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## End-to-end gluing of constant mean curvature hypersurfaces

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**RÉSUMÉ.** — Il a été observé par R. Kusner et prouvé par J. Ratzkin qu'on peut recoller ensemble deux surfaces à courbure moyenne constante ayant deux bouts de même paramètre de Delaunay. Cette procédure de recollement est connu comme «somme connexe bout-à-bout». Dans ce papier, nous donnons une généralisation de cette construction en dimension quelconque dans le but de construire des nouvelles hypersurfaces à courbure moyenne constante à partir des hypersurfaces connues.

**ABSTRACT.** — It was observed by R. Kusner and proved by J. Ratzkin that one can connect together two constant mean curvature surfaces having two ends with the same Delaunay parameter. This gluing procedure is known as a “end-to-end connected sum”. In this paper we generalize, in any dimension, this gluing procedure to construct new constant mean curvature hypersurfaces starting from some known hypersurfaces.

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### 1. Introduction and statement of results

Surfaces of revolution whose mean curvature is constant equal to 1 are classified in 1841 by C. Delaunay [1]. The profile curves of these surfaces (known as Delaunay surfaces) are conic roulettes [2].

In higher dimension, a generalization of the Delaunay classification for hypersurfaces with some symmetries is given in [4]. In particular, for  $n \geq 3$ , there exist a constant mean curvature hypersurface of revolution in  $\mathbb{R}^{n+1}$  denoted by  $n$ -Delaunay hypersurfaces. These hypersurfaces construct a one

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parameter family  $\mathcal{D}_\tau$  for

$$\tau \in (-\infty, 0) \cup (0, \tau_*].$$

Let us denote by  $\mathcal{M}_k$ , for  $k \in \mathbb{N}^*$ , the set of complete noncompact constant mean curvature hypersurfaces in  $\mathbb{R}^{n+1}$  with  $k$  ends asymptotic to  $n$ -Delaunay hypersurface. Then, any element  $\Sigma$  of  $\mathcal{M}_k$  is said to be nondegenerate if the linearized operator (the Jacobi operator)

$$\mathcal{L}_\Sigma : L^2(\Sigma) \longrightarrow L^2(\Sigma)$$

is injective. In general, it is difficult to prove that a given hypersurface is nondegenerate. But, we will prove in the beginning of the paper that any  $n$ -Delaunay hypersurface is nondegenerate.

The structure of the set  $\mathcal{M}_k$  is now fairly well understood. In particular, it is proved in [5] that if  $\Sigma$  is a nondegenerate element of  $\mathcal{M}_k$  and if the Delaunay parameters  $\tau_\ell$  of the ends of  $\Sigma$  satisfy  $\tau_\ell \in [\tau^*, 0) \cup (0, \tau_*]$ , for some  $\tau^*$  depends only in  $n$ , there exists an open set  $\mathcal{U} \subset \mathcal{M}_k$  containing  $\Sigma$  which is a smooth manifold of dimension  $k(n+1)$ . This result generalizes in any dimension and for constant mean curvature hypersurfaces with ends asymptotic to  $n$  Delaunay with negative parameter the result of R. Kusner, R. Mazzeo and D. Pollack [14].

In 1990, a method of construction of constant mean curvature surfaces is elaborated by N. Kapouleas to construct a compact surfaces [9], [10] and a noncompact completes surfaces of  $\mathbb{R}^3$  [8]. This method is based in analogous method used by R. Schoen to construct a constant scalar curvature in a finitely punctured sphere. Later, this technique has been adopted in [15], [16], [3] and generalized in [6]. More precisely, starting from a nondegenerate element of  $\mathcal{M}_{k_1}$  we can add a finite number ( $k_2 \in \mathbb{N}^*$ ) of constant mean curvature hypersurfaces asymptotic to  $n$ -Delaunay to this hypersurface to construct a new nondegenerate element of  $\mathcal{M}_{k_1+k_2}$ .

Recently, a new method of construction of constant mean curvature surfaces which is known as a “end-to-end connected sum” in introduced by J. Ratzkin [19]. This gluing method is crucial in the construction of many examples of compact constant mean curvature surfaces with nontrivial topology [7].

In this paper, we give a generalization of this technique in any dimension in the aim to construct new constant mean curvature hypersurfaces starting from some known hypersurfaces. We start in Section 2 by giving a parameterization of a one parameter family of hypersurfaces of revolution in  $\mathbb{R}^{n+1}$ , which have constant mean curvature normalized to be equal to 1. These hypersurfaces, which were originally studied in [12], generalize the classical

constant mean curvature surfaces in  $\mathbb{R}^3$  which were discovered by Delaunay in [1] in the middle of the 19-th century. Next, we define the Jacobi operator about a  $n$ -Delaunay hypersurface and we give the expression of the geometric Jacobi fields (some solutions of the homogenous problem  $\mathcal{L}_{\mathcal{D}_\tau} w = 0$ ). In Section 3, we recall some important results of the moduli space concerning the Fredholm properties of the Jacobi operator acting in some weighted functional space. Next, we describe the construction by considering two non-degenerate hypersurfaces  $\Sigma_1 \in \mathcal{M}_{k_1}$  and  $\Sigma_2 \in \mathcal{M}_{k_2}$  and we assume that the ends  $E_1 \subset \Sigma_1$  and  $E_2 \subset \Sigma_2$  are asymptotic to the same  $n$ -Delaunay  $\mathcal{D}_\tau$  with parameter  $\tau \in [\tau^*, 0) \cup (0, \tau_*)$ . Then, we can align  $\Sigma_1$  and  $\Sigma_2$  such that the axis of  $E_1$  and of  $E_2$  coincide (with opposite directions). Finally, we assume that  $E_1$  or  $E_2$  is a "regular end" then we can translate one of these hypersurface along this axis and we prove

**THEOREM 1.1.** — *Let  $\Sigma_1 \in \mathcal{M}_{k_1}$  and  $\Sigma_2 \in \mathcal{M}_{k_2}$  two nondegenerate constant mean curvature hypersurfaces described as above. There exists a family of hypersurfaces which is a connected sum of  $\Sigma_1$  and  $\Sigma_2$ . These hypersurfaces can be perturbed into a constant mean curvature hypersurface which is element of  $\mathcal{M}_{k_1+k_2-2}$ .*

The principal advantage of this construction (in addition to the fact that it can be used to construct new examples of compact or complete noncompact constant mean curvature hypersurfaces) is the simplicity of its proof.

