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Asymptotic Solutions of nonlinear difference equations[(*)]

I.P. van den Berg[(1)]

Abstract. — We study the asymptotics of first-order nonlinear difference equations. In particular we present an asymptotic functional equation for potential asymptotic behaviour, and a theorem stating sufficient conditions for the existence of an actual solution with such asymptotic behaviour.

Résumé. — We study the asymptotics of first-order nonlinear difference equations. In particular we present an asymptotic functional equation for potential asymptotic behaviour, and a theorem stating sufficient conditions for the existence of an actual solution with such asymptotic behaviour.

1. Introduction

We study first-order difference equations of the type

$$Y(X+1) = F(X, Y(X)), \tag{1.1}$$

where F is supposed to be continuously partially differentiable in Y. The main theorem of this article (Theorem 2.2) gives sufficient conditions in terms of F and F_2' to determine whether (1.1) possesses solutions with asymptotic behaviour $\widehat{Y}(X)$; such a sequence $\widehat{Y}$, that we call the *approximate solution*, should satisfy the so-called *asymptotic functional equation*

$$\lim_{X\to\infty} \frac{F\left(X, \widehat{Y}(X)\right) - \widehat{Y}(X)}{\widehat{Y}(X)\left(\left|F_2'\left(X, \widehat{Y}(X)\right)\right| - 1\right)} = 0. \tag{1.2}$$

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So $\widehat{Y}(X)$ is a kind of weak, asymptotic, fixed point for F. The other conditions express that the solutions of (1.1) in a neighbourhood of $\widehat{Y}(X)$ contract (attract or repel each other), faster than $\widehat{Y}(X)$ moves itself. Under some additional regularity conditions there will be an actual solution $\widetilde{Y}$ of (1.1) such that $\widetilde{Y}(X) \sim \widehat{Y}(X)$ for $X \to \infty$.

The main theorem appears to be more general than other existence theorems in the study of the asymptotics of first-order difference equations. Firstly, the class of equations under consideration is larger than the classes which are usually studied, which are the class of linear equations [20][18][19], sometimes allowing for certain types of perturbations [3][16][11], and the class of analytic equations [14][13][15].

Secondly, in almost all cases the study of the asymptotic directions of interest – the asymptotic fixed points for the function F defining the equation – is well separated from the study of sufficient conditions for the existence of an actual solution in such asymptotic directions. In fact it is often taken for granted that the equation is rewritten in such a way that one has to prove the existence of a solution tending to 0. In contrast, Theorem 2.2 indicates to which extent the exact equation for fixed points $F\left(X, \widehat{Y}(X)\right) - \widehat{Y}(X) = 0$ may be relaxed, i.e. to (1.2) or alternatively $F\left(X, \widehat{Y}(X)\right) - \widehat{Y}(X) = o\left(\widehat{Y}(X)\left(\left|F_2'\left(X, \widehat{Y}(X)\right)\right| - 1\right)\right)$, and gives at the same time an asymptotic condition on $\widehat{Y}(X)$ and on $F_2'\left(X, \widehat{Y}(X)\right)$, such that there exists a solution $\widetilde{Y}$ of (1.1) asymptotic to $\widehat{Y}$. In a sense, these conditions allow for a convenient transformation (see Section 4.1) such that $\widehat{Y}$ becomes exactly 0. A definite subclass of our equations is thus transformed in singular perturbations, and we mention here some relation to [12].

Our approach is confined to first-order difference equations in one real variable. As such it is less developed than most of the theories encountered in the literature mentioned above. These theories consider mainly equations of higher order and/or in more variables, which may be complex. Notably the analytic theory gives more precision. For instance, in the case of analytic equations one may look for a *formal solution* Y_0 in terms of a power series, for which one may show that it acts as the asymptotic expansion of an actual solution [14], [13], [15], or else for expansions in terms of factorial series [18]. Still, an approach only based on first-order approximations, both in its conditions and in its proofs is of some interest. Depending on additional properties of F, such an approximation may or may not be extended to give full expansions.

Because we consider approximations of first order and impose few regularity on F, our results are somewhat similar to some results on perturbation of linear equations like in [11] and [3]. There in principle no regularity is required with respect to differentiation. However the general perturbation result of [11, Section 7.6] needs a strong condition on the boundedness of the perturbation. In our setting this would mean that $\sum_{X\geqslant A}\left|F_2'\left(X,\widehat{Y}(X)\right)\right|$ is bounded for some integer A, while we consider uncertainties of the form $o\left(\widehat{Y}(X)\left(\left|F_2'\left(X,\widehat{Y}(X)\right)\right|-1\right)\right)$, which may be unbounded. The article [3] considers essentially perturbations of linear homogeneous equations and their non-zero eigenvalues, while in our setting $F_2'\left(X,\widehat{Y}(X)\right)$ may be asymptotically zero, or infinite. Still, we need that always $\left|F_2'\left(X,\widehat{Y}(X)\right)\right|<1$, or always $\left|F_2'\left(X,\widehat{Y}(X)\right)\right|>1$. In a sense this means that $\widehat{Y}$ lies in an attractive tube, or in an repulsive tube.

Our work is inspired by the so-called *river-phenomenon* for differential equations $Y'(X)=F(X,Y)$. Computer graphics of the phase-portrait of several types of differential equations show contractions of trajectories, an optical phenomenon similar to rivers and their confluents on a map. The phenomenon, often so strong as to seem to contradict the property of uniqueness of solutions – a property shared by all the equations in question – , is observed for such familiar equations as linear equations with constant coefficients, Riccati equations and the Van der Pol equation. Attempts to modelling were made in among others [10][1][5]. As shown in [5] (see also [4]) a large class of rivers satisfies an asymptotic functional equation similar to (1.2), i.e.

$$\lim_{X\to\infty}\frac{F\left(X,\widehat{Y}(X)\right)}{\widehat{Y}(X)F_2'\left(X,\widehat{Y}(X)\right)}=0. \tag{1.3}$$

There are similarities between the asymptotics of differential equations and of difference equations, but also some differences. This is already seen comparing the associated asymptotic functional equations (1.2) and (1.3). A notable difference appears in the proofs of the existence of an actual solution $\widetilde{Y}$ asymptotic to $\widehat{Y}$. In the continuous case one may use rescalings at liberty and has a priori possibility to go back and forth in time. On the contrary, in the discrete case the fixed time-step puts a bound on the factor of rescaling (it makes no sense to reduce the scale to such an extent as to observe only one time-moment) and going back in time must be carefully prepared (see Propositions 4.12, and 4.14, and the proof of the nonstandard existence theorem 4.1 in Section 5).

The conditions of Theorem 2.2 for the existence of solutions in the asymptotic directions given by the equation (1.2) are actually more general than for corresponding existence theorems in the articles on rivers of differential equations. In fact weaker forms of contraction are allowed, up to almost parallelness. In a second paper [7] we present necessary and sufficient criteria to verify whether such forms of contraction occur in a well-defined region around a solution obtained through Theorem 2.2.

The article is written in the context of Nonstandard Analysis. To our opinion the use of infinitesimal and infinitely large numbers facilitates asymptotic calculations and reasoning. For an introduction to the axiomatic form IST of nonstandard analysis we use, and for terminology and notations, we refer to [17][8]. We denote by $\oslash$ the external set of all infinitesimal numbers, $\pounds$ the external set of all limited numbers and @ the set of all positive appreciable (i.e. limited, but not infinitesimal) numbers. These symbols will be used just as $o(\cdot)$ and $O(\cdot)$ in classical asymptotics; for example we may write $x = \oslash$ instead of $x \simeq 0$.

The article has the following structure. In Section 2 we give a formal definition for approximate solutions, state the existence theorem and present some geometric motivation, finishing with some examples. In Section 3 we state some general lemmas which enable to deduce in a sense global asymptotic behaviour from behaviour known locally. In Section 4 we formulate nonstandard equivalents for the existence theorem and for the definition of an approximate solution and carry out an approximate rescaling. Within this rescaling, we show that the set of solutions within an appropriate tube exhibits a sort of uniform behaviour, which may be of repulsive or attractive nature; in the latter case we distinguish between very weak attraction, leading to a sort of almost-parallelness of solutions and a stronger form of attraction. In Section 5 we derive from these behaviours the existence of a solution asymptotic to the approximate solution. We adapt the proof to the various types of behaviour observed in Section 4.

2. Approximate solutions and the existence theorem

Conventions. —

1. Unless it is said explicitly to be otherwise, we always consider difference equations (1.1) of the form

$$Y(X+1) = F(X, Y(X)),$$

where F is a real-valued function defined and of class C^1 in the second variable on some set $U \subset \mathbb{N} \times \mathbb{R}$ such that the projection on $\mathbb{N}$

contains a set of the from $\{X \in \mathbb{N} \,|\, X \geqslant A_0\}$ with $A_0 \in \mathbb{N}$. We say that a sequence Y is a solution if $Y(X)$ is defined and satisfies (1.1) on some set $\{X \in \mathbb{N} \,|\, X \geqslant A_1\}$ with $A_1 \in \mathbb{N}$, or $Y(X)$ is defined on some set $\{X \in \mathbb{N} \,|\, A_2 \leqslant X \leqslant A_3\}$, with $A_2, A_3 \in \mathbb{N}, A_2 < A_3$ and satisfies (1.1) on $\{X \in \mathbb{N} \,|\, A_2 \leqslant X \leqslant A_3 - 1\}$; it is supposed that such an interval is maximal.

2. The symbol $\sim$ is used in the sense of classical asymptotics. Indeed, let Y and Z be two sequences. Then $Y(X) \sim Z(X)$ for $X \to \infty$ if and only if $\lim_{X\to\infty} Y(X)/Z(X) = 1$.

