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Representations of PGL(2) of a local field and harmonic cochains on graphs

Paul Broussous[(1)]

Dedicated to Colin Bushnell on his 60th birthday

Abstract. — We give combinatorial models for non-spherical, generic, smooth, complex representations of the group $G = \mathrm{PGL}(2, F)$, where F is a non-Archimedean locally compact field. More precisely we carry on studying the graphs $(\tilde{X}_k)_{k\geqslant 0}$ defined in a previous work. We show that such representations may be obtained as quotients of the cohomology of a graph $\tilde{X}_k$, for a suitable integer k, or equivalently as subspaces of the space of discrete harmonic cochains on such a graph. Moreover, for supercuspidal representations, these models are unique.

Résumé. — Nous donnons des modèles combinatoires des représentations lisses, complexes, génériques, non-sphériques du groupe $G = \mathrm{PGL}(2, F)$, où F est un corps localement compact non-archimédien. Plus précisément nous reprenons l'étude des graphes $(\tilde{X}_k)_{k\geqslant 0}$ inaugurée dans un précédent travail. Nous montrons que de telles représentations se réalisent comme quotients de la cohomologie d'un graphe $\tilde{X}_k$ pour un k bien choisi, ou, de façon équivalente, dans un espace de formes harmoniques discrètes sur un tel graphe. Pour les représentations supercuspidales, ces modèles sont de plus uniques.

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Introduction

Let F be a non-archimedean local field and G be the locally compact group $\mathrm{PGL}(N, F)$, where $N \geqslant 2$ is an integer. In [1] the author constructed a projective tower of simplicial complexes fibered over the Bruhat-Tits building X of G. He addressed the question of understanding the structure of the cohomology spaces of these complexes as G-modules. In this article we give a conceptual treatment of the case $N = 2$. In that case X is a homogeneous tree and (a slightly modified version of) the projective tower is formed of graphs $\tilde{X}_n$, $n \geqslant 0$, acted upon by G.

Let π be an irreducible generic and non-spherical smooth complex representation of G. We show there is a natural G-equivariant map

$$\tilde{\Psi}_\pi \ : \ \pi^\vee \longrightarrow \mathcal{H}_\infty(\tilde{X}_{n(\pi)}, \mathbb{C}) \ ,$$

where $n(\pi)$ is an integer related to the conductor of π, $\tilde{\pi}$ denotes the contragredient representation of π, and $\mathcal{H}_\infty(\tilde{X}_{n(\pi)}, \mathbb{C})$ denotes the space of smooth discrete harmonic cochains on $\tilde{X}_{n(\pi)}$. Our construction is based on the existence of *new vectors* for irreducible generic representations whose proof is due to Casselman in the case $N = 2$ [5] (see [7] for the general case).

The G-space $\mathcal{H}_\infty(\tilde{X}_{n(\pi)}, \mathbb{C})$ is naturally isomorphic to the contragredient representation of the cohomology space $H^1_c(\tilde{X}_{n(\pi)}, \mathbb{C})$ (cohomology space with compact support and complex coefficients). We show that $\tilde{\Psi}_\pi$ corresponds to a non-zero natural G-equivariant map:

$$\Psi_\pi \ : \ H^1_c(\tilde{X}_{n(\pi)}, \mathbb{C}) \longrightarrow \pi \ .$$

In other words π is naturally a quotient of $H^1_c(\tilde{X}_{n(\pi)}, \mathbb{C})$. When π is supercuspidal the surjective map Ψ splits and π embeds in $H^1_c(\tilde{X}_{n(\pi)}, \mathbb{C})$. We show that this model is unique:

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi, H^1_c(\tilde{X}_{n(\pi)}, \mathbb{C})) = 1 \ .$$

The proof of that fact roughly goes as follows. Using a *geodesic Radon transform* on the space of 1-cochains with finite support on $\tilde{X}_k$, $k \geqslant 0$, we construct an intertwining operator:

$$j_k \ : \ H^1_c(\tilde{X}_k, \mathbb{C}) \longrightarrow c\text{-}\mathrm{ind}_T^G 1_T$$

(the compactly induced representation of the trivial character of the diagonal torus T of G). The point is that this map is injective. Moreover we have:

$$\bigcup_{k \geqslant 0} \mathrm{Im}(j_k) = c\text{-}\mathrm{ind}_T^G 1_T \ .$$

We show that $c\text{-ind}_T^G 1_T$ naturally embeds in a space of Whittaker functions on G and we may then rely on the uniqueness of Whittaker model for $\mathrm{PGL}(2,F)$.

That such a combinatorial realization of the generic non-spherical irreducible representations of $\mathrm{PGL}(2,F)$ is feasible was actually conjectured by Pierre Cartier more than thirty years ago [4] (but he did not introduce any simplicial structure). Later on Cartier's student Ahumada Bustamante [3] studied the action of the full automorphism group $\Gamma = \mathrm{Aut}(X)$ of the tree X on pairs of vertices at distance $k+1$ (i.e. on edges of $\tilde{X}_k$). Using an equivalent language, he proved that, under the action of Γ, the space $\mathcal{H}_2(\tilde{X}_k,\mathbb{C})$ of L^2 harmonic cochains splits into two irreducible components $\mathcal{H}_2^{\pm}(\tilde{X}_k,\mathbb{C})$, formed of even and odd harmonic cochains respectively.

The present article is not a completion of the previous work [1] where we computed the supercuspidal part of the cohomology space $H_c^1(\tilde{X}_2,\mathbb{C})$. (Be aware that the notation slightly differs. In particular, $\tilde{X}_2$ is the space $\tilde{X}_1$ of [1])). Even though the results are compatible, here we do not determine the structure of the spaces $H_c^1(\tilde{X}_k,\mathbb{C})$, $k \geqslant 0$, as G-modules. In a forthcoming work [2] we shall work out this structure and its links with *types theory.*

I would like to thank V. Sécherre and A. Bouaziz for pointing out some embarrassing mistakes in previous versions of this work. I thank G. Henniart, A. Gaborieau and A. Raghuram for stimulating discussions.

The notes are organized as follows. In §1 we shall discuss the link between cohomology and harmonic cochains on graphs. §§2-4 are about the construction of the maps Ψ_π, $\tilde{\Psi}_\pi$ and some of their properties. The Radon transform is defined and studied in §5 in order to prove our uniqueness result (theorem (5.3.2)).