## 2. Delaunay hypersurfaces

### 2.1. Parameterization:

It will be more interesting to consider an isothermal type parameterization for which will be more convenient for analytical purposes. Hence, we looking for hypersurfaces of revolution which can be parameterized by

$$X(s, \theta) = (|\tau| e^{\sigma(s)} \theta, \kappa(s)), \quad (2.1)$$

for  $(s, \theta) \in \mathbb{R} \times S^{n-1}$ . The constant  $\tau$  being fixed, the functions  $\sigma$  and  $\kappa$  are determined by asking that the hypersurface parameterized by  $X$  has constant mean curvature equal to  $H$  and also by asking that the metric associated to the parameterization is conformal to the product metric on  $\mathbb{R} \times S^{n-1}$ , namely

$$(\partial_s \kappa)^2 = \tau^2 e^{2\sigma} (1 - (\partial_s \sigma)^2). \quad (2.2)$$

We choose the orientation of the hypersurface parameterized by  $X$  so that, the unit normal vector field is given by

$$N := \left( -\frac{\partial_s \kappa}{|\tau| e^\sigma} \theta, \partial_s \sigma \right). \quad (2.3)$$

This time, using (2.2) the first fundamental form  $g$  of the hypersurface parameterized by  $X$  is given by

$$g = \tau^2 e^{2\sigma} (ds \otimes ds + d\theta_i \otimes d\theta_j),$$

and its second fundamental form  $b$  is given by

$$b = (\partial_s^2 \kappa \partial_s \sigma - \partial_s \kappa (\partial_s^2 \sigma + (\partial_s \sigma)^2)) ds \otimes ds + \partial_s \kappa d\theta_i \otimes d\theta_j.$$

Therefore, the mean curvature  $H$  of the hypersurface parameterized by  $X$  is given by

$$H = \frac{1}{n\tau^2 e^{2\sigma}} ((n-1) \partial_s \kappa - \partial_s \kappa (\partial_s^2 \sigma + (\partial_s \sigma)^2) + \partial_s^2 \kappa \partial_s \sigma).$$

This is a rather intricate second order ordinary differential equation in the functions  $\sigma$  and  $\tau$  which has to be complimented by the equation (2.2). In order to simplify our analysis, we use of (2.2) to get rid of the factor  $\tau^2 e^{2\sigma}$  in the above equation. This yields

$$\partial_s \sigma \partial_s^2 \kappa = \partial_s \kappa \left( 1 - n + \partial_s^2 \sigma + (\partial_s \sigma)^2 + n H \partial_s \kappa (1 - (\partial_s \sigma)^2)^{-1} \right).$$

Now, we can differentiate (2.2) with respect to  $s$ , and we obtain

$$\partial_s \kappa \partial_s^2 \kappa = \tau^2 e^{2\sigma} \partial_s \sigma (1 - \partial_s^2 \sigma - (\partial_s \sigma)^2).$$

The difference between the last equation, multiplied by  $\partial_s \sigma$ , and the former equation, multiplied by  $\partial_s \kappa$ , yields

$$\partial_s^2 \sigma + (1-n)(1 - (\partial_s \sigma)^2) + n H \partial_s \kappa = 0. \quad (2.4)$$

Hence, in order to find constant mean curvature hypersurfaces of revolution, we have to solve (2.2) together with (2.4).

Let us define

$$\tau_* := \frac{1}{n} (1-n)^{\frac{n-1}{n}}.$$

For all  $\tau \in (-\infty, 0) \cup (0, \tau_*]$ , we define  $\sigma_\tau$  to be the unique smooth nonconstant solution of

$$(\partial_s \sigma)^2 + \tau^2 \left( e^\sigma + \iota e^{(1-n)\sigma} \right)^2 = 1, \quad (2.5)$$

with initial condition  $\partial_s \sigma(0) = 0$  and  $\sigma(0) < 0$ . Next, we define the function  $\kappa_\tau$  to be the unique solution of

$$\partial_s \kappa = \tau^2 \left( e^{2\sigma} + \iota e^{(2-n)\sigma} \right), \quad \text{with} \quad \kappa(0) = 0. \quad (2.6)$$

Here,  $\iota$  is the sign of  $\tau$ .

In particular, the hypersurface parameterized by

$$X_\tau(s, \theta) := (|\tau|e^{\sigma_\tau(s)}\theta, \kappa_\tau(s)),$$

for  $(s, \theta) \in \mathbb{R} \times S^{n-1}$ , is an embedded constant mean curvature hypersurface of revolution when  $\tau$  belongs  $(0, \tau_*]$ , this hypersurface will be referred to as the “ $n$ -unduloid” of parameter  $\tau$ . In the other case, if  $\tau < 0$ , this hypersurface is only immersed and will be referred to as the “ $n$ -nodoid” of parameter  $\tau$ .

*Remark 2.1.* — Thanks to the Hamiltonian structure of (2.5), the function  $s \mapsto \sigma(s)$  is periodic. Let denote by  $s_\tau$  this period. Then, it is proved in [6]

$$s_\tau = -\frac{n}{n-1} \log \tau^2 + \mathcal{O}(1) \quad (2.7)$$

as  $\tau$  tends to 0.

## 2.2. The Jacobi operator about a $n$ -Delaunay

It is well known [20] that the linearized mean curvature operator about  $\mathcal{D}_\tau$ , which is usually referred to as the Jacobi operator, is given by

$$\mathcal{L}_\tau := \Delta_\tau + |A_\tau|^2,$$

where  $\Delta_\tau$  is the Laplace-Beltrami operator and  $|A_\tau|^2$  is the square of the norm of the shape operator  $A_\tau$  on  $\mathcal{D}_\tau$ .

Let us define the function  $\varphi_\tau := |\tau|e^{\sigma_\tau}$ . We find the expression of the Jacobi operator in term of the function  $\varphi_\tau$

$$\mathcal{L}_\tau := \varphi_\tau^{-n} \partial_s (\varphi_\tau^{n-2} \partial_s) + \varphi_\tau^{-2} \Delta_{S^{n-1}} + n + n(n-1) \tau^{2n} \varphi_\tau^{-2n}. \quad (2.8)$$

It will be convenient to define the conjugate operator

$$L_\tau := \varphi_\tau^{\frac{n+2}{2}} \mathcal{L}_\tau \varphi_\tau^{\frac{2-n}{2}}, \quad (2.9)$$

which is explicitly given in terms of the function  $\varphi_\tau$  by

$$L_\tau = \partial_s^2 + \Delta_{S^{n-1}} - \left( \frac{n-2}{2} \right)^2 + \frac{n(n+2)}{4} \varphi_\tau^2 + \frac{n(3n-2)}{4} \tau^{2n} \varphi_\tau^{2-2n}. \quad (2.10)$$

Since the operators  $\mathcal{L}_\tau$  and  $L_\tau$  are conjugate, the mapping properties of one of them will easily translate for the other one. With slight abuse of terminology, we shall refer to any of them as the Jacobi operator about  $\mathcal{D}_\tau$ .

Now, we give the following definition.

DEFINITION 2.2. — *Let us denote by  $\theta \mapsto e_j(\theta)$ , for  $j \in \mathbb{N}$ , the eigenfunctions of the Laplace-Beltrami operator on  $S^{n-1}$ , which will be normalized to have  $L^2$  norm equal to 1 and correspond to the eigenvalue  $\lambda_j$ . That is*

$$-\Delta_{S^{n-1}} e_j = \lambda_j e_j,$$

and

$$\lambda_0 = 0, \quad \lambda_1 = \dots = \lambda_n = n - 1, \quad \lambda_{n+1} = 2n, \dots \quad \text{and} \quad \lambda_j \leq \lambda_{j+1}.$$

We ended this section by giving only the expression of some Jacobi fields, i.e., solution of the homogeneous problem

$$\mathcal{L}_\tau \omega = 0,$$

since these Jacobi fields follow from a rigid motion or by changing the De-launay parameter  $\tau$ . More details are given in [5].

- For  $\tau \in (-\infty, 0) \cup (0, \tau_*)$ , we define  $\Phi_\tau^{0,+}$  to be the Jacobi field corresponding to the translation of  $\mathcal{D}_\tau$  along the  $x_{n+1}$  axis

$$\Phi_\tau^{0,+} := \varphi^{\frac{n-4}{2}} \partial_s \varphi.$$

It is easy to check that  $\Phi_\tau^{0,+}$  only depends on  $s$  and is periodic. Then, this Jacobi field is bounded.

