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## Limit trees and generic discriminants of minimal surface singularities<sup>(\*)</sup>

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**ABSTRACT.** — According to R. Bondil the dual graph of the minimal resolution of a minimal normal surface singularity determines the generic discriminant of that singularity. In this article we give with combinatorial arguments the link between the limit trees and the generic discriminants of minimal normal surface singularities. The weighted limit trees of a minimal surface singularity determine the generic discriminant of that singularity.

**RÉSUMÉ.** — D’après R. Bondil, le graphe dual de la résolution minimale d’une singularité minimale de surface normale détermine le discriminant générique de cette singularité. Par des arguments combinatoires, nous donnons dans cet article le lien entre les arbres limites et les discriminants génériques des singularités minimales de surfaces normales. Les arbres limites pondérés d’une singularité minimale de surface normale détermine le discriminant générique de cette singularité.

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### Introduction

Minimal normal surface singularities are the rational surface singularities with reduced fundamental cycle. These singularities were studied by Spivakovsky [10], Theo De Jong and Van Straten [7] and recently by R. Bondil [3], [4]. By using a result of Spivakovsky in [10], R. Bondil gives in [3] the algebraic structure of the generic discriminants of minimal normal surface singularities. However in their study of the deformation theory of minimal surface singularities Theo De Jong and Van Straten introduced the

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notion of limit trees for these singularities (see [7]). It is shown in [7] that any limit tree of a minimal surface singularity determines the dual graph of the minimal resolution of that singularity ([7], Lemma (1.16)). In [4] R. Bondil showed with induction arguments that the limit trees of a minimal surface singularity are intimately bound with its generic discriminant. We give in this article a combinatorial point of view on this relation. The interest of a weighted limit tree of a minimal surface singularity is that it determines both the generic discriminant and the dual graph (of the minimal resolution) of that singularity. By using the notion of limit trees we can give examples of different minimal surface singularities with equisingular generic discriminants.

The generic discriminants of normal surface singularities are defined in section 1. In section 2 we will recall a characterization of the dual graphs of minimal surface singularities. We introduce in section 3 some new integer invariants (cf. Notation 3.3) on the vertices of minimal graphs. We will use them in section 5. Theorem 3.5 gives the algebraic structure of the generic discriminants of minimal surface singularities. The limit trees of minimal surface singularities are defined in section 4. The main result of the article is Theorem 5.5.

## 1. Generic discriminants of normal surface singularities

Let  $(S, 0)$  be a germ of normal complex surface singularity and take a representative  $S$  embedded in  $\mathbb{C}^N$ . For any  $(N - 2)$ -dimensional subspace  $D$  in  $\mathbb{C}^N$ , we consider the linear projection  $\mathbb{C}^N \rightarrow \mathbb{C}^2$  with kernel  $D$  and denote by  $p_D : (S, 0) \rightarrow \mathbb{C}^2$  the restriction of this projection to  $(S, 0)$ .

Considering a small representative  $S$  of the germ  $(S, 0)$  and restricting to the  $(N - 2)$ -dimensional subspaces  $D$  such that  $p_D$  is finite, we define as in [11] the *polar curve*  $C(D)$  of the projection  $p_D$  as the closure in  $S$  of the critical locus of the restriction of  $p_D$  to  $S \setminus \{0\}$ . It is a reduced analytic curve. It is shown in [11] that it makes sense to say that for an open dense subset of the Grassmannian of  $(N - 2)$ -linear subspaces of  $\mathbb{C}^N$  the polar curves  $C(D)$  are equisingular in terms of strong simultaneous resolution (cf. [5] for this notion). It also turns out that this equisingularity class depends only on the analytic type of the germ  $(S, 0)$  (cf. [11], page 430).

*The discriminant* of the finite projection  $p_D$  is (the germ at 0 of) the reduced analytic curve of  $(\mathbb{C}^2, 0)$ , image of the polar curve  $C(D)$ .

We can state the following result (cf. [5], Proposition VI.2, [11], page 352, 462).

**THEOREM 1.1.** — *There is an open dense subset  $\mathcal{W}$  of the Grassmannian of  $(N - 2)$ -linear subspaces of  $\mathbb{C}^N$  such that the discriminants  $\Delta_{p_D}$ ,  $D \in \mathcal{W}$  obtained are equisingular (germs of) plane curves.*

We refer to [5], [12], [13] for the concept of equisingularity of reduced plane curves. As explained in [5] the equisingularity class of the discriminant  $\Delta_{p_D}$ ,  $D \in \mathcal{W}$  is uniquely defined by the saturation ring  $\hat{\mathcal{O}}_{C(D),0}$  of the polar curve  $C(D)$ . Also note that the equisingularity class of the discriminant  $\Delta_{p_D}$ ,  $D \in \mathcal{W}$  depends only on the analytic type of the germ  $(S, 0)$ . We will denote by  $\Delta_{S,0}$  the equisingularity class of the discriminant of a generic projection  $p_D$  and call it *the generic discriminant* of the normal surface singularity  $(S, 0)$ .

