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Giovanni Gaiffi, Michele Grassi

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Lie Algebra bundles on s-Kähler manifolds, with applications to Abelian varieties

Giovanni Gaiffi, Michele Grassi[(1)]

Abstract. — We prove that one can obtain natural bundles of Lie algebras on rank two s-Kähler manifolds, whose fibres are isomorphic respectively to $\mathbf{so}(s+1,s+1)$, $\mathbf{su}(s+1,s+1)$ and $\mathbf{sl}(2s+2,\mathbb{R})$. These bundles have natural flat connections, whose flat global sections generalize the Lefschetz operators of Kähler geometry and act naturally on cohomology. As a first application, we build an irreducible representation of a rational form of $\mathbf{su}(s+1,s+1)$ on (rational) Hodge classes of Abelian varieties with rational period matrix.

Résumé. — Nous prouvons que on peut obtenir fibrés naturels des algèbres de Lie $\mathbf{so}(s+1,s+1)$, $\mathbf{su}(s+1,s+1)$ et $\mathbf{sl}(2s+2,\mathbb{R})$ sur variétés s-Kähler de rang 2. Ces fibrés ont connexions naturelles dont les sections globales généralisent les opérateurs de Lefschetz de la géométrie de Kähler et agissent d'une façon naturelle sur la cohomologie. Pour première application nous construisons une représentation irréductible d'une forme rationnelle de $\mathbf{su}(s+1,s+1)$ sur les classes de Hodge (rationelles) de variétés abéliennes dont la matrice des periodes est rationelle.

1. Introduction

In this paper we prove that one can obtain natural bundles of Lie algebras on rank two s-Kähler manifolds, with fibres isomorphic respectively to $\mathbf{so}(s+1,s+1)$, to $\mathbf{su}(s+1,s+1)$ and to $\mathbf{sl}(2s+2,\mathbb{R})$. These bundles have natural flat connections, whose flat global sections act naturally on cohomology.

An s-Kähler structure is a direct generalization (with s distinct degenerate "Kähler forms") of the notion of Kähler structure, to which it reduces

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(1) Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo, 5, 56125 Pisa – Italy.
gaiffi@dm.unipi.it, grassi@dm.unipi.it

when $s = 1$. The original motivation for the introduction in [G1] of s-Kähler manifolds (and almost s-Kähler manifolds, which are geometrically less rigid) was the geometric study of the analytical theory of maps from (open subsets of) $\mathbb{R}^s$ to a given manifold. Then it was realized that this theory in the case $s = 2$ is well suited for the study of Mirror Symmetry (see [G2],[G3]). Starting from [GG1] we decided to embark in a systematic study of the natural algebraic structures arising from this geometry.

In the paper [GG2] we found a natural Lie superalgebra bundle on rank three (almost) 2-Kähler manifolds. In [G2] it was conjectured that precisely this type of bundles could provide the natural background on which to build Field Theories; these are rich from the representation theoretic point of view and, once quantized using the language of [G1], were conjectured to be the right playing field for the search of an M-theory (see for example [DOPW] for a similar approach to the Standard Model in particle physics, and [CDGP], [SYZ] and [KS] for some background on T-duality and its mathematical implications). More specifically, the families of "toric-type" compact manifolds $\mathbb{X}^m_{k_1,k_2}$ of [G3] (see Definition 3.11 on page 11 of that paper) interpolate between mirror dual Calabi-Yau manifolds on the boundary of their deformation space. To these manifolds, when $m = 3$ one can apply the results of [GG2], while when $m = 2$ one can apply the constructions of the present paper.

Another application of the algebraic bundles on s-Kähler manifolds is to Higgs bundles and Hitchin systems (see for example [HT] and also [BMP] and [KS]), which follows from the existence of semi-flat tori fibrations.

