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On the analytic non-integrability of the Rattleback problem

H. R. Dullin[1], A.V. Tsygvintsev[2]

Abstract. — We establish the analytic non-integrability of the nonholonomic ellipsoidal rattleback model for a large class of parameter values. Our approach is based on the study of the monodromy group of the normal variational equations around a particular orbit. The imbedding of the equations of the heavy rigid body into the rattleback model is discussed.

Résumé. — Nous montrons la non-intégrabilité du modèle ellipsoïdal nonholonome de Rattleback pour une classe large des paramètres. Notre approche est basée sur l'étude de groupe de monodromie des équations variationnelles le long d'une orbite particulière. Le prolongement des équations de rotation d'un corps rigide dans les équations de Rattleback est discuté.

1. Introduction

The rattleback's amazing mechanical behaviour can be described as follows : when spun on a flat horizontal surface in the clockwise direction this top continues to spin in the same direction until it consumes the initial spin energy; when, however, spun in the counterclockwise direction, the spinning soon ceases, the body briefly oscillates, and then reverses its spin direction and thus spins in the clockwise direction until the energy is consumed. The first mathematical model of this phenomena belongs to Walker (1896) who studied the linearized equations of motion and concluded that the completely stable motion is possible in only one (say clockwise) spin direction. This classical explanation of the rattleback's behavior is incomplete since

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it does not reflect the global dynamical effects explaining the transfer of trajectories from the vicinity of the unstable solution to the stable one. To analyze thoroughly this question we propose to study the nonholonomic equations of the rattleback in the complex domain and particularly to study *the existence of additional analytic (meromorphic) first integrals*. It has been observed in many mechanical systems that non existence of analytic first integrals is usually associated to complicated chaotic behavior of trajectories of the system. We mention that actually only numerical evidence for chaos in the rattleback systems has been observed [1].

In our case the rattleback represents a full ellipsoidal body whose center of mass P coincides with its geometric one. All vectors are defined in the body fixed axes (with origin in P) coinciding the with principal geometric axes of the ellipsoid

$$E(r) = \frac{r_1^2}{b_1^2} + \frac{r_2^2}{b_2^2} + \frac{r_3^2}{b_3^2} = 1 \,. \tag{1.1}$$

Let $\langle \, , \, \rangle$ denote the euclidean scalar product in $\mathbb{R}^3$ and let $|| \cdot ||$ be the corresponding norm. In addition, let $[\, , \,]$ denote the cross product in $\mathbb{R}^3$.

As shown in [3], the nonholonomic equations of motion can be written in the following form

$$\Theta \frac{d\omega}{dt} + m\,[r, [\frac{d\omega}{dt}, r]] = -[\omega, \Theta\omega] - m\,[r, [\omega, [\omega, r]]] + mg\,[r, \gamma] - m\,[r, [\omega, \frac{dr}{dt}]] \,, \tag{1.2a}$$

$$\frac{d\gamma}{dt} = [\gamma, \omega] \,, \tag{1.2b}$$

where

$$\Theta = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & 0 \\ \Sigma_{12} & \Sigma_{22} & 0 \\ 0 & 0 & \Sigma_{33} \end{bmatrix} , \tag{1.3}$$

is the inertia matrix; m – mass of the body; g – gravitational constant; $\omega = (\omega_1, \omega_2, \omega_3)^T$ – angular velocity; $\gamma = (\gamma_1, \gamma_2, \gamma_3)^T$ – unit vector normal to the body's surface at the point of contact; $r = (r_1, r_2, r_3)^T$– position of contact point.

To observe the rattleback behavior, one has to impose the following condition on Θ (see e.g. [1]) and b_i

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} I_1 \cos^2 \delta + I_2 \sin^2 \delta & (I_1 - I_2) \cos \delta \sin \delta \\ (I_1 - I_2) \cos \delta \sin \delta & I_1 \sin^2 \delta + I_2 \cos^2 \delta \end{bmatrix}, \tag{1.4}$$

$$\Sigma_{33} = I_3, \quad b_1 \neq b_2 \,,$$

where I_1, I_2, I_3 are the principal components of the inertia tensor whose principal horizontal axes are rotated by the angle $\delta > 0$ with respect to the r_1-r_2-axes of the ellipsoid.

Unless otherwise mentioned below we will assume that

$$\Sigma_{ij} > 0, \quad \Sigma_{3,3} < \Sigma_{1,1} + \Sigma_{2,2}, \quad b_1 > b_2 > 0, \quad b_3 > 0, \quad m > 0, \quad g > 0\,. \tag{1.5}$$

Using (1.1), the vector $r = R(\gamma)$ is defined by solving $\gamma = -\nabla E/||\nabla E||$ for r, which gives

$$r_i = R_i(\gamma) = \frac{-b_i^2 \gamma_i}{\sqrt{b_1^2\gamma_1^2 + b_2^2\gamma_2^2 + b_3^2\gamma_3^2}}, \quad i = 1, 2, 3\,. \tag{1.6}$$

Solving (1.2a) for $\dot{\omega}$ and using (1.6) we transform the nonholonomic rattleback equations to the standard form

$$\frac{d\omega}{dt} = F(\omega, \gamma), \tag{1.7a}$$

$$\frac{d\gamma}{dt} = [\gamma, \omega]\,, \tag{1.7b}$$

where $F = (F_1, F_2, F_3)^T$ is a vector field rational in the variables ω, γ and $s = \sqrt{b_1^2\gamma_1^2 + b_2^2\gamma_2^2 + b_3^2\gamma_3^2}$, see the appendix.

Let $(\omega, \gamma, s) \in \mathbb{C}^7$. Then the vector field (1.7a), as a function of (ω, γ, s), is analytic in the domain $\mathcal{D} = \mathbb{C}^7 \backslash (\mathcal{S} \cup \mathcal{L})$ where $\mathcal{L} = \{(\omega, \gamma, s) \in \mathbb{C}^7 \,:\, s = 0\}$ and $\mathcal{S} \subset \mathbb{C}^7$ is the surface on which the determinant of the matrix U given below is zero

$$U = \Theta - m[r, [r, \cdot]] = \Theta + m\big(\langle r, r\rangle \mathrm{Id} - r \otimes r\big)\,. \tag{1.8}$$

This matrix appears when we solve (1.2a) to find $\dot{\omega}$. For an arbitrary non-zero vector $u \in \mathbb{R}^3$

$$\begin{aligned}\langle u, Uu\rangle &= \langle u, \Theta u\rangle + m\langle u, u\rangle\langle r, r\rangle - m\langle u, r\rangle^2 \\ &= \langle u, \Theta u\rangle + m||u \times r||^2 > 0\,,\end{aligned} \tag{1.9}$$

so that U is positive definite and hence $\det(U) = 0$ never occurs in the mechanical case.

The system (1.7a) always possesses two first integrals (see [3]) :

$$H = \frac{m}{2}||[\omega, R(\gamma)]||^2 + \frac{1}{2}\langle \omega, \Theta\omega\rangle - mg\langle R(\gamma), \gamma\rangle = h, \quad h \in \mathbb{R}\,, \tag{1.10}$$

– the energy;

$$G = \langle \gamma, \gamma\rangle = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = l, \quad l \in \mathbb{R}_+\,, \tag{1.11}$$

– the geometric integral.

We see that $H(\omega, \gamma, s)$ (after introduction of s with the help of (1.6)) and $G(\gamma)$ are rational functions of variables ω, γ, s viewed as independent variables in $\mathbb{C}^7$. The natural question arises whether the equations (1.7a) can have a third first integral which can be written as an analytic (meromorphic) in $\mathbb{C}^7$ function of complex variables (ω, γ, s) (viewed as independent) and which is functionally independent of H and G.

Therefore, the question whether there exist *analytic* first integrals of the *rational in* (ω, γ, s) system (1.7) makes sense.