In the sequel we shall use the following notation:

$\mathfrak{o} = \mathfrak{o}_F$ is the ring of integers of F,

$\mathfrak{p} = \mathfrak{p}_F$ is the maximal ideal of $\mathfrak{o}$,

$v = v_F$ is the normalized additive valuation on F,

$k = k_F$ is the residue class field $\mathfrak{o}/\mathfrak{p}$,

$q = |k_F|$ is the cardinal of k,

$|\ | = |\ |_F$ is the multiplicative valuation on $F^\times$ normalized in such a way that $|\varpi|_F = 1/q$ for any generator ϖ of the ideal $\mathfrak{p}$.

The contragrediente of a representation $\mathcal{V}$ is denoted by $\tilde{\mathcal{V}}$ or $\mathcal{V}^\vee$.

1. Proper G-graphs and harmonic cochains

1.1. In this section we let G be any locally profinite group and Y be a locally finite directed graph (each vertex belongs to a finite number of edges). We write Y^0 (resp. Y^1) for the set of vertices (resp. edges) of Y. We have the map $Y^1 \longrightarrow Y^0$, $a \mapsto a^+$ (resp. $a \mapsto a^-$), where for any edge a we denote by a^+ and a^- its head and tail respectively. We assume that G acts on Y and preserves the structure of directed graph. For all $s \in Y^0$, $a \in Y^1$, we have incidence numbers $[a : s] \in \{-1, 1, 0\}$ satisfying $[g.a : g.s] = [a : s]$, for all $g \in G$; these are defined by $[a : a^+] = +1$, $[a : a^-] = -1$, and $[a : s] = 0$ if $s \notin \{a^+, a^-\}$. Finally we assume that the action of G on Y is proper: for all $s \in Y^0$, the stabilizer $G_s := \{g \in G \ ; \ g.s = s\}$ is open and compact.

1.2. We let $H^1_c(Y, \mathbb{C})$ denote the cohomology space of the CW-complex Y with compact support and complex coefficients. Recall that it may be calculated as follows. Let $C_0(Y, \mathbb{C})$ (resp. $C_1(Y, \mathbb{C})$) be the $\mathbb{C}$-vector space with basis Y^0 (resp. Y^1). Let $C^i_c(Y, \mathbb{C})$, $i = 0, 1$, be the $\mathbb{C}$-vector space of 1-cochains with finite support : $C^i_c(Y, \mathbb{C})$ is the subspace of the algebraic dual of $C_i(Y, \mathbb{C})$ formed of those linear forms whose restrictions to the basis Y^i have finite support. The coboundary map

$$d \ : \ C^0_c(Y, \mathbb{C}) \longrightarrow C^1_c(Y, \mathbb{C})$$

is given by $df(a) = f(a^+) - f(a^-)$. Then

$$H^1_c(Y, \mathbb{C}) \simeq C^1_c(Y, \mathbb{C})/dC^0_c(Y, \mathbb{C}) \ . \tag{1.2.1}$$

The group G acts on $C_i(Y, \mathbb{C})$ and $C^i_c(Y, \mathbb{C})$. Since the action of G on Y is proper, these spaces are smooth G-modules. The coboundary map is G-equivariant and the isomorphism (1.2.1) is G-equivariant. So $H^1_c(Y, \mathbb{C})$ is smooth as a G-module; it is not admissible in general.

1.3. For $i = 0, 1$, we have a natural pairing:

$$\langle -, - \rangle \ : \ C^i(Y, \mathbb{C}) \times C^i_c(Y, \mathbb{C}) \longrightarrow \mathbb{C} \ ,$$

where $C^i(Y, \mathbb{C})$ is the space of i-cochains with arbitrary support. The pairings are given by:

$$\langle f, g \rangle = \sum_{x \in Y^i} f(x) g(x) \ , \ i = 0, 1 \ .$$

Via these pairings we may identify the algebraic dual $C_c^i(Y,\mathbb{C})^*$ of $C_c^i(Y,\mathbb{C})$ with $C^i(Y,\mathbb{C})$, $i=0,1$. The contragredient representation $C_c^i(Y,\mathbb{C})^\vee$ identifies with the space of smooth linear forms in $C^i(Y,\mathbb{C})$. A straightforward computation gives:

$$\langle f, dg\rangle = \langle d^*f, g\rangle \ , \ f\in C^1(Y,\mathbb{C}) \ , \ g\in C_c^0(Y,\mathbb{C}) \ , \tag{1.3.1}$$

where $d^*\colon C^1(Y,\mathbb{C})\longrightarrow C^0(Y,\mathbb{C})$ is defined by

$$d^*f(s) = \sum_{a\in Y^1,\ s\in a} [a:s]f(a) \ , \ s\in Y^0 \ .$$

Of course this latter sum has a finite number of terms. An element of the kernel of d^* is called a *harmonic cochain* on Y. We denote by $\mathcal{H}(Y,\mathbb{C}) = \ker(d^*)$ the space of harmonic cochains. It is naturally acted upon by G. The smooth part of $\mathcal{H}(Y,\mathbb{C})$ under the action of G, i.e. the space of *smooth harmonic cochains* is denoted by $\mathcal{H}_\infty(Y,\mathbb{C})$. The following lemma follows from equality (1.3.1).

(1.3.2) Lemma. — *The algebraic dual of $H_c^1(Y,\mathbb{C})$ naturally identifies with $\mathcal{H}(Y,\mathbb{C})$. Under this isomorphism, the contragredient representation of $H_c^1(Y,\mathbb{C})$ corresponds to $\mathcal{H}_\infty(Y,\mathbb{C})$.*

2. The projective tower of graphs

2.1 In this section, we recall the construction of [1]. The notation is slightly modified. We denote by X the Bruhat-Tits building of G (cf. [8] chap. II, §1). This is a 1-dimensional simplicial complex (a $(q+1)$-homogeneous tree). Let $k\geqslant 0$ be an integer. An *(oriented) k-path* in X is an injective sequence $(s_0,\dots,s_k)$ of vertices in X such that, for $i=0,\dots,k-1$, $\{s_i,s_{i+1}\}$ is an edge of X.We define an oriented graph $\tilde{X}_k$ as follows. Its vertex set (resp. edge set) is the set of k-paths (resp. $(k+1)$-paths) in X. The structure of oriented graph is given by:

$$a^+ = (t_1,\dots,t_{k+1}),\ a^- = (t_0,\dots,t_k),\ \text{if } a=(t_0,\dots,t_{k+1}) \ .$$

The group G acts on $\tilde{X}_k$. If $k\geqslant 1$, $\tilde{X}_k$ is a simplicial complex and the G-action is simplicial. For $k=0$ the action preserves the graph structure. For all k, it preserves the orientation of $\tilde{X}_k$. Recall ([1] Lemma 4.1) that the simplicial complexes $\tilde{X}_k$, $k\geqslant 1$ are connected. The directed graph $\tilde{X}_0$

is obtained from X by doubling the edges (with the same vertex set); it is obviously connected.