**DEFINITION 1.2.** — *Let  $(C_1, 0)$ ,  $(C_2, 0)$  be two analytically irreducible plane curve germs. The contact between  $(C_1, 0)$  and  $(C_2, 0)$  is defined as the number of point blow-ups necessary to separate these two branches.*

## 2. Minimal normal surface singularities

The class of minimal normal surface singularities can be defined as the subclass of rational surface singularities with reduced tangent cone. The reader can find in [8] the definition of minimal singularities in any dimension. Let us quote the following result from [8].

**THEOREM 2.1.** — *A normal surface singularity is minimal if and only if it is rational with reduced fundamental cycle (with the terminology of [2]).*

Let  $(S, 0)$  be a normal surface singularity and  $\pi : X \rightarrow (S, 0)$  a resolution of the singularity. We denote by  $\Gamma$  the dual graph associated to the exceptional curve configuration  $\pi^{-1}(0) = \cup_{i=1}^n E_i$  in the usual way. For rational surface singularities it is well known that all the irreducible components of the exceptional curve are smooth rational curves and the dual graph  $\Gamma$  is a tree. Also note that it takes some computation to check whether a given weighted tree is the dual graph of a rational surface singularity (cf. [9]). For any dual graph  $\Gamma$  we will denote by  $w_i = -E_i^2$  the weight of the vertex  $i \in \Gamma$  ( $E_i^2$  is the self-intersection of the corresponding component  $E_i$  on  $X$ ) and we will denote by  $v_i$  the valence of the vertex  $i \in \Gamma$ , i.e, the number of edges attached to  $i$ .

The following statement holds [10].

**PROPOSITION 2.2.** — *The graph  $\Gamma$  is the dual graph of a minimal normal surface singularity if and only if,  $\Gamma$  is a tree and  $w_i \geq v_i$  for each vertex  $i \in \Gamma$ .*

In this work we will mainly use the dual graphs of the minimal resolutions of minimal normal surface singularities. We will simply say that the graph  $\Gamma$  is a *minimal graph*.

A vertex  $E$  of a minimal graph  $\Gamma$  will be called a *Tyurina vertex* if  $w_E = v_E$  (see [10], Definition 3.1). We denote by  $\Gamma_{TC}$  the set of vertices  $E$  which are not Tyurina, i.e.,  $w_E > v_E$ . Such vertices will be called *non-Tyurina*.

### 3. Generic discriminants of minimal surface singularities

By using a result of Spivakovsky ([10], Theorem 5.4) R. Bondil gives in [4] the algebraic structure of the generic discriminants of minimal normal surface singularities. To state these results we introduce some further terminology.

Let  $\pi : X \rightarrow (S, o)$  be the minimal resolution of the minimal surface singularity  $(S, 0)$ , where  $\pi^{-1}(0) = \cup_{i=1}^n E_i$  is the exceptional divisor with components  $E_i$ . Let  $\Gamma$  be the corresponding minimal graph (we will frequently abuse notation and write  $E_i \in \Gamma$  to indicate the vertex of  $\Gamma$  corresponding to the component  $E_i$ ). The following notions were introduced in [10].

**DEFINITION 3.1.** — *The depth of the vertex  $E$  is  $s_E = 1 + \text{dist}(E, \Gamma_{TC})$ , where  $\text{dist}(E, \Gamma_{TC})$  is the distance of  $E$  to  $\Gamma_{TC}$ .*

*A vertex  $k$  is called a central vertex if there are at least two vertices  $i, j$  adjacent to  $k$  such that  $s_i - 1 = s_k = s_j - 1$ .*

*Let  $i, j$  be two adjacent vertices of  $\Gamma$ . The edge  $(i, j)$  is a central arc if  $s_i = s_j$ .*

We then define a  $\mathbb{Q}$ -cycle on the minimal resolution  $X$  of  $(S, 0)$  by  $Z_\Omega = \sum_{i \in \Gamma} s_i E_i - K$  where  $K$  is the numerically canonical  $\mathbb{Q}$ -cycle uniquely defined by the condition: for all  $i \in \Gamma$ ,  $K.E_i = -2 - E_i^2$  (since the intersection product on  $\cup E_i$  is negative definite).

We quote Spivakovsky's result ([10], Theorem 5.4).

**THEOREM 3.2.** — *Let  $(S, 0)$  be a minimal normal surface singularity. There is an open dense subset  $\mathcal{V}$  of the open set  $\mathcal{W}$  of Theorem 1.1, such that for all  $D \in \mathcal{V}$  the strict transform  $C'(D)$  of the polar curve  $C(D)$  on  $X$ :*

*a) is a multi-germ of curves intersecting each component  $E_i$  transversally in exactly  $m_i := -Z_\Omega.E_i$  points,*

*b) goes through the point of intersection of  $E_i$  and  $E_j$  if and only if  $s_i = s_j$  (point corresponding to a central arc of the minimal graph  $\Gamma$ ). Furthermore, the curves  $C'(D)$ ,  $D \in \mathcal{V}$  do not share other common points (base points) and these base points are simple, i.e., the curves  $C'(D)$  are separated when one blows up these points once.*

We give here an explicit expression of the number  $m_E = -Z_{\Omega}.E$  of the branches of the generic polar curve strict transform which intersect the component  $E$  (of the exceptional fibre of the minimal resolution). Let us define first the following new integer invariants.