The paper is organized as follows. In Chapter 2 we describe one particular solution Γ of the Rattleback problem and its complex neighborhood B_Γ. Chapter 3 discusses some properties of the normal variational equations and the monodromy group generators of this solution. Chapter 4 examines the case in which the normal variations equations have logarithmic branching points. Under these conditions we show the non-existence of additional meromorphic first integrals (Theorem 4.1). Finally, Chapter 5 contains the proof of our main theorem about the analytic non-integrability :

THEOREM 1.1. — *Assume that the conditions (1.5) and (5.3) hold. Then the geometric first integral G is the only integral that is functionally independent of the energy H and that is analytic in the complex domain B_Γ of the variables (ω, γ).*

1.1. Rigid body limiting case

The rattleback equations (1.2a)-(1.2b) formally contain the equations of the heavy rigid body in the singular limiting case $m \to 0$, $mg \not\to 0$. Moreover the functions $R_i(\gamma)$ are constants denoted by r_i again, now designating the position of the center of mass in the body frame. By a rotation the tensor of inertia Θ can be diagonalised so that Σ_{12} can be set to zero in this case, thus violating (1.5). Under these assumptions the system is Hamiltonian with another well known integral

$$L = \langle \gamma, \Theta\omega \rangle = \gamma_1\omega_1 I_1 + \gamma_2\omega_2 I_2 + \gamma_3\omega_3 I_3 \,. \tag{1.12}$$

2. The invariant manifold

We consider the vector field (1.7) as a function of six variables (ω, γ). Let $l > 0$ (in the mechanical case we can set $l = 1$). For $(\omega, \gamma) \in \mathbb{R}^6$ the square root in s is assumed to be always positive. It is straightforward to check that the system (1.7) has the invariant manifold $M = \{(\omega, \gamma) \in \mathbb{R}^6 : \omega_1 = \omega_2 = 0, \gamma_3 = 0\}$. The mechanical sense of the motion on M is quite clear : it corresponds to rolling (or oscillating) of the body on the line of intersection of its surface with $r_3 = 0$ plane. We note that this invariant manifold exists because of the assumption that the r_3 ellipsoidal axis coincides with one of the principal inertia body axes.

Complexifying (1.7) we denote by $U_{l,h} \subset M \subset \mathbb{C}^6$ its orbit in M corresponding to fixed $(l, h) \in \mathbb{C}^2$ defined as the intersection of the two algebraic surfaces

$$\omega_3{}^2 \left(\frac{m\,(b_1^4\gamma_1^2 + b_2^4\gamma_2^2)}{b_1^2\gamma_1^2 + b_2^2\gamma_2^2} + \Sigma_{33} \right) + 2\,mg\,\sqrt{b_1^2\gamma_1^2 + b_2^2\gamma_2^2} = 2\,h \in \mathbb{R}, \tag{2.1a}$$

$$\gamma_1^2 + \gamma_2^2 = l \in \mathbb{R}\,. \tag{2.1b}$$

We assume now that $h \neq 0$ and $l = 0$. Obviously the solution obtained in this way cannot be real; it is non-mechanical.

Let $\Gamma \subset \mathbb{C}^6 = (\omega, \gamma)$ be the complex curve defined by

$$\omega_3 = p, \quad \gamma_1 = \frac{p^2 - \alpha^2}{\beta},\ \gamma_2 = i\frac{p^2 - \alpha^2}{\beta},\ \omega_1 = \omega_2 = \gamma_3 = 0,\ p \in \mathbb{C}\backslash\{-\alpha, +\alpha\}\,, \tag{2.2}$$

where

$$\alpha^2 = \frac{2h}{m(b_1^2 + b_2^2) + \Sigma_{3,3}}, \quad \beta = -\frac{2mg\sqrt{b_1^2 - b_2^2}}{m(b_1^2 + b_2^2) + \Sigma_{3,3}}, \quad h \in \mathbb{C}\,, \tag{2.3}$$

This is the parametrization of $U_{0,h}$.

Let $P_0 = (\omega_0, \gamma_0) \in \Gamma$ such that $s(\gamma_0) \neq 0$.

The function $s = \sqrt{b_1^2\gamma_1^2 + b_2^2\gamma_2^2 + b_3^2\gamma_3^2}$, once the square-root branching is fixed, is analytic in a small neighborhood $U_{P_0} \subset \mathbb{C}^6$ of P_0. We can analytically continue s and all its derivatives along Γ with the help of the parametrization above : $s((\omega,\gamma) \in \Gamma) = s(p) = \sigma \dfrac{p^2 - \alpha^2}{\beta}$ where $\sigma = \sqrt{b_1^2 - b_2^2} > 0$. In particular, it shows that s is single-valued on Γ and hence is analytic in a small neighborhood $B_\Gamma \subset \mathbb{C}^6$ of it. Thus, the vector field (1.7) restricted to B_Γ is an analytic function of the complex variables (ω, γ) and has the invariant curve Γ.

We use the reparametrisation $b_i^2 = \rho_i(\rho_1 - \rho_2)$, with $\rho_1 > \rho_2$, $\rho_i =$ const > 0, which makes $\sigma = \rho_1 - \rho_2$ polynomial on the solutions we are considering. In particular for this solution $r = (-\rho_1, -\mathrm{i}\rho_2, 0)^T$, which is a constant.

Since in the *rigid body limiting case* r is not proportional to γ the invariant manifold M only exists under the additional assumption that $r_3 = 0$.

3. The variational equations and its monodromy group

The relation between the parameters p and t can be easily found by substitution of the parametrization (2.2) in one of the equations (1.7). This gives

$$dp = i\frac{p^2 - \alpha^2}{2} dt\,. \tag{3.1}$$

Remark 3.1. — The change of time (3.1) has infinitely many sheets. Nevertheless, one can always replace the vector field $F(X)$, $X = (\omega, \gamma)^T$ defined in (1.7) by $\tilde{F} = (F/(2^{-1}i(p^2 - \alpha^2)), 1)^T$ i.e. consider the new autonomous system of differential equations

$$\frac{dX}{d\tau} = \frac{\tilde{F}(X)}{2^{-1}i(p^2 - \alpha^2)}, \quad \frac{dp}{d\tau} = 1\,, \tag{3.2}$$

which will have the same particular solution Γ defined by (2.2) where p is replaced by τ.

Obviously, an autonomous analytic first integral of (1.7a) gives a first integral of the same type for the vector field (3.2). One sees also that $\tilde{F}$ is analytic in the neighborhood of Γ. The further analysis based on the variational equations of (1.7) or (3.2) will be essentially the same.

Let $\tilde{\omega}_i$, $\tilde{\gamma}_i$, $i = 1, 2, 3$ be the variations of variables ω_i, γ_i, respectively. We use the following notation for coordinates of the variation vector $V = (\tilde{X}, \tilde{Y})^T = ((\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\gamma}_3), (\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\omega}_3))^T$. In these variables, the variational equations of (1.7a) along Γ, after the substitution (3.1), have the block diagonal form

$$\frac{dX}{dp} = \begin{bmatrix} M & O \\ O & N \end{bmatrix} X, \quad X \in T_\Gamma B_\Gamma\,. \tag{3.3}$$

where $M(p)$, $N(p) \in GL(3, \mathbb{C}(p))$ and O is the zero 3×3 matrix.

One can show that the derivatives of the first integrals H and G with respect to the variables ω_1, ω_2, γ_3 vanishes along Γ. So, the linear first integrals $\tilde{H} = \langle dH(\Gamma), \tilde{Y}\rangle$ and $\tilde{G} = \langle dG, \tilde{Y}\rangle$ are not useful in solving the first block system

$$\frac{d\tilde{X}}{dp} = M\tilde{X}\,, \tag{3.4}$$

whereas the second subsystem $\dfrac{d\tilde{Y}}{dp} = N\tilde{Y}$ can be completely solved in radicals with the help of $\tilde{H}$ and $\tilde{G}$.

