velocity to the earth rotation velocity around the axis e_3 .

The above system is a simplified version of a general one given in [4] to modelise the magneto-hydrodynamical process in the earth's core which is believed to support a self-excited dynamo process generating the earth's magnetic field.

Our first motivation follows from Taylor Proudman Theorem [10], that the three-dimensional fluid is inclined to behave as a two-dimensional one. The second motivation follows from the fact that large turbulent eddies move preferably in horizontal layers and do rarely move vertically, which makes vertical turbulent diffusion negligible compared to the horizontal one (see [3] and references therein). So, it makes sense to consider an anisotropic diffusive process. Without loss of generality, $-e_3$ is supposed to be the time independent component of the earth magnetic field B defined in [4] by

$$B = -e_3 + \theta b,$$

where θ denotes the Reynolds' number. According to [5], the magnetic Prandtl number P_m defined as the ratio of the dynamic viscosity by the

magnetic viscosity is equal to 10^{-6} in the case of the earth's core. However, to simplify the mathematical computation we will take $\nu_h = \eta_h$. Moreover, we remark that

$$\operatorname{curl}(u \times e_3) = \partial_3 u$$

and thanks to the divergence free condition,

$$\operatorname{curl} b \times e_3 = \partial_3 b - \nabla b_3.$$

Since the proofs are based in taking the L^2 -scaler product either in (u, b) or in their associated dyadic blocs $(\Delta_h^q u, \Delta_h^q b)$ (see section 2 for notation), terms like ∇f , for any function f , have no contribution in the L^2 -energy estimate. Thus, we can reformulate the $(MHD_{\nu_h}^\varepsilon)$ system to obtain the following noted also $(MHD_{\nu_h}^\varepsilon)$

$$(MHD_{\nu_h}^\varepsilon) \left\{ \begin{array}{l} \partial_t u - \nu_h \Delta_h u + u \cdot \nabla u - b \cdot \nabla b + \frac{1}{\varepsilon} \partial_3 b + \frac{1}{\varepsilon} u \times e_3 \\ \quad = -\nabla P \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \partial_t b - \nu_h \Delta_h b + u \cdot \nabla b - b \cdot \nabla u + \frac{1}{\varepsilon} \partial_3 u = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} b = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ (u, b)|_{t=0} = (u_0, b_0), \end{array} \right.$$

where

$$P = p + b_3.$$

This system can be written in the following abstract form

$$(S^\varepsilon) \left\{ \begin{array}{l} \partial_t U + Q(U, U) + a_{2,h}(D)U + L^\varepsilon(U) = {}^t(-\nabla p, 0) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} b = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ U|_{t=0} = U_0 \end{array} \right.$$

where

$$U = \begin{pmatrix} u \\ b \end{pmatrix},$$

the quadratic term Q is defined by

$$Q(U, U) = \begin{pmatrix} u \cdot \nabla u - b \cdot \nabla b \\ u \cdot \nabla b - b \cdot \nabla u \end{pmatrix},$$

the viscous term is

$$a_{2,h}(D)U = -\nu_h \Delta_h U$$

and the linear perturbation L^ε is given by

$$L^\varepsilon(U) = \frac{1}{\varepsilon} L(U) := \frac{1}{\varepsilon} \begin{pmatrix} \partial_3 b + u \times e_3 \\ \partial_3 u \end{pmatrix}.$$

We note that L^ε is a skew-symmetric linear operator. This skew-symmetry is an important property for the existence results since the perturbation disappears in the energy estimate.

In the case of anisotropic diffusion process, existence, uniqueness and asymptotic behavior of some magneto hydrodynamical systems have been studied by several authors ([1], [2], [11]). According to our knowledge, no paper is concerned by the anisotropic case. Nevertheless, anisotropic Navier-Stokes equation was studied in [3], where it was proved the local existence for arbitrary initial data and global existence for small initial data in $H^{0,s}(\mathbb{R}^3)$ for $s > 1/2$ and uniqueness for $s > 3/2$. In [8], the author closed the gap between existence and uniqueness and proved that uniqueness holds when existence does; that is for $s > 1/2$.

In this paper, dealing with all coupled nonlinearities and singular perturbation, we establish local well-posedness (existence and uniqueness) for arbitrary initial data, global existence for small initial data in $H^{0,s}(\mathbb{R}^3)$ for $s > 1/2$ and we investigate the asymptotic behavior of the solution as the Rossby number ε tends to zero.

Precisely, we establish global existence for small initial data and local existence of solution on uniform time for arbitrary initial data. Namely, solution given by the following theorems.

THEOREM 1.1. — *Let $s > 1/2$ be a real number and $U_0 = (u_0, b_0) \in H^{0,s}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = 0$ and $\operatorname{div} b_0 = 0$. There exists a positive time T such that for all $\varepsilon > 0$, there exists a unique solution U^ε of $(MHD_{\nu_h}^\varepsilon)$ satisfying $U^\varepsilon \in L_T^\infty(H^{0,s}(\mathbb{R}^3))$ and $\nabla_h U^\varepsilon \in L_T^2(H^{0,s}(\mathbb{R}^3))$. Moreover, U^ε satisfies the following energy estimate*

$$\|U^\varepsilon(t, \cdot)\|_{H^{0,s}(\mathbb{R}^3)}^2 + \nu_h \int_0^t \|\nabla_h U^\varepsilon(\tau, \cdot)\|_{H^{0,s}(\mathbb{R}^3)}^2 d\tau \leq \|U_0\|_{H^{0,s}}^2. \quad (1.1)$$

Furthermore, there exists a constant c such that if $\|U_0\|_{H^{0,s}(\mathbb{R}^3)} \leq c\nu_h$, then the solution is global.

To prove Theorem 1.1, we will rearrange the nonlinear terms in a suitable way to apply Lemma 2.2. The proof, where the divergence free condition plays a crucial role, uses the energy method, an appropriate dyadic decomposition of the frequency space, compactness argument and classical analysis.

The following theorem gives an uniqueness result.

THEOREM 1.2. — *Let U^ε and V^ε be two solutions of $(MHD_{\nu_h}^\varepsilon)$ on $[0, T]$ belonging to*

$$L_T^\infty(H^{0,s}(\mathbb{R}^3)) \cap L_T^2(H^{1,s}(\mathbb{R}^3)),$$

where $s > 1/2$. If U^ε and V^ε have the same initial data, then $U^\varepsilon = V^\varepsilon$.

To prove Theorem 1.2, we will estimate the $H^{0,-1/2}$ norm of the difference $U^\varepsilon - V^\varepsilon$ instead of the $H^{0,1/2}$ one. This will be done via Lemma 3.3 and standard arguments.

Once the existence and the uniqueness results are proved we turn to the asymptotic behavior of the solution when the initial data belongs to $H^{0,s}(\mathbb{R}^3)$, $s > 2$. This will be the aim of the following theorem.

THEOREM 1.3. — *Let $s > 2$, $U_0 = (u_0, b_0) \in H^{0,s}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = 0$ and $\operatorname{div} b_0 = 0$. Let (U^ε) be the family of solution given by Theorem 1.1, then*

$$U^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{in} \quad L_T^2(\mathcal{C}^{-3/2}(\mathbb{R}^3)). \quad (1.2)$$

That is, for χ in $\mathcal{D}(\mathbb{R})$, $U_R^\varepsilon = \chi(|\nabla_h|/R)U^\varepsilon$ and $\tilde{U}_R^\varepsilon = U^\varepsilon - U_R^\varepsilon$, it holds

$$\limsup_{\varepsilon \rightarrow 0} \|U_R^\varepsilon\|_{L_T^4(L^\infty)} \xrightarrow{R \rightarrow +\infty} 0 \quad (1.3)$$

$$\limsup_{\varepsilon \rightarrow 0} \|\tilde{U}_R^\varepsilon\|_{L_T^2(H^{1-\eta,0} \cap H^{0,s'})} \xrightarrow{R \rightarrow +\infty} 0, \quad \forall s' < s \text{ and } \eta > 0. \quad (1.4)$$

To prove Theorem 1.3, we will use a Strichartz type inequality and standard Fourier analysis results. In fact, dispersive effects are of great importance in the study of nonlinear partial differential equations, since they yield decay estimates on waves when the domain is the whole space \mathbb{R}^3 , chosen here essentially for mathematical convenience.