DEFINITION 2.1. — *A sequence* $\widehat{Y}$ *is called an* approximate solution *of (1.1) if*

1. *There exist* $A \in \mathbb{N}, B \neq 0$ *such that*
   (a) *Either* $(\forall X \geqslant A)\left(\widehat{Y}(X) < 0\right)$ *or* $(\forall X \geqslant A)\left(\widehat{Y}(X) > 0\right)$.
   (b) *In case* $(\forall X \geqslant A)\left(\widehat{Y}(X) < 0\right)$ *one has* $(X, Y) \in U$ *for all* $X \geqslant A$ *and all* Y *such that* $(1+B)\widehat{Y}(X)) \leqslant Y \leqslant (1-B)\widehat{Y}(X)$, *and in case* $(\forall X \geqslant A)\left(\widehat{Y}(X) > 0\right)$ *one has* $(X, Y) \in U$ *for all* $X \geqslant A$ *and all* Y *such that* $(1-B)\widehat{Y}(X)) \leqslant Y \leqslant (1+B)\widehat{Y}(X)$.
   (c) *Either* $(\forall X \geqslant A)\left(\left|F_2'\left(X, \widehat{Y}(X)\right)\right| < 1\right)$ *or* $(\forall X \geqslant A)$ $\left(\left|F_2'\left(X, \widehat{Y}(X)\right)\right| > 1\right)$.
2. $\dfrac{F\left(X,\widehat{Y}(X)\right)-\widehat{Y}(X)}{\widehat{Y}(X)} = o\left(\left|F_2'\left(X, \widehat{Y}(X)\right)\right| - 1\right)$ *for* $X \to \infty$.
3. $\dfrac{\widehat{Y}(X+1)-\widehat{Y}(X)}{\widehat{Y}(X)} = o\left(\left|F_2'\left(X, \widehat{Y}(X)\right)\right| - 1\right)$ *for* $X \to \infty$.
4. $Y(X) \sim \widehat{Y}(X)$ *for* $X \to \infty$ *implies* $(|F_2'(X, Y(X))| - 1) \sim \left(\left|F_2'\left(X, \widehat{Y}(X)\right)\right| - 1\right)$ *for* $X \to \infty$.

THEOREM 2.2 (Existence theorem). — *Let* $\widehat{Y}$ *be an approximate solution of (1.1). Then (1.1) has a solution* $\widetilde{Y}$ *such that* $\widetilde{Y}(X) \sim \widehat{Y}(X)$ *for* $X \to \infty$.

Comments. —

1. The condition of Definition 2.1.2 is equivalent to the asymptotic functional equation (1.2) and can be used to determine approximate solutions. Observe that (1.2) is satisfied if $\widehat{Y}$ satisfies the simpler equation

$F\left(X,\widehat{Y}(X)\right)-\widehat{Y}(X)=0$. However, it may be impossible to find an exact solution of $F\left(X,\widehat{Y}(X)\right)=\widehat{Y}(X)$ in practice. In fact (1.2) expresses that there is no need for an exact fixed point of F, and that a tolerance is allowed of the form

$$F\left(X,\widehat{Y}(X)\right)-\widehat{Y}(X)=o\left(\widehat{Y}(X)\left(\left|F_2'\left(X,\widehat{Y}(X)\right)\right|-1\right)\right)\text{ for} X\to\infty.$$

2. The condition 2 of Definition 2.1 has a clear geometric interpretation. Applied to a true solution $\widetilde{Y}$ we obtain

$$\frac{F\left(X,\widetilde{Y}(X)\right)-\widetilde{Y}(X)}{\widetilde{Y}(X)}=o\left(\left|F_2'\left(X,\widetilde{Y}(X)\right)\right|-1\right) \tag{2.1}$$

for $X\to\infty$. To get a more precise description of the configuration of the set of solutions, we will use the terminology of nonstandard analysis. We suppose F,U and $\widehat{Y}$ to be standard. We use the nonstandard characterization of a standard approximate solution presented in Proposition 4.2 of Section 4. We consider the behaviour of the solutions on the external set

$$H\equiv\left\{(\omega,Y)\,\middle|\,\omega\simeq\infty, Y=(1+\varnothing)\widehat{Y}(\omega)\right\}.$$

The set H is sometimes called the *asymptotic halo* of $\widehat{Y}$. It is a subset of U by Proposition 4.2. Using Proposition 4.2 we may rewrite (2.1) for unlimited ω to

$$\frac{\widetilde{Y}(\omega+1)-\widetilde{Y}(\omega)}{\widetilde{Y}(\omega)}=\varnothing\cdot\left(\left|F_2'\left(\omega,\widetilde{Y}(\omega)\right)\right|-1\right). \tag{2.2}$$

The left-hand side of (2.2) measures the relative growth of the solution at ω. Now under the regularity condition given by Proposition 4.2 (corresponding to the regularity condition given by definition 2.1.4 the term $\left|F_2'\left(\omega,\widetilde{Y}(\omega)\right)\right|-1$ measures the relative growth of the difference of two solutions, say Φ and Ψ, such that $(\omega,\Phi(\omega)),(\omega,\Psi(\omega))\in H$. Indeed, we have for some Y lying between $\Phi(\omega)$ and $\Psi(\omega)$

$$\begin{aligned}&\frac{|\Psi(\omega+1)-\Phi(\omega+1)|-|\Psi(\omega)-\Phi(\omega)|}{|\Psi(\omega)-\Phi(\omega)|}\\ =\;&\left|\frac{F(\omega,\Psi(\omega))-F(\omega,\Phi(\omega))}{\Psi(\omega)-\Phi(\omega)}\right|-1\\ =\;&|F_2'(\omega,Y)|-1,\end{aligned}$$

whereas

$$|F_2'(\omega,Y)|-1=(1+\varnothing)\left(\left|F_2'\left(\omega,\widehat{Y}(\omega)\right)\right|-1\right)=(1+\varnothing)\left(\left|F_2'\left(\omega,\widetilde{Y}(\omega)\right)\right|-1\right).$$

So if (2.1) is satisfied, the relative growth of the true solution $\widetilde{Y}$ at ω is infinitely small with respect to the relative growth of the difference of two solutions which at ω are contained in the asymptotic halo of $\widetilde{Y}$. This behaviour of strong contraction of solutions around a moderately growing solution is also typical of slow-fast systems. Notice that, for all unlimited arguments, this contraction is always of the same nature, attraction if $\left|F_2'\left(\omega, \widetilde{Y}(\omega)\right)\right| < 1$ and repulsion if $\left|F_2'\left(\omega, \widetilde{Y}(\omega)\right)\right| > 1$. In the first case we speak of an *attractive solution* and in the second case we speak of a *repulsive solution.* The regularity condition 4.2.3, corresponding to the regularity condition given by definition 2.1.3, thus expresses that an approximate solution should have infinitely small relative growth with respect to the relative growth of the difference of two true solutions contained in its asymptotic halo.

As regards to the standard conditions expressed by Definition 2.1, one may remark the following, still in the case of a standard equation and a standard approximate solution. The (standard) set U is to be interpreted as a tube around $\widehat{Y}$, which contains the asymptotic halo. Applying appropriate permanence principles, one may show that the above observations hold in a weakened sense on U. Indeed, a solution $\widetilde{Y}$ which is staying within the tube is attractive for solutions within the tube if $(\forall X \geqslant A)\left(\left|F_2'\left(X, \widehat{Y}(X)\right)\right| < 1\right)$ and repulsive for solutions within the tube if $(\forall X \geqslant A)\left(\left|F_2'\left(X, \widehat{Y}(X)\right)\right| > 1\right)$. This observation may also be reversed. The asymptotic functional equation (1.1), together with the regularity conditions of Definition 2.1.3 and 2.1.4 imply that the tube U is either attractive or repulsive, which is sufficient for some solution to slip through. However it is our opinion that the asymptotic estimations, already rather delicate in the case of the nonstandard proof, may turn out to be still more difficult.

Well-known simple special cases of attractiveness and repulsiveness are linear difference equations with constant coefficients of type $Y(X+1) = AY(X)+R(X)$. Here $F_2'(X,Y) = A$ is constant, and may also be seen as the eigenvalue of the corresponding homogeneous equation $Y(X+1) = AY(X)$. If $|A| < 1$ the particular solution is asymptotically stable and if $|A| > 1$ the particular solution is asymptotically unstable.

Not every (standard) solution asymptotic to a (standard) approximate solution satisfies the equations (2.1) and (2.2). This is due to the fact that solutions of first-order difference equations may have oscillatory behaviour. Still, in such cases the reasoning above remains valid taking two steps instead of one. Indeed, in [7] it is proved that

$$\frac{\widetilde{Y}(\omega+2)-\widetilde{Y}(\omega)}{\widetilde{Y}(\omega)} = \varnothing \cdot \left(\left|F_2'\left(\omega, \widetilde{Y}(\omega)\right)\right| - 1\right).$$

The article [7] addresses also the property of uniqueness of solutions in the asymptotic directions given by the asymptotic functional equation (1.2), which appears to be satisfied in the repulsive case, but clearly not in the attractive case.