2.2. For any integer $n \geqslant 1$, we write $\Gamma_0(\mathfrak{p}^n)$ for the image in G of the following subgroup of $\mathrm{GL}(2,F)$:

$$\tilde{\Gamma}_0(\mathfrak{p}^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,F);\ a,\ d \in \mathfrak{o}^\times,\ b \in \mathfrak{o},\ c \in \mathfrak{p}^n \right\}$$

We let $\Gamma_o(\mathfrak{p}^0)$ be the image in $\mathrm{PGL}(2,F)$ of the standard maximal compact subgroup of $\mathrm{GL}(2,F)$:

$$\tilde{\Gamma}_0(\mathfrak{p}^0) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathfrak{o});\ ad - bc \in \mathfrak{o}^\times \right\}$$

For all $n \geqslant 0$, $\Gamma_o(\mathfrak{p}^{n+1})$ is the stabilizer in G of some edges of $\tilde{X}_n$. We fix such an edge a_o.

3. The construction

3.1. We start by recalling Casselman's result. We fix an irreducible complex smooth representation $(\pi, \mathcal{V})$ of G. We assume that:

(3.1.1) π has no non-zero vector fixed by $\Gamma_0(\mathfrak{p}^0)$,

(3.1.2) π is generic, i.e. it is not of the form $\chi \circ \det$, where χ is a character of $F^\times/(F^\times)^2$ and $\det : G \longrightarrow F^\times/(F^\times)^2$ is the map induced by the determinant map: $\mathrm{GL}(2,F) \longrightarrow F^\times$.

We have the following result ([5] Theorem 1):

(3.1.3) Theorem (Casselman). — *i) For k large enough, the space of fixed vectors $\mathcal{V}^{\Gamma_o(\mathfrak{p}^{k+1})}$ is non-zero.*

ii) Let $n(\pi) \geqslant 0$ be such that $\mathcal{V}^{\Gamma_o(\mathfrak{p}^{n(\pi)+1})} \neq \{0\}$ and $\mathcal{V}^{\Gamma_o(\mathfrak{p}^{n(\pi)})} = \{0\}$. Then for all $k \geqslant n(\pi)$, we have:

$$\dim_{\mathbb{C}} \mathcal{V}^{\Gamma_o(\mathfrak{p}^{k+1})} = k - n(\pi) + 1\ .$$

3.2. For all $a \in \tilde{X}^1_{n(\pi)}$ (resp. $s \in \tilde{X}^0_{n(\pi)}$), we write Γ_a (resp. Γ_s) for the stabilizer of a (resp. s) in G. In particular we have $\Gamma_{a_o} = \Gamma_o(\mathfrak{p}^{n(\pi)+1})$.

(3.2.1) LEMMA. — *i) For all $a \in \tilde{X}^1_{n(\pi)}$, we have* $\dim \mathcal{V}^{\Gamma_a} = 1$.

ii) Let $a \in \tilde{X}^1_{n(\pi)}$ and $s \in \tilde{X}^0_{n(\pi)}$ with $s \in a$. Then for all $v \in \mathcal{V}^{\Gamma_a}$, we have

$$\sum_{k \in \Gamma_s/\Gamma_a} kv = 0 \ .$$

Point i) is obvious. In ii), the vector $\sum_{k \in \Gamma_s/\Gamma_a} kv$ is fixed by Γ_s. So it must be zero since Γ_s is conjugate to $\Gamma_o(\mathfrak{p}^{n(\pi)})$.

Let us fix a non-zero vector $v_o \in \mathcal{V}^{\Gamma_{a_o}}$; v_o is unique up to a scalar in $\mathbb{C}^\times$. If a is any edge of $X_{n(\pi)}$, we put

$$v_a = gv_o, \text{ where } a = ga_o \ . \tag{3.2.2}$$

This is indeed possible since G acts transitively on $\tilde{X}^1_{n(\pi)}$. Moreover, since v_o is fixed by Γ_{a_o}, v_a does not depend on the choice of $g \in G$ such that $a = ga_o$. Let $\tilde{\mathcal{V}}$ be the contragredient representation of $\mathcal{V}$. We define a map:

$$\tilde{\Psi}_\pi \ : \ \tilde{\mathcal{V}} \longrightarrow C^1(X_{n(\pi)}, \mathbb{C})$$

by $\tilde{\Psi}_\pi(\varphi)(a) = \varphi(v_a)$. From (3.2.2) we have that $\tilde{\Psi}_\pi$ is G-equivariant.

(3.2.3) LEMMA. — *i) The image of $\tilde{\Psi}_\pi$ lies in $\mathcal{H}_\infty(X_{n(\pi)}, \mathbb{C})$.*

ii) The map $\tilde{\Psi}_\pi$ is injective.

For i), it suffices to prove that $\mathrm{Im}(\tilde{\Psi}_\pi) \subset \mathcal{H}(\tilde{X}_{n(\pi)}, \mathbb{C})$. So we must prove that for all $\varphi \in \tilde{\mathcal{V}}$, $(\varphi(v_a))_{a \in \tilde{X}^1_{n(\pi)}}$ is a harmonic cochain on $\tilde{X}_{n(\pi)}$, that is:

$$\sum_{s \in a} [a : s]\varphi(v_a) = 0, \text{ for all } s \in X^0_{n(\pi)} \ .$$

Let s be any vertex of $\tilde{X}_{n(\pi)}$. Write $\bar{s}$ for the convex hull in X of the set of vertices of X occuring in the path s (this is a segment lying in some apartment). We know that the pointwise stabilizer of $\bar{s}$ in G (that is the stabilizer Γ_s of s in G) acts transitively on the set of apartments of X containing $\bar{s}$. It follows that Γ_s acts transitively on

$$A_s^+ = \{a \in \tilde{X}^1_{n(\pi)};\ a^+ = s\} \text{ and } A_s^- = \{a \in \tilde{X}^1_{n(\pi)};\ a^- = s\}\ .$$

Fix some $a_s^+ \in A_s^+$ and $a_s^- \in A_s^-$. Then

$$\sum_{s\in a}[a:s]\varphi(v_a) = \varphi\Big(\sum_{a\in A_s^+} v_a - \sum_{a\in A_s^-} v_a\Big)$$

$$= \varphi\Big(\sum_{k\in\Gamma_s/\Gamma_{a_s^+}} kv_{a_s^+} - \sum_{k\in\Gamma_s/\Gamma_{a_s^-}} kv_{a_s^-}\Big) = 0\ ,$$

thanks to lemma (3.2.1).