As seen from the parametrization (2.2), the equations (3.4) are the normal variational equations (see [7] for definition) of the system (1.7) along the orbit Γ. The following lemma is a consequence of the Ziglin's Lemma from [7]

LEMMA 3.2. — *Let us suppose that the system (1.7a) has a third first integral $H_3(\omega, \gamma)$ which is analytic (meromorphic) in the neighborhood B_Γ and functionally independent of H and G. Then the monodromy group of the normal variational equations* (3.4) *have a non trivial polynomial (rational) invariant.*

We can put the linear system (3.4) into the Fuchsian form with help of the rational transformation

$$\tilde{X} = \operatorname{diag}(p, p, p^2 - \alpha^2)x\,, \tag{3.5}$$

where $x = (x_1, x_2, x_3)^T \in \mathbb{C}^3$. The new system takes the following form

$$\frac{dx}{dp} = T(p)x = \left(\frac{A}{p-\alpha} + \frac{A}{p+\alpha} + \frac{B}{p}\right)x, \quad p \in \mathcal{R} = \bar{\mathbb{C}} \setminus \{-\alpha, +\alpha, 0, \infty\}\,. \tag{3.6}$$

Clearly, the monodromy groups of the systems (3.4) and (3.6) are equivalent. The constant 3×3 matrices A, B are given in Appendix A.

Remark 3.3. — One sees that the above linear equations are invariant under the transformation $p \mapsto -p$. That in particular allows to reduce the number of finite singularities up to two via introducing the new time $p = \tau^2$. Nevertheless, we prefer to keep this symmetry inside the equations and to use it later in the study of the monodromy group of (3.6).

PROPOSITION 3.4. — *The general solution of (3.6) is meromorphic in a neighborhood of $p = 0$.*

The proof follows from the fact that the matrix $M(p)$ in (3.4) is holomorphic in the neighborhood of $p = 0$ as seen from (2.2), (1.2a). In particualr, we have

$$\mathrm{Spectr}(B) = \{0, -1, -1\}\,. \tag{3.7}$$

That can be verified directly with help of formulas for B given in Appendix A.

We will need some technical results. The following proposition will play a crucial role for our non-integrability results below.

PROPOSITION 3.5. — *The characteristic polynomials $P(x)$, $P_\infty(x)$ of the residue matrices A at $p = \pm\alpha$ and $A_\infty = -2A - B$ at $p = \infty$ always have non-real roots.*

Proof. — A direct calculation shows that $P(x)$ is of the form

$$P(x) = x^3 + x^2 + \theta_1 x + \theta_0\,, \tag{3.8}$$

where with the definitions

$$\Psi_n = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix}, \Psi_d = \Psi_n - (\Sigma_{11} + \Sigma_{22} - \Sigma_{33})\mathrm{Id},$$
$$v = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} -\rho_1 \\ -\mathrm{i}\rho_2 \end{pmatrix}, \tag{3.9}$$

the coefficients of P are given by

$$\theta_d = \det \Psi_d + m\langle v, \Psi_d v\rangle \tag{3.10a}$$

$$\theta_d\theta_1 = \det \Psi_n + m\{\langle v, \Psi_n v\rangle + \rho_3(\Sigma_{11}\rho_1 - \Sigma_{22}\rho_2 + \mathrm{i}\Sigma_{12}(\rho_1 + \rho_2))\} \tag{3.10b}$$

$$\theta_d\theta_0 = \theta_d\theta_1 - m\rho_3(\rho_1 - \rho_2)(\Sigma_{11} + \Sigma_{22} - \Sigma_{33})\,. \tag{3.10c}$$

Therefore

$$\operatorname{Im} \frac{1}{\theta_0 - \theta_1} = \frac{2\Sigma_{12}\rho_1\rho_2}{\rho_3(\rho_1 - \rho_2)(\Sigma_{11} + \Sigma_{22} - \Sigma_{33})}, \tag{3.11}$$

which is obviously non-zero according to (1.5). A direct calculation shows that $P_\infty(x)$ is of the form

$$P_\infty(x) = x^3 - 4x^2 + \chi_1 x + \chi_0 \tag{3.12}$$

In the notation of the previous proposition and the Appendix the linear coefficient is given by

$$\chi_1 = 5 + 4a_1^2 + 4a_2a_3 + 4(c_3 - \mathrm{i}c_1)/\beta \tag{3.13}$$

The coefficients of $P(x)$ and $P_\infty(x)$ are related by

$$12(\theta_0 - \theta_1) + \chi_0 + 3\chi_1 = -9\,. \tag{3.14}$$

Therefore

$$\operatorname{Im} \frac{1}{\chi_0 + 3\chi_1 + 9} = -\frac{\Sigma_{12}\rho_1\rho_2}{6\rho_3(\rho_1 - \rho_2)(\Sigma_{11} + \Sigma_{22} - \Sigma_{33})} \tag{3.15}$$

□

In the rigid body case the matrices A and B become

$$A = \begin{pmatrix} 0 & \mathrm{i}(\Sigma_{33} - \Sigma_{22})/\Sigma_{11} & 0 \ \mathrm{i}(\Sigma_{11} - \Sigma_{33})/\Sigma_{22} & 0 & 0 \\ -1/\beta & -\mathrm{i}/\beta & -1 & & \end{pmatrix} \tag{3.16}$$

$$B = \begin{pmatrix} -1 & 0 & -2\mathrm{i}mgr_2/\Sigma_{11} \\ 0 & -1 & 2\mathrm{i}mgr_1/\Sigma_{22} \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.17}$$

Therefore

$$\operatorname{Spectr}(A) = \left\{ -1, -\sqrt{\frac{(\Sigma_{11} - \Sigma_{33})(\Sigma_{22} - \Sigma_{33})}{\Sigma_{11}\Sigma_{22}}}, \sqrt{\frac{(\Sigma_{11} - \Sigma_{33})(\Sigma_{22} - \Sigma_{33})}{\Sigma_{11}\Sigma_{22}}} \right\}. \tag{3.18}$$

For the Kovalesvkaya case $\Sigma_{11} = \Sigma_{22}$, $\Sigma_{33} = 2\Sigma_{11}$ the eigenvalues become $\{-1, 1, 1\}$ and $\{-2, 3, 3\}$ at infinity with a non-trivial Jordan block.

The following proposition shows that the system (3.6) is not solvable in the Lappo-Danilevsky sense (see [2]).

Proposition 3.6. — *The residue matrices A and B do not have common eigenvectors. In particular, they do not commute.*

Proof. — The proof is a straightforward, but since the proposition will play an important role later, we indicate some basic steps of it. A direct computation gives for the eigenvectors $\mathcal{V}_i$ and the corresponding eigenvalues λ_i of B :

$$\begin{array}{ll} \mathcal{V}_1 = (1, v_1, v_2)^T, & \lambda_1 = 0, \\ \mathcal{V}_2 = (1, 0, 0)^T, & \lambda_1 = -1, \\ \mathcal{V}_3 = (0, 1, 0)^T, & \lambda_1 = -1\,, \end{array} \tag{3.19}$$

where v_i are some known expressions.

One verifies that the matrix $R = (v_1, v_2, v_3)$ is not singular once the condition (1.5) holds. Under the same conditions it is easy to show that $\mathcal{V}_2$ and $\mathcal{V}_3$ are not eigenvectors of A. Here is one method to prove the same statement for $\mathcal{V}_1$. We consider the conjugation $\tilde{A} = R^{-1}AR$ of A by R. It is sufficient to show then that $\tilde{A}_{1,2} \neq 0$ or $\tilde{A}_{1,3} \neq 0$. That can be done quite easily once the condition (1.5) is fixed. Otherwise, if $C = [A, B]$, then the non-commutativity property follows directly from $C_{3,1} = 1/\beta \neq 0$. □

Fixing a basepoint $e \in \mathcal{R}$, one defines the monodromy group G of the system (3.6) as the image of the fundamental group $\pi(\mathcal{R}, e)$ given by the analytical continuations of the fundamental matrix solution $\Sigma(p)$, $\Sigma(e) = \mathrm{Id}$ of (3.6) along all closed paths $\Gamma \in \pi(\mathcal{R}, e)$ starting from e. One verifies that $\mathrm{tr}(A), \mathrm{tr}(B) \in \mathbb{Z}$. Hence, $G \subset SL(3, \mathbb{C})$. According to Proposition (3.4), the group $G = \langle M_+, M_- \rangle$ is generated by the local monodromy transformations M_+, M_- around the singularities $p = \alpha$ and $p = -\alpha$, respectively.