3. We point out some analogy with the rivers of differential equations $Y' = F(X,Y)$, as mentioned in the introduction. There the approximate solutions satisfy the asymptotic functional equation (1.3). This equation rewritten to

$$F\left(X, \widehat{Y}(X)\right) = o\left(\widehat{Y}(X) F_2'\left(X, \widehat{Y}(X)\right)\right) \text{for} X \to \infty$$

allows a tolerance of the form $o\left(\widehat{Y}(X) F_2'\left(X, \widehat{Y}(X)\right)\right)$ in the determination of approximate zeroes of F. It may be shown that a true river solution $\widetilde{Y}$ also verifies (1.3). Then one finds

$$\frac{\widetilde{Y}'(X)}{\widetilde{Y}(X)} = o\left(F_2'\left(X, \widetilde{Y}(X)\right)\right) \text{for} X \to \infty.$$

This again expresses slow growth of the river with respect to the growth of the difference of two solutions, say Φ and Ψ, on the asymptotic halo $\widetilde{Y}$, for assuming some regularity on F_2' we find in a similar way as above for $\omega \simeq \infty$

$$\frac{\Psi'(\omega)-\Phi'(\omega)}{\Psi(\omega)-\Phi(\omega)} = \frac{F(\omega,\Psi(\omega)) - F(\omega,\Phi(\omega))}{\Psi(\omega)-\Phi(\omega)} = (1+\varnothing) F_2'\left(\omega, \widetilde{Y}(\omega)\right),$$

whereas $\widetilde{Y}'(\omega)/\widetilde{Y}(\omega) = \varnothing \cdot \left(F_2'\left(\omega, \widetilde{Y}(\omega)\right)\right)$.

Examples 2.3. — We give some examples which show how the existence theorem can be used to solve difference equations asymptotically. The first example shows that the usual formula of exact polynomial solutions of linear difference equations with constant coefficients may be extended to the asymptotics of such equations. This example serves only to illustrate that linear equations, that may be solved by other methods, belong also to the class of equations covered by the existence theorem. A second class of relevance is formed by the polynomial equations. For simplicity we present two classes of quadratic equations. However, one might develop a general approach with the help of Newton polygons. Indeed, if $F(X,Y)$ is polynomial, then $F(X,Y) - Y$ is also polynomial and thus one disposes of a general method to search for approximate zeroes. Still one needs to verify whether the approximate zeroes correspond to approximate solutions in the sense of Definition 2.1. It is not excluded that such a verification can be substituted by a simpler, algebraic verification on the basis of inequalities, as can be found in [9] for the case of polynomial differential equations. One may expect that a sort of generalized Newton-polygon method still is effective if some coefficients of the polynomial are bounded C^1 functions instead of constants. We will not develop this point here, but we present a quadratic equation with coefficients of the like, which is C^1 in the second variable, but not C^2. As such, it constitutes an example of an equation, outside the realm of methods based on analytic functions, but within the reach of Theorem 2.2.

2.1. Difference equations with constant coefficients

PROPOSITION 2.4. — *Consider*

$$Y(X+1) = AY(X) + R(X) \tag{2.3}$$

with $|A| \neq 1$ *and* $R(X) \sim BX^r$ *for* $X \to \infty$ *with* $B \neq 0$ *and* $r \in \mathbb{R}$*. Then (2.3) has a solution* $\widetilde{Y}(X) \sim \frac{B}{1-A}X^r$ *for* $X \to \infty$.

It is easy to verify that $\widehat{Y}(X) = \frac{B}{1-A}X^r$ is an approximate solution. We omit further details. Of course, if R is a polynomial we obtained the principal term given by the exact solution of the usual resolution method.

2.2. Some quadratic equations

PROPOSITION 2.5. — *Consider*

$$Y(X+1) = Y(X)^2 - X^a \tag{2.4}$$

1. *If* $a > 0$ *the equation (2.4) has a solution* $\widetilde{Y_1}(X) \sim X^{a/2}$ *for* $X \to \infty$ *and a solution* $\widetilde{Y_2}(X) \sim -X^{a/2}$ *for* $X \to \infty$*. Both solutions are repulsive.*
2. *If* $a < 0$ *the equation (2.4) has a repulsive solution* $\widetilde{Y_1}(X) \to 1$ *for* $X \to \infty$ *and several attractive solutions asymptotic to* $-X^a$ *for* $X \to \infty$.

To prove Proposition 2.5, note that the sequences $\widehat{Y_1}(X) = X^{a/2}$ for $X \to \infty$ and $\widehat{Y_2}(X) = -X^{a/2}$ are solutions of the associated asymptotic functional equation (1.2). The verification of the remaining conditions of the existence theorem is straightforward. The repulsion follows from the equality $F_2'\left(X, \widehat{Y_1}(X)\right) = \left|F_2'\left(X, \widehat{Y_2}(X)\right)\right| = 2X^{a/2}$.

In Proposition 2.5 the sequences $\widehat{Y_1}(X) = 1$ for $X \to \infty$ and $\widehat{Y_2}(X) = -X^{a/2}$ are solutions of the associated asymptotic functional equation. The verification of the remaining conditions of the existence theorem is again straightforward. The repulsiveness of $\widetilde{Y_1}$ follows from the equality $F_2'\left(X, \widehat{Y_1}(X)\right) = 2$. The attractiveness of the standard solutions asymptotic to $\widehat{Y_2}$ follows from $\left|F_2'\left(X, \widehat{Y_2}(X)\right)\right| = 2X^a$.

PROPOSITION 2.6. — *Consider*

$$Y(X+1) = Y(X)^2 + Y(X) - X^a \tag{2.5}$$

1. *If $-2 < a < 0$ the equation (2.5) has a repulsive solution $\widetilde{Y_1}(X) \sim X^{a/2}$ for $X \to \infty$ and several attractive solutions asymptotic to $-X^{a/2}$ for $X \to \infty$.*

2. *If $a > 0$ the equation (2.5) has a solution $\widetilde{Y_1}(X) \sim X^{a/2}$ for $X \to \infty$ and a solution $\widetilde{Y_2}(X) \sim -X^{a/2}$ for $X \to \infty$. Both solutions are repulsive.*

For a proof of Proposition 2.6, notice that the sequences $\widehat{Y_1}(X) = X^{a/2}$ for $X \to \infty$ and $\widehat{Y_2}(X) = -X^{a/2}$ are obvious solutions of the associated asymptotic functional equation (1.2). The verification of the remaining conditions of the existence theorem is straightforward. The repulsiveness of $\widetilde{Y_1}(X)$ follows from the equality $F_2'\left(X, \widehat{Y_1}(X)\right) = 2X^{a/2} + 1$ and the attractiveness of the solutions asymptotic to $\widehat{Y_2}$ from the equality $\left|F_2'\left(X, \widehat{Y_2}(X)\right)\right| = -2X^{a/2} + 1$. Note that, if $i = 1, 2$,

$$\frac{\widehat{Y}_i(X+1) - \widehat{Y}_i(X)}{\widehat{Y}_i(X)} \sim \frac{a}{2X} = o\left(\left|F_2'\left(X, \widehat{Y}_i(X)\right)\right| - 1\right) \qquad \text{for} X \to \infty$$

only for $a > -2$, so the existence theorem cannot be applied for $a \leqslant -2$. In the latter case there are no solutions asymptotic to $\pm X^{a/2}$. To show this, let $b = -2a$ be standard. Suppose $\widetilde{Y}$ is a (standard) solution asymptotic to $\pm X^{a/2}$. Let $\omega \simeq \infty$. Put $x = X/\omega$ and $\widetilde{y}(x) = \omega^b \widetilde{Y}(\omega x)$. Then equation (2.5) becomes

$$\frac{\widetilde{y}(x+1/\omega) - \widetilde{y}(x)}{1/\omega} = \frac{\widetilde{y}(x)^2 - 1/x^b}{\omega^{b-1}}.$$

If $\widetilde{Y}$ were asymptotic to $\pm 1/X^b$, one would have $\widetilde{y}^2(x) \simeq 1/x^b$ for all appreciable x. Because $b \geqslant 1$, this would imply

$$\frac{\widetilde{y}(x+1/\omega) - \widetilde{y}(x)}{1/\omega} \simeq 0$$

for all appreciable x. But then $\widetilde{y}(x) \simeq \widetilde{y}(1)$ for all appreciable x, a contradiction. By Transfer, the result holds for all $b \geqslant 1$, i.e. $a \leqslant -2$.

Part (2) of Proposition 2.6 follows from the above considerations, where the repulsiveness of $\widetilde{Y_2}$ is a consequence of the equality $\left|F_2'\left(X, \widehat{Y_2}(X)\right)\right| = 2X^{a/2} - 1$.

2.3. An equation of class C^1

The function $F(X,Y) = Y^2 + \sin Y \cdot |\sin Y| - X^a$ is C^1 in the second variable, but not C^2. In the same way as in Proposition 2.5 (1) one shows that, if $a > 0$, the equation $Y(X+1) = F(X, Y(X))$ has two repulsive solutions asymptotic to $\pm X^{a/2}$.

3. General lemmas on asymptotic behaviour

The next lemma "presses" a solution of a difference equation which is infinitely close to zero on some initial part of the infinitely large numbers onto a standard solution tending to zero. The lemma is formulated for difference equations (1.1), defined by a function F that needs only to be continuous in the second variable. Its proof uses several fundamental principles of nonstandard reasoning. One of these is the Monadic Transfer Principle [6] which, though being more general, implies that a halic, absolute external property (a property of the form $\forall^{st} x I(x)$, with I an internal property with x as its only free parameter) valid for all unlimited numbers smaller than some given unlimited number ω, or valid for all numbers larger than some given unlimited number ω, is in fact valid for all unlimited numbers. Such a property is for instance "$y \simeq 0$", i.e. "$(\forall^{st} n)(n \in \mathbb{N} \rightarrow y < 1/n)$". The case of a sequence Y for which it can be proved that $Y(X) \simeq 0$ for all unlimited X, if it is only known that there exists an unlimited number ω such that $Y(X) \simeq 0$ for all unlimited $X \leqslant \omega$, had already been considered by Robinson in [21, p. 79].

LEMMA 3.1. — *Let* $U = ([A_0, \infty) \cap \mathbb{N}) \times [-B_0, B_0]$ *with* $A_0 \in \mathbb{N}, B_0 > 0$ *standard. Let* $F : U \longrightarrow \mathbb{R}$ *be a function which is standard and continuous in the second variable. Consider the difference equation* $Y(X+1) = F(X, Y(X))$. *Assume there exist* $\omega \simeq \infty$ *and a solution* Y *such that* $Y(X) \simeq 0$ *for all* $X \simeq \infty$ *with* $X \leqslant \omega$. *Then there exists a standard solution* $\widetilde{Y}$ *such that* $\lim_{X \to \infty} \widetilde{Y}(X) = 0$.