The G-equivariant map $\tilde{\Psi}_\pi$ is necessarily injective since it is non-zero and since the representation π is irreducible.

Passing to contragredient representations, we get an intertwining operator:

$$\tilde{\tilde{\Psi}}_\pi\ :\ \tilde{\tilde{H}}{}^1_c(\tilde{X}_{n(\pi)},\mathbb{C}) \longrightarrow \tilde{\tilde{\mathcal{V}}}\ .$$

Recall that for any smooth G-module $\mathcal{W}$, we have a canonical injection $\mathcal{W} \longrightarrow \tilde{\tilde{\mathcal{W}}}$. It is surjective if and only if $\mathcal{W}$ is admissible. In particular $\mathcal{V}$ and $\tilde{\tilde{\mathcal{V}}}$ are canonically isomorphic since π is irreducible, whence admissible.

(3.2.4) THEOREM.— *The map $\tilde{\tilde{\Psi}}_\pi$ restricts to a non-zero intertwining operator $\Psi_\pi\ : H^1_c(\tilde{X}_{n(\pi)},\mathbb{C}) \longrightarrow \mathcal{V} \simeq \tilde{\tilde{\mathcal{V}}}$, given by:*

$$\Psi_\pi(\bar{\omega}) = \sum_{a\in\tilde{X}^1_{n(\pi)}} \omega(a)v_a\ .$$

where for $\omega \in C^1_c(\tilde{X}_{n(\pi)},\mathbb{C})$, $\bar{\omega}$ denotes the image of ω in $H^1_c(\tilde{X}_{n(\pi)},\mathbb{C})$.

In particular, the representation $(\pi,\mathcal{V})$ is naturally a quotient of the cohomology space $H^1_c(\tilde{X}_{n(\pi)},\mathbb{C})$.

The theorem follows from a straightforward computation based on Lemma (3.2.1)(ii) and is left to the reader.

Remark. — Assume that π is supercuspidal. Then it is projective in the category of smooth complex representations of G. So as a corollary of Theorem (3.2.4) we get an injective map $(\pi, \mathcal{V}_\pi) \longrightarrow H^1_c(X_{n(\pi)}, \mathbb{C})$.

4. Some properties of the map $\tilde{\Psi}_\pi$.

4.1. We keep the notation as in the last section. If Y is any directed graph, we write $\mathcal{H}_c(Y, \mathbb{C})$ for the subspace of $\mathcal{H}(Y, \mathbb{C})$ of harmonic cochains with finite support and $\mathcal{H}_2(Y, \mathbb{C})$ for the subspace of L^2-harmonic cochains, that is cochains $f \in \mathcal{H}(Y, \mathbb{C})$ satisfying

$$\sum_{a \in Y^1} |f(a)|^2 < \infty \ .$$

Note that any element of $\mathcal{H}_c(X_{n(\pi)}, \mathbb{C})$ is smooth.

(4.1.1) Proposition. — *i) If π is a supercuspidal representation then* $\mathrm{Im}\tilde{\Psi}_\pi$ *is contained in $\mathcal{H}_c(X_{n(\pi)}, \mathbb{C})$.*

ii) If π is a square-integrable representation then $\mathrm{Im}\tilde{\Psi}_\pi$ *lies in*

$\mathcal{H}_2(X_{n(\pi)}, \mathbb{C})$.

Assume π supercuspidal. Let $\lambda \in \tilde{\mathcal{V}}$. Then $\tilde{\Psi}_\pi(\lambda)(a) = \lambda(gv_{a_o})$ for all $a = ga_o \in X^1_{n(\pi)}$. Since π is supercuspidal, the coefficient $g \in G \mapsto \lambda(gv_{a_o})$ has compact support C. Choose a finite number of compact open subgroups K_i, $i \in I$, of G and elements $g_i \in G$, $i \in I$, such that C lies in the union of the g_iK_i, $i \in I$. Then the support of the harmonic cochain $\tilde{\Psi}_\pi(\lambda)$ lies in

$$\bigcup_{i \in I} g_iK_ia_o = \bigcup_{i \in I} g_iK_i/(K_i \cap \Gamma_{a_o})a_o \ ,$$

a finite set.

Now assume that π is square-integrable. With the notation as above, the coefficient $g \mapsto \lambda(gv_{a_o})$ is square-integrable. Consider the Haar measure on G such that Γ_{a_o} has volume 1. Then

$$\int_G |\lambda(gv_{a_o})|^2 dg = \sum_{a \in X^1_{n(\pi)}} |\lambda(v_a)|^2 < \infty \ ,$$

as required.

(4.1.2) Corollary. — *If π is supercuspidal, the map $\tilde{\Psi}_\pi : (\tilde{\mathcal{V}}, \pi^\vee) \longrightarrow \mathcal{H}_c(X_{n(\pi)}, \mathbb{C})$ induces a non-zero (whence injective) map $\bar{\Psi}_\pi : (\tilde{\mathcal{V}}, \pi^\vee) \longrightarrow H^1_c(X_{n(\pi)}, \mathbb{C})$.*

The map $\bar{\Psi}_\pi$ is

$$\lambda \mapsto \tilde{\Psi}_\pi(\lambda) \bmod dC^0_c(X_{n(\pi)}, \mathbb{C}) \ .$$

It suffices to prove that

$$\mathcal{H}_c(X_{n(\pi)}, \mathbb{C}) \cap dC^0_c(X_{n(\pi)}, \mathbb{C}) = \{0\} \ .$$

If f lies in the intersection then $d^* f = 0$ and $f = dg$ for some $g \in C^0_c(X_{n(\pi)}, \mathbb{C})$. We then have $d^* \bar{f} = 0$ (where $\bar{f}(a)$ is the complex conjugate of $f(a)$), and

$$\sum_{a \in X^1_{n(\pi)}} |f(a)|^2 = \langle \bar{f}, f \rangle = \langle \bar{f}, dg \rangle = \langle d^* \bar{f}, g \rangle = 0 \ .$$

Hence $f = 0$.