4. On the non-existence of additional meromorphic first integrals in the logarithmic branching case

The next theorem will show that the logarithmic branching of solutions of the normal variational equations (3.6) is not compatible with existence of a new *meromorphic* first integral of the rattleback problem (1.7).

THEOREM 4.1. — *Let us assume that the general solution of the variational equations (3.6) has logarithmic branching around one of the singularities $p = \pm\alpha$. Then the geometric first integral G and the energy H are the only first integrals meromorphic in the complex domain B_Γ of the variables (ω, γ).*

Proof. — By a suitable linear transormation of the general solution of (3.6), we can always reduce one of the monodromy matrices around $p = \alpha$ or $p = -\alpha$ to its Jordan form. In the case that the general solution has logarithmic branching points, at least one of the monodromy matrices M_+, M_- has a non-trivial Jordan block.

Since $p = \pm\alpha$ enter in (3.6) in a symmetric way, it is sufficient to consider the case of $p = \alpha$. Firstly, we assume that M_+ is of the form

$$M_+ = \begin{bmatrix} q & 1 & 0 \\ 0 & q & 1 \\ 0 & 0 & q \end{bmatrix}. \tag{4.1}$$

Since $M_+ \in SL(3, \mathbb{C})$, one has $q^3 = 1$.

Let

$$\mathrm{Spectr}(A) = (\lambda_1, \lambda_2, \lambda_3)\,. \tag{4.2}$$

As follows from (3.6) (see e.g. [4])

$$\mathrm{Spectr}(M_+) = \mathrm{Spectr}(M_-) = \{e^{2\pi i\lambda_1}, e^{2\pi i\lambda_2}, e^{2\pi i\lambda_3}\}\,. \tag{4.3}$$

Thus, in the case (4.1) all eigenvalues of A must be real (even rational) numbers. This is impossible according to Proposition (3.5).

We suppose now that M_+ is of the form

$$M_+ = \begin{bmatrix} u & 1 & 0 \\ 0 & u & 0 \\ 0 & 0 & u^{-2} \end{bmatrix}, \quad u \in \mathbb{C}^*\,. \tag{4.4}$$

In this case we prove the following lemma.

Lemma 4.2. — *Let $R(x)$, $x = (x_1, x_2, x_3)^T$ be a rational invariant of G. Then $R = R(x_2, x_3)$.*

Proof. — We write $R(x)$, $\deg R = m$ as follows

$$R = \frac{\sum\limits_{i=0}^{l} x_3^{l-i}\, P_i(x_1, x_2)}{\sum\limits_{j=0}^{k} x_3^{k-j} Q_j(x_1, x_2)}, \quad l, k \in \{0, 1, 2, \ldots\}, \quad m = l - k\,, \tag{4.5}$$

where $P_i(x_1, x_2)$, $Q_j(x_1, x_2)$ are homogeneous polynomials of degrees i and j, respectively. Let us assume that at least one of the polynomials P_i or Q_j depends on x_1. So, for example, we will find $0 \leqslant \rho, s \leqslant l$ such that the term $x_3^{\rho} P_s(x_1, x_2)$, $\partial P/\partial x_1 \neq 0$, $\deg(P_s) = s$ is a semi-invariant of M_+, i.e. it satisfies

$$\tilde{M}_+^n\, P_s(x_1, x_2) = c^n\, P_s(x_1, x_2), \quad c \in \mathbb{C}^*, \quad \forall\, n \in \mathbb{N}, \quad \mathbb{M}_+ = \begin{bmatrix} u & 1 \\ 0 & u \end{bmatrix}. \tag{4.6}$$

Since $P_s(x_1, x_2)$ is homogeneous of degree s, we can put it in the form

$$P_s(x_1, x_2) = \alpha_0 x_1^s + \alpha_1 x_1^{s-1} x_2 + \cdots + \alpha_s x_2^s \,. \tag{4.7}$$

We observe that $\tilde{M}_+^n (x_1, x_2)^T = (u^n x_1 + n u^{n-1} x_2, u^n x_2)^T$. Consequently, the polynomial $\tilde{M}_+^n\, P_s(x_1, x_2)$ will have its coefficient of x_2^s equal to $\theta_n = \sum_{r=0}^{s} \alpha_r n^{s-r} u^{s(n-1)+r}$. Let q be the smallest index such that $\alpha_q \neq 0$. Then the coefficient before $x_1^q x_2^{s-q}$ in $\tilde{M}_+^n\, P_s(x_1, x_2)$ is $\eta_n = u^{nq} \alpha_q$. Clearly, as $n \to \infty$, the asymptotic behaviors of θ_n and η_n are different. So, the equality (4.6) cannot be true. That finishes the proof of Lemma 4.2. $\square$

Lemma 4.3. — *All rational homogeneous invariants of M_+ are functions of the invariant $x_2^2 x_3$.*

The proof follows easily from the condition that $\mathrm{Spectr}(A)$ is non-real.

Lemma 4.4. — *Let us assume that the monodromy group $G = \langle M_-, M_+ \rangle$ has a rational homogeneous invariant $R(x)$. Then $x_2^2 x_3$ is the invariant of G and $R = (x_2^2 x_3)^m$ for a certain $m \in \mathbb{Z}^*$.*

Proof. — Let $R(x_1, x_2, x_3)$ be a rational homogeneous invariant of G written as

$$R = \frac{\displaystyle\sum_{i_1+i_2+i_3=M} a_{i_1,i_2,i_3}\, x_1^{i_1} x_2^{i_2} x_3^{i_3}}{\displaystyle\sum_{j_1+j_2+j_3=N} b_{j_1,j_2,j_3}\, x_1^{j_1} x_2^{j_2} x_3^{j_3}}, \quad i_k, j_p \in \{0, 1, 2, \ldots\}\,, \tag{4.8}$$

where M, N are non-negative integer numbers and all coefficients a_i, b_i are different from zero.

Firstly, we assume that R reduces to the single term

$$R = x_1^{l_1} x_2^{l_2} x_3^{l_3}, \quad l_i \in \mathbb{Z}, \quad (l_1, l_2, l_3) \neq (0, 0, 0)\,. \tag{4.9}$$

Then $l_1 = 0$ according to Lemma 4.2, and, as follows from Lemma 4.3, there exists $m \in \mathbb{Z}^*$ such that

$$R = (x_2^2 x_3)^m\,, \tag{4.10}$$

that in turn proves the result.

If R is not of the form (4.9), it can be written as follows

$$R = \frac{a x_1^{r_1} x_2^{r_2} x_3^{r_3} + b x_1^{q_1} x_2^{q_2} x_3^{q_3} + \cdots}{c x_1^{p_1} x_2^{p_2} x_3^{p_3} + \cdots}, \quad r_i, q_j, p_k \in \{0, 1, 2, \ldots\}\,, \tag{4.11}$$

where $a, b, c \neq 0$ and $(r_1, r_2, r_3) \neq (q_1, q_2, q_3)$, $\sum r_i = \sum q_i = M$.

We note that, as seen from (4.11), the terms $x_1^{r_1}x_2^{r_2}x_3^{r_3}$ and $x_1^{q_1}x_2^{q_2}x_3^{q_3}$ are multiplied by the same constant under the action of M_+. Therefore, the division by $x_1^{q_1}x_2^{q_2}x_3^{q_3}$ shows that $r(x) = x_1^{k_1}x_2^{k_2}x_3^{k_3}$, $k_i = r_i - q_i$, $(k_1, k_2, k_3) \neq (0,0,0)$, $\sum k_i = 0$ is the invariant of M_+. We have $k_1 = 0$ according to Lemma 4.3 and hence $r(x) = (x_2/x_3)^{k_2}$, $k_2 \neq 0$. This is impossible according to Lemma 4.3.