- Since we have  $n$  directions orthogonal to  $x_{n+1}$ , there are  $n$  linearly independent Jacobi fields which are obtained by translating  $\mathcal{D}_\tau$  in a direction orthogonal to its axis. We get for  $j = 1, \dots, n$

$$\Phi_\tau^{j,+} := \left( \varphi^{\frac{n}{2}} \pm |\tau|^n \varphi^{-\frac{n}{2}} \right) e_j.$$

Again, we see that  $\Phi_\tau^{j,+}$  is periodic (hence bounded) for all  $j = 1, \dots, n$ .

- For  $j = 1, \dots, n$ , we define

$$\Phi_\tau^{j,-}(s, \theta) := \varphi^{\frac{n-4}{2}} \left( \varphi \partial_s \varphi + \kappa \partial_s \kappa \right) e_j$$

to be the Jacobi field corresponding to the rotation of the axis of  $\mathcal{D}_\tau$ . Observe that  $\Phi_\tau^{j,-}$  is not bounded, but grows linearly.

- Finally, The Jacobi field corresponding to a change of parameter  $\tau \in (-\infty, 0) \cup (0, \tau_*)$  is given by

$$\Phi_\tau^{0,-} := \varphi^{\frac{n-4}{2}} (\partial_\tau \kappa \partial_s \varphi - \partial_\tau \varphi \partial_s \kappa).$$

Because of the rotational invariance of the operator  $L_\tau$ , we can introduce the eigenfunction decomposition with respect to the cross-sectional Laplace-Beltrami operator  $\Delta_{S^{n-1}}$ . In this way, we obtain the sequence of operators

$$L_{\tau,j} = \partial_s^2 - \lambda_j - \left( \frac{n-2}{2} \right)^2 + \frac{n(n+2)}{4} \varphi^2 + \frac{n(3n-2)}{4} \tau^{2n} \varphi^{2-2n} \quad (2.11)$$

for  $j \in \mathbb{N}$ . By definition, the *indicial roots* of the operator  $L_{\tau,j}$  characterize the rate of growth (or rate of decay) of the solutions of the homogeneous equation

$$L_{\tau,j} \omega = 0$$

at infinity (see [14]). Observe that the explicit knowledge of some Jacobi fields yields some information about the indicial roots of the operator  $L_\tau$ . Indeed, since the Jacobi fields  $\Phi_\tau^{j,\pm}$ , described below, are at most linearly growing, the associated indicial roots are all equal to 0. Hence, we conclude that for all  $\tau \in (-\infty, 0) \cup (0, \tau_*]$

$$\gamma_j(\tau) = 0, \quad \text{for } j = 0, \dots, n.$$

The situation is completely different when  $j \geq n+1$ . Indeed, its proved in [5]:

**PROPOSITION 2.3.** — *There exists  $\tau^* < 0$ , depending only on  $n$ , such that for all  $\tau \in [\tau^*, 0) \cup (0, \tau_*]$ ,*

$$\gamma_j(\tau) > 0, \quad \text{for all } j \geq n+1.$$

**Remark 2.4.** — Thanks to the last analysis, it is easy to see that the  $n$ -Delaunay is a nondegenerate element of  $\mathcal{M}_k$ .



### 3. End-to-End connected sum

#### 3.1. Moduli space theory

Assume that we are given  $\Sigma$  a complete noncompact, constant mean curvature hypersurfaces in  $\mathbb{R}^{n+1}$ , with  $k$  ends which are modeled after  $n$ -Delaunay hypersurfaces.

We recall that, for all  $\tau \in (-\infty, 0) \cup (0, \tau_*]$ , the Delaunay hypersurface  $\mathcal{D}_\tau$  is parameterized by

$$X_\tau = (\varphi_\tau \theta, \kappa_\tau),$$

where  $\kappa_\tau$  and  $\varphi_\tau$  are defined respectively in subsection 2.1 and section 2.2.

Now, we denote by  $E_1, \dots, E_k$  the ends of the hypersurface  $\Sigma$ . We require that these ends are asymptotic to half  $n$ -unduloids or a half  $n$ -nodoids. More precisely, we require that, up to some rigid motion, each end can be parameterized as the normal graph of some exponentially decaying function over some Delaunay hypersurface, i.e. up to some rigid motion, the end  $E_\ell$  is parameterized by

$$Y_\ell := X_{\tau_\ell} + \omega_\ell N_{\tau_\ell}, \quad (3.12)$$

where  $Y_\ell$  is defined in  $(0, +\infty) \times S^{n-1}$  and where the function  $\omega_\ell$  is exponentially decaying as well as all its derivatives. Hence, for all  $k \in \mathbb{N}$ , there exists  $c_k > 0$  such that

$$|\nabla^k \omega_\ell| \leq c_k e^{-\gamma_{n+1}(\tau_\ell)s} \quad (3.13)$$

on  $(0, +\infty) \times S^{n-1}$ .

The Jacobi operator about the end  $E_\ell$  is close to the Jacobi operator about the  $n$ -Delaunay hypersurface  $\mathcal{D}_{\tau_\ell}$ . The content of the following Lemma is to make this result quantitatively precise.

LEMMA 3.1. — *The Jacobi operator about  $\Sigma$ , restricted to the end  $E_\ell$  is given by*

$$\mathcal{L}_\Sigma := \Delta_\Sigma + |A_\Sigma|^2 = \mathcal{L}_{\tau_\ell} + L_\ell,$$

where the  $L_\ell$  is a second order linear operator whose coefficients and their derivatives are bounded by a constant times  $e^{-\gamma_{n+1}(\tau_\ell)s}$ .

*Proof.* — This follows at once from the fact that the coefficients of the first and second fundamental forms associated to the end  $E_\ell$  are equal to the coefficients of the first and second fundamental form of  $\mathcal{D}_{\tau_\ell}$  up to some functions which are exponentially decaying like  $e^{-\gamma_{n+1}(\tau_\ell)s}$ .  $\square$

Now, we decompose  $\Sigma$  into slightly overlapping pieces which are a compact piece  $K$  and the ends  $E_\ell$ . Then, we define the following functional space

DEFINITION 3.2. — *For all  $r \in \mathbb{N}$ ,  $\delta \in \mathbb{R}$  and all  $\alpha \in (0, 1)$ , the function space  $\mathcal{C}_\delta^{r,\alpha}(\Sigma)$  is defined to be the space of functions  $w \in \mathcal{C}_{loc}^{r,\alpha}(\Sigma)$  for which the following norm is finite*

$$\|w\|_{\mathcal{C}_\delta^{r,\alpha}(\Sigma)} := \sum_{\ell=1}^k \|w \circ Y_\ell\|_{\mathcal{E}_\delta^{r,\alpha}((0,+\infty) \times S^{n-1})} + \|w\|_{\mathcal{C}^{r,\alpha}(K)},$$

where the space

$$\mathcal{E}_\delta^{\ell,\alpha}([s_0, +\infty) \times S^{n-1})$$

is the set of functions  $\mathcal{C}_{loc}^{\ell,\alpha}$  which are defined on  $[s_0, +\infty) \times S^{n-1}$  and for which the following norm is finite :

$$\|\omega\|_{\mathcal{E}_\delta^{\ell,\alpha}(\mathbb{R} \times S^{n-1})} := \sup_{s \geq s_0} |e^{-\delta s} \omega|_{\mathcal{C}^{\ell,\alpha}([s, s+1] \times S^{n-1})}.$$

Here,  $|\cdot|_{\mathcal{C}^{\ell,\alpha}([s, s+1] \times S^{n-1})}$  denotes the usual Hölder norm in  $[s, s+1] \times S^{n-1}$ .