5. The geodesic Radon tranform

5.1. In corollary (4.1.2) we saw that each supercuspidal irreducible representation of G may be realized as a G-invariant subspace of $H^1_c(X_k, \mathbb{C})$ for a certain k. In order to prove that this model is unique, we need to embed the space $H^1_c(X_k, \mathbb{C})$ in a standard G-module which contains supercuspidal irreducible representations with multiplicity 1. This standard G-module is the space of locally constant functions with compact support on the set of all *oriented apartments* of X_k endowed with a certain topology (see below). This will be done via a Radon transform that "integrates" 1-cochains over each apartment of X_k.

Recall that an apartment of X is a doubly infinite geodesic of X, that is the image in X of an injective sequence of vertices $(s_k)_{k \in \mathbb{Z}}$ such that for all $k \in \mathbb{Z}$, $\{s_k, s_{k+1}\}$ is an edge of X.

An *oriented apartment* $\tilde{A}$ of X is by definition a pair (A, ϵ), where A is an apartment of X and ϵ is an orientation of A as a simplicial complex. Our group G acts on oriented apartments via $g.(A, \epsilon) = (gA, g\epsilon)$, where $g\epsilon$ is the unique orientation on A satisfying $[ga : gs]_{g\epsilon} = [a : s]_\epsilon$, for all $a \in A^1$, $s \in A^0$. The group G acts transitively on the set of oriented apartments.

Let T be the diagonal torus of G (the image of the diagonal torus of $\mathrm{GL}(2,F)$ in G). The G-set $\tilde{\mathcal{A}}$ of oriented apartments in X is isomorphic to G/T. Indeed the stabilizer of an oriented apartement (A,ϵ), that is the set of $g\in G$ such that $gA=A$ and $g\epsilon=\epsilon$, is conjugate to T. We endow $\tilde{\mathcal{A}}$ with the topology corresponding to the quotient topology of G/T. In particular for any $\tilde{A}$ in $\tilde{\mathcal{A}}$ and any open (compact) subgroup K of G, $K.\tilde{A}=\{k\tilde{A};\ k\in K\}$ is an open (compact) neighbourhood of $\tilde{A}$ in $\tilde{\mathcal{A}}$. Let $\tilde{A}\in\tilde{\mathcal{A}}$ and k be a non-negative integer. By definition the (oriented) apartment of $\tilde{X}_k$ corresponding to $\tilde{A}$ is the 1-dimensional subsimplicial complex $\tilde{A}_k$ of $\tilde{X}_k$ whose edges (resp. vertices) are the $(k+1)$-paths a (resp. k-paths s) in X contained in $\tilde{A}$ and such that the orientations of a (resp. s) and $\tilde{A}$ are compatible. We denote by $\tilde{\mathcal{A}}_k$ the set of apartments of $\tilde{X}_k$. Then $\tilde{A}\mapsto\tilde{A}_k$ is a G-equivariant bijection between $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}_k$ which allows us to identify both G-sets.

Let k be a non-negative integer and a be an edge of $\tilde{X}_k$. Define a subset $\tilde{\mathcal{A}}_a$ of $\tilde{\mathcal{A}}$ by

$$\tilde{\mathcal{A}}_a=\{\tilde{A}\in\tilde{\mathcal{A}}\ ;\ a\in\tilde{A}^1\}\ .$$

(5.1.1) Lemma. — *i) With the notation as above, we have $\tilde{\mathcal{A}}_a=\Gamma_a\tilde{A}_o$, for any $\tilde{A}_o$ in $\tilde{\mathcal{A}}_a$. In particular $\tilde{\mathcal{A}}_a$ is a compact open subset of $\tilde{\mathcal{A}}$.*

ii) The set $\{\tilde{\mathcal{A}}_a\ ;\ k\geqslant 0,\ a\in\tilde{X}_k^1\}$ is a basis of the topology of $\tilde{\mathcal{A}}$ formed of compact open subsets.

The equality $\tilde{\mathcal{A}}_a=\Gamma_a\tilde{A}_o$ follows from the fact that Γ_a acts transitively on the apartments of X containing the path a. For ii), we must prove that any open subset Ω of $\tilde{\mathcal{A}}$ contains $\tilde{\mathcal{A}}_a$ for some $k\geqslant 0$ and $a\in\tilde{X}_k^1$. Replacing Ω by $g\Omega$ for some $g\in G$ we may assume that it contains the oriented apartment $\tilde{A}_{\mathrm{st}}$ corresponding to the coset $1.T\in G/T$. For $r\geqslant 1$, let K_r be the image in G of the following congruence subgroup of $\mathrm{GL}(2,F)$:

$$\Big\{\begin{pmatrix} a & b\\ c & d\end{pmatrix};\ a,\ d\in 1+\mathfrak{p}^r,\ b,\ c\in\mathfrak{p}^r\Big\}\ .$$

Take r large enough so that $K_r\tilde{A}_{\mathrm{st}}\subset\Omega$. Let T^o be the maximal compact open subgroup of T. It stabilizes A_{st} pointwise, whence it fixes $\tilde{A}_{\mathrm{st}}$. So $K_rT^o\tilde{A}_{\mathrm{st}}\subset\Omega$. The subgroup K_rT^o is Γ_a for some $a\in\tilde{A}^1_{\mathrm{st}}$ and we are done.

5.2. Fix $k\geqslant 0$. We define a (geodesic) Radon transform

$$\mathcal{R}=\mathcal{R}_k\ :\ C^1_c(\tilde{X}_k,\mathbb{C})\longrightarrow\mathcal{F}(\tilde{\mathcal{A}})\ ,$$

where $\mathcal{F}(\tilde{\mathcal{A}})$ is the space of functions on $\tilde{\mathcal{A}}$, by

$$\mathcal{R}(\omega)(\tilde{A}) = \sum_{a\in\tilde{A}^1} \omega(a)\ .$$

Note that the image of a 1-cochain ω whose support is reduced to a single edge a_o is $\omega(a_o)\mathrm{Char}_{\tilde{\mathcal{A}}_{a_o}}$, where Char denotes a characteristic function. So the image of $\mathcal{R}$ actually lies in the space $C_c^0(\tilde{\mathcal{A}})$ of locally constant functions with compact support on $\tilde{\mathcal{A}}$. Clearly $\mathcal{R}$ is G-equivariant.