Thus, we can assume R to have the form (4.10). This immediately fixes the monodromy transformation M_-, also preserving R, as follows

$$M_- = \begin{bmatrix} u_1 & n & k \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{bmatrix}, \quad u_1u_2u_3 = 1, \quad u_i, n, k \in \mathbb{C}\,. \tag{4.12}$$

We will show that $u_1 = u_2 = u$ and $u_3 = u^{-2}$.

Indeed, since $\mathrm{Spectr}(M_+) = \mathrm{Spectr}(M_-) = \{u, u, u^{-2}\}$, we have necessarily $u_1 = u$ or $u_2 = u$. Let $u_1 = u$, $u_2 = u^{-2}$, $u_3 = u$. With help of R we obtain $u^{-3m} = 1$ and so u is the root of unity. Let now $u_1 = u^{-2}$, $u_2 = u$, $u_3 = u$. Then $u^{3m} = 1$ and we conclude as above. Hence, the monodromy transformation M_- takes the form

$$M_- = \begin{bmatrix} u & n & k \\ 0 & u & 0 \\ 0 & 0 & u^{-2} \end{bmatrix}. \tag{4.13}$$

The rational function $R = x_2^2x_3$ is clearly invariant under the action of M_- and M_+. $\square$

We shall follow the approach close to that of Tannakian (see e.g. [6]).

Let $\Sigma(p)$ be the fundamental matrix solution of the equations (3.6) and $\Sigma^{-1}(p) = (\Sigma_1, \Sigma_2, \Sigma_3)^T$ where Σ_i are linearly independent vector functions. It is known (see e.g. [5], p. 246) that if $R(x) = x_2^2x_3$ is a polynomial invariant of the monodromy group G then

$$I(p,x) = R(\Sigma(p)^{-1}x) = \langle \Sigma_2(p), x\rangle^2 \langle \Sigma_3(p), x\rangle\,, \tag{4.14}$$

will be a first integral of (3.6) invariant under the action of G, i.e. single-valued as a function of $p \in \mathcal{R}$. One can express this fact by stating that all coefficients $a_{i,j,k}(p)$ in the expression for I,

$$I = \sum_{i+j+k=3} a_{i,j,k}(p)\, x_1^i x_2^j x_3^k\,, \tag{4.15}$$

are rational functions of p.

Since the polynomial I given by (4.15) can be factorized as (4.14), there exist two 3-vector functions $\mathcal{A}(p)$, $\mathcal{B}(p)$, algebraically dependent on p, such that I becomes

$$I(p,x) = \langle \mathcal{A}(p), x\rangle^2 \langle \mathcal{B}(p), x\rangle \,. \tag{4.16}$$

The structure of the monodromy group generated by the transformations (4.4), (4.13) suggests that the system (3.6) has two linearly independent particular solutions $X_{1,2}(p)$ of the following form

$$X_1(p) = (p-\alpha)^{\lambda_1}(p+\alpha)^{\lambda_1} R_1(p), \quad e^{2\pi i \lambda_1} = u, \quad \lambda_1 \in \mathrm{Spectr}(A)\,, \tag{4.17}$$

$$X_2(p) = (p-\alpha)^{\lambda_1}(p+\alpha)^{\lambda_1} R_2(p) + u^{-1} X_1(p)\left(\frac{\log(p-\alpha)}{2\pi i} + n\frac{\log(p+\alpha)}{2\pi i}\right), \tag{4.18}$$

where $R_{1,2}(p)$ are vector functions rationally dependent on p.

Lemma 4.5. — *We have* $\langle \mathcal{A}, R_1\rangle = \langle \mathcal{B}, R_1\rangle = 0$.

Proof. — Plugging the solution (4.17) into the integral (4.16) we obtain

$$I(p, X_1) = (p-\alpha_1)^{3\lambda_1}(p+\alpha)^{3\lambda_1}\langle \mathcal{A}, R_1\rangle^2 \langle \mathcal{B}, R_1\rangle = c = \mathrm{const}\,. \tag{4.19}$$

In the case $c \neq 0$, since $\mathcal{A}, \mathcal{B}, R_1$ are algebraic with respect to p, the last equality implies $\lambda_1 \in \mathbb{Q}$. According to (4.4)

$$\mathrm{Spectr}(A) = \{\lambda, \lambda + k, -2\lambda - k - 1\}\,, \tag{4.20}$$

for certain $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}$.

So, it is clear that if $\lambda_1 \in \mathbb{R}$, then $\mathrm{Spectr}(A)$ has to be real. According to Proposition 3.5 we therefore get $\langle \mathcal{A}, R_1\rangle^2 \langle \mathcal{B}, R_1\rangle = 0$ for all p.

We shall consider the case $\langle \mathcal{A}, R_1\rangle = 0$ and $\langle \mathcal{B}, R_1\rangle \neq 0$. Obviously, the shift $\tilde{I}(p,x) = I(p, x + X_1(p))$ of the first integral I is again a first integral of the system (3.6) and

$$\tilde{I} = I + (p-\alpha)^{\lambda_1}(p+\alpha)^{\lambda_1}\langle \mathcal{B}, R_1\rangle \langle \mathcal{A}, x\rangle^2 = \mathrm{const}\,. \tag{4.21}$$

Since the equations (3.6) are homogeneous, with help of (4.21), (4.16), we derive the first integral

$$J_1 = \frac{\langle \mathcal{B}, x\rangle}{(p-\alpha)^{\lambda_1}(p+\alpha)^{\lambda_1}(\mathcal{B}, R_1)} = (p-\alpha)^{-\lambda_1}(p+\alpha)^{-\lambda_1}\langle \tilde{B}, x\rangle\,, \tag{4.22}$$

where $\tilde{B}$ is algebraic on p.

Finally, combining J_1 with I, one gets

$$\begin{aligned} J_2 &= \sqrt{\langle \mathcal{B}, R_1\rangle (p-\alpha)^{\lambda_1}(p+\alpha)^{\lambda_1}}\langle \mathcal{A}, x\rangle \\ &= (p-\alpha)^{\lambda_1/2}(p+\alpha)^{\lambda_1/2}\langle \tilde{A}, x\rangle\,, \end{aligned} \tag{4.23}$$

– the first integral of (3.6) with $\tilde{A}(p)$ algebraic on p.

Obviously, J_1 and J_2 are functionally independent. Indeed, the vectors $\mathcal{A}(p)$, $\mathcal{B}(p)$ are independent as being proportional to the lines $\Sigma_{2,3}(p)$ of the matrix $\Sigma^{-1}(p)$ whose determinant is not identically zero. Finally, in order to find the third linear first integral, we apply the Liouville theorem to the fundamental matrix solution of (3.6) formed by the columns $X_1(p)$, $X_2(p)$ and the arbitrary solution $x(p)$:

$$\begin{aligned} &\Sigma(p) = (X_1(p), X_2(p), x), \\ &\det(\Sigma) = (p-\alpha)^{2\lambda_1}(p+\alpha)^{2\lambda_1}\det(R_1, R_2, x) = \text{const}\cdot\, a(p)\,, \end{aligned} \tag{4.24}$$

where $a(p)$ is a rational function of p in view of $\operatorname{tr}(A), \operatorname{tr}(B) \in \mathbb{Z}$.

One derives from (4.24) the following first integral of (3.6)

$$J_3 = (p-\alpha)^{-2\lambda_1}(p+\alpha)^{-2\lambda_1}\langle \mathcal{C}, x\rangle\,, \tag{4.25}$$

with $\mathcal{C}(p)$ algebraic on p.