2. Proof of existence results

2.1. Anisotropic Littlewood-Paley theory

To introduce the anisotropic Sobolev spaces, we use an anisotropic dyadic decomposition of the frequency space and we introduce, for any function a , the following operators of localization in Fourier space :

$$\begin{aligned} \Delta_j^h a &= \mathcal{F}^{-1}(\varphi(2^{-j}|\xi_h|)\mathcal{F}(a)) \quad \text{for } j \in \mathbb{Z}, \\ \Delta_q^v a &= \mathcal{F}^{-1}(\varphi(2^{-q}|\xi_3|)\mathcal{F}(a)) \quad \text{for } q \in \mathbb{N}, \\ \Delta_{-1}^v a &= \mathcal{F}^{-1}(\vartheta(|\xi_3|)\mathcal{F}(a)) \end{aligned}$$

and

$$\Delta_q^v a = 0 \quad \text{for } q \leq -2.$$

The functions φ and ϑ represent a dyadic partition of unity in \mathbb{R} . This means that these functions are regular non-negative and satisfy

$$\begin{aligned} \text{supp}(\vartheta) &\subset B(0, 4/3), \\ \text{supp}(\varphi) &\subset \mathcal{C}(0, 3/4, 8/3) \end{aligned}$$

and for all $t \in \mathbb{R}$,

$$\vartheta(t) + \sum_{q \geq 0} \varphi(2^{-q}t) = 1.$$

Here,

$$\mathcal{C}(0, 3/4, 8/3) = \{\xi, \xi \in \mathbb{R}^3, 3/4 \leq |\xi| \leq 8/3\}.$$

We define, also, for any function u , the operators $S_q^v u$ and $S_j^h u$ by

$$S_q^v u = \sum_{q' \leq q-1} \Delta_{q'}^v u$$

and

$$S_j^h u = \sum_{j' \leq j-1} \Delta_{j'}^h u.$$

By this way, we note that we consider an homogeneous decomposition in the horizontal variable and an inhomogeneous one in the vertical.

We define the corresponding Sobolev spaces by the following.

DEFINITION 2.1. — *Let s and s' be two real numbers and u a tempered distribution. Let*

$$\|u\|_{H^{s,s'}} = \left(\sum_{j,q} 2^{2(j s + q s')} \|\Delta_j^h \Delta_q^v u\|_{L^2}^2 \right)^{1/2}.$$

The space $H^{s,s'}(\mathbb{R}^3)$ is the closure of $\mathcal{D}(\mathbb{R}^3)$ for the above semi-norm.

The interest of the dyadic decomposition lies in the fact that any vertical derivative of a function localized in vertical frequencies of size 2^q acts as a multiplication by 2^q . We refer to [8] for a precise construction of anisotropic Sobolev spaces.

2.2. Global existence result for small initial data

In this section, the following lemma, a detailed proof of which is given in [3], will be useful to estimate the nonlinear part.

LEMMA 2.2. — *For any real numbers $s_0 > 1/2$ and $s \geq s_0$ a constant C exists such that for any divergence free vectors fields a and b , we have*

$$\begin{aligned} |(\Delta_q^v(a \cdot \nabla b)|\Delta_q^v b)_{L^2}| &\leq C d_q 2^{-2qs} \|b\|_{H^{1/2,s}} \left(\|a\|_{H^{1/2,s_0}} \|\nabla_h b\|_{H^{0,s}} \right. \\ &+ \|a\|_{H^{1/2,s}} \|\nabla_h b\|_{H^{0,s_0}} + \|\nabla_h a\|_{H^{0,s_0}} \|b\|_{H^{1/2,s}} + \|\nabla_h a\|_{H^{0,s}} \|b\|_{H^{1/2,s}} \left. \right) \end{aligned}$$

where the positive numbers d_q verify $\sum_{q \in \mathbb{Z}} d_q = 1$.

To prove Theorem 1.1, apply the operator Δ_q^v to $(\mathcal{S}^\varepsilon)$, denote $U_q = (\Delta_q^v u^\varepsilon, \Delta_q^v b^\varepsilon)$ and use an L^2 energy estimate to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|U_q^\varepsilon(t)\|_{L^2} + \nu_h \|\nabla_h U_q^\varepsilon(t)\|_{L^2} &\leq |(\Delta_q^v(u^\varepsilon \cdot \nabla u^\varepsilon)|u_q^\varepsilon)_{L^2}| \\ &+ |(\Delta_q^v(u^\varepsilon \cdot \nabla b^\varepsilon)|b_q^\varepsilon)_{L^2}| \\ &+ |(\Delta_q^v(b^\varepsilon \cdot \nabla b^\varepsilon)|u_q^\varepsilon)_{L^2}| \\ &+ |(\Delta_q^v(b^\varepsilon \cdot \nabla u^\varepsilon)|b_q^\varepsilon)_{L^2}|. \end{aligned}$$

In order to estimate the left hand side of the above inequality, rearrange the nonlinearity as

$$\begin{aligned} (b^\varepsilon \cdot \nabla b^\varepsilon | u^\varepsilon)_{L^2} + (b^\varepsilon \cdot \nabla u^\varepsilon | b^\varepsilon)_{L^2} &= (b^\varepsilon \cdot \nabla(u^\varepsilon + b^\varepsilon) | (u^\varepsilon + b^\varepsilon))_{L^2} \\ &- (b^\varepsilon \cdot \nabla b^\varepsilon | b^\varepsilon)_{L^2} - (b^\varepsilon \cdot \nabla u^\varepsilon | u^\varepsilon)_{L^2}. \end{aligned}$$

Note that the $H^{s,s'}$ norm of any vector field is the Euclidean norm of the $H^{s,s'}$ norms of its components. That is why

$$\begin{aligned} \|u^\varepsilon\|, \|b^\varepsilon\| &\leq \|U^\varepsilon\|, \\ \|\nabla_h u^\varepsilon\|, \|\nabla_h b^\varepsilon\| &\leq \|\nabla_h U^\varepsilon\| \end{aligned}$$

and so on.

Use Lemma 2.2, in the case $s = s_0$, to deduce that

$$\frac{1}{2} \frac{d}{dt} \|U_q^\varepsilon\|_{L^2}^2 + \nu_h \|\nabla_h U_q^\varepsilon\|_{L^2}^2 \leq C d_q 2^{-2qs_0} \|U^\varepsilon\|_{H^{1/2,s_0}}^2 \|\nabla_h U^\varepsilon\|_{H^{1/2,s_0}}. \quad (2.1)$$

Multiply by 2^{2qs_0} and take the sum over q to obtain

$$\frac{1}{2} \frac{d}{dt} \|U^\varepsilon\|_{H^{0,s_0}}^2 + \nu_h \|\nabla_h U^\varepsilon\|_{H^{0,s_0}}^2 \leq \|U^\varepsilon\|_{H^{1/2,s_0}}^2 \|\nabla_h U^\varepsilon\|_{H^{1/2,s_0}}.$$

An interpolation inequality yields

$$\|U^\varepsilon\|_{H^{1/2},s_0}^2 \leq \|U^\varepsilon\|_{H^{0,s_0}} \|\nabla_h U^\varepsilon\|_{H^{0,s_0}}.$$

So, one concludes that if $\|U_0\|_{H^{0,s_0}}$ is sufficiently small then $\|U^\varepsilon(t)\|_{H^{0,s_0}}^2$ is a decreasing function. More precisely, if

$$\|U_0\|_{H^{0,s_0}} \leq \frac{\nu_h}{2C}$$

then for any positive time t , one gets

$$\|U^\varepsilon(t)\|_{H^{0,s_0}}^2 + \nu_h \int_0^t \|\nabla_h U^\varepsilon(\tau)\|_{H^{0,s_0}}^2 d\tau \leq \|U_0\|_{H^{0,s_0}}^2. \quad (2.2)$$

Finally, a standard compactness argument gives the global existence result.

2.3. Local existence result for arbitrary initial data

Now, we turn to the local existence result of $(MHD_{\nu_h}^\varepsilon)$. Decompose the frequency space in order to investigate the high vertical frequencies then the low vertical-high horizontal ones.