Let  $\chi_\ell$  be a cutoff function which is equal to 0 on  $E_\ell \cap K$  and equal to 1 on  $Y_\ell((c_\ell, +\infty) \times S^{n-1})$  for some  $c_\ell > 0$  chosen large enough. We define the deficiency space  $\mathcal{W}(\Sigma)$  by

$$\mathcal{W}(\Sigma) := \bigoplus_{\ell=1}^k \text{Span} \{ \chi_\ell \Phi_\ell^{j,\pm} \mid j = 0, \dots, n \}.$$

The analogue of the following result for  $n = 2$  is usually known as the “Linear Decomposition Lemma” (see [18] and [15]).

PROPOSITION 3.3 [5]. — *We assume that  $\tau_\ell \in [\tau^*, 0) \cup (0, \tau_*]$ ,  $\delta \in (-\inf_{n+1} \gamma_{n+1}(\tau_\ell), 0)$ ,  $\alpha \in (0, 1)$  and  $\Sigma$  is nondegenerate. Let  $\mathcal{N}(\Sigma)$  the trace of the kernel of  $\mathcal{L}_\Sigma$  over the deficiency space  $\mathcal{W}_\Sigma$ , then  $\mathcal{N}(\Sigma)$  is a  $k(n+1)$  dimensional subspace of  $\mathcal{W}(\Sigma)$  which satisfies*

$$\text{Ker}(\mathcal{L}_\Sigma) \subset \mathcal{C}_{-\delta}^{2,\alpha}(\Sigma) \oplus \mathcal{N}(\Sigma).$$

If  $\mathcal{K}(\Sigma)$  is a  $k(n+1)$  dimensional subspace of  $\mathcal{W}(\Sigma)$  such that

$$\mathcal{W}(\Sigma) = \mathcal{K}(\Sigma) \oplus \mathcal{N}(\Sigma),$$

we have

$$\mathcal{L}_\Sigma : \mathcal{C}_\delta^{2,\alpha}(\Sigma) \oplus \mathcal{K}(\Sigma) \longrightarrow \mathcal{C}_\delta^{0,\alpha}(\Sigma)$$

is an isomorphism.

### 3.2. Regular ends of constant mean curvature hypersurfaces

We need to introduce the following

DEFINITION 3.4. — *Let  $M$  be a constant mean curvature 1 hypersurface with  $k$  ends of Delaunay type. We will say that the end  $E$  of  $M$ , which is asymptotic to some Delaunay hypersurface  $\mathcal{D}_\tau$ , is regular if there exists a Jacobi field  $\Psi$  which is globally defined on  $M$  and which is asymptotic to  $\Phi_\tau^{0,-}$  (the Jacobi field on  $\mathcal{D}_\tau$  which corresponds to the change of the Delaunay parameter  $\tau$ ) on  $E$ .*

Of course,  $n$ -Delaunay hypersurfaces have regular ends. In the case where  $M$  is a nondegenerate element of the moduli space of  $k$  ended constant mean curvature hypersurfaces, we have the following characterization of regular ends:

PROPOSITION 3.5. — *Assume that  $M$  is a nondegenerate element of  $\mathcal{M}_k$  and let  $E$  be one of the ends of  $M$ . Then, the following propositions are equivalent :*

- (i) *The end  $E$  is regular.*
- (ii) *There exists  $t_0 > 0$  and  $(M_t)_{t \in (-t_0, t_0)}$  a one parameter family of constant mean curvature 1 hypersurfaces in  $\mathcal{M}_k$  such that*

$$M_0 = M$$

*and the Delaunay parameter of the end of  $M_t$  which is close to  $E$  is equal to  $\tau - t$ .*

*Proof.* — The fact that (ii) implies (i) should be clear. The fact that (i) implies (ii) just follows from the application of the implicit function Theorem and the analysis of the moduli space  $\mathcal{M}_k$ . Indeed, as explained in the end of the paper [5], the Jacobi fields span the tangent space to the moduli space  $\mathcal{M}_k$  at the point  $M$ . Therefore, we simply have

$$M_t = \exp_M(t\psi)$$

as the one parameter family of constant mean curvature 1 hypersurfaces close to  $M$  which have the right properties.  $\square$

### 3.3. Connecting two constant mean curvature hypersurfaces together

Assume that we are given two nondegenerate complete noncompact constant mean curvature hypersurfaces  $M_i \in \mathcal{M}_{k_i}$ , for  $i = 1, 2$ . We denote by

$E_{i,1}, \dots, E_{i,k_i}$  the ends of the hypersurface  $M_i$ , each of which are assumed to be asymptotic to a  $n$ -Delaunay hypersurfaces  $\mathcal{D}_{\tau_{i,j}}$ ,  $j = 1, \dots, k_i$ .

We further assume that  $M_1$  and  $M_2$  have one end with the same Delaunay parameter. For example, let us assume that the Delaunay parameter of  $E_{1,1}$  and  $E_{2,1}$  are the same, both equal to

$$\tau := \tau_{1,2} = \tau_{2,1}.$$

Further more we assume that  $\tau \neq \tau_*$  i.e.  $E_{1,1}$  and  $E_{1,2}$  are not asymptotic to a cylinders.

Given  $m \in \mathbb{N}$ , we can use a rigid motion to ensure that the end  $E_{1,1}$  is parameterized by

$$Y_{1,1}(s, \theta) := X_\tau(s, \theta) + w_1(s + m s_\tau, \theta) N_\tau(s, \theta),$$

for all  $(s, \theta) \in (-m s_\tau, +\infty) \times S^{n-1}$  and that the end  $E_{2,1}$  is parameterized by

$$Y_{2,1}(s, \theta) := X_\tau(s, \theta) + w_2(s - m s_\tau, \theta) N_\tau(s, \theta),$$

for all  $(s, \theta) \in (-\infty, m s_\tau) \times S^{n-1}$ . Though this is not explicit in the notation,  $Y_{i,1}$  does depend on  $m$ . In addition, we know from (3.12) and (3.13) that the functions  $w_1$  and  $w_2$  are exponentially decaying as  $s$  tends to  $+\infty$  (resp.  $-\infty$ ). More precisely,

$$w_1 \in \mathcal{E}_{-\gamma_{n+1}(\tau)}^{2,\alpha}((0, +\infty) \times S^{n-1})$$

and also that

$$w_2 \in \mathcal{E}_{\gamma_{n+1}(\tau)}^{2,\alpha}((-\infty, 0) \times S^{n-1}),$$

where the function spaces  $\mathcal{E}_\delta^{2,\alpha}$  are introduced in Definition 3.2.