In the case the vectors $\tilde{A}, \tilde{B}, \mathcal{C}$ are linearly independent, finding x from the linear system $J_1 = c_1, J_2 = c_2, J_3 = c_3$, $c_{1,2,3} \in \mathbb{C}$, we see that the general solution of (3.6) does not contain logarithmic branching and so we get a contradiction. Let us assume that $\tilde{A}, \tilde{B}, \mathcal{C}$ are linearly dependent. Then $\mathcal{C} = l_1\tilde{A} + l_2\tilde{B}$ where l_1, l_2 are certain algebraic functions of p and the following relation holds in view of (4.22), (4.23), (4.25)

$$(p-\alpha)^{-\frac{5}{2}\lambda_1}(p+\alpha)^{-\frac{5}{2}\lambda_1}c_1 l_1 + (p-\alpha)^{-\lambda_1}(p+\alpha)^{-\lambda_1}c_2 l_2 = c_3\,. \tag{4.26}$$

Since J_1, J_2 are functionally independent, the last expression shows that $\lambda_1 \in \mathbb{Q}$ and the proof is finished with help of (4.20). The case $\langle \mathcal{A}, R_1\rangle \neq 0$, $\langle \mathcal{B}, R_1\rangle = 0$ is treated in the analogous way. $\square$

With help of the Lemma 4.5 one obtains $I(p, x + X_2) = I(p, x + (p-\alpha)^{\lambda_1}(p+\alpha)^{\lambda_1}R_2)$ which is a first integral of (3.6). As before we show $\langle \mathcal{A}, R_2\rangle = \langle \mathcal{B}, R_2\rangle = 0$. Since $\mathcal{A}$, $\mathcal{B}$ are linearly independent, it follows from (4.17), (4.18) that $X_2 = \theta X_1$ for a certain function $\theta = \theta(p) \neq \text{const}$. This situation can be ruled out by the substitution of X_1, X_2 into the system (3.6) that gives a contradiction. According to Lemma 3.2 the proof of Proposition 4.1 is finished. $\square$

5. Non-existence of additional analytic first integrals

Our aim now is to prove the non-existence of new analytic first integrals of the rattleback problem (1.7) under one of the following hyperbolicity conditions

$$\mathrm{Spectr}(M_\pm) = \{s_{1,2,3} \in \mathbb{C} \,:\, 0 < |s_1| < 1, |s_2| > 1, |s_3| > 1, s_2 \neq s_3\}\,, \quad (5.1)$$

or

$$\mathrm{Spectr}(M_\pm) = \{s_{1,2,3} \in \mathbb{C} \,:\, |s_1| > 1, 0 < |s_2| < 1, 0 < |s_3| < 1, s_2 \neq s_3\}\,, \quad (5.2)$$

which represent certain restrictions on the eigenvalues of the characteristic polynomial (3.8). Recalling that $\sum_i \lambda_i = -1$ and $s_1 s_2 s_3 = 1$ we deduce from the relations $e^{2\pi i \lambda_i} = s_i$ that at least one of the conditions (5.1) or (5.2) is satisfied when

$$\lambda_1, \lambda_2, \lambda_3 \notin \mathbb{R} \quad \text{and} \quad \mathrm{Im}\,\lambda_i - \mathrm{Im}\,\lambda_j \neq 0, \quad \forall\, i \neq j\,, \qquad (5.3)$$

where λ_i, $i = 1,2,3$ are three roots of the cubic algebraic equation

$$x^3 + x^2 + \theta_1 x + \theta_0 = 0\,, \qquad (5.4)$$

whose coefficients depend on $(\Sigma_{i,j},\, b_i,\, m)$ and are defined by (3.10).

From now on, we will assume that the condition (5.3) is fulfilled and that the case (5.1) holds.

Let $\mathcal{L}$ be the space of linear forms $l = \langle L, x\rangle$, $L \in \mathbb{C}^n$ dual to $\mathbb{C}^n$. To each $M \in GL(n, \mathbb{C})$ we associate the linear automorphism $M \,:\, \mathcal{L} \to \mathcal{L}$ according to $M\,.\,l = \langle M^T L, x\rangle$. The next results shows how the hyperbolicity of G implies the reducibility of its polynomial invariants.

Proposition 5.1. — *Let the monodromy group $G = \langle M_+, M_-\rangle$ have a polynomial homogeneous invariant $P(x_1, x_{,2}\,, x_3)$. Then one of the following situations holds :*

a) $P = x_1^\rho x_2^l L(x_1, x_2, x_3)$, $\rho, l \in \mathbb{N}$,

b) $P = x_1^\mu x_3^k M(x_1, x_2, x_3)$, $\mu, k \in \mathbb{N}$,

c) $P = x_1^\eta N(x_1, x_2, x_3)$, $\eta \in \mathbb{N}$ and $M_\pm\,.\,x_1 = s_1 x_1$,
where $L, M, N \in \mathbb{C}[x_1, x_2, x_3]$.

Proof. — Since P is invariant under the action of M_+ and (5.1) holds, all different monomials entering in P contain a positive degree of x_1. Therefore, P factorizes as follows

$$P = x_1^\rho\, P_1(x_1, x_2, x_3), \quad \rho \in \mathbb{N}\,, \qquad (5.5)$$

where P_1 is a homogeneous polynomial not divisible by x_1.

Let $\mathcal{E}$ be the set of all pairwise non-colliniear forms dividing P. In particular we have already $x_1 \in \mathcal{E}$. We denote $\mathcal{E}_0 \subset \mathbb{C}P^2$ the set of directions of all elements from $\mathcal{E}$ equipped with the naturally defined action of G. The G-invariance of P implies clearly the invariance of directions from $\mathcal{E}_0$ under the action of G.

Exchanging the roles of M_+ and M_-, one finds $e \in \mathcal{E}$ – the eigenform of M_-, $M_- \,.\, e = s_1 e$. Let $e = e_1 x_1 + e_2 x_2 + e_3 x_3$. Then, considering the orbit $M_+^n \,.\, e$, $n \geqslant 1$ and using (5.1) together with card $\mathcal{E}_0 < \infty$, one sees that if $e_1 \neq 0$ then $e_2 = e_3 = 0$ i.e. $e = x_1$ (case c). If $e_1 = 0$, then either $e_2 e_3 = 0$ and so $e \in \{x_2, x_3\}$ (cases a-b) or $e_2 e_3 \neq 0$ so that $\exists\, m \in \mathbb{N}$ such that $s_2^m = s_3^m$. The last equation implies $\lambda_2 - \lambda_3 \in \mathbb{R}$ which is impossible according to (5.3). This concluded the proof. □

Proposition 5.2. — *The monodromy transformations M_+ and M_-, taken in any basis, are permutationally conjugated i.e. $\exists\, C \in GL(3, \mathbb{C})$ such that*

$$CM_+C^{-1} = M_-, \quad CM_-C^{-1} = M_+ \,. \tag{5.6}$$

Proof. — We take a basepoint $e \in \mathcal{R}$ on the positive imaginary axis $\operatorname{Im} p > 0$. Let $\Sigma(p)$, $\Sigma(e) = \mathrm{Id}$ be the normalised fundamental matrix solution of (3.6) and let G be the corresponding monodromy group. Since the equations (3.6) are invariant under the change of time $p \to -p$, $\tilde{\Sigma}(p) = \Sigma(-p)$ is again a fundamental matrix solutions of (3.6). Let Γ_1, Γ_2 be two loops starting from e and going around the singularities $p = \alpha$ and $p = -\alpha$, respectively. We define the loops $\tilde{\Gamma}_i = -\Gamma_i$, $i = 1, 2$ (symmetric to $\Gamma_{1,2}$ with respect to origin), starting from the point $\tilde{e} = -e$ and having the same orientation as $\Gamma_{1,2}$. Obviously, $\tilde{\Sigma}(\tilde{e}) = \mathrm{Id}$ and we can define the monodromy group $\tilde{G}$ using $\tilde{\Sigma}$ in the usual way. One sees that $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$ define now the monodromy transformations around $p = -\alpha$ and $p = \alpha$, respectively. The result follows then from the fact that G and $\tilde{G}$ are always conjugated. □

Remark 5.3. — It follows from the previous proposition that if $C^2 \neq \mathrm{Id}$ then it is a centraliser of G in $GL(3, \mathbb{C})$.