Proof. — Besause a solution is an internal sequence, it is defined at least on some interval $\{A_1, \cdot\cdot, \omega\}$ with A_1 standard. By the Cauchy Principle it satisfies $|Y(X)| \leqslant B_0$ on some interval $\{A_2, \cdot\cdot, \omega\}$ with $A_2 \geqslant A_1$, A_2 standard. Then Y has a shadow $\widetilde{Y}$, which is a standard sequence such that $\widetilde{Y}(X) \simeq Y(X)$ for all standard $X \geqslant A_2$. For such a standard X one has, applying the nonstandard characterization of continuity

$$\widetilde{Y}(X+1) \simeq Y(X+1) = F(X, Y(X)) \simeq F(X, \widetilde{Y}(X)).$$

By the Carnot Principle (standard numbers which are infinitely close are equal) $\widetilde{Y}(X+1) = F(X, \widetilde{Y}(X))$. By the Transfer Principle this equality holds for every $X \geqslant A_2$, so $\widetilde{Y}$ is a solution of the difference equation. Because $\widetilde{Y}(X) \simeq Y(X)$ for all standard $X \geqslant A_2$, by Robinson's Lemma (or the more general Fehrele Principle) there is $\nu \simeq \infty$ such that $\widetilde{Y}(X) \simeq Y(X)$ for all X with $A_2 \leqslant X \leqslant \nu$. We may assume that $\nu \leqslant \omega$. Then $\widetilde{Y}(X) \simeq 0$ for all $X \simeq \infty$ such that $X \leqslant \nu$. By the Monadic Transfer Principle [6] the property $\widetilde{Y}(X) \simeq 0$ holds for all $X \simeq \infty$. Then $\lim_{X \to \infty} \widetilde{Y}(X) = 0$ by the nonstandard characterization of the limit. □

Comments. — The lemma holds for other families of sequences or functions, for example continuous curves. Essential is that the family is closed under uniform convergence on compact intervals.

As a corollary we obtain the following existence theorem on the asymptotics of difference equations (1.1) satisfying the convention mentioned at the beginning of Section 2.

COROLLARY 3.2. — *Let (1.1) be standard and* $\widehat{Y}$ *be a standard approximate solution. Assume there exist* $\omega \simeq \infty$ *and a solution* Y *such that* $Y(X) = (1 + \varnothing)\widehat{Y}(X)$ *for all* $X \simeq \infty$ *with* $X \leqslant \omega$. *Then (1.1) has a standard solution* $\widetilde{Y}$ *such that* $\widetilde{Y}(X) \sim \widehat{Y}(X)$ *for* $X \to \infty$.

The type of reasoning within "tubes" of standard thickness of the next lemma is very common in nonstandard singular perturbation theory, see for instance [2]. The lemma gives a stepwise criterion on sequences which is sometimes helpful in proving that the solutions of difference equations satisfy the condition of Lemma 3.1. However we stress the point that the criterion alone is not sufficient.

Lemma 3.3. — *Let $B_0 > 0$ be standard. Let $\xi, \omega \simeq \infty$, $\xi < \omega$. Let Y be a sequence. Assume that $Y(\xi) \simeq 0$ and that for all X with $\xi \leqslant X \leqslant \omega$ we have the property that $|Y(X)| \leqslant B_0$ implies that*

$$\begin{cases} Y(X+1) \text{ is defined and } |Y(X+1)| < |Y(X)| & \text{if } Y(X) \not\simeq 0 \\ Y(X+1) \text{ is defined and } Y(X+1) \simeq 0 & \text{if } Y(X) \simeq 0. \end{cases}$$

Then $Y(X)$ is defined for all X with $\xi \leqslant X \leqslant \omega$ and satisfies $Y(X) \simeq 0$.

Proof. — Clearly $|Y(X+1)| \leqslant B_0$ if $|Y(X)| \leqslant B_0$. Because $|Y(\xi)| \leqslant B_0$ is defined, by induction $Y(X)$ is defined for all X with $\xi \leqslant X \leqslant \omega$ and satisfies $|Y(X)| \leqslant B_0$. Define

$$\begin{aligned} Z(X) &= \max_{\xi \leqslant N \leqslant X} |Y(N)| \\ M(X) &= \min \{ N \,|\, \xi \leqslant N, |Y(N)| = Z(X) \} . \end{aligned}$$

Suppose $Y(X) \not\simeq 0$ for some X with $\xi \leqslant X \leqslant \omega$. Then $Y(M(X)) = Z(X) \not\simeq 0$. Because $|Y(M(X)-1)| \leqslant B_0$, we have $Y(M(X)) \simeq 0$ if $Y(M(X)-1) \simeq 0$ and $|Y(M(X))| < |Y(M(X)-1)|$ if $Y(M(X)-1)) \not\simeq 0$. In both cases we have a contradiction. Hence $Y(X) \simeq 0$ for all X with $\xi \leqslant X \leqslant \omega$. □

4. Translation into nonstandard terms and rescaling

To prove the existence theorem it suffices to consider standard equations and standard approximate solutions. To be precise, one has the following equivalent version of the existence theorem.

Theorem 4.1. — (Nonstandard existence theorem) *Let (1.1) be standard and $\widehat{Y}$ be a standard approximate solution. Then (1.1) has a standard solution $\widetilde{Y}$ such that $\widetilde{Y}(X) \sim \widehat{Y}(X)$ for $X \to \infty$.*

Proof. — Assume Theorem 2.2 holds. Let (1.1) be standard and $\widehat{Y}$ be a standard approximate solution. Then (1.1) has a solution $\overline{Y}$ such that

$\overline{Y}(X) \sim \widehat{Y}(X)$ for $X \to \infty$. By the Transfer Principle (1.1) has a standard solution $\widetilde{Y}$ such that $\widetilde{Y}(X) \sim \widehat{Y}(X)$ for $X \to \infty$. Conversely, by weakening the nonstandard existence theorem, we obtain that for every standard difference equation (1.1) and standard approximate solution $\widehat{Y}$ there exists a solution $\widetilde{Y}$ such that $\widetilde{Y}(X) \sim \widehat{Y}(X)$ for $X \to \infty$. The latter, weakened property being stated in classical terms, Theorem 2.2 follows from the Transfer Principle. □

The asymptotic properties of the solutions of a standard difference equation close to a standard approximate solution depend essentially on their behaviour for infinitely large values of X. For these values we will apply a rescaling, enabling convenient estimates and near-equalities.

To be able to work only with infinitely large values of X, we must give a nonstandard characterization for standard approximate solutions in terms of unlimited X. This will be done in the next proposition, the proof of which also relies on the Transfer Principle.

Proposition 4.2. — *Let (1.1) be standard. Then $\widehat{Y}(X)$ is an approximate solution of (1.1) if and only if*

1. $(\forall \omega \simeq \infty)\left(\widehat{Y}(\omega) < 0\right)$ *or* $(\forall \omega \simeq \infty)\left(\widehat{Y}(\omega) > 0\right)$,
   $\left\{(\omega, Y) \,\middle|\, \omega \simeq \infty, Y = (1+\varnothing)\widehat{Y}(\omega)\right\} \subset U$,
   and $(\forall \omega \simeq \infty)\left(\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| < 1\right)$
   or $(\forall \omega \simeq \infty)\left(\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| > 1\right)$.
2. $(\forall \omega \simeq \infty)\left(\frac{F\left(\omega, \widehat{Y}(\omega)\right) - \widehat{Y}(\omega)}{\widehat{Y}(\omega)} = \varnothing \cdot \left(\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| - 1\right)\right)$.
3. $(\forall \omega \simeq \infty)\left(\frac{\widehat{Y}(\omega+1) - \widehat{Y}(\omega)}{\widehat{Y}(\omega)} = \varnothing \cdot \left(\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| - 1\right)\right)$.
4. $(\forall \omega \simeq \infty)$
$$\left(\left|F_2'\left(\omega, (1+\varnothing)\widehat{Y}(\omega)\right)\right| - 1 = (1+\varnothing)\left(\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| - 1\right)\right).$$

Proof. — The equivalence between Definition 2.1.1 and Proposition 4.2.3 follows directly from the Cauchy Principle. The equivalences between Definition 2.1.2 and Proposition 4.2.2, and of Definition 2.1.3 and Proposition 4.2.3 follow simply from the nonstandard characterization of the limit. The equivalence of Definition 2.1.4 and Proposition 4.2.4 follows directly from the External Function Criterion [6], see also Proposition A2 of [5]. □

4.1. Rescaling around an approximate solution

We start by rescaling the equation around the approximate solution. We introduce some convenient notation and derive some useful estimates. Then we study the solutions of the rescaled equation within an appropriate tube. We distinguish between repulsive behaviour within the tube, very weak attractive, almost parallel behaviour, and attractive, not almost parallel behaviour. We prove the existence theorem separately for the latter case, where a joint proof of the remaining cases can be given using the general lemma 3.1.

Remark 4.3. — If the difference equation (1.1) and the approximate solution $\widehat{Y}$ are standard, we always assume that the natural number A of Definition 2.1.1 is standard.

Notation 4.4. — Let (1.1) be standard. Let $\omega \in \mathbb{N}, \omega \geqslant A$ and $\widehat{Y}$ be a standard approximate solution.