Given  $s > -m s_\tau$ , we define the truncated hypersurface

$$M_1(s) := M_1 - Y_{1,1}((s, +\infty) \times S^{n-1})$$

and given  $s < m s_\tau$  we define the truncated hypersurface

$$M_2(s) := M_2 - Y_{2,1}((-\infty, s) \times S^{n-1}).$$

Again,  $M_i(s)$  depends on  $m$ . Now let  $s \longrightarrow \xi(s)$  be a cutoff function such that  $\xi \equiv 0$  for  $s \geq 1$  and  $\xi \equiv 1$  for  $s \leq -1$ . We define the hypersurface  $\tilde{M}_m$  to be

$$\tilde{M}_m := M_1(-1) \cup C(1) \cup M_2(1), \quad (3.14)$$

where, for all  $s \in (0, m s_\tau)$ , the cylindrical type hypersurface  $C(s)$  is the image of  $[-s, s] \times S^{n-1}$  by

$$(s, \theta) \longrightarrow \xi(s) Y_{1,1}(s, \theta) + (1 - \xi(s)) Y_{2,1}(s, \theta). \quad (3.15)$$

By construction  $\tilde{M}_m$  is a smooth hypersurface whose mean curvature is identically equal to 1 except in the annulus  $C(1)$  where the mean curvature is close to 1 and tends to 1 as  $m$  tends to  $+\infty$ . Indeed, in  $C(m s_\tau)$ , the hypersurface  $\tilde{M}_m$  is a normal graph over the  $n$ -Delaunay hypersurface  $\mathcal{D}_\tau$  for some function  $w_3$ . But the estimates we have on  $w_1$  and  $w_2$  imply that

$$w_3 = \mathcal{O}_{\mathcal{C}^{2,\alpha}}(e^{-\gamma_{n+1}(\tau)(s+m s_\tau)}) + \mathcal{O}_{\mathcal{C}^{2,\alpha}}(e^{\gamma_{n+1}(\tau)(s-m s_\tau)}) \quad (3.16)$$

on  $C(m s_\tau)$ . The mean curvature of  $\tilde{M}_m$  in  $C(m s_\tau)$  can then be computed using Taylor's expansion

$$H_{\tilde{M}_m} = H_{\mathcal{D}_\tau} + \mathcal{O}_{\mathcal{C}^{0,\alpha}}(w_3).$$

It then follows from (3.16) that the mean curvature of  $\tilde{M}_m$  can be estimated by

$$\| H_{\tilde{M}_m} - 1 \|_{\mathcal{C}^{0,\alpha}(C(1))} \leq c e^{-m \gamma_{n+1}(\tau) s_\tau}. \quad (3.17)$$

### 3.4. Mapping properties of the Jacobi operator about $\tilde{M}_m$

In this subsection, we develop the linear analysis which will be needed in the next subsection in order to perturb  $\tilde{M}_m$  into a constant mean curvature hypersurface. In particular, our aim is to find function spaces on which the mapping properties of the Jacobi operator about  $\tilde{M}_m$  do not depend (too much) on  $m$ .

We further assume that the Delaunay parameters of the ends of  $M_1$  and  $M_2$  are greater than or equal to  $\tau^*$  which has been introduced in Proposition 2.3. In particular, we can define

$$\tilde{\delta} := \inf_{i=1,2} \inf_{j=1,\dots,k_i} \gamma_{n+1}(\tau_{i,j}),$$

and we know that  $\tilde{\delta} > 0$ . Furthermore, the moduli space theory, as described in subsection 3.1 applies. We fix

$$-\tilde{\delta} < \delta < 0.$$

We have already assumed that the hypersurfaces  $M_i$  are nondegenerate. Using the notations of subsection 3.1, we define the deficiency subspace  $\mathcal{W}(M_i)$  associated to  $M_i$  and we decompose

$$\mathcal{W}(M_i) = \mathcal{N}(M_i) \oplus \mathcal{K}(M_i),$$

where  $\mathcal{N}(M_i)$  and  $\mathcal{K}(M_i)$  are both  $(n+1)k_i$  dimensional, in such a way that the Jacobi operator about  $M_i$

$$\mathcal{L}_{M_i} : \mathcal{C}_\delta^{2,\alpha}(M_i) \oplus \mathcal{K}(M_i) \longrightarrow \mathcal{C}_\delta^{0,\alpha}(M_i) \quad (3.18)$$

is an isomorphism. The elements of  $\mathcal{K}(M_i)$  are supported in the complement of a compact set of  $M_i$ . Recall that, on each end  $E_{i,j}$  there are  $2(n+1)$  locally defined Jacobi fields  $\Phi_{E_{i,j}}^{k,\pm}$ , where  $k = 0, \dots, n$ . These Jacobi fields are *a priori* only defined for  $s$  large enough, say  $s \geq s_{i,j}$  and are asymptotic to the corresponding Jacobi fields on  $\mathcal{D}_\tau$

$$\Phi_{E_{i,j}}^{k,\pm} = \Phi_\tau^{k,\pm} + \mathcal{O}_{\mathcal{C}^{2,\alpha}}(e^{\delta s})$$

for any  $\delta \in (-\gamma_{n+1}(\tau_{i,j}), 0)$ . By definition,  $\mathcal{W}(M_i)$  is the vector space spanned by  $\chi_{i,j} \Phi_{E_j^i}^{k,\pm}$  for  $j = 1, \dots, k_i$  and  $k = 0, \dots, n$ , where  $\chi_{i,j}$  are cutoff functions which are equal to 0 on  $M_i - E_{i,j}$ , still equal to 0 on  $E_{i,j}$  for  $s \in (0, s_{i,j})$  and identically equal to 1 on the  $E_{i,j}$ , when  $s \geq s_{i,j} + 1$ . Finally, recall that any globally defined Jacobi field which is at most linearly growing on the ends of  $M_i$  belongs to

$$\mathcal{C}_\delta^{2,\alpha}(M_i) \oplus \mathcal{N}(M_i).$$

Observe that, on a given end  $E_{i,j}$ , all Jacobi fields, except the Jacobi field  $\Phi_{E_{i,j}}^{0,-}$  which corresponds to a change of Delaunay parameter, come from the action of the group of rigid motions. Hence, these Jacobi fields are all globally defined on  $M_i$ . By assumption, the Jacobi field  $\Phi_{E_{1,1}}^{0,-}$  also corresponds to a globally defined Jacobi field on  $M_1$ , this is precisely the meaning of the fact that  $E_{1,1}$  is a regular end. Hence, for each  $k = 0, \dots, n$ , the space  $\mathcal{N}(M_1)$  contains an element which is equal to  $\chi_{1,1} \Phi_{E_{1,1}}^{k,\pm}$  on  $E_{1,1}$ . As a consequence, one can choose  $\mathcal{K}(M_1)$  in such a way that it does not contain any  $\chi_{1,1} \Phi_{E_{1,1}}^{k,\pm}$ , for  $k = 0, \dots, n$ . Since elements of  $\mathcal{K}(M_1)$  are not supported on the end  $E_{1,1}$ , they can be extended to  $\tilde{M}_m$ .

A similar analysis can be performed on  $M_2$ , with the Jacobi fields defined on the end  $E_{2,1}$ . However, this time we have not assumed that the end  $E_{2,1}$  was a regular end. Hence, one can still choose  $\mathcal{K}(M_2)$  in such a way that it does not contain any  $\chi_{2,1} \Phi_{E_{2,1}}^{k,\pm}$ , for  $k = 1, \dots, n$  and also does not contain  $\chi_{2,1} \Phi_{E_{2,1}}^{0,+}$ . But there might exist an element of  $\mathcal{K}(M_2)$  which is equal to  $\chi_{2,1} \Phi_{E_{2,1}}^{0,-}$  on  $E_{2,1}$ . Therefore, all elements of  $\mathcal{K}(M_2)$  can be extended to  $\tilde{M}_m$  except possibly the elements which are collinear to  $\chi_{2,1} \Phi_{E_{2,1}}^{0,-}$  on  $E_{2,1}$ .