For the high vertical frequencies term, one multiplies inequality (2.1) by 2^{2qs_0} then takes the sum for q greater than $n - 1$ in order to obtain after integration in time

$$\begin{aligned} \|(\text{Id} - S_n^v)U^\varepsilon\|_{L_T^\infty(H^{0,s_0})} &+ 2\nu_h \|\nabla_h(\text{Id} - S_n^v)U^\varepsilon\|_{L_T^2(H^{0,s_0})}^2 \\ &\leq C \|(\text{Id} - S_n^v)U^\varepsilon\|_{H^{0,s_0}}^2 \\ &+ C \int_0^T \|U^\varepsilon(t)\|_{H^{1/2},s_0}^2 \|\nabla_h U^\varepsilon(t)\|_{H^{0,s_0}} dt. \end{aligned} \quad (2.3)$$

To study the low vertical-high horizontal frequency, let m and n tow cutoff integers such that $m \geq n$ and define $U_{m,n}^\varepsilon$ by

$$U_{m,n}^\varepsilon = (\text{Id} - S_m^h)S_n^v U^\varepsilon.$$

Denote, respectively, by Q^h and Q^v the horizontal and the vertical part of the nonlinear term defined by

$$Q^h(U, U) = \begin{pmatrix} u_h \cdot \nabla_h u - b_h \cdot \nabla_h b \\ u_h \cdot \nabla_h b - b_h \cdot \nabla_h u \end{pmatrix}$$

and

$$Q^v(U, U) = \begin{pmatrix} u_3 \cdot \partial_3 u - b_3 \cdot \partial_3 b \\ u_3 \cdot \partial_3 b - b_3 \partial_3 u \end{pmatrix}.$$

By standard L^2 energy estimate, one has

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|U_{m,n}^\varepsilon\|_{L^2}^2 + \nu_h \|\nabla_h U_{m,n}^\varepsilon\|_{L^2}^2 &= ((\text{Id} - S_m^h) S_n^v(Q^h(U^\varepsilon, U^\varepsilon)) |U_{m,n}^\varepsilon)_{L^2} \\ &+ ((\text{Id} - S_m^h) S_n^v(Q^v(U^\varepsilon, U^\varepsilon)) |U_{m,n}^\varepsilon)_{L^2}, \end{aligned}$$

where

$$\begin{aligned} ((\text{Id} - S_m^h) S_n^v(Q^h(U^\varepsilon, U^\varepsilon)) |U_{m,n}^\varepsilon)_{L^2} &= ((\text{Id} - S_m^h) S_n^v(u_h^\varepsilon \cdot \nabla_h u^\varepsilon) |u_{m,n}^\varepsilon)_{L^2} \\ &- ((\text{Id} - S_m^h) S_n^v(b_h^\varepsilon \cdot \nabla_h b^\varepsilon) |u_{m,n}^\varepsilon)_{L^2} \\ &+ ((\text{Id} - S_m^h) S_n^v(u_h^\varepsilon \cdot \nabla_h b^\varepsilon) |b_{m,n}^\varepsilon)_{L^2} \\ &- ((\text{Id} - S_m^h) S_n^v(b_h^\varepsilon \cdot \nabla_h u^\varepsilon) |b_{m,n}^\varepsilon)_{L^2} \end{aligned}$$

and

$$\begin{aligned} ((\text{Id} - S_m^h) S_n^v(Q^v(U^\varepsilon, U^\varepsilon)) |U_{m,n}^\varepsilon)_{L^2} &= ((\text{Id} - S_m^h) S_n^v(u_3^\varepsilon \cdot \partial_3 u^\varepsilon) |u_{m,n}^\varepsilon)_{L^2} \\ &- ((\text{Id} - S_m^h) S_n^v(b_3^\varepsilon \cdot \partial_3 b^\varepsilon) |u_{m,n}^\varepsilon)_{L^2} \\ &+ ((\text{Id} - S_m^h) S_n^v(u_3^\varepsilon \cdot \partial_3 b^\varepsilon) |b_{m,n}^\varepsilon)_{L^2} \\ &- ((\text{Id} - S_m^h) S_n^v(b_3^\varepsilon \cdot \partial_3 u^\varepsilon) |b_{m,n}^\varepsilon)_{L^2}. \end{aligned}$$

By product law in Sobolev spaces, one infers that

$$\begin{aligned} &\|(\text{Id} - S_m^h) S_n^v(Q^h(U^\varepsilon, U^\varepsilon)(\cdot, x_3))\|_{H^{-1/2}(\mathbb{R}_{x_h}^2)} \\ &\leq C \|U_h^\varepsilon(\cdot, x_3)\|_{H^{1/2}(\mathbb{R}_{x_h}^2)} \|\nabla_h U^\varepsilon(\cdot, x_3)\|_{L^2(\mathbb{R}_{x_h}^2)}. \end{aligned}$$

Since $s_0 > 1/2$, one obtains

$$\|(\text{Id} - S_m^h) S_n^v(Q^h(U^\varepsilon, U^\varepsilon))\|_{L^2(\mathbb{R}_{x_3}; H^{-1/2}(\mathbb{R}^2))} \leq C \|U_h^\varepsilon\|_{H^{1/2, s_0}} \|\nabla_h U^\varepsilon\|_{H^{0, s_0}}.$$

By Cauchy-Schwarz inequality, one has

$$\left| ((\text{Id} - S_m^h) S_n^v(Q^h(U^\varepsilon, U^\varepsilon)) |U_{m,n}^\varepsilon)_{L^2(\mathbb{R}^3)} \right| \leq C \|U^\varepsilon\|_{H^{1/2, s_0}}^2 \|\nabla_h U^\varepsilon\|_{H^{0, s_0}}.$$

To estimate $Q^v(U^\varepsilon, U^\varepsilon)$, use the divergence free condition; that is

$$\partial_3 u_3^\varepsilon = -\text{div}_h u_h^\varepsilon$$

and

$$\partial_3 b_3^\varepsilon = -\text{div}_h b_h^\varepsilon.$$

These equalities allow one to reformulate the vertical nonlinearity as

$$Q^v(U^\varepsilon, U^\varepsilon) = Q_1^v(U^\varepsilon, U^\varepsilon) + Q_2^v(U^\varepsilon, U^\varepsilon),$$

where

$$Q_1^v(U^\varepsilon, U^\varepsilon) = \begin{pmatrix} \partial_3(u_3^\varepsilon u^\varepsilon) - \partial_3(b_3^\varepsilon b^\varepsilon) \\ \partial_3(u_3^\varepsilon b^\varepsilon) - \partial_3(b_3^\varepsilon u^\varepsilon) \end{pmatrix}$$

and

$$Q_2^v(U^\varepsilon, U^\varepsilon) = \begin{pmatrix} u^\varepsilon \operatorname{div}_h u_h^\varepsilon - b^\varepsilon \operatorname{div}_h b_h^\varepsilon \\ b^\varepsilon \operatorname{div}_h u_h^\varepsilon - u^\varepsilon \operatorname{div}_h b_h^\varepsilon \end{pmatrix}.$$

The nonlinear term, $((\operatorname{Id} - S_m^v)S_n^v(Q_2^v(U^\varepsilon, U^\varepsilon))|U_{m,n}^\varepsilon)_{L^2}$ can be estimated exactly as the term $((\operatorname{Id} - S_m^v)S_n^v(Q^h(U^\varepsilon, U^\varepsilon))|U_{m,n}^\varepsilon)_{L^2}$.