Proposition 5.4. — *Let us assume that the monodromy group G of the normal variational equations (3.6) has a polynomial homogeneous invariant and that the conditions (1.5), (5.3) hold. Then G is diagonalizable.*

Proof. — One first assumes that a)-b) from Proposition 5.1 hold.

Then we have three possible cases : $\mathcal{E} = \{x_1, x_2\}$ (A), $\mathcal{E} = \{x_1, x_3\}$ (B) or $\mathcal{E} = \{x_1, x_2, x_3\}$ (C).

The group G acts on $\mathcal{E}_0$ by permutations. Hence, exchanging if necessary x_2 and x_3, in A-B we can put

$$M_+ = \mathrm{diag}(s_1, s_2, s_3),\ M_- = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ a & b & \sigma_3 \end{bmatrix},\ \sigma_i \in \mathrm{Spectr}(M_+),\ a, b \in \mathbb{C}\,. \tag{5.7}$$

Indeed, $\mathcal{E}$ always contains an eigenform of M_- corresponding to the stable eigenvalue $s_1 \in \mathrm{Spectr}(M_+) = \mathrm{Spectr}(M_-)$. In particular $\sigma_1 = s_1$ or $\sigma_2 = s_1$.

Using (5.7) one can calculate the unique (modulo multiplication by a constant) nonsingular matrix T such that $T^{-1}\, M_-\, T = \mathrm{diag}(s_1, s_2, s_3)$. Then, as follows from Proposition 5.2, $T^{-1}\, M_+\, T = M_-$. The last equation, together with (5.3) and some elementary calculations implies $a = b = 0$ that proves the result.

If the case C holds, then the matrix $M_- \in SL(3, \mathbb{C})$ is either diagonal, so the proposition is proved, or is a permutation of the eigendirections of M_+ fixing one of them. In this last case, $\mathrm{Spectr}(M_-)$ will contain necessarily a pair of eigenvalues with equal absolute values. This contradicts to (5.3).

We consider now the case c) from Proposition 5.1. One puts P into the form

$$P = x_1^{\rho}(x_1 P_1(x_1, x_2, x_3) + P_2(x_2, x_3))\,, \tag{5.8}$$

where P_1, P_2 are homogeneous polynomials and $P_2 \neq \mathrm{const}$ since $|s_1| < 1$.

Let $\tilde{M}_+ = \mathrm{diag}(s_2, s_3)$ and let $\tilde{M}_- = (m_{ij})_{2\leqslant i,j\leqslant 3}$ denotes the restriction of the linear operator M_- to the x_2, x_3-plane.

It is clear from (5.8) that P_2 is a polynomial semi-invariant for both $\tilde{M}_+$ and $\tilde{M}_+$. If P_2 contains two different monomials $x_2^n x_3^m$ and $x_2^p x_3^q$, $n + m = p + q = \deg P - \rho$ then $s_2^r = s_3^r$ for $r = n - p = q - m \neq 0$ that contradicts to (5.3).

If x_2^N (resp. x_3^N) is the only monomial entering in P_2, then x_2 (resp. x_3) is the eigenform of $\tilde{M}_-$.

Thus, exchanging if necessary x_2 and x_3, it is sufficient to consider the case

$$M_+ = \mathrm{diag}(s_1, s_2, s_3), \quad M_- = \begin{bmatrix} s_1 & 0 & 0 \\ a & m_{22} & 0 \\ b & m_{32} & m_{33} \end{bmatrix}. \tag{5.9}$$

With help of (5.6), (5.3) and some elementary algebraic computations one shows that $m_{22} = s_2$, $m_{33} = s_3$.

We introduce the matrices $U = M_+^{-1} M_-$ and $K = U - \mathrm{Id}$. One verifies that $\mathrm{Spectr}(U) = \{1, 1, 1\}$ and that if $\mathrm{rang}(K) = 2$ then M_- does not have any polynomial invariants. That can be done by transforming U to its Jordan form. If $\mathrm{rang}(K) = 0$ then $M_- = M_+$ and the proposition is proved. The condition $\mathrm{rang}(I - \mathrm{Id}) = 1$ implies in turn : $a = 0$ (i), $b = m_{32} = 0$ (ii) or $m_{32} = 0$ (iii). In the cases (i), (ii), M_- has the eigenform equal to x_2 or x_3 that corresponds to the case considered before. If (iii) holds, it is sufficient to apply again the conjugacy conditions (5.6) to obtain a contradiction with (5.3).

Finally, we consider the case then $P_2 = c\, x_2^t x_3^l$, $t, l \in N$, $c \in \mathbb{C}^*$. Then $\tilde{M}_-$ either preserves or permutes x_2, x_3-eigendirections of $\tilde{M}_+$. In the first case the proceed as above. In the second one, it is sufficient to verify by a direct computation that $\mathrm{Spectr}(M_-)$ will contain in this case a pair of eigenvalues $\pm s$, $s \in \mathbb{C}^*$ so that the condition (5.3) is violated. The proof of Proposition 5.4 is finished. $\square$

The next proposition shows that in our case the Fuchsian system (3.6) never has a diagonal monodromy group.

PROPOSITION 5.5. — *Under the conditions (1.5) and (5.3) the monodromy group G of the normal variational equations (3.6) is not diagonalizable.*

Proof. — Let us assume that G is diagonalizable : $M_+ = \mathrm{diag}(s_1, s_2, s_3)$, $M_- = \mathrm{diag}(s_{i_1}, s_{i_2}, s_{i_1})$, $s_k = e^{2\pi i \lambda_k}$, $k = 1, 2, 3$ where (i_1, i_2, i_3) is a certain permutation of $(1, 2, 3)$.

That implies existence of three independent solutions of (3.6) of the form

$$X_k(p) = (p - \alpha)^{\lambda_k} (p + \alpha)^{\lambda_{i_k}} N_k(p), \quad k = 1, 2, 3\,, \tag{5.10}$$

with vector functions N_k rational on p.

Let Y_i, $i = 1, 2, 3$ be three linearly independent eigenvectors of A with corresponding eigenvalues λ_i (we remind that λ_i are pairwise different in view of 5.3)). One deduces from (5.10) the following formulas containing the rational vector functions R_k

$$X_k = (p - \alpha)^{\lambda_{g_k}} (p + \alpha)^{\lambda_{u_k}} R_k(p), \quad k = 1, 2, 3 \tag{5.11}$$

with $g_k\ u_k \in \{1, 2, 3\}$ and where now $R_k = Y_{g_k} + Y_{g_k}^+ (p - \alpha) + \cdots$ in the neighborhood of $p = \alpha$ and $R_k = Y_{u_k} + Y_{g_k}^- (p + \alpha) + \cdots$ in the neighborhood of $p = -\alpha$, $(k = 1, 2, 3)$. Since the system (3.6) is invariant under the change $p \to -p$, $X_k(-p)$ is also its solution. It yields, together with condition (5.3), $g_k = u_k$ for $k = 1, 2, 3$.

Under the same condition (5.3), as a simple argument shows, Y_{g_1}, Y_{g_2}, Y_{g_3} must be pairwise different. Since $M_+ = M_-$ and $M_+ M_- M_\infty = \mathrm{Id}$, the similar property holds for the point $p = \infty$.

Thus, one can represent X_k as below

$$X_k = (p - \alpha)^{\lambda_k} (p + \alpha)^{\lambda_k} p^{n_k} P_k(p), \quad k = 1, 2, 3\,, \tag{5.12}$$

where $n_k \in \mathrm{Spectr}(B) = \{0, -1, -1\}$; P_k are polynomial vector functions such that $P_k(\alpha) = P_k(-\alpha) = Y_k$ for $k = 1, 2, 3$ and $P_k(0)$ are eigenvectors of B.