We now explain how to extend the function  $\chi_{2,1} \Phi_{E_{2,1}}^{0,-}$  to a function which is globally defined on  $\tilde{M}_m$  and is equal to a Jacobi field on  $M_1(-1)$ . This construction again uses the fact that  $E_{1,1}$  is a regular end. We parameterize  $C(m s_\tau)$  by (3.15). As already mentioned,  $\Phi_{E_{2,1}}^{0,-}$  is asymptotic to  $\Phi_\tau^{0,-}$ . This means that  $\forall (s, \theta) \in [-m s_\tau, m s_\tau] \times S^{n-1}$ ,

$$\Phi_{E_{2,1}}^{0,-}(s, \theta) = \Phi_\tau^{0,-}(m s_\tau - s) + \mathcal{O}_{C^{2,\alpha}}(e^{\delta(m s_\tau - s)}).$$

Similarly,  $\Phi_{E_{1,1}}^{0,-}$  is asymptotic to  $\Phi_\tau^{0,-}$ . This means that

$$\forall (s, \theta) \in [-m s_\tau, m s_\tau] \times S^{n-1}, \quad \Phi_{E_{1,1}}^{0,-}(s, \theta) = \Phi_\tau^{0,-}(m s_\tau + s) + \mathcal{O}_{C^{2,\alpha}}(e^{\delta(m s_\tau + s)}).$$

Now, there exists  $c_m \in \mathbb{R}$  such that

$$\Phi_\tau^{0,-}(m s_\tau - s) - \Phi_\tau^{0,-}(m s_\tau + s) = c_m \Phi_\tau^{0,+}(s)$$

since the function of the left hand side is a Jacobi field on  $\mathcal{D}_\tau$  which is bounded and only depends on  $s$ . In addition, it is easy to see that

$$|c_m| \leq c_\tau m,$$

where the constant  $c_\tau$  only depends on  $\tau$ .

Now, on  $M_1$  the Jacobi fields  $\Phi_{E_{1,1}}^{0,-}$  and  $\Phi_{E_{1,1}}^{0,+}$  are globally defined. Observe that we also have

$$\Phi_{E_{1,1}}^{0,+}(s, \theta) = \Phi_\tau^{0,+}(s) + \mathcal{O}_{C^{2,\alpha}}(e^{\delta(m s_\tau + s)})$$

on  $[-m s_\tau, m s_\tau] \times S^{n-1}$ . This being understood, we use the cutoff function  $\xi$  which has already been defined in (3.15) to define the function  $\tilde{\Phi}_{E_{2,1}}^{0,-}$  which is equal to  $\Phi_{E_{2,1}}^{0,-}$  on  $E_{2,1} \cap M_2(1)$ , which is equal to  $\Phi_{E_{1,1}}^{0,-} + c_m \Phi_{E_{1,1}}^{0,+}$  on  $M_1(-1)$  and which interpolates smoothly between these definitions in  $C(1)$ , namely

$$\tilde{\Phi}_{E_{2,1}}^{0,-} := \xi (\Phi_{E_{1,1}}^{0,-} + c_m \Phi_{E_{1,1}}^{0,+}) + (1 - \xi) \Phi_{E_{2,1}}^{0,-}$$

on  $C(1)$ . Granted (3.15) and (3.17) one easily checks that

LEMMA 3.6. — *Let  $\mathcal{L}_{\tilde{M}_m}$  denote the Jacobi operator about  $\tilde{M}_m$ . Then,*

$$\mathcal{L}_{\tilde{M}_m} \tilde{\Phi}_{E_{2,1}}^{0,-} = \mathcal{O}_{C^{0,\alpha}(C(1))}(e^{\delta m s_\tau})$$

on  $C(1)$ .

We now define  $\tilde{\mathcal{K}}(M_2)$  to be the space of functions of  $\mathcal{K}(M_2)$  which have been extended on all  $\tilde{M}_m$  either by 0 or by replacing  $\Phi_{E_{2,1}}^{0,-}$  by  $\tilde{\Phi}_{E_{2,1}}^{0,-}$ . We set

$$\mathcal{K}(\tilde{M}_m) := \mathcal{K}(M_1) \oplus \tilde{\mathcal{K}}(M_2).$$

We agree that the norm on  $\mathcal{K}(\tilde{M}_m)$  is the product norm on  $\mathcal{K}(M_1) \times \tilde{\mathcal{K}}(M_2)$ .

Granted the decomposition of  $\tilde{M}_m$  given in (3.14), we define the weighted spaces :

**DEFINITION 3.7.** — *Given  $\delta \in \mathbb{R}$ ,  $k \in \mathbb{N}$  and  $\alpha \in (0, 1)$ , we define  $\mathcal{D}_\delta^{k,\alpha}(\tilde{M}_m)$  to be the subspace of functions  $u \in \mathcal{C}_{loc}^{k,\alpha}(\tilde{M}_m)$  for which the following norm*

$$\begin{aligned} \|u\|_{\mathcal{D}_\delta^{k,\alpha}(\tilde{M}_m)} &:= \|u\|_{\mathcal{C}_\delta^{k,\alpha}(M_1(-1))} + \|u\|_{\mathcal{C}_\delta^{k,\alpha}(M_2(1))} \\ &+ e^{-\delta \tilde{m} s_\tau} \|u\|_{\mathcal{C}^{k,\alpha}((-2,2) \times S^{n-1})} \end{aligned}$$

*is finite. Here,  $\mathcal{C}_\delta^{k,\alpha}(M_i(s))$  is the restriction of functions of  $\mathcal{C}_\delta^{k,\alpha}(M_i)$  to  $M_i(s)$  and this space is endowed with the induced norm.*

We can now state the

**PROPOSITION 3.8.** — *Assume that  $\delta \in (-\tilde{\delta}, 0)$  is fixed and  $E_{1,1}$  is a regular end of  $M_1$ . There exist  $m_0 > 0$  and  $c > 0$  and, for all  $m \geq m_0$ , one can find an operator*

$$G_m : \mathcal{D}_\delta^{0,\alpha}(\tilde{M}_m) \longrightarrow \mathcal{D}_\delta^{2,\alpha}(\tilde{M}_m) \oplus \mathcal{K}(\tilde{M}_m),$$

*such that  $w := G_m(f)$  solves  $\mathcal{L}_{\tilde{M}_m} w = f$  on  $\tilde{M}_m$ . Furthermore,*

$$\|w\|_{\mathcal{D}_\delta^{2,\alpha}(\tilde{M}_m) \oplus \mathcal{K}(\tilde{M}_m)} \leq c \|f\|_{\mathcal{D}_\delta^{0,\alpha}(\tilde{M}_m)}.$$