To study the other term, one observes that

$$\operatorname{Id} - S_m^h = 2^{-m}(\operatorname{Id} - S_m^h)(\chi_1(D)\partial_1 + \chi_2(D)\partial_2),$$

where χ_1 and χ_2 are homogeneous functions of degree 0. Then, it is clear that, when $m \geq n$,

$$\begin{aligned} ((\operatorname{Id} - S_m^v)S_n^v(Q_1^v(U^\varepsilon, U^\varepsilon))|U_{m,n}^\varepsilon)_{L^2} &\leq 2^{n-m} \|U_3^\varepsilon U^\varepsilon\|_{L^2(\mathbb{R}^3)} \|\nabla_h U_{m,n}^\varepsilon\|_{L^2(\mathbb{R}^3)} \\ &\leq C \|U^\varepsilon\|_{H^{1/2, s_0}}^2 \|\nabla_h U_{m,n}^\varepsilon\|_{H^{0, s_0}} \\ &\leq C \|U^\varepsilon\|_{H^{1/2, s_0}}^2 \|\nabla_h U^\varepsilon\|_{H^{0, s_0}}. \end{aligned}$$

Finally, by integration in time, one has

$$\begin{aligned} \|U_{m,n}^\varepsilon\|_{L_T^\infty(H^{0, s_0})}^2 &+ 2\nu_h \|\nabla_h U_{m,n}^\varepsilon\|_{L_T^2(H^{0, s_0})}^2 \\ &\leq C \|U_{m,n}^\varepsilon(0)\|_{H^{0, s_0}}^2 \\ &+ C \int_0^T \|U^\varepsilon(t)\|_{H^{1/2, s_0}}^2 \|\nabla_h U^\varepsilon(t)\|_{H^{0, s_0}} dt. \end{aligned} \quad (2.4)$$

To conclude the proof of the local part, one begins by observing that since U_0 belongs to $L^2(\mathbb{R}^3)$ the basic L^2 energy estimate holds

$$\|U^\varepsilon(t)\|_{L^2}^2 + 2\nu_h \int_0^t \|\nabla_h U^\varepsilon(\tau)\|_{L^2}^2 d\tau \leq \|U_0^\varepsilon\|_{L^2}^2. \quad (2.5)$$

Thus, it is obvious that

$$\|S_n^v S_m^h U^\varepsilon(t)\|_{H^{0, s_0}}^2 + 2\nu_h \int_0^t \|\nabla_h S_n^v S_m^h U^\varepsilon(\tau)\|_{H^{0, s_0}}^2 d\tau \leq C 2^{2m+2n s_0} T \|U_0^\varepsilon\|_{L^2}^2.$$

Using an interpolation inequality, estimations (2.3) and (2.4) one infers that

$$\begin{aligned} (2\nu_h)^{1/2} \|U^\varepsilon\|_{L_T^4(H^{1/2, s_0})}^2 &+ 2\nu_h \|\nabla_h U^\varepsilon\|_{L_T^2(H^{0, s_0})}^2 \leq C \left(\|(\operatorname{Id} - S_n^v)U_0^\varepsilon\|_{H^{0, s_0}}^2 \right. \\ &+ \|S_n^v(\operatorname{Id} - S_m^h)U_0^\varepsilon\|_{H^{0, s_0}}^2 + 2^{2m+2n s_0} T \|U_0\|_{L^2}^2 \\ &\left. + \int_0^T \|U^\varepsilon(t)\|_{H^{1/2, s_0}}^2 \|\nabla_h U^\varepsilon(t)\|_{H^{0, s_0}} dt \right). \end{aligned}$$

It follows that

$$\begin{aligned} \nu_h^{1/2} \|U^\varepsilon\|_{L_T^4(H^{1/2, s_0})}^2 + \nu_h \|\nabla_h U^\varepsilon\|_{L_T^2(H^{0, s_0})}^2 &\leq C \left(\|(\text{Id} - S_n^v)U_0^\varepsilon\|_{H^{0, s_0}}^2 + \right. \\ &\quad \left. \|S_n^v(\text{Id} - S_m^h)U_0^\varepsilon\|_{H^{0, s_0}}^2 + 2^{2m+2ns_0}T\|U_0\|_{L^2}^2 + \frac{C}{\nu_h} \|U^\varepsilon(t)\|_{L_T^4(H^{1/2, s_0})}^4 \right). \end{aligned}$$

Choose, in this order, an integer n , then an integer m greater than n and finally a strictly positive real number T_0 such that

$$\begin{aligned} \|(\text{Id} - S_n^v)U_0^\varepsilon\|_{H^{0, s_0}}^2 &\leq \eta, \\ \|S_n^v(\text{Id} - S_m^h)U_0^\varepsilon\|_{H^{0, s_0}}^2 &\leq \eta \end{aligned}$$

and

$$2^{2m+2ns_0}T_0\|U_0^\varepsilon\|_{L^2}^2 \leq \eta$$

to obtain, for any $T \leq T_0$,

$$\nu_h^{1/2} \|U^\varepsilon\|_{L_T^4(H^{1/2, s_0})}^2 + \nu_h \|\nabla_h U^\varepsilon\|_{L_T^2(H^{0, s_0})}^2 \leq C\eta + \frac{C}{\nu_h} \|U^\varepsilon(t)\|_{L_T^4(H^{1/2, s_0})}^4.$$

So, for any positive real number η , such that

$$0 < \eta < \frac{\nu_h^2}{4C^2}$$

a positive real number T_1 exists, such that

$$\nu_h^{1/2} \|U^\varepsilon\|_{L_{T_1}^4(H^{1/2, s_0})}^2 + \nu_h \|\nabla_h U^\varepsilon\|_{L_{T_1}^2(H^{0, s_0})}^2 \leq \eta.$$

That proves Theorem 1.1.

3. Proof of uniqueness result

In this section we denote by $\langle x \rangle$ the quantity $(1 + |x|^2)^{1/2}$ and by Λ_3 the operator defined by

$$\Lambda_3 = (1 - \partial_3^2)^{1/2},$$

that is, the operator of multiplication by $\langle \xi_3 \rangle$ in the frequency space. Clearly, Λ_3 is an isometry from $H^{s, s'}$ to $H^{s, s'-1}$ for all real numbers s and s' .

The following lemmas will be useful in the sequel. The first one and the second one are proved in [7], the third one is proved in [8].

LEMMA 3.1. — *Let $s, t, s', t' \in \mathbb{R}$ and $\alpha \in [0, 1]$. If $f \in H^{s, s'} \cap H^{t, t'}$ then f belongs to $H^{\alpha s + (1-\alpha)t, \alpha s' + (1-\alpha)t'}$ and*

$$\|fg\|_{H^{\alpha s + (1-\alpha)t, \alpha s' + (1-\alpha)t'}} \leq \|f\|_{H^{s, s'}}^\alpha \|g\|_{H^{t, t'}}^{1-\alpha}.$$

LEMMA 3.2. — *Let $s, t < 1$, $s + t > 0$ and $s', t' < 1/2$, $s' + t' > 0$. If f belongs to $H^{s,s'}$ and g belongs to $H^{t,t'}$ then fg belongs to $H^{s+t-1, s'+t'-1/2}$ and there exists a constant C such that*

$$\|fg\|_{H^{s+t-1, s'+t'-1/2}} \leq C \|f\|_{H^{s,s'}} \|g\|_{H^{t,t'}}.$$

LEMMA 3.3. — *Let $s, t < 1$, $s + t > 0$ and $s' > 1/2$. If f belongs to $H^{s,s'}$ and g belongs to $H^{s,-1/2}$ then fg belongs to $H^{s+t-1, -1/2}$ and there exists a constant C such that*

$$\|fg\|_{H^{s+t-1, -1/2}} \leq C \|f\|_{H^{s,s'}} \|g\|_{H^{t, -1/2}}.$$

The originality of Lemma 3.3, as said in [8], is to give a product law for the anisotropic Sobolev spaces where the regularities in the vertical direction are supercritical for one of the terms and subcritical for the other.

3.1. Proof of Theorem 1.2

Let $U^\varepsilon = {}^t(u^\varepsilon, b^\varepsilon)$ and $V^\varepsilon = {}^t(v^\varepsilon, c^\varepsilon)$ be two solutions of $(MHD_{\nu_h}^\varepsilon)$ satisfying Theorem 1.1 and having the same initial data. It is explained, in Remark 2, that this makes sense.