(5.2.1) Lemma. — *For all $f\in C_c^0(\tilde{X}_k,\mathbb{C})$, we have $\mathcal{R}(df)=0$.*

Indeed if $\tilde{A}\in\tilde{\mathcal{A}}$, we have

$$\mathcal{R}(df)(\tilde{A}) = \sum_{a\in\tilde{A}^1} f(a^+)-f(a^-)$$

$$= \sum_{a\in\tilde{A}^1} f(a^+) - \sum_{a\in\tilde{A}^1} f(a^-) = \sum_{s\in\tilde{A}^0} f(s) - \sum_{s\in\tilde{A}^0} f(s) = 0\ ,$$

since the map $\tilde{A}^1\longrightarrow\tilde{A}^0$, $a\mapsto a^+$ (resp. $a\mapsto a^-$) is a bijection.

(5.2.2) Proposition. — *i) The following sequence of G-modules is exact:*

$$C_c^0(\tilde{X}_k,\mathbb{C}) \xrightarrow{d} C_c^1(\tilde{X}_k,\mathbb{C}) \xrightarrow{\mathcal{R}} C_c^0(\tilde{\mathcal{A}})\ .$$

In other words $\mathcal{R}$ induces an injective map:

$$\bar{\mathcal{R}}\ :\ H_c^1(\tilde{X}_k,\mathbb{C})\longrightarrow C_c^0(\tilde{\mathcal{A}})\ .$$

ii) Moreover we have:

$$\bigcup_{k\geqslant 0}\bar{\mathcal{R}}_k(H_c^1(\tilde{X}_k,\mathbb{C})) = C_c^0(\tilde{\mathcal{A}})\ .$$

Point ii) follows from the fact that $C_c^0(\tilde{\mathcal{A}})$ is generated as a $\mathbb{C}$-vector space by the functions $\mathrm{Char}_{\tilde{\mathcal{A}}_a}$, where $a\in\tilde{X}_k^1$ and $k\geqslant 0$.

To prove i) we need to introduce some more concepts. A path p in $\tilde{X}_k$ is a sequence of edges a_u, $u=0,\dots,l-1$, such that for $u=0,\dots,l-2$, a_u and

a_{u+1} share a vertex. Abusing the notation we shall write $p = (x_0, x_1, \dots, x_l)$, where for $u = 0, \dots, l-1$, $\{a_u^+, a_u^-\} = \{x_u, x_{u+1}\}$, keeping in mind that when $k = 0$ two neighbour vertices do not determine a unique edge. We define "incidence coefficients" $[p : a_u] \in \{\pm 1\}$, $u = 0, \dots, l-1$, by $[p : a_u] = 1$ if and only if $a_u^- = x_u$ and $a_u^+ = x_{u+1}$.

Let $\omega \in C_c^1(\tilde{X}_k, \mathbb{C})$ be in the kernel of $\mathcal{R}$. The "integral" of ω along p is by definition

$$\int_p \omega = \sum_{u=0,\dots,l-1} [p : a_u]\omega(a_u) \ .$$

(5.2.3) LEMMA.— *With the notation as above, if p is a loop, i.e. if a_0 and a_{l-1} share the vertex $x_l = x_0$, then*

$$\int_p \omega = 0 \ .$$

We first show that the lemma implies proposition (5.2.2). Fix $s_o \in \tilde{X}_k^0$ and $\alpha_o \in \mathbb{C}$. For any $x \in \tilde{X}_k^0$, we set

$$f(x) = \alpha_o + \int_p \omega \ ,$$

where $p = (x_0, \dots, x_l)$ is any path satisfying $x_0 = s_o$ and $x_l = x$. We claim that $f(x)$ does not depend on the choice of p. Indeed let $q = (y_0, \dots, y_m)$ be another path satisfying the same assumptions and set $q{-}p = (z_0, \dots, z_{m+l+1})$, where $z_u = x_u$, for $u = 0, \dots, l$, and $z_u = y_{m+l+1-u}$, for $u = l+1, \dots, l+1+m$. Then one easily checks that

$$(\alpha_o + \int_q \omega) - (\alpha_o + \int_p \omega) = \int_{q-p} \omega = 0 \ ,$$

since $q - p$ is a loop.

Let $a \in \tilde{X}_k^1$ and $p = (x_o, \dots, x_l)$ be a path such that $x_0 = s_o$ and $x_l = a^-$. Set $q = (x_o, \dots, x_l, a^+)$. Then

$$f(a^+) - f(a^-) = \int_q \omega - \int_p \omega = [q : a]\omega(a) = \omega(a) \ .$$

So we must now prove that one can choose s_o and α_o so that $f \in C^0(\tilde{X}_k, \mathbb{C})$ has finite support. Let S_k be the support of ω in $\tilde{X}_k^1$ and set

$$S := \bigcup_{a \in S_k} \mathrm{cvx}(a) \subset X \ ,$$

where $\mathrm{cvx}(a)$ denotes the convex hull of a in (the geometric realization of) X. Then S is a bounded subset of X. Let t be a vertex in S and δ be an integer large enough so that $S \subset X(t,\delta)$, where $X(t,\delta)$ is the subtree of X whose vertices are at combinatorial distance from t less than or equal to δ. Then the complementary set ${}^{c}X(t,\delta)$ of $X(t,\delta)$ in X has the following property; for any k-path $a \subset {}^{c}X(t,\delta)$, there exists a half-apartment A^+ such that $a \subset A^+ \subset {}^{c}X(t,\delta)$. Set

$$S'_k = \{s \in \tilde{X}_k^0 \ ; \ \mathrm{cvx}(s) \cap X(t,\delta) \neq \emptyset\}.$$

Choose s_o outside S'_k and set $\alpha_o = 0$. We are going to prove that the support of f is contained in S'_k. Let x be in $\tilde{X}_k^0 \backslash S'_k$. Choose half-apartments A_o^+ and A^+ so that:

$$A^+, \ A_o^+ \subset {}^{c}S \text{ and } \mathrm{cvx}(s_o) \subset A_o^+, \ \mathrm{cvx}(x) \subset A^+ \ .$$

Consider vertices s'_o and x' of $\tilde{X}_k$ whose convex hulls in X lie in A_o^+ and A^+ respectively. One may choose s'_o (resp. x') away enough from the origin of A_o^+ (resp. of A^+) so that there exists an apartment B of X containing both $\mathrm{cvx}(s'_o)$ and $\mathrm{cvx}(x')$.