1. Let Y be a sequence. We write for $x \in \mathbb{N}$

$$\begin{aligned} y_\omega(x) &= \frac{Y(\omega+x)-\widehat{Y}(\omega)}{\widehat{Y}(\omega)} \\ y_\omega &= y_\omega(0). \end{aligned}$$

2. We write

$$\begin{aligned} \epsilon_\omega &= \frac{F\left(\omega,\widehat{Y}(\omega)\right)-\widehat{Y}(\omega)}{\widehat{Y}(\omega)\left(\left|F_2'\left(\omega,\widehat{Y}(\omega)\right)\right|-1\right)} \\ \delta_\omega(u) &= \begin{cases} \frac{F\left(\omega,(1+u)\widehat{Y}(\omega)\right)-F\left(\omega,\widehat{Y}(\omega)\right)}{u\widehat{Y}(\omega)} & \text{if} \quad u\neq 0 \\ F_2'\left(\omega,\widehat{Y}(\omega)\right) & \text{if} \quad u=0 \end{cases} \\ g_\omega &= \left|F_2'\left(\omega,\widehat{Y}(\omega)\right)\right|-1 \\ \gamma_\omega &= \left|g_\omega\right|. \end{aligned}$$

Proposition 4.5. — *Let (1.1) be standard. Let $\omega \geqslant A$, and $\widehat{Y}$ be a standard approximate solution. Let Y be a solution such that $(\omega, Y(\omega)) \in U$. Then $y_{\omega+1} = f(\omega, y_\omega)$, where*

$$f(\omega,y_\omega)=\frac{\delta_\omega(y_\omega)}{1+\widehat{y}_\omega(1)}y_\omega+\frac{\epsilon_\omega g_\omega-\widehat{y}_\omega(1)}{1+\widehat{y}_\omega(1)}. \tag{4.1}$$

Proof. — Observe that $\widehat{y}_\omega(1) = \left(\widehat{Y}(\omega+1) - \widehat{Y}(\omega)\right)/\widehat{Y}(\omega)$, so $1+\widehat{y}_\omega(1) = \widehat{Y}(\omega+1)/\widehat{Y}(\omega) \neq 0$. One has

$$
\begin{aligned}
&\mathrm{y}_{\omega+1} = \frac{Y(\omega+1) - \widehat{Y}(\omega+1)}{\widehat{Y}(\omega+1)} \\
&= \frac{F(\omega, Y(\omega)) - F\left(\omega, \widehat{Y}(\omega)\right) + F\left(\omega, \widehat{Y}(\omega)\right) - \widehat{Y}(\omega) + \widehat{Y}(\omega) - \widehat{Y}(\omega+1)}{\widehat{Y}(\omega+1)} \\
&= \frac{F\left(\omega, \widehat{Y}(\omega) + \frac{Y(\omega)-\widehat{Y}(\omega)}{\widehat{Y}(\omega)}\widehat{Y}(\omega)\right) - F\left(\omega, \widehat{Y}(\omega)\right)}{\frac{Y(\omega)-\widehat{Y}(\omega)}{\widehat{Y}(\omega)}\widehat{Y}(\omega)} \frac{Y(\omega) - \widehat{Y}(\omega)}{\widehat{Y}(\omega)} \frac{\widehat{Y}(\omega)}{\widehat{Y}(\omega+1)} \\
&+ \left(\frac{F\left(\omega, \widehat{Y}(\omega)\right) - \widehat{Y}(\omega)}{\widehat{Y}(\omega)\left(\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| - 1\right)} \left(\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| - 1\right)\right. \\
&\left. - \frac{\widehat{Y}(\omega+1) - \widehat{Y}(\omega)}{\widehat{Y}(\omega)}\right) \frac{\widehat{Y}(\omega)}{\widehat{Y}(\omega+1)} \\
&= \frac{\delta_\omega(y_\omega)}{1+\widehat{y}_\omega(1)} y_\omega + \frac{\epsilon_\omega g_\omega - \widehat{y}_\omega(1)}{1+\widehat{y}_\omega(1)} \\
&= f(\omega, y_\omega). \qquad \square
\end{aligned}
$$

We derive some estimates for the introduced quantities, which imply that equation (1.1) is asymptotically linear close to the approximate solution.

Proposition 4.6. — *Let (1.1) be standard. Let $\omega \simeq \infty$, and $\widehat{Y}$ be a standard approximate solution. Then the following estimates hold:*

$$u \simeq 0 \textit{ implies } |\delta_\omega(u)| = 1 + (1+\varnothing) g_\omega \tag{4.2}$$

$$\widehat{y}_\omega(1) = \varnothing \cdot \gamma_\omega \tag{4.3}$$

$$\epsilon_\omega \simeq 0. \tag{4.4}$$

The estimate (4.2) follows simply from the mean value theorem and Proposition 4.2. The remaining estimates follow directly from Proposition 4.2(3) and 4.2(2).

Proposition 4.7. — *Let (1.1) be standard. Let $\widehat{Y}$ be an approximate solution.*

1. *If $\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| < 1$ for all $\omega \simeq \infty$, there exists standard $b > 0$ such that for all $\omega \simeq \infty$ and u with $|u| \leqslant b$*

$$1 - 2\gamma_\omega \leqslant |\delta_\omega(u)| \leqslant 1 - \frac{1}{2}\gamma_\omega.$$

2. *If* $\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| > 1$ *for all* $\omega \simeq \infty$*, there exists standard* $b > 0$ *such that for all* $\omega \simeq \infty$ *and* u *with* $|u| \leqslant b$

$$1 + \frac{1}{2}\gamma_\omega \leqslant |\delta_\omega(u)| \leqslant 1 + 2\gamma_\omega.$$

Proof. —

1. Let $\omega \simeq +\infty$. Remark that $-1 \leqslant g_\omega < 0$, so $0 < \gamma_\omega \leqslant 1$. It follows from (4.2) that for all $u \simeq 0$ it holds that $|\delta_\omega(u)| = 1 - (1 + \varnothing)\gamma_\omega$, hence clearly $1 - 2\gamma_\omega \leqslant |\delta_\omega(u)| \leqslant 1 - \frac{1}{2}\gamma_\omega$. By the Cauchy Principle there exists $\beta \gtrapprox 0$ such that $1 - 2\gamma_\omega \leqslant |\delta_\omega(u)| \leqslant 1 - \frac{1}{2}\gamma_\omega$ for all $|u| \leqslant \beta$. Define

$$b_\omega = \max\left\{\beta \leqslant 1 \,\middle|\, (\forall\, |u| \leqslant \beta)\left(1 - 2\gamma_\omega \leqslant |\delta_\omega(u)| \leqslant 1 - \frac{1}{2}\gamma_\omega\right)\right\}.$$

   By the Cauchy Principle there exists standard A_0 such that b_ω is defined for $\omega \geqslant A_0$; because $b_\omega \gtrapprox 0$ for all $\omega \simeq \infty$, we may assume by the Fehrele Principle that $b_\omega \gtrapprox 0$ for all $\omega \geqslant A_0$. Then $b \equiv^\circ \inf\{b_\omega \,|\omega \geqslant A_0\}/2 > 0$. This implies the affirmation.

2. Analogous, observing that for all $\omega \simeq \infty$ and $u \simeq 0$ one has $|\delta_\omega(u)| = 1 + (1 + \varnothing)\gamma_\omega$, hence certainly $1 + \frac{1}{2}\gamma_\omega \leqslant |\delta_\omega(u)| \leqslant 1 + 2\gamma_\omega$. □

We note that the lower bound in Proposition 4.7 is trivially satisfied if $\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| \leqslant \frac{1}{2}$. However we will use this lower bound only in the case that $\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| \simeq 1$ for all $\omega \simeq \infty$.

4.2. General properties of attractive behaviour

We present some upper bounds and approximations valid in the attractive case. In the next paragraph on nearly parallel attractive behaviour we will also consider some lower bounds.

LEMMA 4.8. — *Let (1.1) be standard. Let* $\widehat{Y}$ *be a standard approximate solution. Assume* $\left|F_2'(\omega, \widehat{Y}(\omega)\right| < 1$ *for all* $\omega \simeq \infty$*. Then there exists standard* $b > 0$ *such that for all* $\omega \simeq \infty$ *and all solutions* Y *defined at* ω *with* $|y_\omega| \leqslant b, y_\omega \not\simeq 0$ *it holds that* $Y(\omega + 1)$ *is defined and*

$$|y_{\omega+1}| < \left(1 - \frac{1}{4}\gamma_\omega\right)|y_\omega|.$$

Proof. — From (4.1) and Proposition 4.7 we conclude that there exists standard $b > 0$ such that for all $\omega \simeq \infty$ and for solutions Y with $|y_\omega| \leqslant b$

$$|y_{\omega+1}| \leqslant \frac{1 - \frac{1}{2}\gamma_\omega}{1 + \widehat{y}_\omega(1)} |y_\omega| + \frac{|\epsilon_\omega|\,\gamma_\omega}{1 + \widehat{y}_\omega(1)} + \frac{|\widehat{y}_\omega(1)|}{1 + \widehat{y}_\omega(1)}.$$

If $y_\omega \not\simeq 0$ we may rewrite this to

$$|y_{\omega+1}| \leqslant \frac{1 - \frac{1}{2}\gamma_\omega + \frac{|\epsilon_\omega|\gamma_\omega + \left|\widehat{y}_\omega(1)\right|}{|y_\omega|}}{1 + \widehat{y}_\omega(1)} |y_\omega|.$$

Using (4.3) and (4.4) we find

$$\frac{1 - \frac{1}{2}\gamma_\omega + \frac{|\epsilon_\omega|\gamma_\omega + \left|\widehat{y}_\omega(1)\right|}{|y_\omega|}}{1 + \widehat{y}_\omega(1)} = \frac{1 - \frac{1}{2}\gamma_\omega + \varnothing\gamma_\omega}{1 + \varnothing\gamma_\omega}.$$

Hence certainly

$$|y_{\omega+1}| < \left(1 - \frac{1}{4}\gamma_\omega\right) |y_\omega|. \qquad \square$$