Denote by $W^\varepsilon = {}^t(w^\varepsilon, \beta^\varepsilon)$ the difference $W^\varepsilon = U^\varepsilon - V^\varepsilon$. One has

$$\left\{ \begin{array}{l} \partial_t W^\varepsilon + \nu_h \nabla_h W^\varepsilon + Q(W^\varepsilon, W^\varepsilon + 2U^\varepsilon) + \frac{1}{\varepsilon} L(W^\varepsilon) \\ \quad = {}^t(-\nabla p^\varepsilon, 0) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \text{div } w^{\varepsilon'} = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \text{div } \beta^\varepsilon = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3. \end{array} \right. \quad (3.1)$$

Without loss of generality, assume that $s < 1$. Suppose that U^ε and V^ε belong to $C_T^0(H^{0,r})$ for all $r < s$, a fact on which we will return in Remark 1. By this assumption, one is able to multiply equation (3.1) by $\Lambda_3^{-1}W^\varepsilon$ then integrate on $(\varepsilon', t) \times \mathbb{R}^3$, let ε tends to zero and use the continuity in time of $\|W^\varepsilon\|_{H^{0,-1/2}}$ to obtain

$$\begin{aligned} \|W^\varepsilon\|_{H^{0,-1/2}}^2 &+ 2\nu_h \int_0^t \|\nabla_h W^\varepsilon\|_{H^{0,-1/2}}^2 d\tau \\ &= -2 \int_0^t \int Q(W^\varepsilon, W^\varepsilon + 2U^\varepsilon) \Lambda_3^{-1} W^\varepsilon dx d\tau, \end{aligned} \quad (3.2)$$

where

$$Q(W^\varepsilon, W^\varepsilon + 2U^\varepsilon) = \left(\begin{array}{l} u^\varepsilon \cdot \nabla w^\varepsilon + w^\varepsilon \cdot \nabla v^\varepsilon - b^\varepsilon \cdot \nabla \beta^\varepsilon - \beta^\varepsilon \cdot \nabla c^\varepsilon \\ u^\varepsilon \cdot \nabla \beta^\varepsilon + w^\varepsilon \cdot \nabla c^\varepsilon - b^\varepsilon \cdot \nabla w^\varepsilon - \beta^\varepsilon \cdot \nabla v^\varepsilon \end{array} \right).$$

Here, note that for every multi-index s and sufficiently smooth function U , one has

$$\int \partial^\alpha U L^\varepsilon (\partial^\alpha U^\varepsilon) = 0.$$

To estimate the left hand side of equation (3.2), one restricts itself to the two first terms, the same holds for the others since that

$$\|u^\varepsilon\|, \quad \|b^\varepsilon\| \leq \|U^\varepsilon\|$$

and so on.

Following [8], decompose the nonlinearity as follows

$$\int (u^\varepsilon \cdot \nabla w^\varepsilon + w^\varepsilon \cdot \nabla v^\varepsilon) \cdot \Lambda_3^{-1} w^\varepsilon dx = \sum_{i=1}^4 L_i,$$

where

$$\begin{aligned} L_1 &= \int u_h^\varepsilon \cdot \nabla_h w^\varepsilon \cdot \Lambda_3^{-1} w^\varepsilon dx, \\ L_2 &= \int u_3^\varepsilon \cdot \partial_3 w^\varepsilon \cdot \Lambda_3^{-1} w^\varepsilon dx, \\ L_3 &= \int w_h^\varepsilon \cdot \nabla_h v^\varepsilon \cdot \Lambda_3^{-1} w^\varepsilon dx \end{aligned}$$

and

$$L_4 = \int w_3^\varepsilon \cdot \partial_3 v^\varepsilon \cdot \Lambda_3^{-1} w^\varepsilon dx.$$

To estimate L_1 , note that

$$|L_1| \leq \|u_h^\varepsilon \cdot \nabla_h w^\varepsilon\|_{H^{-1/2, -1/2}} \|\Lambda_3^{-1} w^\varepsilon\|_{H^{1/2, 1/2}},$$

apply Lemma 3.3 to the first factor, use properties of Λ_3 and Lemma 3.1 for the second to obtain

$$|L_1| \leq C \|U^\varepsilon\|_{H^{1/2, s}} \|W^\varepsilon\|_{H^{0, -1/2}}^{1/2} \|W^\varepsilon\|_{H^{1, -1/2}}^{3/2}.$$

The same holds for L_3 , just begin by

$$|L_3| \leq \|w_h^\varepsilon \cdot \nabla_h v^\varepsilon\|_{H^{-3/4, -1/2}} \|\Lambda_3^{-1} w^\varepsilon\|_{H^{3/4, 1/2}}$$

to obtain

$$|L_3| \leq C \|V^\varepsilon\|_{H^{1/2, s}} \|W^\varepsilon\|_{H^{0, -1/2}}^{1/2} \|W^\varepsilon\|_{H^{1, -1/2}}^{3/2}.$$

For L_4 , remark that

$$|L_4| \leq \|w_3^\varepsilon \partial_3 v^\varepsilon\|_{H^{-1/2, (2s-3)/4}} \|\Lambda_3^{-1} w^\varepsilon\|_{H^{1/2, (3-2s)/4}},$$

use Lemma 3.2, decreasing inclusion of H^s spaces and properties of Λ_3 to infer that

$$|L_4| \leq C \|v^\varepsilon\|_{H^{1/2,s}} \|w_3^\varepsilon\|_{H^{0,1/2}} \|w^\varepsilon\|_{H^{1/2,-1/2}}.$$

Classical computation and divergence free condition imply that

$$\|w_3^\varepsilon\|_{H^{0,1/2}} \leq \|w^\varepsilon\|_{H^{0,-1/2}} + \|w^\varepsilon\|_{H^{1,-1/2}}.$$

By Lemma 3.1, one infers that

$$|L_4| \leq C \|V^\varepsilon\|_{H^{1/2,s}} (\|W^\varepsilon\|_{H^{0,-1/2}}^{3/2} \|W^\varepsilon\|_{H^{1,-1/2}}^{1/2} + \|W^\varepsilon\|_{H^{0,-1/2}}^{1/2} \|W^\varepsilon\|_{H^{1,-1/2}}^{3/2}).$$

To investigate L_2 , use Parseval's formula and Fourier analysis to obtain

$$L_2 = \frac{i}{(2\pi)^6} \int \int \frac{\eta_3}{\langle \xi_3 \rangle} \widehat{v}_3^\varepsilon(\xi - \eta) \widehat{w}^\varepsilon(\eta) \widehat{w}^\varepsilon(-\xi) d\xi d\eta.$$

By the change of variables $(\xi, \eta) \longleftrightarrow (-\nu, -\xi)$, rewrite L_2 in the the following form

$$L_2 = \frac{i}{2(2\pi)^6} \int \int \left(\frac{\eta_3}{\langle \xi_3 \rangle} - \frac{\xi_3}{\langle \eta_3 \rangle} \right) \widehat{v}_3^\varepsilon(\xi - \eta) \widehat{w}^\varepsilon(\eta) \widehat{w}^\varepsilon(-\xi) d\xi d\eta.$$

For η_3 and ξ_3 in \mathbb{R} , one simply checks that

$$\frac{\eta_3}{\langle \xi_3 \rangle} - \frac{\xi_3}{\langle \eta_3 \rangle} = \frac{\eta_3 - \xi_3}{\langle \xi_3 \rangle} + \frac{(\eta_3 - \xi_3)\xi_3(\eta_3 - \xi_3)}{\langle \eta_3 \rangle \langle \xi_3 \rangle (\langle \eta_3 \rangle + \langle \xi_3 \rangle)}$$

and deduces that

$$|L_2| \leq \frac{1}{2(2\pi)^6} \sum_j \int \int |\xi_3 - \eta_3| \left(\frac{1}{\langle \xi_3 \rangle} + \frac{1}{\langle \eta_3 \rangle} \right) |\widehat{v}_3^\varepsilon(\xi - \eta)| |\widehat{w}_j^\varepsilon(\eta)| |\widehat{w}_j^\varepsilon(-\xi)| d\xi d\eta.$$

Use again the change of variables $(\xi, \eta) \longleftrightarrow (-\eta, -\xi)$ to infer

$$|L_2| \leq \frac{1}{(2\pi)^6} \sum_j \int \int \frac{|\xi_3 - \eta_3|}{\langle \xi_3 \rangle} |\widehat{v}_3^\varepsilon(\xi - \eta)| |\widehat{w}_j^\varepsilon(\eta)| |\widehat{w}_j^\varepsilon(-\xi)| d\xi d\eta.$$