First case: the vertices s'_o and x' induce the same orientation on B (see figure 1). Let $\tilde{B}$ be the corresponding oriented apartment. Let $\tilde{A}_o^+$ be the oriented half-apartment whose orientation is induced by s_o and $\tilde{A}^+$ be the oriented half-apartment whose orientation is induced by x. Let $p(s_o, s'_o)$ (resp. $p(x', x)$ and $p(s'_o, x')$) be the unique injective path in $\tilde{X}_k$ joining s_o to s'_o (resp. x' to x, s'_o to x') and such that $p(s_o, s'_o) \subset \tilde{A}_o^+$ (resp. $p(x', x) \subset \tilde{A}^+$, $p(s'_o, x') \subset \tilde{B}$). By concatenation, we get a path $p(s_o, x) = p(s_o, s'_o) + p(s'_o, x') + p(x', x)$ joining s_o and x. Since those vertices of $\tilde{X}_k$ occuring in $p(s_o, s'_o)$ or $p(x', x)$ do not lie in S, we have

$$\int_{p(s_o,s'_o)} \omega = \int_{p(x,x')} \omega = 0 \ .$$

Moreover

$$\int_{p(s'_o,x')} \omega = \pm \mathcal{R}(\omega)(\tilde{B}) = 0 \ ,$$

by assumption. So

$$\int_{p(s_o,x)} \omega = 0 \text{ and } f(x) = \alpha_o + \int_{p(s_o,x)} \omega = 0 \ ,$$

as required.

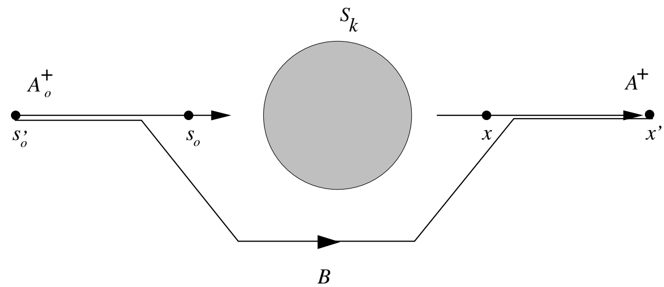

Figure 1

Second case: the vertices s_o' and x' induce different orientations on B (see figure 2). Then one can choose a third vertex $x'' \in S_k'$ such that s_o' and x'' (resp. x' and x'') lie in some common apartment B_1 (resp. B_2) and induce the same orientation on that apartment. We denote by $\tilde{B}_1$ and $\tilde{B}_2$ the corresponding oriented apartments. Then, with the notation as in the first case, one easily shows that

$$f(x) = f(s_o) + \int_{p(s_o, s_o')} \omega + \pm\mathcal{R}(\omega)(\tilde{B}_1) + \pm\mathcal{R}(\omega)(\tilde{B}_2) + \int_{p(x',x)} \omega = 0\ .$$

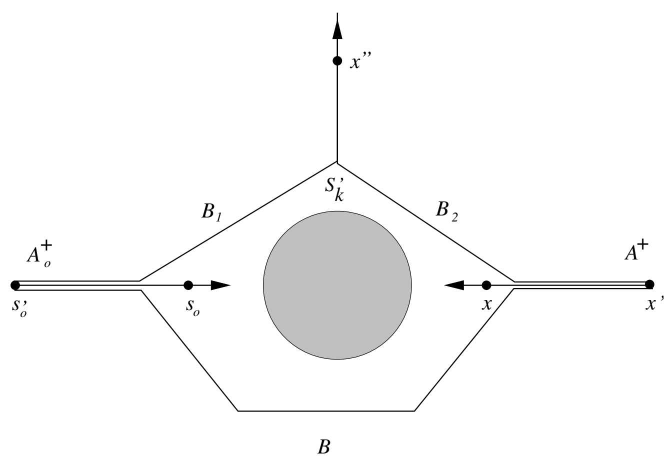

Figure 2

Proof of lemma (5.2.3). — It is based on the following easy lemma whose proof is left to the reader.

(5.2.4) Lemma. — *Let U be either $\mathbb{Z}$, $\mathbb{N}$ or a finite interval of integers. Let $p = (x_u)_{u \in U}$ be a path in $\tilde{X}_k$. Assume that p satisfies one of the following properties:*

(P1) *For all $u \in U$ such that $u+1 \in U$, we have $a_u^+ = a_{u+1}^-$;*

(P2) *For all $u \in U$ such that $u+1 \in U$, we have $a_u^- = a_{u+1}^+$.*

Then p is injective and there is an apartment $\tilde{A}$ containing p. In particular p cannot be a loop.

Remark. — A path p satisfies (P1) or (P2) if and only if the sequence of incidence numbers $([p : a_u])_u$ is constant.

Let $p = (x_0, \ldots, x_l)$ be a loop. We consider the index u as an element of $\mathbb{Z}/l\mathbb{Z}$. According to the previous lemma, the set V of indices $u \in \mathbb{Z}/l\mathbb{Z}$ such that we have neither $a_u^+ = a_{u+1}^-$ nor $a_u^- = a_{u+1}^+$ is non-empty. Moreover it has cardinal at least 2. Let us first consider the case $\sharp V = 2$ (this case indeed occurs when $k = 0$). We may for instance assume that

$$a_0^- = a_{l-1}^- = x_0 \text{ and } a_{u_o-1}^+ = a_{u_o}^+ = x_{u_o} \ ,$$

for some $u_o \in \mathbb{Z}/l\mathbb{Z}\backslash\{0\}$. Hence we must have

$$a_{u+1}^- = a_u^+, \ u = 0, \ldots, u_o - 2 \ \ a_u^- = a_{u+1}^+, \ \ u = u_o, \ldots, l-2 \ .$$

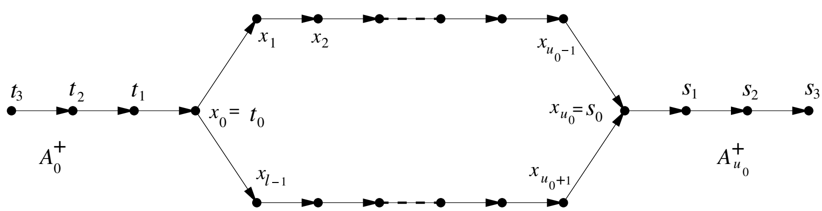

Figure 3

Choose a half-apartment A_o^+, whose vertex set is $\{s_0, s_1, \ldots, s_u, \ldots\}$, satisfying:

- for all $u \geqslant 0$, there is an edge b_u of A_o^+, such that $b_u^+ = s_u$, $b_u^- = s_{u+1}$;
- $s_0 = x_0 = b_0^+$.