*Proof.* — Given  $f \in \mathcal{D}_\delta^{0,\alpha}(\tilde{M}_m)$ , we want to solve the equation

$$\mathcal{L}_{\tilde{M}_m} w = f$$

on  $\tilde{M}_m$ . To this aim, we use the cutoff function  $\xi$  defined in (3.15) and we start to solve

$$\mathcal{L}_{M_1} w_1 = \xi f$$

on  $M_1$  and also

$$\mathcal{L}_{M_2} w_2 = (1 - \xi) f$$



on  $M_2$ . The existence of  $w_i$  follows at once from (3.18) and we have the estimate

$$\|w_i\|_{\mathcal{C}_\delta^{2,\alpha}(M_i) \oplus \mathcal{K}(M_i)} \leq c \|f\|_{\mathcal{D}_\delta^{0,\alpha}(\tilde{M}_m)}, \quad (3.19)$$

where the constant  $c > 0$  does not depend on  $m$ . Observe that on  $E_{2,1}$  the function  $w_2$  can be decomposed as

$$w_2 := v_2 + a_2 \Phi_{E_{2,1}}^{0,-}, \quad (3.20)$$

with  $v_2$  decays exponentially. We define on  $C(m s_\tau)$  a cutoff function  $\chi_1$  which is identically equal to 1 for  $s \leq m s_\tau - 2$  and identically equal to 0 for  $s \geq m s_\tau - 1$ . We also define a cutoff function  $\chi_2$  which is identically equal to 1 for  $s \geq -m s_\tau + 2$  and identically equal to 0 for  $s \leq -m s_\tau + 1$ . This being understood, we define a function  $w$  as follows :

- $w$  is equal to  $w_2$  on  $M_2(m s_\tau)$ ,
- $w$  is equal to  $w_1 + a_2 \tilde{\Phi}_{E_{2,1}}^{0,-}$  on  $M_1(-m s_\tau)$ ,
- $w := \chi_1 w_1 + \chi_2 v_2 + a_2 \tilde{\Phi}_{E_{2,1}}^{0,-}$ , on  $C(m s_\tau)$ .

To begin with observe that

$$\|w\|_{\mathcal{D}_\delta^{2,\alpha}(\tilde{M}_m) \oplus \mathcal{K}(\tilde{M}_m)} \leq c \|f\|_{\mathcal{D}_\delta^{0,\alpha}(\tilde{M}_m)}$$

for some constant which does not depend on  $m$ . This estimate follows at once from (3.19) and (3.20). We now estimate

$$\mathcal{L}_{\tilde{M}_m} w - f.$$

Obviously, this quantity is equal to 0 on  $M_1(1 - m s_\tau) \cup M_2(m s_\tau - 1)$ . Hence, it remains to evaluate it on  $C(m s_\tau - 1)$ .

**Case 1** First assume that  $-m s_\tau + 2 \leq s \leq -1$ , then

$$\mathcal{L}_{\tilde{M}_m} w - f = \mathcal{L}_{M_1} w - f = \mathcal{L}_{M_1} v_2$$

since  $\mathcal{L}_{M_1} w_1 = f$  and  $\mathcal{L}_{M_1} \tilde{\Phi}_{E_{2,1}}^{0,-} = 0$  in this set. Further observe that  $\mathcal{L}_{M_2} v_2 = 0$  in this set, hence we conclude that

$$\mathcal{L}_{\tilde{M}_m} w - f = \mathcal{L}_{M_1} w - f = (\mathcal{L}_{M_1} - \mathcal{L}_{M_2}) v_2.$$

We use the fact that  $E_{1,2}$  and  $E_{2,1}$  are graphs over the same Delaunay hypersurface  $\mathcal{D}_\tau$ , hence we can write

$$\mathcal{L}_{M_1} = \mathcal{L}_\tau + L_1,$$

where  $L_1$  is a second order partial differential operator whose coefficients, computed at  $s$ , are bounded by a constant times  $e^{-\gamma_{n+1}(\tau)(s+m s_\tau)}$  in  $\mathcal{C}^{0,\alpha}((s-1, s+1) \times S^{n-1})$ . Similarly, we can write

$$\mathcal{L}_{M_2} = \mathcal{L}_\tau + L_2,$$

where  $L_2$  is a second order partial differential operator whose coefficients, computed at  $s$ , are bounded by a constant times  $e^{\gamma_{n+1}(\tau)(s-m s_\tau)}$  in  $\mathcal{C}^{0,\alpha}((s, s+1) \times S^{n-1})$ . Using this, it is easy to show that, there exists a constant  $c > 0$  which does not depend on  $m$  such that

$$\|\mathcal{L}_{\tilde{M}_m} w - f\|_{\mathcal{C}^{0,\alpha}((s-1, s+1) \times S^{n-1})} \leq c e^{-\gamma_{n+1}(\tau)(s+m s_\tau)} e^{-\delta(s-m s_\tau)} \quad (3.21)$$

for all  $s \in [-m s_\tau + 2, -2]$ .

**Case 2** Now, assume that  $-m s_\tau + 1 \leq s \leq -m s_\tau + 2$ . In this set, we have to take into account the effect of the cutoff function  $\chi_2$ , in addition to the estimate established in the previous case. This yields

$$\|\mathcal{L}_{\tilde{M}_m} w - f\|_{\mathcal{C}^{0,\alpha}((-m s_\tau + 1, -m s_\tau + 2) \times S^{n-1})} \leq c e^{2\delta m s_\tau}. \quad (3.22)$$

**Case 3** Finally, assume that  $s \in [-1, 0]$ . Then, we have to take into account the effect of the cutoff function  $\xi$ , in addition to the estimate established in the first case. This yields

$$\|\mathcal{L}_{\tilde{M}_m} w - f\|_{\mathcal{C}^{0,\alpha}((-1, 0) \times S^{n-1})} \leq c e^{-(\gamma_{n+1}(\tau) - \delta) m s_\tau}. \quad (3.23)$$

**Case 4** The cases where  $s \in [0, m s_\tau - 1]$  can be treated similarly.

Collecting the previous estimates, we conclude that

$$\|\mathcal{L}_{\tilde{M}_m} w - f\|_{\mathcal{D}_\delta^{0,\alpha}(\tilde{M}_m)} \leq c (e^{-\gamma_{n+1} m s_\tau} + e^{2\delta m s_\tau}) \|f\|_{\mathcal{D}_\delta^{0,\alpha}(\tilde{M}_m)}.$$

So far, we have produced an operator  $G_0$ , defined by  $G_0(f) = w$ , which is uniformly bounded as  $m$  tends to  $+\infty$  and which satisfies

$$\|\mathcal{L}_{\tilde{M}_m} \circ G_0 - I\| \leq c (e^{-\gamma_{n+1} m s_\tau} + e^{2\delta m s_\tau}).$$

The result then follows from a simple perturbation argument, provided  $m$  is chosen large enough.  $\square$

### 3.5. The perturbation argument

In this subsection, we construct a constant mean curvature hypersurface by perturbing  $\tilde{M}_m$ .