By the divergence free condition and the reversed computation of the first step, one establishes that

$$|L_2| \leq \int (|D_1| \mathcal{F}^{-1}(|\widehat{v}_1^\varepsilon|) + |D_2| \mathcal{F}^{-1}(|\widehat{v}_2^\varepsilon|)) w^\varepsilon \cdot \Lambda_3^{-1} w^\varepsilon dx,$$

where $|D_j|$ denotes the operator of multiplication in the frequency space by $|\xi_j|$. As $|D_j| \mathcal{F}^{-1}(|\widehat{v}_j^\varepsilon|)$ and $\partial_j \mathcal{F}^{-1}(|\widehat{v}_j^\varepsilon|)$ have the same $H^{r,r'}$ norm for all r, r' and j , the same argument used for L_3 implies that

$$|L_2| \leq C \|V^\varepsilon\|_{H^{1/2,s}} \|W^\varepsilon\|_{H^{0,-1/2}}^{1/2} \|W^\varepsilon\|_{H^{1,-1/2}}^{3/2}.$$

Collect the $|L_i|$ estimations, for all nonlinear terms, and use inequality

$$ab \leq \frac{1}{4}a^4 + \frac{3}{4}b^{4/3}$$

to obtain

$$\begin{aligned} & \|W^\varepsilon\|_{H^{0,-1/2}}^2 + 2\nu_h \int_0^t \|\nabla_h W^\varepsilon\|_{H^{0,-1/2}}^2 d\tau \\ & \leq C \int_0^t \|W^\varepsilon\|_{H^{0,-1/2}}^2 (\|U^\varepsilon\|_{H^{1/2,s}}^4 + \|V^\varepsilon\|_{H^{1/2,s}}^4 + \|V^\varepsilon\|_{H^{1/2,s}}^{4/3}) d\tau. \end{aligned} \quad (3.3)$$

By interpolation, one has

$$\|U^\varepsilon\|_{H^{1/2,s}} \leq \|U^\varepsilon\|_{H^{0,s}}^{1/2} \|U^\varepsilon\|_{H^{1,s}}^{1/2}$$

and then $\|U^\varepsilon\|_{H^{1/2,s}} \in L_T^4$. The same holds for $\|V^\varepsilon\|_{H^{1/2,s}}$. So, the function h defined by

$$h(t) = \|U^\varepsilon\|_{H^{1/2,s}}^4 + \|V^\varepsilon\|_{H^{1/2,s}}^4 + \|V^\varepsilon\|_{H^{1/2,s}}^{4/3}$$

belongs to L_T^4 .

Gronwall's lemma applied to (3.3) implies that $W^\varepsilon = 0$. This completes the proof.

Remark 1. — It makes sense to multiply equation (3.1) by $\Lambda_3^{-1}W^\varepsilon$. Indeed, $\Lambda_3^{-1}W^\varepsilon$ belongs to $L_T^2(H^{1,1+s})$ which is a subset of $L_T^2(H^{1,3/2})$. Moreover, $\Delta_h U^\varepsilon$ belongs to $L_T^2(H^{1,s-2})$, $L(U^\varepsilon)$ belongs to $L_T^2(H^{1,s-1})$. On the other hand, since U^ε belongs to $L_T^4(H^{1/2,s})$ Lemma 3.2 implies that $U^\varepsilon \cdot \nabla U^\varepsilon$ belongs to $L_T^2(H^{-1,-s})$. So, $\Delta_h U^\varepsilon + U^\varepsilon \cdot \nabla U^\varepsilon + \frac{1}{\varepsilon}L(U^\varepsilon)$ belongs to $L_T^2(H^{-1,-3/2})$. Consequently, by the $(MHD)_{\nu_h}^\varepsilon$,

$$\partial_t U^\varepsilon \in L_T^2(H^{-1,-3/2}).$$

Remark 2. — The regularity made on U^ε invoked by hypothesis is insufficient by itself to define a trace of U^ε at time $t = 0$. Nevertheless, the facts that $\partial_t U^\varepsilon$ belongs $L_T^2(H^{-1,-3/2})$ and U^ε belongs to $L_T^2(H^{1,s})$ imply by the interpolation theory, developed in [9], that U^ε belongs to $C_T^0(H^{0,(2s-3)/4})$. Lemma 3.1 and the fact that U^ε belongs to $L_T^\infty(H^{0,s})$ imply that

$$\forall r < s, \quad U^\varepsilon \in C_T^0(H^{0,r}).$$

4. Proof of convergence result

The “linearized” equation associated to the system $(\mathcal{S}^\varepsilon)$ is

$$(L\mathcal{S}^\varepsilon) \left\{ \begin{array}{l} \partial_t U^\varepsilon - \nu_h \Delta_h U^\varepsilon + \frac{1}{\varepsilon} L(U^\varepsilon) = 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x^3 \\ \operatorname{div} u^\varepsilon = 0 \\ \operatorname{div} b^\varepsilon = 0 \\ U^\varepsilon|_{t=0} = U_0(x). \end{array} \right.$$

In Fourier variables $\xi \in \mathbb{R}^3$, we obtain

$$\partial_t \mathcal{F}(U^\varepsilon) + \nu_h |\xi_h|^2 \mathcal{F}(U^\varepsilon) + \frac{1}{\varepsilon} A(\xi) \mathcal{F}(U^\varepsilon) = 0.$$

Hence, we are led to study the following family of operators

$$\begin{aligned} \mathcal{G}^\varepsilon : f &\longmapsto \int_{\mathbb{R}_\xi^3} \mathcal{F}(f)(\xi) e^{-t(\nu_h |\xi_h|^2 + i\frac{a(\xi)}{\varepsilon}) + ix \cdot \xi} d\xi \\ &= \int_{\mathbb{R}_y^3 \times \mathbb{R}_\xi^3} f(y) e^{-t(\nu_h |\xi_h|^2 + i\frac{a(\xi)}{\varepsilon}) + i(x-y) \cdot \xi} d\xi dy. \end{aligned}$$

Notice that the phase function $a(\xi)$ is such that

$$a(\xi) \in \left\{ \pm \frac{\xi_3}{|\xi|} (1 + \sqrt{1 + 4|\xi|^2}), \pm \frac{\xi_3}{|\xi|} (1 - \sqrt{1 + 4|\xi|^2}) \right\}.$$

So, it is almost stationary when ξ_3 is almost equal to 0 as well as when $|\xi|$ goes to $+\infty$.

For some $0 < r < \min(R, R')$, let us define the domain $\mathcal{C}_{r,R,R'}$ by

$$\mathcal{C}_{r,R,R'} = \left\{ \xi \in \mathbb{R}^3; R' \geq |\xi_3| \geq r; |\xi_h| \leq R \right\}$$

and consider a cut-off function ψ , which is radial with respect to horizontal variable ξ_h and whose value is 1 near $\mathcal{C}_{r,R,R'}$.

First, we study the case when $\mathcal{F}(f)$ is supported in $\mathcal{C}_{r,R,R'}$. We have

$$\mathcal{G}^\varepsilon f(t, x) = \left(K(t/\varepsilon, \nu t, \cdot) \star f \right)(x),$$

where the kernel K is defined by

$$K(t, \tau, z) = \int_{\mathbb{R}^3} \psi(\xi) e^{ita(\xi) + iz \cdot \xi - \tau |\xi_h|^2} d\xi.$$

As in [3], we recall the following property of K :

LEMMA 4.1. — *For all r, R and R' satisfying $0 < r < \min(R, R')$, there exists a constant $C(r, R, R')$ such that*

$$\|K(t, \tau, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C(r, R, R') \min\{1, t^{-\frac{1}{2}}\}.$$

Proof. — The proof follows the lines of a stationary phase method. First, using the rotation invariance in ξ_h , we restrict to the case $z_2 = 0$. Next, if we denote

$$\alpha(\xi) = -\partial_{\xi_2} a(\xi),$$

we remark that

$$|\alpha(\xi)| \geq C(r, R, R') |\xi_2|,$$

where C is a strictly positive constant depending only on r, R and R' . Then, for all $\xi \in \mathcal{C}_{r, R, R'}$, we introduce the differential operator \mathcal{L} defined by

$$\mathcal{L} = \frac{1}{1 + t\alpha^2(\xi)} (1 + i\alpha(\xi)\partial_{\xi_2})$$

which acts on the ξ_2 variables and satisfies $\mathcal{L}(e^{ita}) = e^{ita}$.