*Notation 3.3.* — We denote by  $n_T^E$  (resp.  $n_{TC}^E$ ) the number of Tyurina (resp. non-Tyurina) vertices adjacent to the vertex  $E$ .

Let  $E$  be a Tyurina vertex with depth  $s_E$ . We will denote by  $n_+^E$  (resp.  $n_-^E$ ) the number of vertices  $F$  adjacent to  $E$  such that  $s_F = s_E + 1$  (resp.  $s_F = s_E - 1$ ) and  $n_{\pm}^E$  will be the number of vertices  $F$  adjacent to  $E$  such that  $s_F = s_E$ .

Note that if  $E$  is a Tyurina vertex with  $n_{\pm}^E \geq 2$  then  $E$  is a central vertex (cf. Definition 3.1). If  $n_{\pm}^E \neq 0$  then  $E$  is an endpoint of  $n_{\pm}^E$  central arcs.

We state ([1], Corollary 3.1.1.).

**COROLLARY 3.4.** — *The following formulas hold:*

$m_E = 2(w_E - v_E - 1) + n_{TC}^E$  if  $E$  is non-Tyurina and  
 $m_E = 2(n_{\pm}^E - 1) + n_{\pm}^E$  if  $E$  is Tyurina.

It follows from Theorem 3.2 that the generic polar curve has components of multiplicity equal at most two (cf. [1], [3]). In fact the minimal resolution of the minimal surface singularity  $(S, 0)$  is a resolution of the generic polar curve (cf. Theorem 3.2). By the projection formula for the intersection number the multiplicity  $e(C, 0)$  of any component  $C$  of the generic polar curve is  $e(C, 0) = Z.\tilde{C}$  where  $Z = \sum_{i=1}^n E_i$  is the fundamental cycle and  $\tilde{C}$  is the strict transform of the component  $C$ . By Theorem 3.2 we have  $Z.\tilde{C} \leq 2$ . We recall that an  $A_n$ -curve is a curve analytically isomorphic to the plane curve defined by  $x^2 + y^{n+1} = 0$ . Note that any reduced curve of multiplicity equal to two is an  $A_n$ -curve for some  $n$ .

The generic discriminant of a minimal surface singularity with minimal graph  $\Gamma$  is determined in the following way: let  $\zeta$  denote the set of central arcs and central vertices of the minimal graph  $\Gamma$ . For any element  $x \in \zeta$  we consider a set  $C(x)$  of chains  $c = [E_p, E_q] \subset \Gamma$  containing the central element  $x$  and such that  $E_p, E_q$  are non-Tyurina vertices and the depth function is monotonically increasing with step one on the chains  $(E_p, x), (E_q, x)$ . We denote by  $l(c)$  the number of vertices in the chain  $[E_p, E_q]$ , i.e.  $l(c) = \text{dist}(E_p, E_q) + 1$ .

Note that the set of chains  $C(x)$  depends on the number of branches (of the generic polar curve strict transform) which intersect the central element  $x$  (we refer to [1], [3], [4] for more details). For central elements  $x, y$  we will denote by  $(x, y)$  the minimal chain joining them in the minimal graph  $\Gamma$ .

The following theorem by R. Bondil [3], [4] gives the algebraic structure of the generic discriminants of the minimal surface singularities with minimal graph  $\Gamma$ .

**THEOREM 3.5.** — *For any non-Tyurina vertex  $E_i \in \Gamma$  we denote by  $\delta_{E_i}$  a germ of curve defined by  $2(w_{E_i} - v_{E_i}) - 2$  distinct lines.*

*For any central element  $x \in \zeta$  and  $c \in C(x)$  we consider an  $A_{I(c)}$ -curve.*

*a) The generic discriminant of the minimal surface singularity with minimal graph  $\Gamma$  is the union*

$$\Delta_{S,0} = \bigcup \delta_{E_i} \cup \bigcup_{x \in \zeta, c \in C(x)} A_{I(c)}.$$

*The contact (cf. Definition 1.2) between any line in  $\delta_{E_i}$  and any component  $A_{I(c)}$  is one. The contact between two distinct components  $A_{I(c)}$  and  $A_{I(c')}$  where  $c \in C(x)$ ,  $c' \in C(y)$ ,  $x, y \in \zeta$  is the minimum depth in the chain  $(x, y)$ .*

This theorem gives the equisingularity class of the generic discriminant. In fact we can obtain the multiplicity sequence (we refer to [6], page 507 for this notion) of each branch of the generic discriminant and we know the contacts between the branches. Then we can calculate the intersection number of any two branches by using the Max Noether's formula (cf. [6], page 518). It is not hard to obtain the Puiseux pairs of each branch. We then recall [13].