PROPOSITION 3.9. — *There exists  $m_0 \in \mathbb{N}$  such that, for all  $m \geq m_0$  the hypersurface  $\tilde{M}_m$  can be perturbed into a constant mean curvature 1 hypersurface.*

*Proof.* — As usual, we consider  $M_w$  a normal graph over  $\tilde{M}_m$  for some small function  $w \in \mathcal{D}_\delta^{2,\alpha}(\tilde{M}_m) \oplus \mathcal{K}(\tilde{M}_m)$ . Now, on each end  $E_{i,j}$  of  $\tilde{M}_m$ , when  $j \neq 1$ , we decompose

$$w = v + \chi_{i,j} \sum_{k=0}^n a_{k,\pm} \Phi_{E_{i,j}}^{k,\pm}$$

and consider the normal graph for the function  $v$  over the hypersurface

$$\exp_{E_{i,j}}(\chi_{i,j} \sum_{k=0}^n a_{k,\pm} \Phi_{E_{i,j}}^{k,\pm}).$$

Now, the equation we have to solve reads

$$H(\tilde{M}_m) + \mathcal{L}_{\tilde{M}_m} w + \mathcal{Q}_m(w) = 1, \quad (3.24)$$

where  $\mathcal{L}_{\tilde{M}_m}$  is the Jacobi operator about  $\tilde{M}_m$  and  $\mathcal{Q}_m$  contains all the nonlinear terms resulting from the Taylor expansion of the mean curvature  $H(M_w)$ . We fix  $-\delta < \delta < 0$ . Using the result of Proposition 3.8, our problem reduces to finding a fixed point for

$$w \longrightarrow G_m \left( 1 - H(\tilde{M}_m) - \mathcal{Q}_m(w) \right), \quad (3.25)$$

in the space  $\mathcal{D}_\delta^{2,\alpha}(\tilde{M}_m) \oplus \tilde{\mathcal{K}}(\tilde{M}_m)$ . Using Proposition 3.8 and (3.17), we see that

$$\|G_m(1 - H(\tilde{M}_m))\|_{\mathcal{D}_\delta^{0,\alpha}(\tilde{M}_m)} \leq c e^{-(\gamma_{n+1}(\tau) + \delta) m s_\tau}.$$

Now, we claim that

$$\|G_m(\mathcal{Q}_m(w))\|_{\mathcal{D}_\delta^{0,\alpha}(\tilde{M}_m)} \leq c m^2 e^{-m \delta s_\tau} \|w\|_{\mathcal{C}_\delta^{2,\alpha}(\tilde{M}_m)}^2. \quad (3.26)$$

This follows at once from the fact that the operator  $\mathcal{Q}_m$  is quadratic in  $w$  and from the fact that  $w$  can be decomposed as

$$w = v + \Phi,$$

where  $v \in \mathcal{D}_\delta^{2,\alpha}(\tilde{M}_m)$  and  $\Phi \in \mathcal{K}(\tilde{M}_m)$ . Now, because of the modification of  $\chi_{2,1} \Phi_{E_{2,1}}^{0,-}$  into  $\chi_{2,1} \tilde{\Phi}_{E_{2,1}}^{0,-}$ , we see that on  $M_1$  and away from the

ends  $E_{1,j}$ , the function  $\Phi$  is bounded, in  $\mathcal{C}^{2,\alpha}$  norm, by a constant times  $m \|w\|_{\mathcal{D}_\delta^{2,\alpha}(\tilde{M}_m) \oplus \mathcal{K}(\tilde{M}_m)}$ . Similarly, on each end  $E_{1,j}$ , for  $j \neq 1$ , the function  $\Phi$  is a linear combination of the  $\Phi_{E_{1,j}}^{k,\pm}$ , the coefficients of which are bounded by a constant times  $m \|w\|_{\mathcal{D}_\delta^{2,\alpha}(\tilde{M}_m) \oplus \mathcal{K}(\tilde{M}_m)}$ . Using these observations, we get

$$\|(1 - \chi_2) w\|_{\mathcal{C}^{2,\alpha}(M_1) \oplus \mathcal{K}(M_1)} \leq c m \|w\|_{\mathcal{D}_\delta^{2,\alpha}(\tilde{M}_m) \oplus \mathcal{K}(\tilde{M}_m)}$$

from which we obtain easily the bound

$$\|G_m(\mathcal{Q}_m(w))\|_{\mathcal{C}_\delta^{0,\alpha}(M_1(-m s_\tau))} \leq c m^2 \|w\|_{\mathcal{D}_\delta^{2,\alpha}(\tilde{M}_m) \oplus \mathcal{K}(\tilde{M}_m)}^2.$$

On  $M_2(m s_\tau)$  we simply have

$$\|(1 - \chi_1) w\|_{\mathcal{C}^{2,\alpha}(M_2) \oplus \mathcal{K}(M_2)} \leq c \|w\|_{\mathcal{D}_\delta^{2,\alpha}(\tilde{M}_m) \oplus \mathcal{K}(\tilde{M}_m)}$$

from which we obtain easily the bound

$$\|G_m(\mathcal{Q}_m(w))\|_{\mathcal{C}_\delta^{0,\alpha}(M_2(m s_\tau))} \leq c \|w\|_{\mathcal{D}_\delta^{2,\alpha}(\tilde{M}_m) \oplus \mathcal{K}(\tilde{M}_m)}^2.$$

Next, we evaluate  $\mathcal{Q}_m(w)$  in  $C(m s_\tau)$ . Again the key observation is that the function  $\Phi$  is bounded by a constant times  $m \|w\|_{\mathcal{D}_\delta^{2,\alpha}(\tilde{M}_m) \oplus \mathcal{K}(\tilde{M}_m)}$  on  $C(m s_\tau)$ . Hence,

$$\|w\|_{\mathcal{C}^{2,\alpha}((s,s+1) \times S^{n-1})} \leq c \left( m + \left( \frac{\cosh s}{\cosh(m s_\tau)} \right)^{-\delta} \right) \|w\|_{\mathcal{D}_\delta^{2,\alpha}(\tilde{M}_m) \oplus \mathcal{K}(\tilde{M}_m)}$$

for each  $s \in [-m s_\tau + 1, m s_\tau - 1]$ . And this implies easily that

$$e^{-\delta m s_\tau} \|(\cosh s)^\delta G_m(\mathcal{Q}_m(w))\|_{\mathcal{C}^{0,\alpha}} \leq c m^2 e^{-\delta m s_\tau} \|w\|_{\mathcal{D}_\delta^{2,\alpha}(\tilde{M}_m) \oplus \mathcal{K}(\tilde{M}_m)}^2,$$

on  $(-m s_\tau, m s_\tau) \times S^{n-1}$ . The proof of (3.26) is therefore complete.

Finally, it is not hard to check that

$$\|G_m(\mathcal{Q}_m(w_1) - \mathcal{Q}_m(w_2))\|_{\mathcal{D}_\delta^{2,\alpha}(\tilde{M}_m) \oplus \mathcal{K}(\tilde{M}_m)} \leq \frac{1}{2} \|w_1 - w_2\|_{\mathcal{D}_\delta^{2,\alpha}(\tilde{M}_m) \oplus \mathcal{K}(\tilde{M}_m)}. \quad (3.27)$$

provided  $m$  is chosen large enough. We leave the details to the reader.

The previous analysis shows that, if  $\delta \in (-\tilde{\delta}/2, 0)$ , then, there exists a constant  $c > 0$  and  $m_0 \geq 0$  such that, for all  $m \geq m_0$ , the mapping defined in (3.25) is a contraction from the ball of radius  $c e^{-m(\delta + \gamma_{n+1}) s_\tau}$

in  $\mathcal{D}_\delta^{2,\alpha}(\tilde{M}_m) \oplus \mathcal{K}(\tilde{M}_m)$  into itself. In particular, this mapping has a unique fixed point  $w_m$  in this ball. As explained, the graph of the function  $w_m$  is a constant mean curvature 1 hypersurface which is close to  $\tilde{M}_m$ .  $\square$

*Remark 3.10.* — Two  $n$ -Delaunay hypersurfaces with the same Delaunay parameter will be a trivial candidate for the construction. But, in [6] we construct many other constant mean curvature hypersurfaces which satisfy the gluing hypotheses of Theorem 1.

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