Integrating by parts, we obtain

$$K(t, \tau, z) = \int_{\mathbb{R}^3} {}^t\mathcal{L}\left(\psi(\xi)e^{-\tau|\xi_h|^2}\right) e^{ita(\xi) + iz \cdot \xi} d\xi.$$

Easy computation yields

$$\begin{aligned} {}^t\mathcal{L}(\psi(\xi)e^{-\tau|\xi_h|^2}) &= \left(\frac{1}{1 + t\alpha^2} - i(\partial_{\xi_2}\alpha) \frac{1 - t\alpha^2}{(1 + t\alpha^2)^2} \right) \psi(\xi)e^{-\tau|\xi_h|^2} \\ &\quad - \frac{i\alpha}{1 + t\alpha^2(\xi)} \partial_{\xi_2}(\psi(\xi)e^{-\tau|\xi_h|^2}). \end{aligned}$$

Using the fact that ξ is in a fixed annulus of \mathbb{R}^3 , and $\psi \in \mathcal{D}(\mathbb{R}^3)$, we get

$$\|K(t, \tau, \cdot)\|_{L^\infty} \leq C(r, R, R') \int_{\mathbb{R}} \frac{d\xi_2}{1 + t\xi_2^2}$$

which proves the lemma. \square

Denote by w^ε the solution of the free linear system associated to $(\mathcal{S}^\varepsilon)$ defined by

$$(PLF^\varepsilon) \begin{cases} \partial_t w^\varepsilon - \nu_h \Delta_h w^\varepsilon + \frac{1}{\varepsilon} L(w^\varepsilon) = f & \text{in } \mathbb{R}_t \times \mathbb{R}_x^3 \\ w^\varepsilon(0) = w_0. \end{cases}$$

Lemma 4.1 yields, in a standard way, the following Strichartz estimate (see [6]).

COROLLARY 4.2. — For all constants r , R and R' such that $0 < r < \min(R, R')$, let $\mathcal{C}_{r,R,R'}$ be the domain defined above. Then a constant $C'(r, R, R')$ exists such that if

$$\text{supp}(\mathcal{F}(w_0)) \cup \text{supp}(\mathcal{F}(f)) \subset \mathcal{C}_{r,R,R'}$$

then the solution w^ε of (PLF^ε) with the forcing term f and initial data w_0 satisfies

$$\|w^\varepsilon\|_{L^4(\mathbb{R}_+, L^\infty)} \leq C'(r, R, R') \varepsilon^{1/4} \left(\|w_0\|_{L^2} + \|f\|_{L^1(\mathbb{R}_+, L^2)} \right).$$

Notice that the constant $C'(r, R, R')$ does not depend on ε . Using the above estimate, we are able to prove the convergence result.

Proof of Theorem 1.3. — Denote by F^ε the quadratic part, the system $(\mathcal{S}^\varepsilon)$ can be reformulated in the following form

$$(\mathcal{S}^\varepsilon) \quad \begin{cases} \partial_t U^\varepsilon - \nu_h \Delta_h U^\varepsilon + \frac{1}{\varepsilon} L(U^\varepsilon) = F^\varepsilon & \text{in } \mathbb{R}_t \times \mathbb{R}_x^3 \\ \text{div } u^\varepsilon = 0 & \text{in } \mathbb{R}_x^3 \\ \text{div } b^\varepsilon = 0 & \text{in } \mathbb{R}_x^3 \\ U^\varepsilon|_{t=0} = U_0(x). \end{cases}$$

Define U_R^ε and \tilde{U}_R^ε respectively by

$$U_R^\varepsilon = \chi(|\nabla_h|/R) U^\varepsilon$$

and

$$\tilde{U}_R^\varepsilon = U^\varepsilon - U_R^\varepsilon.$$

To study the asymptotic behavior of \tilde{U}_R^ε , one begins by noting that, for all $\eta > 0$,

$$\begin{aligned} \|\tilde{U}_R^\varepsilon\|_{H^{1-\eta,0}}^2 &= \int_{\mathbb{R}^3} (1 - \chi(|\xi_h|/R))^2 |\xi_h|^{2(1-\eta)} |\mathcal{F}(U^\varepsilon)(\xi)|^2 d\xi \\ &\leq \int_{\xi_3} \int_{|\xi_h| \geq R} \frac{1}{|\xi_h|^{2\eta}} |\xi_h|^2 |\mathcal{F}(U^\varepsilon)(\xi_h, \xi_3)|^2 d\xi \\ &\leq R^{-2\eta} \int_{\xi_3} \int_{|\xi_h| \geq R} |\xi_h|^2 |\mathcal{F}(U^\varepsilon)(\xi_h, \xi_3)|^2 d\xi. \end{aligned}$$

It follows that, for all $R > 1$

$$\int_0^T \|\tilde{U}_R^\varepsilon(t)\|_{H^{1-\eta,0}}^2 dt \leq R^{-2\eta} \int_0^T \|\nabla_h U^\varepsilon(t)\|_{L^2}^2 dt.$$

By the energy estimate (2.5), one obtains

$$\int_0^T \|\tilde{U}_R^\varepsilon(t)\|_{H^{1-\eta,0}}^2 dt \leq R^{-2\eta} \frac{1}{2\nu_h} \|U_0\|_{H^{0,s}}^2.$$

Moreover, for all $0 < \eta < 1$ and $R > 1$, it holds that

$$\int_0^T \|\tilde{U}_R^\varepsilon(t)\|_{L^2}^2 dt \leq \int_0^T \|\tilde{U}_R^\varepsilon(t)\|_{H^{1-\eta,0}}^2 dt.$$

So, by interpolation inequality, one infers that

$$\forall \eta > 0, \forall s' < s, \limsup_{\varepsilon \rightarrow 0} \|\tilde{U}_R^\varepsilon\|_{L_T^2(H^{1-\eta,0} \cap H^{0,s'})} \xrightarrow{R \rightarrow +\infty} 0. \quad (4.1)$$

To estimate U_R^ε , note that

$$\partial_t U_R^\varepsilon - \nu_h \Delta_h U_R^\varepsilon + \frac{1}{\varepsilon} L(U_R^\varepsilon) = F_R^\varepsilon,$$

where

$$F_R^\varepsilon := \chi(|\nabla_h|/R) F^\varepsilon.$$

Apply the Fourier transform to obtain

$$\mathcal{F}(U_R^\varepsilon)(t, \xi) = \mathcal{F}(H_R^\varepsilon)(t, \xi) + \mathcal{F}(K_R^\varepsilon)(t, \xi),$$

where

$$\mathcal{F}(H_R^\varepsilon)(t, \xi) := \exp(t\tilde{A}(\varepsilon, \xi)) \mathcal{F}(U_{0,R})(\xi)$$

and

$$\mathcal{F}(K_R^\varepsilon)(t, \xi) := \int_0^t \exp((t-\tau)\tilde{A}(\varepsilon, \xi)) \mathcal{F}(F_R^\varepsilon)(\tau, \xi) d\tau.$$

The expression of the operator \tilde{A} is given by the above equation.

For $0 < r < R < \min(R, R')$, one can decompose $\mathcal{F}(K_R^\varepsilon)$ as follows

$$\begin{aligned} \mathcal{F}(K_R^\varepsilon)(t, \xi) &= \chi(\xi_3/r) \mathcal{F}(K_R^\varepsilon)(t, \xi) \\ &+ (1 - \chi(\xi_3/r)) \chi(\xi_3/R') \mathcal{F}(K_R^\varepsilon)(t, \xi) \\ &+ (1 - \chi(\xi_3/R')) \mathcal{F}(K_R^\varepsilon)(t, \xi). \end{aligned}$$

To investigate $\chi(|D_3|/r) K_R^\varepsilon(t)$, note that

$$\begin{aligned} \|\chi(|D_3|/r) K_R^\varepsilon(t)\|_{L^\infty(\mathbb{R}^3)} &\leq \|\mathcal{F}(\chi(|D_3|/r) K_R^\varepsilon(t))\|_{L^1(\mathbb{R}^3)} \\ &\leq \int_0^t \int_{\mathbb{R}^3} \chi(\xi_3/r) \chi(|\xi_h|/R) |\mathcal{F}(F^\varepsilon)(\tau, \xi)| d\xi \\ &\leq rR^3 \int_0^t \|\mathcal{F}(U^\varepsilon(\tau, \cdot) \otimes U^\varepsilon(\tau, \cdot))\|_{L^\infty} d\tau \int_{\mathbb{R}^3} \chi(\eta_3) \chi(|\eta_h|) d\eta. \end{aligned}$$