**THEOREM 3.6.** — *Two germs of plane curves  $X = \cup_{i \in I} X_i$  and  $X' = \cup_{j \in J} X'_j$  are equisingular if and only if there exists a bijection  $\psi : I \longrightarrow I'$  between their branches which preserves Puiseux characteristic pairs and intersection numbers.*

Note that different minimal surfaces singularities can have equisingular generic discriminants.

*Example 3.7.* — Let  $\Gamma_1$  and  $\Gamma_2$  be the following minimal graphs.

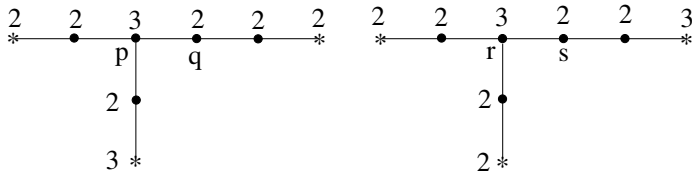


Figure 1. — Minimal graphs  $\Gamma_1$  and  $\Gamma_2$

The corresponding graph weighted by the depth function is  $\Gamma_{12}$ .

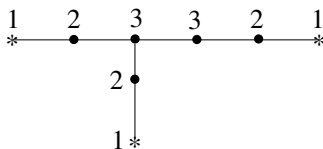


Figure 2. — Graph  $\Gamma_{12}$

Here  $p, r$  are central vertices and  $(p, q), (r, s)$  are central arcs.

The generic discriminant of the minimal singularities with dual graphs  $\Gamma_1, \Gamma_2$  is  $\Delta = \delta_2 \cup A_6 \cup A_5$ , where the contact between  $A_6$  and  $A_5$  is 3 and  $\delta_2$  denotes the union of two distinct lines.

The following data was introduced in [7]. We will use it in the next section.

**DEFINITION 3.8.** — *For any pair  $E_i, E_j$  of vertices of a minimal graph  $\Gamma$  we denote by  $(E_i, E_j)$  the (minimal) chain of  $\Gamma$  joining them, i.e. the geodesic in  $\Gamma$ . It is unique since the minimal graph  $\Gamma$  is a tree. The length  $l(E_i, E_j)$  of the chain  $(E_i, E_j)$  is the number of vertices in the chain  $(E_i, E_j)$  (including the endpoints). For different vertices  $E_i, E_j, E_k$  the overlap  $\rho(E_i, E_j; E_k)$  of the chains  $(E_i, E_k), (E_j, E_k)$  is the number of vertices in  $(E_i, E_k) \cap (E_j, E_k)$ .*

#### 4. Definition limit trees

In [7] Theo De Jong and Duco Van Straten introduced the notion of limit trees for minimal normal surface singularities. We point out that in [7] limit trees were defined by using the *height function* (cf. [7], Definition 1.10 (c)) on vertices of dual graphs of rational surface singularities. This height function is studied more systematically for any rational surface singularity as “desingularization depth” in [9]. For minimal surface singularities this height function corresponds exactly to the depth function defined above ([1], Proposition 4.1.1). The reader should check that this height corresponds to the number of point blow-ups necessary to make the corresponding exceptional component “appear”. Note that for any Tyurina vertex  $E \in \Gamma$  with depth  $s_E = k + 1, k \geq 1$  there exists at least one vertex  $F$  adjacent to  $E$  such that  $s_F = k$  (cf. [1], Remark 4.0.5).



DEFINITION 4.1. — Let  $\Gamma$  be a minimal graph. A limit equivalence relation  $\sim$  is an equivalence relation on the vertices of  $\Gamma$  satisfying the following conditions:

- a) vertices  $E$  with depth  $s_E = 1$  belong to different equivalence classes.
- b) for every vertex  $E$  in  $\Gamma$  with depth  $s_E = 1 + k$ ,  $k \geq 1$  take exactly one vertex  $F$  adjacent to  $E$  with  $s_F = k$  and  $E \sim F$ .

Then the tree  $T = \Gamma / \sim$  is called a limit tree associated to  $\Gamma$ .

The limit equivalence relation is not unique in general. A given minimal graph can have distinct limit trees, depending on the limit equivalence chosen.

Example 4.2. — Let  $\Gamma$  be the following minimal dual graph with the depths for the vertices.

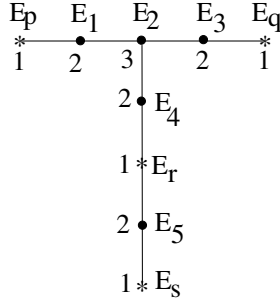


Figure 3. — A minimal graph  $\Gamma$  weighted by the depth function

For the equivalence classes  $\tilde{E}_p = \{E_p, E_1\}$ ,  $\tilde{E}_q = \{E_2, E_3, E_q\}$ ,  $\tilde{E}_r = \{E_4, E_r\}$ ,  $\tilde{E}_s = \{E_5, E_s\}$  the limit tree is  $T_1$ :