PROPOSITION 4.9. — *Let (1.1) be standard. Let $\widehat{Y}$ be a standard approximate solution. Assume $\left|F_2'(\omega, \widehat{Y}(\omega)\right| < 1$ for all $\omega \simeq \infty$. Then there exists standard $b > 0$ such that for all $\omega \simeq \infty$ and all solutions Y defined at ω with $|y_\omega| \leqslant b$ it holds that $Y(\omega + 1)$ is defined and*

$$\begin{cases} y_{\omega+1} \simeq 0 & \text{if} \quad y_\omega \simeq 0 \\ |y_{\omega+1}| < |y_\omega| & \text{if} \quad y_\omega \not\simeq 0. \end{cases}$$

Proof. — If $y_\omega \simeq 0$ it follows from (4.1), using the estimates (4.2)(4.4)(4.3) and the fact that g_ω is limited, that $Y(\omega + 1)$ is defined and

$$y_{\omega+1} = \frac{\pounds}{1 + \varnothing} \cdot \varnothing + \frac{\varnothing \cdot \pounds + \varnothing}{1 + \varnothing} = \varnothing.$$

The case $y_\omega \not\simeq 0$ follows from Lemma 4.8. $\square$

PROPOSITION 4.10. — *Let (1.1) be standard. Let $\widehat{Y}$ be a standard approximate solution. Assume $\left|F_2'(\omega, \widehat{Y}(\omega)\right| < 1$ for all $\omega \simeq \infty$. Then there exists standard $b > 0$ such that for all $\xi \simeq \infty$ and all solutions Y defined at ξ with $|y_\xi| \leqslant b$ it holds that $Y(X)$ is defined for all $X \geqslant \xi$ and*

$$\begin{cases} y_{X+1} \simeq 0 & \text{if} \quad y_X \simeq 0 \\ |y_{X+1}| < |y_X| & \text{if} \quad y_X \not\simeq 0. \end{cases} \tag{4.5}$$

Proof. — Let b be as in Proposition 4.9 and $\xi \simeq \infty$. By this proposition and Lemma 3.3 the property (4.5) holds on any interval $\{\xi, \cdot\cdot, \omega\}$ with $\omega \simeq \infty$. Then it holds for all $X \geqslant \xi$. □

4.3. Almost parallel attractive behaviour

Let (1.1) be standard. Let $\widehat{Y}$ be a standard approximate solution. It was already mentioned that the value of γ_X, i.e. $\left|F_2'\left(X, \widehat{Y}(X)\right)\right| - 1$, in a sense determines the behaviour of the set of solutions close to $\widehat{Y}$, distinguishing between attractiveness if $\gamma_X < 0$ and repulsiveness if $\gamma_X > 0$. As is to be expected, one observes a sort of intermediate, "nearly parallel" behaviour for values of γ_X very close to 0. It appears that such a near parallelness occurs when the series $\sum_{X \geqslant A} \gamma_X$ is convergent. Then $\sum_{X \geqslant \omega} \gamma_X \simeq 0$ for all $\omega \simeq \infty$, hence in particular $\gamma_\omega \simeq 0$, i.e. $\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| \simeq 1$, for all $\omega \simeq \infty$. For the purpose of proving the existence theorem it is sufficient to study the phenomenon of near parallelness only in the attractive case (of course, it is not necessary that always $\left|F_2'\left(X, \widehat{Y}(X)\right)\right| < 1$ for the series $\sum_{X \geqslant A} \gamma_X$ to be convergent). Proposition 4.13 implies that the solutions are nearly parallel to the approximate solution in a precise sense. Indeed, in the rescaling of Proposition 4.5, there exists a tube of standard width around the approximate solution such that within this tube, the distance between two solutions remains unchanged, up to an infinitesimal. Such a near-parallelness will express stability, but not asymptotic stability of the solution $\widetilde{Y}$ of Theorem 4.1, once its existence is proved. Notice that the condition that $\sum_{X \geqslant A} \gamma_X$ is convergent reduces in the case of linear equations with constant coefficients to $\gamma_X = 0$, and then we obtain the two types of equations with only stability $Y(X+1) = Y(X) + R(X)$ and $Y(X+1) = -Y(X) + R(X)$. For general linear equations $Y(X+1) = G(X)Y(X) + R(X)$ the condition that $\sum_{X \geqslant A} \gamma_X$ is convergent corresponds to the well-known criterion for non-asymptotic stability that $\prod_{X \geqslant A} |G(X)|$ converges to a non-zero limit [11].

In order to prove Proposition 4.13, we present first some lower bounds and estimations in relation to the "near parallelness".

LEMMA 4.11. — *Let (1.1) be standard. Let $\widehat{Y}$ be a standard approximate solution. Assume $\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| < 1, \left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| \simeq 1$ for all $\omega \simeq \infty$. Then there exists standard $b > 0$ such that for all $\omega \simeq \infty$ and for all u with $0 \lnapprox |u| \lnapprox b$*

$$|f(\omega, (1 + 4\gamma_\omega)u|) > |u|\,,$$

with

$$sgnf(\omega,(1+4\gamma_\omega)u) = \operatorname{sgn} y_\omega \cdot \operatorname{sgn} F_2'\left(\omega, \widehat{Y}(\omega)\right).$$

Proof. — By Proposition 4.7 there exists standard $b > 0$ such that $|\delta_\omega(y_\omega)| \geqslant 1 - 2\gamma_\omega$ for all $\omega \simeq \infty$ and all $|u| \leqslant b$. Note that $\gamma_\omega \simeq 0$, so if $|u| \lessapprox b$, one has $|(1+4\gamma_\omega)u| \leqslant b$. We consider first the case $F_2'\left(\omega, \widehat{Y}(\omega)\right) > 0$. It follows from (4.1) that

$$f(\omega,(1+4\gamma_\omega)u) - u = \frac{\delta_\omega((1+4\gamma_\omega)u)(1+4\gamma_\omega)u}{1+\widehat{y}_\omega(1)} - u - \frac{\epsilon_\omega\gamma_\omega + \widehat{y}_\omega(1)}{1+\widehat{y}_\omega(1)}.$$

Now if $0 \lessapprox u \lessapprox b$ we have $\delta_\omega((1+4\gamma_\omega)u)(1+4\gamma_\omega)u \geqslant (1-2\gamma_\omega)(1+4\gamma_\omega)u$, from which we derive that

$$\begin{aligned} f(\omega,(1+4\gamma_\omega)u) - u &\geqslant \frac{2\gamma_\omega - 8\gamma_\omega^2 - \widehat{y}_\omega(1) - \frac{\epsilon_\omega\gamma_\omega + \widehat{y}_\omega(1)}{u}}{1+\widehat{y}_\omega(1)} u \\ &= \frac{2\gamma_\omega + \varnothing\gamma_\omega - \frac{\varnothing\gamma_\omega + \varnothing\gamma_\omega}{@}}{1+\varnothing\gamma_\omega} u. \end{aligned}$$

So $f(\omega,(1+4\gamma_\omega)u) > u$.

If $-b \lessapprox u \lessapprox 0$ we have $\delta_\omega((1+4\gamma_\omega)u)(1+4\gamma_\omega)u \leqslant (1-2\gamma_\omega)(1+4\gamma_\omega)u$, from which we derive in the same manner that $f(\omega,(1+4\gamma_\omega)u) < u$. The case $F_2'\left(\omega, \widehat{Y}(\omega)\right) < 0$ is treated similarly, showing that $f(\omega,(1+4\gamma_\omega)u) < -u$ if $0 \lessapprox u \lessapprox b$ and $f(\omega,(1+4\gamma_\omega)u) > -u$ if $-b \lessapprox u \lessapprox 0$. Combining all cases we obtain the lemma. □

PROPOSITION 4.12. — *Let (1.1) be standard. Let $\widehat{Y}$ be a standard approximate solution. Assume $\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| < 1, \left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| \simeq 1$ for all $\omega \simeq \infty$. Then there exists standard $b > 0$ such that for all $\omega \simeq \infty$ and for all u with $|u| \lessapprox b$ there exists a solution Y of (1.1) such that $Y(\omega)$ and $Y(\omega+1)$ are defined and satisfy $|y_\omega| \lessapprox b$ and $y_{\omega+1} = u$. Moreover,*

$$\begin{cases} y_\omega \simeq 0 & \text{if} \quad y_{\omega+1} \simeq 0 \\ |y_{\omega+1}| < |y_\omega| < |y_{\omega+1}|(1+4\gamma_\omega) & \text{if} \quad y_{\omega+1} \not\simeq 0. \end{cases}$$

Proof. — Let b be standard as in Proposition 4.7.1. and Lemma 4.11. Let $\omega \simeq \infty$ and f be given by (4.1). Assume first that $0 \lessapprox |u| \lessapprox b$. By Lemma 4.11

$$\begin{gathered} \inf\{f(\omega,y) \mid |y| \leqslant |u|(1+4\gamma_\omega)\} < -|u| \\ < |u| < \sup\{f(\omega,y) \mid |y| \leqslant |u|(1+4\gamma_\omega)\}. \end{gathered}$$

By continuity of f in the second variable there exists v with $|v| \leqslant |u|\,(1+4\gamma_\omega)$ such that $f(\omega, v) = u$. Let Y be a solution such that $y_\omega = v$. Then $Y(\omega+1)$ is defined and $y_{\omega+1} = u$. Combining Proposition 4.9 and Lemma 4.11 one obtains that

$$|y_{\omega+1}| < |y_\omega| < |y_{\omega+1}|\,(1+4\gamma_\omega).$$

Note that in particular $|y_\omega| \simeq |y_{\omega+1}|$.