By the fact that

$$\begin{aligned} \|\mathcal{F}(U^\varepsilon(\tau, \cdot) \otimes U^\varepsilon(\tau, \cdot))\|_{L^\infty} &\leq \|U^\varepsilon(\tau, \cdot) \otimes U^\varepsilon(\tau, \cdot)\|_{L^1} \\ &\leq \|U^\varepsilon(\tau, \cdot)\|_{L^2}^2 \\ &\leq \|U_0\|_{L^2}^2, \end{aligned}$$

one has, for all $R > 0$,

$$\limsup_{\varepsilon \rightarrow 0} \|\chi(|D_3|/r)K_R^\varepsilon\|_{L_T^\infty(L^\infty(\mathbb{R}^3))} \xrightarrow{r \rightarrow 0} 0.$$

To investigate the term $(1 - \chi(\xi_3/r))\chi(\xi_3/R')\mathcal{F}(K_R^\varepsilon)(t, \xi)$, one begins by using Corollary 4.2 to obtain

$$\begin{aligned} &\|(1 - \chi(|D_3|/r))\chi(|D_3|/R')K_R^\varepsilon\|_{L_T^4(L^\infty)} \\ &\leq C\varepsilon^{1/4}\|(1 - \chi(|D_3|/r))\chi(|D_3|/R')F_R^\varepsilon\|_{L_T^1(L^2)}, \end{aligned}$$

where $C = C(r, R, R')$.

To prove that $\|(1 - \chi(|D_3|/r))\chi(|D_3|/R')F_R^\varepsilon\|_{L_T^1(L^2)}$ is bounded, use classical computation and elementary properties of Fourier transform to obtain

$$\begin{aligned} &\|(1 - \chi(|D_3|/r))\chi(|D_3|/R')F_R^\varepsilon\|_{L^2}^2 \\ &= c \int_{\mathbb{R}^3} (1 - \chi(\xi_3/r))^2 \chi(\xi_3/R')^2 |\mathcal{F}(F_R^\varepsilon)(t, \xi)|^2 d\xi \\ &\leq c(R + R')^2 \int_{\mathbb{R}^3} (1 - \chi(\xi_3/r))^2 \chi(\xi_3/R')^2 \chi(|\xi_h|/R)^2 |\mathcal{F}(U^\varepsilon \otimes U^\varepsilon)(t, \xi)|^2 d\xi \\ &\leq c(R + R')^2 \|\mathcal{F}(U^\varepsilon \otimes U^\varepsilon)(t, \cdot)\|_{L^\infty}^2 \int_{\mathbb{R}^3} (1 - \chi(\xi_3/r))^2 \chi(\xi_3/R')^2 \chi(|\xi_h|/R)^2 d\xi \\ &\leq c(R + R')^2 \|U^\varepsilon \otimes U^\varepsilon(t, \cdot)\|_{L^1}^2 \int_{\mathbb{R}^3} (1 - \chi(\xi_3/r))^2 \chi(\xi_3/R')^2 \chi(|\xi_h|/R)^2 d\xi \\ &\leq c(R + R')^2 R^2 R' \|U^\varepsilon(t, \cdot)\|_{L^2}^4 \int_{\mathbb{R}^3} \chi(\xi_3)^2 \chi(|\xi_h|)^2 d\xi \\ &\leq C(R, R') \|U^\varepsilon(t, \cdot)\|_{L^2}^4 \\ &\leq C(R, R') \|U_0\|_{H^{0,s}}^4. \end{aligned}$$

It follows that

$$\limsup_{\varepsilon \rightarrow 0} \|(1 - \chi(|D_3|/r))\chi(|D_3|/R')\chi(|\nabla_h|/R)K^\varepsilon\|_{L_T^4(L^\infty)} = 0.$$

To investigate $(1 - \chi(\xi_3/R'))\mathcal{F}(K_R^\varepsilon)$, note that

$$\begin{aligned}
 & \|\mathcal{F}^{-1}((1 - \chi(\xi_3/R'))\mathcal{F}(K_R^\varepsilon)(t, \xi))\|_{L^\infty(\mathbb{R}^3)} \\
 & \leq \int_0^t \|\mathcal{F}^{-1}((1 - \chi(\xi_3/R'))\chi(|\xi_h|/R)\mathcal{F}(F^\varepsilon)(\tau, \xi))\|_{L^\infty} d\tau \\
 & \leq \int_0^t \|(1 - \chi(\xi_3/R'))\chi(|\xi_h|/R)\mathcal{F}(F^\varepsilon)(\tau, \xi)\|_{L^1} d\tau \\
 & \leq \int_0^t \int_{\mathbb{R}^3} (1 - \chi(\xi_3/R'))\chi(|\xi_h|/R)|\mathcal{F}(F^\varepsilon)(\tau, \xi)| d\tau \\
 & \leq \int_0^t \int_{\mathbb{R}^3} \frac{1 - \chi(\xi_3/R')}{|\xi_3|^{s-1}} \chi(|\xi_h|/R) |\xi_3|^{s-1} |\mathcal{F}(F^\varepsilon)(\tau, \xi)| d\tau \\
 & \leq RI_{R,R'}^\varepsilon(t, s-1) + I_{R,R'}^\varepsilon(t, s),
 \end{aligned}$$

where

$$I_{R,R'}^\varepsilon(t, \beta) = \int_0^t \int_{\mathbb{R}^3} P_{R,R'}(\xi) |\xi_3|^\beta |\mathcal{F}(U^\varepsilon \otimes U^\varepsilon)(\tau, \xi)| d\xi d\tau$$

and

$$P_{R,R'}(\xi) := \frac{1 - \chi(\xi_3/R')}{|\xi_3|^{s-1}} \chi(|\xi_h|/R).$$

It is clear that

$$\int_{\mathbb{R}^3} P_{R,R'}(\xi) d\xi \xrightarrow{R' \rightarrow \infty} 0.$$

Recall the elementary inequality

$$\forall a > 0 \quad \text{and} \quad x, y \in \mathbb{R} \quad |x|^a \leq 2^a(|x-y|^a + |y|^a)$$

to deduce that, for all $a > 0$ and for all $V \in H^{0,a}(\mathbb{R}^3)$, that

$$|\xi_3|^a |\mathcal{F}(V \otimes V)(\xi)| \leq 2^{a+1} \|V\|_{H^{0,a}} \|V\|_{L^2}.$$

So, for all $\beta \in \{s, s-1\}$, it holds that

$$\begin{aligned}
 |\xi_3|^\beta |\mathcal{F}(U^\varepsilon \otimes U^\varepsilon)(\tau, \xi)| & \leq 2^{\beta+1} \|U^\varepsilon\|_{H^{0,\beta}} \|U^\varepsilon\|_{L^2} \\
 & \leq 2^{\beta+1} \|U^\varepsilon\|_{H^{0,\beta}}^2.
 \end{aligned}$$

By the energy estimate (2.5), one obtains

$$\|\mathcal{F}^{-1}((1 - \chi(\xi_3/R'))\mathcal{F}(K_R^\varepsilon)(t, \xi))\|_{L^\infty(\mathbb{R}^3)} \leq 2^s (R+2) T \|U_0\|_{H^{0,s}}^2 \int_{\mathbb{R}^3} P_{R,R'}(\xi) d\xi.$$

Finally, one deduces that

$$\limsup_{\varepsilon \rightarrow 0} \|\mathcal{F}^{-1}((1 - \chi(\xi_3/R'))\mathcal{F}(K_R^\varepsilon)(t, \xi))\|_{L_T^\infty(L^\infty)} \xrightarrow{R' \rightarrow +\infty} 0.$$

By (2.5), (1.3) and (1.4) it holds that

$$\limsup_{\varepsilon \rightarrow 0} \|U_R^\varepsilon\|_{L_T^4(L^\infty)} \xrightarrow{R \rightarrow +\infty} 0. \quad (4.2)$$

Use (4.1), (4.2) and the injection $L^\infty(\mathbb{R}^3) \cup L^2(\mathbb{R}^3) \hookrightarrow \mathcal{C}^{-3/2}(\mathbb{R}^3)$ to obtain

$$U^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0, \text{ in } L_T^2(\mathcal{C}^{-3/2}(\mathbb{R}^3))$$

which ends the proof of Theorem 1.3. \square

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