Figure 4. — Limit tree  $T_1$

And for the equivalence classes  $\tilde{E}_p = \{E_p, E_1\}$ ,  $\tilde{E}_q = \{E_3, E_q\}$ ,  $\tilde{E}_r = \{E_2, E_4, E_r\}$ ,  $\tilde{E}_s = \{E_5, E_s\}$ , the limit tree is  $T_2$ :

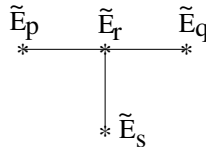


Figure 5. — Limit tree  $T_2$

DEFINITION 4.3. — *It is clear that any equivalence class contains exactly one vertex  $E$  of depth one (it is a non-Tyurina vertex) so that we will denote this equivalence class as vertex  $\tilde{E}$  in the limit tree  $T$ .*

*For any adjacent vertices  $\tilde{E}_p, \tilde{E}_q$  in the limit tree  $T$  we denote  $l_T(\tilde{E}_p, \tilde{E}_q) := l(E_p, E_q)$ . For different vertices  $\tilde{E}_p, \tilde{E}_q, \tilde{E}_r$  in  $T$  such that  $\tilde{E}_r$  is adjacent to  $\tilde{E}_p$  and  $\tilde{E}_q$  we denote  $\rho_T(\tilde{E}_p, \tilde{E}_q; \tilde{E}_r) := \rho(E_p, E_q; E_r)$  (where  $l$  and  $\rho$  are the functions defined in Definition 3.8, page 43). The degree  $d_T \tilde{E}_p$  of any vertex  $\tilde{E}_p$  in  $T$  is defined to be  $d_T \tilde{E}_p = w_{E_p} - v_{E_p}$ . We will use the notation  $(T, l_T, \rho_T, d_T)$  to denote exactly that data and  $(T, l_T, \rho_T, d_T)$  will be called a weighted limit tree of the minimal surface singularity with minimal graph  $\Gamma$ .*

Any limit tree  $(T, l_T, \rho_T, d_T)$  of the minimal graph  $\Gamma$  has the following property (cf. [7]). If  $(\tilde{p}, \tilde{r})$  and  $(\tilde{q}, \tilde{r})$  are adjacent edges in  $T$  then the following inequalities hold:

$$\rho_T(p, q; r) \leq \rho_T(q, r; p), \quad \rho_T(p, q; r) \leq \rho_T(r, p; q).$$

We recall Lemma 1.16 of [7]:

PROPOSITION 4.4. — *The weighted limit tree  $(T, l_T, \rho_T, d_T)$  determines the minimal graph  $\Gamma$ .*

## 5. Generic discriminants via limit trees

Let  $\Gamma = (E_i)_{1 \leq i \leq n}$  be a minimal graph. We may assume that for any  $i, 1 \leq i \leq N$  the vertex  $E_i$  is non-Tyurina, i.e.  $w_{E_i} > v_{E_i}$  and for  $N + 1 \leq i \leq n$  the vertex  $E_i$  is Tyurina, i.e.  $w_{E_i} = v_{E_i}$ .

PROPOSITION 5.1. — *The multiplicity  $e(\Delta_{S,0}, 0)$  of the generic discriminant of the minimal surface singularity with dual graph  $\Gamma$  is (cf. Notation 3.3, page 41)*

$$e(\Delta_{S,0}, 0) = 2 \sum_{i=1}^N (w_{E_i} - v_{E_i} - 1) + \sum_{i=1}^N n_{TC}^{E_i} + \sum_{i=1}^{N+1} n_{\pm}^{E_i} + 2 \sum_{N+1}^n (n_{\pm}^{E_i} - 1)$$

*Proof.* — The generic polar curve and the generic discriminant have the same multiplicity at 0 (cf. [11]). Using the projection formula for the intersection number and Theorem 3.2, the multiplicity of the generic polar curve  $C(D)$  is  $e(C(D), 0) = -Z_{\Omega} \cdot Z$  where  $Z = \sum_{i=1}^n E_i$  is the fundamental cycle and  $Z_{\Omega} = \sum_{i=1}^n s_i E_i - K$  is as in Theorem 3.2. Then we have

$$e(\Delta_{S,0}, 0) = -\sum_{i=1}^N Z_{\Omega} \cdot E_i - \sum_{i=N+1}^n Z_{\Omega} \cdot E_i$$

By Corollary 3.4 we know that  $-Z_{\Omega} \cdot E_i = 2(w_{E_i} - v_{E_i} - 1) + n_{TC}^{E_i}$  for any  $i = 1, \dots, N$  and  $-Z_{\Omega} \cdot E_i = 2(n_{\pm}^{E_i} - 1) + n_{\pm}^{E_i}$  for any  $i = N + 1, \dots, n$ .

The term  $2\sum_{i=1}^N (w_{E_i} - v_{E_i} - 1)$  in the above Proposition is the contribution of the curves  $\delta_{E_i}$  associated to non-Tyurina vertices  $E_i$  (cf. Theorem 3.5).