Second, let $u \simeq 0$. By the previous argument $f(\omega, \cdot)$ is surjective at least on $[-b/2, b/2]$. Because $|u| < b/2$ we find in the same way as above a solution $\overline{Y}$ such that $|\overline{y}_\omega| \leqslant |b/2|\,(1+4\gamma_\omega)$, $\overline{Y}(\omega+1)$ is defined and $\overline{y}_{\omega+1} = u$. In fact $y_\omega \simeq 0$, else $|y_{\omega+1}| \simeq |y_\omega| \not\simeq 0$. □

Proposition 4.13. — *Let (1.1) be standard. Let $\widehat{Y}$ be a standard approximate solution. Assume $\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| < 1$ for all $\omega \simeq \infty$ and that $\sum_{X \geqslant A} \gamma_X$ is convergent. Then there exists standard $b > 0$ such that for all $\xi, \omega \simeq \infty, \xi < \omega$ and $|u| \leqslant b/2$ there is a solution Y defined for $\xi \leqslant X \leqslant \omega$ such that $y_\omega = u$ and $|y_X| \leqslant b$ for all X with $\xi \leqslant X \leqslant \omega$. Moreover, one has $|y_X| \simeq |u|$ for all X with $\xi \leqslant X \leqslant \omega$.*

Proof. — Observe that $\gamma_\omega \simeq 0$ for all $\omega \simeq \infty$, which implies that $\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| \simeq 1$ for all $\omega \simeq \infty$. Let b be standard as in Proposition 4.7, Lemma 4.11 and Proposition 4.12. By the latter proposition there exists a solution $Y^{(\omega-1)}$ such that $Y^{(\omega-1)}(\omega-1)$ and $Y^{(\omega-1)}(\omega)$ are defined, with $y_\omega^{(\omega-1)} = u$ and $y_{\omega-1}^{(\omega-1)} \leqslant \frac{b}{2}(1+4\gamma_{\omega-1})$. Successively extending this solution downward, we obtain by finite induction a solution $Y^{(\xi)}$ defined for $\xi \leqslant X \leqslant \omega$, such that for all X with $\xi \leqslant X < \omega - 1$ we have $y_{X+1}^{(\xi)} = y_{X+1}^{(X)} = y_{X+1}^{(X+1)}$ and

$$\left|y_X^{(X)}\right| \leqslant \frac{b}{2} \prod_{X \leqslant Z < \omega} (1+4\gamma_Z).$$

By the estimation

$$\begin{aligned}
\prod_{X \leqslant Z < \omega} (1+4\gamma_Z) &= \exp \sum_{X \leqslant Z < \omega} \log(1+4\gamma_Z) \\
&= \exp \sum_{X \leqslant Z < \omega} (1+\oslash)4\gamma_Z \\
&= \exp(1+\oslash)4 \sum_{X \leqslant Z < \omega} \gamma_Z \\
&= \exp(1+\oslash)4.\oslash \\
&= 1+\oslash,
\end{aligned}$$

we obtain $\left|y_X^{(\xi)}\right| \lesssim \frac{b}{2}$ for all X with $\xi \leqslant X < \omega$, so certainly $\left|y_X^{(\xi)}\right| \leqslant b$ for all X with $\xi \leqslant X \leqslant \omega$.

To prove the remaining part of the proposition, assume first that $|u| \not\simeq 0$. On the one hand $\left|y_X^{(\xi)}\right| > \left|y_\omega^{(\xi)}\right| = |u|$ for all X with $\xi \leqslant X < \omega$ on behalf of Proposition 4.9, and on the other hand $\left|y_X^{(\xi)}\right| \lesssim |u|$ for all X with $\xi \leqslant X < \omega$ by the above estimation. Hence $\left|y_X^{(\xi)}\right| \simeq |u|$ for all X with $\xi \leqslant X \leqslant \omega$.

Finally, consider the case $u \simeq 0$. Suppose $y_X^{(\xi)} \not\simeq 0$ for some X with $\xi \leqslant X \leqslant \omega$. We deduce from Proposition 4.12 that

$$\left|y_{X+1}^{(\xi)}\right| > \frac{\left|y_X^{(\xi)}\right|}{1+4\gamma_X}.$$

In a similar way as above we may obtain using finite induction that

$$\left|y_\omega^{(\xi)}\right| > \frac{\left|y_X^{(\xi)}\right|}{\prod\limits_{X\leqslant Z<\omega}(1+4\gamma_Z)} \simeq \left|y_X^{(\xi)}\right| \not\simeq 0,$$

a contradiction. Hence $y_X^{(\xi)} \simeq 0 \simeq u$ for all X with $\xi \leqslant X \leqslant \omega$. □

4.4. Repulsive behaviour

PROPOSITION 4.14. — *Let (1.1) be standard. Let $\widehat{Y}$ be a standard approximate solution. Assume $\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| > 1$ for all $\omega \simeq \infty$. Then there exists standard $b > 0$ such that for all $\omega \simeq \infty$ and for all $|u| \leqslant b$ there exists a solution Y with $|y_{\omega-1}| \leqslant b$ and $y_\omega = u$. Moreover, for such a solution Y it holds that*

$$\begin{cases} y_{\omega-1} \simeq 0 & \text{if} \quad y_\omega \simeq 0 \\ |y_{\omega-1}| < |y_\omega| & \text{if} \quad y_\omega \not\simeq 0. \end{cases}$$

The proof of the proposition uses two lemmas.

LEMMA 4.15. — *Let (1.1) be standard. Let $\widehat{Y}$ be a standard approximate solution. Assume $\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| > 1$ for all $\omega \simeq \infty$. Then there exists standard $b > 0$ such that for all $\omega \simeq \infty$ and for all solutions Y with $|y_\omega| \leqslant b, y_\omega \not\simeq 0$ it holds that $|y_{\omega+1}| > |y_\omega|$. Moreover,*

$$\mathrm{sgn} y_{\omega+1} = \mathrm{sgn} y_\omega \cdot \mathrm{sgn} F_2'\left(\omega, \widehat{Y}(\omega)\right).$$

Proof. — From Proposition 4.7 we derive that there exists standard $b > 0$ such that for all $\omega \simeq \infty$ and all solutions Y with $|y_\omega| \leqslant b$ it holds that $1 + 2\gamma_\omega \geqslant |\delta_\omega(y_\omega)| \geqslant 1 + \frac{1}{2}\gamma_\omega$. We consider first the case $F_2'\left(\omega, \widehat{Y}(\omega)\right) > 0$. We rewrite (4.1) to

$$\begin{aligned} y_{\omega+1} - y_\omega &= \frac{\delta_\omega(y_\omega) - 1 - \widehat{y}_\omega(1) + \frac{\epsilon_\omega \gamma_\omega - \widehat{y}_\omega(1)}{y_\omega}}{1 + \widehat{y}_\omega(1)} y_\omega \\ &= \frac{1 + @\gamma_\omega - 1 + \varnothing\gamma_\omega + \frac{\varnothing\gamma_\omega + \varnothing\gamma_\omega}{@}}{1 + \widehat{y}_\omega(1)} \mathrm{sgn} y_\omega \cdot @ \\ &= \frac{\mathrm{sgn} y_\omega \cdot @\gamma_\omega}{1 + \widehat{y}_\omega(1)}. \end{aligned}$$

So $y_{\omega+1} > y_\omega$ if $y_\omega \gtrapprox 0$ and $y_{\omega+1} < y_\omega$ if $y_\omega \lessapprox 0$. Secondly, we consider the case $F_2'\left(\omega, \widehat{Y}(\omega)\right) < 0$. Then (4.1) yields

$$\begin{aligned} y_{\omega+1} + y_\omega &= \frac{\delta_\omega(y_\omega) + 1 + \widehat{y}_\omega(1) + \frac{\epsilon_\omega \gamma_\omega - \widehat{y}_\omega(1)}{y_\omega}}{1 + \widehat{y}_\omega(1)} y_\omega \\ &= \frac{-1 - @\gamma_\omega + 1 + \varnothing\gamma_\omega + \frac{\varnothing\gamma_\omega + \varnothing\gamma_\omega}{@}}{1 + \widehat{y}_\omega(1)} \mathrm{sgn} y_\omega \cdot @ \\ &= \frac{-\mathrm{sgn} y_\omega \cdot @\gamma_\omega}{1 + \widehat{y}_\omega(1)}. \end{aligned}$$

So $y_{\omega+1} < -y_\omega$ if $y_\omega \gtrapprox 0$ and $y_{\omega+1} > -y_\omega$ if $y_\omega \lessapprox 0$. Combining both cases, we derive the lemma. □

The next lemma is an immediate corollary.

Lemma 4.16. — *Let (1.1) be a standard difference equation. Let $\widehat{Y}$ be a standard approximate solution. Assume $\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| > 1$ for all $\omega \simeq \infty$. Then there exists standard $b > 0$ such that for all $\omega \simeq \infty$*

$$\begin{aligned} &\inf\{y_{\omega+1} \,|Y \text{ is a solution such that } |y_\omega| \leqslant b\} < -b \\ &< b < \sup\{y_{\omega+1} \,|Y \text{ is a solution such that } |y_\omega| \leqslant b\}. \end{aligned}$$

Proof of Proposition 4.14 . — Let $b > 0$ be standard as in Lemma 4.15 and Lemma 4.16. Let $|u| \leqslant b$. Let f be given by (4.1). From Lemma 4.16 it follows, due to the continuity of f, that $[-b, b] \subset f(\omega - 1, [-b, b])$. Then there is v with $|v| \leqslant b$ and $f(\omega - 1, v) = u$. Let Y be a solution such that $y_{\omega-1} = v$. Then $y_\omega = u$. If $y_\omega \not\simeq 0$ one has $|y_{\omega-1}| < |y_\omega|$ by Lemma 4.15. Finally, assume $y_\omega \simeq 0$. If $y_{\omega-1} \not\simeq 0$, one should have $|y_{\omega-1}| < |y_\omega| \simeq 0$, a contradiction. Hence $y_{\omega-1} \simeq 0$.