Let D_k be the order of X_k at infinity. One finds from (5.12) : $D_k = -2\lambda_k - n_k - d_k$ so that $\sum D_k = -2 \sum \lambda_k - \sum n_k - \sum d_k \leqslant -2 \cdot (-1) - (-3) - \sum d_k$ since $\sum n_k \leqslant 0$. Otherwise, we know that $\sum D_k = 4$ (see Proposition 3.5) and hence $\sum d_k \leqslant 1$. Thus, at least one of the vectors P_k is constant and, as easy follows from substitution of (5.12) into (3.6), is a common eigenvector of A and B. This is impossible according to Proposition 3.6 and the proof is achieved. $\square$

In view of Lemma 3.2 the above results can now be summarized by the main Theorem 1.1.

6. Conclusion

The difficulty of the rattleback problem is due to its non-hamiltonian nature. That explains the non-trivial structure of the monodromy group studied in the previous sections. The technical problem in applying our non-integrability Theorem 1.1 are the hyperbolicity conditions (5.3). Of course, one can write these restrictions directly using Cardano formulas that will lead to quite complicated expressions. It may be interesting to consider a concrete example. Let $I_1 = 0.5$, $I_2 = 0.6$, $I_3 = 0.8$, $\delta = 1.3$, $m = 1$, $g = 1$, $b_1 = 1$, $b_2 = 2$, $b_3 = 3$. Then the eigenvalues of the characteristic polynomial (3.8) are

$$\lambda_1 = -0.365 - 0.858i, \quad \lambda_2 = -0.435 + 0.963i, \quad \lambda_3 = -0.200 - 0.106i\,,$$

and the conditions (5.3) are obviously satisfied.

We believe that our non-integrability conditions can be strengthened. Thus, the further detailed analysis of the characteristic polynomial (3.8) is needed.

We note that in the heavy rigid body case, the characteristic polynomial (3.8) always has a real root as seen from (3.18). Indeed, it corresponds to existence of the fourth polynomial first integral (1.12) of the Euler-Poisson equations. In the rattleback case, the interesting remaining problem is the existence of new meromorphic first integrals. Our Theorem 4.1 answers this question only then the variational equations (3.6) have logarithmic singularities. Our intention to avoid the study of the Zariski closure of the monodromy group G was twofold. Firstly, that makes the proofs self-contained and quite elementary. Secondly, we would like to underline the importance of the symmetry conditions (5.6), coming from the mechanical context of the problem and which simplify greatly the non-integrability analysis.

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A. Appendix

The vector field F is given by

$$F(\gamma,\omega;s) = U^{-1}(-[\omega,\Theta\omega] + mg[r,\gamma] - m[r,[\omega,[\omega,r]]] - m[r,[\omega,\frac{dr}{dt}]])\,, \tag{A.1}$$

where U is given by (1.8). For the case of the ellipsoid $s = \sqrt{\sum b_i^2\gamma_i^2}$, $r = (r_1, r_2, r_3)^t$,

$$r_i = -b_i^2\gamma_i\frac{1}{s}\,, \tag{A.2}$$

and

$$\frac{dr_i}{dt} = -b_i^2\frac{d\gamma_i}{dt}\frac{1}{s} + b_i^2\gamma_i\left(\sum b_i^2\gamma_i\frac{d\gamma_i}{dt}\right)\frac{1}{s^3}\,, \tag{A.3}$$

with $d\gamma/dt = [\gamma,\omega]$ as given by (1.7b).

The matrix N is given by

$$N = \begin{pmatrix} 0 & * & * \\ \mathrm{i}\frac{p^2-\alpha}{\beta} & 0 & p \\ -\frac{p^2-\alpha}{\beta} & -p & 0 \end{pmatrix}\,. \tag{A.4}$$

The matrix M is given by

$$M = \begin{pmatrix} a_1 p & a_2 p & c_1 + c_2 \frac{\alpha}{p^2-\alpha} \\ a_3 p & -a_1 p & c_3 + c_4 \frac{\alpha}{p^2-\alpha} \\ -\mathrm{i}\frac{p^2-\alpha}{\beta} & \frac{p^2-\alpha}{\beta} & 0 \end{pmatrix} . \tag{A.5}$$

The coefficients a_i, c_i are related by one quadratic equation, which is equivalent to (3.14). They are given by

$$\theta_d a_1 = 2\frac{mg}{\beta}(\rho_1-\rho_2)(\Sigma_{12}-\mathrm{i}m\rho_1\rho_2)+(\Sigma_{12}-\mathrm{i}m\rho_1\rho_2)(\Sigma_{11}+\Sigma_{22}+m(\rho_1^2-\rho_2^2)) \tag{A.6a}$$

$$\begin{aligned} \theta_d a_2 &= 2\frac{mg}{\beta}(\rho_1-\rho_2)(\Sigma_{22}+m\rho_1^2) \\ &+ (\Sigma_{12}-\mathrm{i}\Sigma_{22}-\mathrm{i}m\rho_1(\rho_1-\rho_2)(\Sigma_{12}+\mathrm{i}\Sigma_{22} \\ &+ \mathrm{i}m\rho_1(\rho_1-\rho_2) \end{aligned} \tag{A.6b}$$

$$\begin{aligned} \theta_d a_3 &= 2\frac{mg}{\beta}(\rho_1-\rho_2)(\Sigma_{11}-m\rho_2^2)- \\ &- (\Sigma_{11}-\mathrm{i}\Sigma_{12}-m\rho_2(\rho_1-\rho_2)) \\ & (\Sigma_{11}+\mathrm{i}\Sigma_{12}+m\rho_2(\rho_1-\rho_2)) \end{aligned} \tag{A.6c}$$

$$\begin{aligned} 2\theta_d c_1 &= 2mg(\Sigma_{12}(\rho_1-\rho_3)+\mathrm{i}\Sigma_{22}(\rho_2-\rho_3)+\mathrm{i}m\rho_1\rho_3(\rho_1-\rho_2)) \\ &- m\beta(\Sigma_{12}(2\rho_1-\rho_2)-\mathrm{i}\Sigma_{22}(\rho_1-\rho_2)) \\ &+ \mathrm{i}m^2\beta\rho_1\rho_3(\rho_1^2-\rho_2^2) \end{aligned} \tag{A.6d}$$

$$\begin{aligned} 2\theta_d c_3 &= 2mg(\Sigma_{11}(\rho_1-\rho_3)+\mathrm{i}\Sigma_{12}(\rho_2-\rho_3)-m\rho_2\rho_3(\rho_1-\rho_2)) \\ &- m\beta(\Sigma_{11}(2\rho_1-\rho_2)-\mathrm{i}\Sigma_{12}(\rho_1-\rho_2)) \\ &- m^2\beta\rho_2\rho_3(\rho_1^2-\rho_2^2) \end{aligned} \tag{A.6e}$$

$$\theta_d c_2 = -m\beta\rho_3(\Sigma_{12}\rho_1+\mathrm{i}\Sigma_{22}\rho_2) \tag{A.6f}$$

$$\theta_d c_4 = m\beta\rho_3(\Sigma_{11}\rho_1+\mathrm{i}\Sigma_{12}\rho_2) \tag{A.6g}$$

The matrix B is given by

$$B = \begin{pmatrix} -1 & 0 & -2\mathrm{i}(c_1-c_2) \\ 0 & -1 & -2\mathrm{i}(c_3-c_4) \\ 0 & 0 & 0 \end{pmatrix} . \tag{A.7}$$

The matrix A is given by

$$A = \begin{pmatrix} -\mathrm{i}a_1 & -\mathrm{i}a_2 & -\mathrm{i}c_2 \\ -\mathrm{i}a_3 & \mathrm{i}a_1 & -\mathrm{i}c_4 \\ -1/\beta & -\mathrm{i}/\beta & -1 \end{pmatrix} . \tag{A.8}$$

Rigid body case :

$$r_1 = -\rho_1, r_2 = -\mathrm{i}\rho_2, r_3 = \rho_3 = 0, \quad m = 0, mg \neq 0, \quad \Sigma_{12} = \delta = 0\,. \quad \text{(A.9)}$$

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