We now point out the following facts.

i) By the definitions of the integers  $n_{TC}^E, n_{\underline{=}}^E$  (cf. Notation 3.3) we can easily see that in the minimal graph  $\Gamma$  the number of distinct central arcs connecting non-Tyurina vertices (resp. Tyurina vertices) is equal to  $\frac{1}{2}\sum_{i=1}^N n_{TC}^{E_i}$  (resp.  $\frac{1}{2}\sum_{i=N+1}^n n_{\underline{=}}^{E_i}$ ).

ii) If a non-Tyurina vertex  $E_p$  is limit equivalent to a Tyurina vertex  $E_i$  then the depth function is monotonically increasing with step one in the chain  $(E_p, E_i)$ . We then have  $n_{\underline{=}}^{E_i} \neq 0$  for any Tyurina vertex  $E_i$ .

iii) Let us take any Tyurina vertex  $E_i$  such that  $n_{\underline{=}}^{E_i} \neq 0$  and any vertex  $E_j$  adjacent to  $E_i$  with  $s_{E_i} = s = s_{E_j}$  so that  $(E_i, E_j)$  is a central arc. Then there exists at least one chain  $(E_p, E_q)$  in  $\Gamma$  of the form shown in Figure 6.

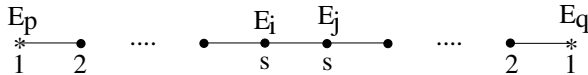


Figure 6. — Chain A

In Figure 6 the vertices  $E_p, E_q$  are non-Tyurina and the depth function is monotonically increasing with step one in the chains  $(E_p, E_i), (E_q, E_j)$ . Such a chain is not unique in general but for any central arc  $(E_i, E_j)$  we will choose only one chain of type A (cf. Figure 6). We will denote it by  $ch(E_i, E_j)$  and  $ch(E_i, E_j) = ch(E_j, E_i)$ .

Note that the strict transform (by the minimal resolution) of a component of the generic polar curve intersects components  $E_i, E_j$  and the image of such a component by the generic projection is a curve of type  $A_{2s} : x^2 + y^{2s+1} = 0$ .

iv) Any Tyurina vertex  $E_i$  such that  $n_{\underline{=}}^{E_i} \geq 2$  is a central vertex in  $\Gamma$ . Then let us take all vertices  $E_{i_1}, \dots, E_{i_k}, (k := n_{\underline{=}}^{E_i})$  adjacent to  $E_i$ , with depths equal to that of  $E_i$  minus one.

Let us fix one of them, e.g. the vertex  $E_{i_1}$ . For any  $E_{i_j} (j = 2, \dots, k)$  we can find in  $\Gamma$  a chain of the form shown in Figure 7. Here  $E_p, E_q$  are non-Tyurina vertices and the depth function is monotonically increasing with step one in the chains  $(E_p, E_{i_j}), (E_q, E_{i_1})$ . Such a chain is not unique in general but we will consider only one of them and denote it by  $ch(E_{i_1}, E_i, E_{i_j})$ . Then

we can define the set of chains

$$\mathcal{C}(E_{i_1}, E_i) := \{ch(E_{i_1}, E_i, E_{i_j}); j = 2, \dots, n_-^{E_i}\}.$$

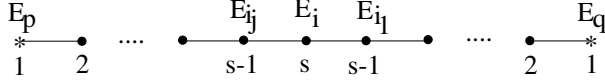


Figure 7. — Chain B

Note that  $\text{Card } \mathcal{C}(E_{i_1}, E_i) = n_-^{E_i} - 1$  and let us recall again that the generic polar curve has some components of type  $A_{2s-1}$  whose strict transforms (by the minimal resolution) intersect the component  $E_i$ . There are  $n_-^{E_i} - 1$  such components.

v) For two adjacent non-Tyurina vertices  $E_p, E_q$  we denote the arc  $(E_p, E_q)$  by  $ch(E_p, E_q)$  and  $ch(E_p, E_q) = ch(E_q, E_p)$ .

We will use the following sets:

$$A(\Gamma_T) := \{\{i, j\}; N+1 \leq i, j \leq n, i \neq j; E_i \text{ is adjacent to } E_j \text{ and } s_{E_i} = s_{E_j}\}$$

and

$$A(\Gamma_{TC}) := \{\{p, q\}; p \neq q, 1 \leq p, q \leq N, E_p \text{ is adjacent to } E_q\}.$$

The reader can easily see that the pairs of integers of  $A(\Gamma_T)$  correspond exactly to the central arcs connecting Tyurina vertices and those of  $A(\Gamma_{TC})$  correspond exactly to the central arcs connecting non-Tyurina vertices.  $\square$

PROPOSITION 5.2. — *The distinct chains of the set*

$$\{ch(E_i, E_j), \{i, j\} \in A(\Gamma_T); \mathcal{C}(E_{i_1}, E_i), i = N+1, \dots, n; ch(E_p, E_q), \{p, q\} \in A(\Gamma_{TC})\}$$

*correspond one-to-one to the edges of a limit tree  $\tilde{\Gamma}$  of the minimal graph  $\Gamma$ .*

*Proof.* — This is trivial if each vertex of the minimal graph  $\Gamma$  is non-Tyurina.