5. Proof of the existence theorem

We recall that the existence theorem is proved by Transfer, once we have proved the nonstandard existence theorem 4.1. As for the latter, we prove separately the attractive not almost parallel case and give a joint proof of the attractive almost parallel case and the repulsive case, using Lemma 3.1.

Proof of the nonstandard existence theorem. —

Case I. $(\forall \omega \simeq \infty)\left(\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| < 1\right)$, $\sum\limits_{X \geqslant A} \gamma_X$ divergent (attractive, not almost parallel case).

Let b, ξ be such as in Proposition 4.10 and Lemma 4.8. By the principle of Cauchy there is standard A' such that for all solutions Y with $|y_{A'}| \leqslant b$ it holds that $|y_X| \leqslant b$ for all X with $X \geqslant A'$. By Transfer there exists a standard solution $\widetilde{Y}$ with $|\widetilde{y}_{A'}| \leqslant b$. We will show that there exists $\omega \simeq \infty$ such that $\widetilde{y}_\omega \simeq 0$. If not, for all $\nu, \omega \simeq \infty, \nu < \omega$ we have $\widetilde{y}_X \not\simeq 0$ for all X with $\nu \leqslant X \leqslant \omega$. Then by Lemma 4.8

$$|\widetilde{y}_\omega| \leqslant |\widetilde{y}_\nu| \prod_{\nu \leqslant X < \omega} \left(1 - \frac{1}{4}\gamma_X\right) \leqslant |\widetilde{y}_\nu| \exp\left(-\frac{1}{4} \sum_{\nu \leqslant X < \omega} \gamma_X\right).$$

By divergence there exist ν, ω such that $\sum_{\nu \leqslant X < \omega} \gamma_X \simeq \infty$, which means that $\widetilde{y}_\omega \simeq 0$, a contradiction. Hence there exists $\omega \simeq \infty$ such that $\widetilde{y}_\omega \simeq 0$. By Proposition 4.10 and Lemma 3.3 we have $\widetilde{y}_X \simeq 0$ for all $X \geqslant \omega$. It follows from the Monadic Transfer Principle [6] that then $\widetilde{y}_X \simeq 0$ for all $X \simeq \infty$. Hence $\widetilde{Y}(X) = (1 + \varnothing)\widehat{Y}(X)$ for all $X \simeq \infty$. We conclude that $\widetilde{Y}(X) \sim \widehat{Y}(X)$ for $X \to \infty$.

Case II. $(\forall \omega \simeq \infty)\left(\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| < 1\right)$, $\sum\limits_{X \geqslant A} \gamma_X$ convergent (attractive, almost parallel case), or $(\forall \omega \simeq \infty)\left(\left|F_2'\left(\omega, \widehat{Y}(\omega)\right)\right| > 1\right)$(repulsive case).

Let $\omega \simeq \infty$. In both cases we prove that there exists a solution $\overline{Y}$ of (1.1) such that $\overline{y}_X \simeq 0$ for all $X \simeq \infty, X \leqslant \omega$.

As for the attractive, almost parallel case, let $b > 0$ be standard, as given by Proposition 4.13. Let $u \simeq 0$. By this same proposition for all $\xi \simeq \infty, \xi < \omega$ there is a solution Y defined for all X with $\xi \leqslant X \leqslant \omega$ such that $y_\omega = u$ and $|y_X| \leqslant b$ for all X with $\xi \leqslant X \leqslant \omega$. By the Cauchy principle there exists standard S and a solution $Y^{(S)}$defined for all X with $S \leqslant X \leqslant \omega$ such that $y_{\omega}^{(S)} = u$ and $\left|y_X^{(S)}\right| \leqslant b$ for all X with $S \leqslant X \leqslant \omega$. Again by Proposition 4.13 one has $y_X^{(S)} \simeq 0$ for all X with $\xi \leqslant X \leqslant \omega$. Because $\xi \simeq \infty$ is arbitrary, we obtain as a corollary that $y_X^{(S)} \simeq 0$ for all $X \simeq \infty, X \leqslant \omega$.

As for the repulsive case, let $b > 0$ be standard, as given by Proposition 4.14. Let $u \simeq 0$. By this same proposition, there exists a solution $Y^{(\omega-1)}$ such that $\left|y_{\omega-1}^{(\omega-1)}\right| \leqslant b$ and $y_{\omega}^{(\omega-1)} = u$. Let $\xi \simeq \infty, \xi < \omega$ be arbitrary. With downward induction, we construct as in the proof of Proposition 4.13 a set of successive extensions $\left\{y^{(X)} \,|\xi \leqslant X \leqslant \omega\right\}$ such that $y^{(X)}$ is defined on $\{X, \cdot\cdot, \omega\}$, $\left|y_X^{(X)}\right| \leqslant b$ and $y_X^{(\xi)} = y_X^{(X)}$ for all X with $\xi \leqslant X < \omega$. By the Cauchy principle there exists standard T and a solution $Y^{(T)}$ such that $y^{(T)}$ is defined on $\{T, \cdot\cdot, \omega\}$, $\left|y_T^{(T)}\right| \leqslant b$ and $y_X^{(T)} = y_X^{(X)}$ for all X with $T \leqslant X < \omega$. This means that $y_{\omega}^{(T)} = u \simeq 0$, and again by Proposition 4.14 the remaining conditions of Lemma 3.3 (read backwards) are satisfied on $\{\xi, \cdot\cdot, \omega\}$. Hence $y_X^{(T)} \simeq 0$ for all X with $\xi \leqslant X \leqslant \omega$. Because in the above lemma $\xi \simeq \infty$ is arbitrary, we obtain as a corollary that $y_X^{(T)} \simeq 0$ for all $X \simeq \infty, X \leqslant \omega$.

So in both cases there exists a solution $\overline{Y}$ of (1.1) such that $\overline{y}_X \simeq 0$ for all $X \simeq \infty, X \leqslant \omega$. This means that $\overline{Y}(X) = (1 + \varnothing)\widehat{Y}(X)$ for all $X \simeq \infty, X \leqslant \omega$. By Corollary 3.2 we conclude that there exists a standard solution $\widetilde{Y}$ such that $\widetilde{Y}(X) \sim \widehat{Y}(X)$ for $X \to \infty$.

Bibliography

[1] Artigue (M.), Gautheron (V.), Isambert (E.). — Une notion non standard d'attracteurs : les fleuves. Mathématiques finitaires & Analyse non standard (Diener, M. and Wallet, G. eds.), Tome 2, Publ. Math. Univ. Paris VII, p. 191-208 (1989).

[2] Benoît (E.), Callot (J.-L.), Diener (F.), Diener (M.) . — Chasse au canard, Collect. Math. 32, no. 1-4, p. 37-119 (1981).

[3] Benzaid (Z.), Lutz (D. A.) . — Asymptotic representation of solutions of perturbed systems of linear difference equations. Stud. Appl. Math. 77, no. 3, p. 195-221 (1987).

[4] Van den Berg (I. P.) . — On solutions of polynomial growth of ordinary differential equations J. Differential Equations, 81 (2), p. 368-402 (1989).

[5] Van den Berg (I. P.) . — Macroscopic rivers. In : Dynamic bifurcations (Benoît, E. ed.), Lect. N. Math. 1493, p. 190-209, Springer (1991).

[6] Van den Berg (I. P.) . — Extended use of IST, Ann. Pure Appl. Logic 58 (1992).

[7] Van den Berg (I. P.) . — Asymptotics of families of solutions of nonlinear difference equations, Logic and Analysis 2, p. 153-185 (2008).

[8] Diener (F.), Diener (M.) (eds.) . — Nonstandard analysis in practice. Universitext, Springer (1995).

[9] Diener (M.) . — Détermination et existence des fleuves en dimension deux, C. R. Acad. Sci. Paris Sér. I Math. 301, no. 20, p. 899-902 (1985).

[10] Diener (M.), Reeb (G.) . — Champs polynômiaux : nouvelles trajectoires remarquables, Bull. Soc. Math. Belg. Sér. A 38, p. 131-150 (1987).

[11] Elaydi (S.) . — An introduction to difference equations. 3rd ed., Springer (2005).

[12] Fruchard (A.), Schäfke (R.) . — Analytic solutions of difference equations with small step size. In memory of W. A. Harris, J. Differ. Equations Appl. 7, no. 5, p. 651-684 (2001).

[13] Harris (W. A. Jr.), Sibuya (Y.) . — Asymptotic solutions of systems of nonlinear difference equations. Arch. Rational Mech. Anal. 15, p. 377-395 (1964).

[14] Harris (W. A. Jr.), Sibuya (Y.) . — On asymptotic solutions of systems of nonlinear difference equations. J. Reine Angew. Math. 222, p. 120-135 (1966).

[15] Immink (G. K.). — Asymptotics of analytic difference equations. Springer Lect. N. Math. 1085 (1984).

[16] Lakshmikantham (V.), Trigiante (D.) . — Theory of difference equations : numerical methods and applications. Second edition. Monographs and Textbooks in Pure and Applied Mathematics, 251. Marcel Dekker, Inc., New York (2002).

[17] Nelson (E.) . — Internal Set Theory, Bull. Amer. Math. Soc. 83, no. 6, p. 1165-1198 (1977).

[18] Nörlund (N. E.) . — Sur les équations linéaires aux différences finies à coefficients rationnels. Acta Math. 40, no. 1, p. 191-249 (1916).

[19] Nörlund (N. E.) . — Mémoire sur le calcul aux différences finies. Acta Math. 44, no. 1, p. 71-212 (1923).

[20] Poincaré (R.) . — Sur les équations linéaires aux différences ordinaires et aux différences finies, American Journal of Mathematics 7, 203-258 (1885).

[21] Robinson (A.) . — Non-standard analysis. 3rd ed., Princ. Univ. Press (1996).