Similarly choose a half-apartment $A_{u_o}^+$, whose vertex set is $\{t_0, t_1, \ldots, t_u, \ldots\}$, satisfying:

- for all $u \geqslant 0$, there is an edge c_u of $A_{u_o}^+$ such that $c_u^+ = s_{u+1}$, $c_u^- = s_u$;
- $t_0 = x_{u_o} = c_0^-$.

Consider the two infinite paths:

$$p_1 = (\ldots, s_u, \ldots, s_1, s_0 = x_0, x_1, x_2, \ldots, x_{u_o-1}, x_{u_o} = t_0, t_1, t_2, \ldots, t_v, \ldots)$$

$$p_2 = (\ldots, s_u, \ldots, s_1, s_0 = x_0, x_{l-1}, x_{l-2}, \ldots, x_{u_o+1}, x_{u_o} = t_0, t_1, t_2, \ldots, t_v, \ldots)$$

By lemma (5.2.4) we can find an apartment $\tilde{A}_1$ (resp. $\tilde{A}_2$) whose vertices are those of p_1 (resp. those of p_2). Since ω has finite support, we can give an obvious meaning to the integrals:

$$\int_{p_1} \omega \text{ and } \int_{p_2} \omega \ .$$

Moreover we have

$$\int_{p_1} \omega = \mathcal{R}(\omega)(\tilde{\mathcal{A}}_1) = 0 \text{ and } \int_{p_2} \omega = \mathcal{R}(\omega)(\tilde{\mathcal{A}}_2) = 0 \ .$$

Finally we have:

$$\int_{p_1} \omega - \int_{p_2} \omega = \sum_{u=0,\ldots,u_o-1} \omega(a_u) - \sum_{u=l-1,l-2,\ldots,u_o} \omega(a_u)$$

$$= \sum_{u=0,\ldots,u_o-1} [p : a_u]\omega(a_u) + \sum_{u=l-1,l-2,\ldots,u_o} [p : a_u]\omega(a_u) = \int_p \omega = 0 \ ,$$

$\square$.

This proof extends to the case $\sharp V > 2$ by introducing for all $u \in V$ a half-apartment starting at the vertex $a_u \cap a_{u+1}$. The details are left to the reader (see figure 4).

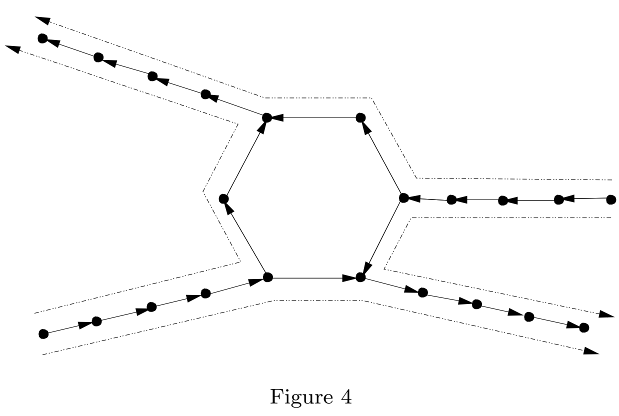

Figure 4

5.3 *Uniqueness of the model in* $H^1_c(\tilde{X}_k, \mathbb{C})$ *for supercuspidals*

Because of the homeomorphism $\tilde{A} \simeq G/T$, the G-module $C^0_c(\tilde{A})$ is isomorphic to $C^0_c(G/T)$, the space of locally constant complex functions with compact support on G/T. The G-module $C^0_c(G/T)$ by definition is the compactly induced representation $c\text{-ind}_T^G 1_T$ of the trivial character 1_T of T to G. So we may rephrase proposition (5.2.2) in the following way.

(5.3.1) Proposition. — *For all $k \geqslant 0$, the Radon transform induces an injective G-equivariant map:*

$$H^1_c(\tilde{X}_k, \mathbb{C}) \longrightarrow c\text{-ind}_T^G 1_T \ ,$$

where c-ind *denotes a compactly induced representation and* 1_T *denotes the trivial character of* T*.*

(5.3.2) Theorem. — *Let $(\pi, \mathcal{V}_\pi)$ be an irreducible supercuspidal representation of G. Then we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G[\mathcal{V}_\pi, H^1_c(\tilde{X}_{n(\pi)}, \mathbb{C})] = 1 \ .$$

This theorem will from the following result due to Waldspurger ([9] Prop. 9', p. 31):

(5.3.3) Theorem (J.-L. Waldspurger). — *Let $(\pi, \mathcal{V}_\pi)$ be an irreducible unitary smooth representation of G. Then we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\mathcal{V}_\pi, 1_T) = 1 \ .$$

Proof of (5.3.2). — Waldspurger's result together with Frobenius reciprocity imply that

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\mathcal{V}_\pi, \mathrm{Ind}_T^G 1_T) = 1 \ ,$$

where $\mathrm{Ind}_T^G 1_T$ is the representation of G smoothly induced from the trivial character of T. This representation is the smooth part of the G-module $C(G/T)$ formed of those locally constant functions on G that are stabilized by an open subgroup of G. Since $c\text{-ind}_T^G 1_T$ naturally embeds as a sub-G-module of $\mathrm{Ind}_T^G 1_T$, we get :

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\mathcal{V}_\pi, c\text{-ind}_T^G 1_T) \leqslant \dim_{\mathbb{C}} \mathrm{Hom}_G(\mathcal{V}_\pi, \mathrm{Ind}_T^G 1_T) \ ,$$

and the theorem follows.

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