Suppose that some vertices are Tyurina.

For any Tyurina vertex  $E_i$  we consider  $n_{\underline{E}_i}^{E_i}$  chains of type A (cf. figure 6) and  $n_{\underline{E}_i}^{E_i} - 1$  chains of type B (cf. figure 7). We can obtain the limit equivalence classes so that in the chain  $ch(E_i, E_j)$  (cf. figure 6) all vertices of the chain  $(E_p, E_i)$  belong to the limit equivalence class of  $E_p$  and all vertices of  $(E_q, E_j)$  belong to the limit equivalence class of  $E_q$ . Again, we can obtain the limit equivalence classes so that in the chain  $ch(E_{i_1}, E_i, E_{i_j})$  (cf. figure 7) all vertices of  $(E_p, E_i)$  belong to the limit equivalence class of  $E_p$  and all vertices of  $(E_q, E_{i_1})$  belong to the limit equivalence class of  $E_q$ . Then the limit tree relative to the limit equivalence classes obtained is that of Proposition 5.2.  $\square$

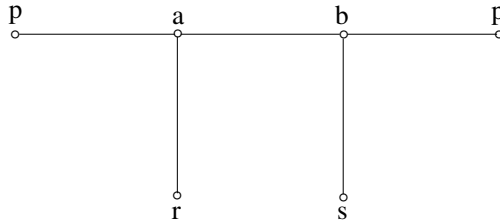
PROPOSITION 5.3. — *Assume that  $(\tilde{p}, \tilde{r})$ ,  $(\tilde{r}, \tilde{s})$ ,  $(\tilde{s}, \tilde{q})$  are some edges of the limit tree  $\tilde{\Gamma}$ . Let us denote by  $c_1$  (resp.  $c_2$ , resp.  $c_3$ ) the central element (central arc or central vertex) of the corresponding chain  $(p, r)$  (resp.  $(r, s)$ , resp.  $(s, q)$ ) in the minimal graph  $\Gamma$ . Then the following equality holds*

$$\min \{depth(c_1, c_2)\} = \min \{depth((c_1, c_2) \cup (c_2, c_3))\}.$$

Here  $(c_i, c_j)$  is the (minimal) chain in  $\Gamma$  joining  $c_i$  and  $c_j$ . The set of the vertices' depths on  $(c_i, c_j)$  is denoted by  $depth(c_i, c_j)$ .

*Proof.* — First note that the subgraph of  $\Gamma$  spanned by  $p, q, r$  and  $s$  is of the following type ([7], page 128, fig. B):

(Here the lines in the graph do not indicate edges of  $\Gamma$ , but rather arbitrary chains, so it is a qualitative picture of the subgraph).



Also note that by hypothesis the case  $a = b$  is not allowed. The vertex  $a$  necessarily belongs to the limit equivalence class of  $r$ . Then as  $(\tilde{p}, \tilde{r})$  is an edge of the limit tree, the central element  $c_1$  lies on the chain  $(p, a)$ . Again  $b$  belongs to the limit equivalence class of  $s$ . Then as  $(\tilde{s}, \tilde{q})$  is an edge of the limit tree, the central element  $c_3$  lies on the chain  $(b, q)$ . The reader can easily see that the central element  $c_2$  lies on the chain  $(a, b)$  because  $a$

belongs to the equivalence class of  $r$  and  $b$  belongs to the equivalence class of  $s$  and  $(\tilde{r}, \tilde{s})$  is an edge of the limit tree. It follows that

$$(c_1, c_3) = (c_1, c_2) \cup (c_2, c_3). \quad \square$$

*Remark 5.4.* — i) It is not hard to show that the above Proposition remains true for chains of length  $k$  in  $\tilde{\Gamma}$ ,  $k \geq 4$ , namely, if  $\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_k$  are vertices of the limit tree  $\tilde{\Gamma}$  such that  $(\tilde{p}_i, \tilde{p}_{i+1})$ ,  $i = 1, \dots, k-1$  is an edge of  $\tilde{\Gamma}$  then

$$\min \{ \text{depth}(c_1, c_k) \} = \min \{ \text{depth}((c_1, c_2) \cup (c_2, c_3) \cup \dots \cup (c_{k-1}, c_k)) \}.$$

ii) The reader can easily see that  $\min \{ \text{depth}(c_1, c_2) \} = \rho_T(\tilde{p}, \tilde{s}; \tilde{r})$  in Proposition 5.3.

iii) We can choose the chains  $\bigcup_{x \in \zeta} C(x)$  in Theorem 3.5 so that these chains correspond one-to-one to the chains of Proposition 5.2.

The previous results lead to the following statement. We will denote by  $e(T)$  the set of edges of the weighted limit tree  $\tilde{\Gamma} = (T, l_T, \rho_T, d_T)$ .

**THEOREM 5.5.** — *The weighted limit tree  $\tilde{\Gamma} = (T, l_T, \rho_T, d_T)$  determines the generic discriminant of the minimal surface singularity with minimal graph  $\Gamma$ . The generic discriminant  $\Delta_{S,0}$  decomposes into:*

$$\Delta_{S,0} = \Delta_{\Sigma_{i=1}^N 2(d_T \tilde{E}_i - 1)} \cup \bigcup_{(\tilde{p}, \tilde{q}) \in e(T)} A_{l_T(\tilde{p}, \tilde{q})}$$

where  $\Delta_{\Sigma_{i=1}^N 2(d_T \tilde{E}_i - 1)}$  is  $\Sigma_{i=1}^N 2(d_T \tilde{E}_i - 1)$  distinct lines in  $(\mathbb{C}^2, 0)$  and  $A_{l_T(\tilde{p}, \tilde{q})}$  is a curve in  $(\mathbb{C}^2, 0)$  of type  $A_{l_T(\tilde{p}, \tilde{q})} : x^2 + y^{l_T(\tilde{p}, \tilde{q})+1} = 0$ .

The contact between any line in  $\Delta_{\Sigma_{i=1}^N 2(d_T \tilde{E}_i - 1)}$  and any branch  $A_{l_T(\tilde{p}, \tilde{q})}$  is one.

For each pair of adjacent edges  $(\tilde{p}, \tilde{r}), (\tilde{r}, \tilde{q})$  the contact between  $A_{l_T(\tilde{p}, \tilde{r})}$  and  $A_{l_T(\tilde{r}, \tilde{q})}$  is exactly  $\rho_T(\tilde{p}, \tilde{q}; \tilde{q})$ .

For non adjacent edges  $(\tilde{p}, \tilde{r}), (\tilde{k}, \tilde{q})$  the contact between  $A_{l_T(\tilde{p}, \tilde{r})}$  and  $A_{l_T(\tilde{k}, \tilde{q})}$  is the minimum of the contacts between adjacent edges on the chain joining them (cf. Theorem 3.5 and Proposition 5.3).

This theorem gives the equisingularity class of the generic discriminant by the same arguments as in section 3 (cf. Theorem 3.6).

*Example 5.6.* — Suppose that the limit tree  $(T, l_T, \rho_T, d_T)$  is:

$$\begin{array}{cccc} \tilde{p} & \tilde{q} & \tilde{r} & \tilde{s} \\ * & * & * & * \end{array}$$

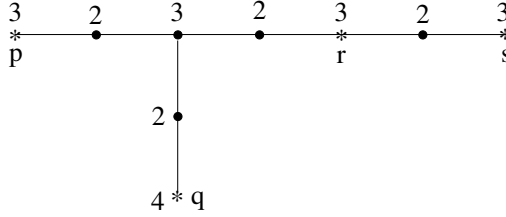
where  $l_T(\tilde{p}, \tilde{q}) = 5$ ,  $l_T(\tilde{q}, \tilde{r}) = 5$ ,  $l_T(\tilde{r}, \tilde{s}) = 3$ ,  $\rho_T(\tilde{p}, \tilde{r}; \tilde{q}) = 3$ ,  $\rho_T(\tilde{q}, \tilde{s}; \tilde{r}) = 1$ ,  $d_T \tilde{p} = 2$ ,  $d_T \tilde{q} = 3$ ,  $d_T \tilde{r} = 1$ ,  $d_T \tilde{s} = 2$ .

The generic discriminant is

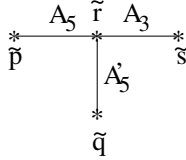
$$\Delta_{S,0} = \Delta_8 \cup A_5 \cup A'_5 \cup A_3.$$

The contact between  $A_5$  and  $A'_5$  is 3. The contact between  $A'_5$  and  $A_3$  is 1 and the contact between  $A_5$  and  $A_3$  is 1.

Note that the minimal graph  $\Gamma$  with limit tree  $(T, l_T, \rho_T, d_T)$  is :



Note also that the following tree is a limit tree of that minimal graph : here  $l_T(\tilde{p}, \tilde{q}) = 5$ ,  $l_T(\tilde{q}, \tilde{r}) = 5$ ,  $l_T(\tilde{r}, \tilde{s}) = 3$ ,  $\rho_T(\tilde{p}, \tilde{s}; \tilde{r}) = 1$ ,  $\rho_T(\tilde{p}, \tilde{q}; \tilde{r}) = 3$ ,  $\rho_T(\tilde{q}, \tilde{s}; \tilde{r}) = 1$ .



*Remark 5.7.* — a) A limit tree of a minimal graph  $\Gamma$  depends in general on the limit equivalence chosen (cf. example 4.2). We point out that for any weighted limit tree  $(T, l_T, \rho_T, d_T)$  of a minimal graph  $\Gamma$  we can find the corresponding set of chains described in Proposition 5.2. Hence any weighted limit tree  $(T, l_T, \rho_T, d_T)$  of a minimal surface singularity determines the generic discriminant of that minimal surface singularity.

b) Different minimal surface singularities with the same multiplicities and limit trees have equisingular generic discriminants.

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