Beyond their intrinsic algebraic interest, the results which we obtain are directly applicable to questions in Complex Geometry and in Algebraic Geometry. The basic reason for this is that a rank two s-Kähler manifold (or a naturally defined double cover of it, in some cases) has a canonical complex structure, with which it becomes Kähler of complex dimension $s + 1$. We prove that on these Kähler manifolds originating from s-Kähler geometry there are natural bundles of unitary and orthogonal Lie algebras of signature $(s+1, s+1)$, which act in various ways on differential forms. It is this natural way of representing "large" and well known unitary Lie algebras on differential forms which opens a wide range of geometric applications. For comparison, one should recall that the corresponding constructions for plain Kähler manifolds produce the "Lefschetz" action of $\mathbf{sl}(2, \mathbb{C})$ on forms and on cohomology, which has a lot of geometric applications and consequences.

The spirit of our investigations is similar to the one of Verbitsky (see e.g. [V1],[V2]), who builds Lefschetz type operators, Lie algebra representations and natural bundles on hyperkähler manifolds. Although the similarity is inspiring, notice that s-Kähler manifolds are $2s+2$ dimensional and hyperkähler only in very special cases, and our structure forms are never symplectic. The algebras that one obtains are as one would expect different: for example our operators L_{jk}, Λ_{jk} (see Definition 1.2 below) when $s=2$ generate at every point an $\mathbf{so}(3,3)$, while the algebra generated by the Verbitsky operators L_i, Λ_j (see [V1] pag. 229) is an $\mathbf{so}(1,4)$.

Let us now introduce more in detail the geometric and algebraic characters which will play a role.

Definition 1.1 ([G1], Definition 7.2). — *A smooth manifold X of dimension $r(s+1)+c$ together with a Riemannian metric $\mathbf{g}$ and 2-forms $\omega_1,...,\omega_s$ is s-*Kähler *(of rank r) if for each point of X there exist an open neighborhood $\mathcal{U}$ of p and a system of coordinates x_i, y_i^j, z_k, $i = 1,...,r$, $j = 1,...,s$, $k = 1,..,c$ on $\mathcal{U}$ such that:*

1) $\forall j \;\; \omega_j = \sum_{i=1}^r dx_i \wedge dy_i^j$,

2) $\mathbf{g}_{(\mathbf{x},\mathbf{y})} = \sum_{i=1}^r dx_i \otimes dx_i + \sum_{i,j} dy_i^j \otimes dy_i^j + \sum_{k=1}^c dz_k \otimes dz_k + \mathbf{O}(2)$.

The coordinates $x_1,...,z_c$ are standard s-Kähler coordinates *in a neighborhood of p and the forms $(dx_1)_p,\dots,(dx_s)_p,(dy_1^1)_p,\dots,(dy_s^r)_p,(dz_1)_p,\dots,(dz_c)_p$ are an* adapted coframe *at p. The forms $\omega_1,...,\omega_s$ are the* structure forms.

For $s=1$ and $c=0$ one recovers the usual notion of Kähler manifold. As in the case of Kähler manifolds or hyperkähler manifolds, one can use the differential forms associated to the structure to build "wedge" operators on forms, and, using their adjoints, one gets natural Lie algebras. When $s=1$ one obtains the classical $\mathbf{sl}(2,\mathbb{C})$ action on the forms of a Kähler manifold (and on its cohomology using the Hodge identities). In the case $s>1$ there is a qualitatively different situation, in that there are more natural differential two-forms than one could initially guess. Indeed, in addition to the structural forms $\omega_1,...,\omega_s$ which generalize directly the Kähler form, there are also "mixed" forms ω_{jk} for any pair of distinct indices $j,k \in \{0,...,s\}$, including the structural ones via the identifications

$$\omega_j = \omega_{0j} \quad \text{for } j \in \{1,...,s\}$$

The precise description of these derived natural forms will be given in the next section. Here however we can already use them to build corresponding "wedge" operators, Lefschetz style:

Definition 1.2. — *For $\phi \in \Omega^*_{\mathbb{C}}X$ and $j,k \in \{0,\dots s\}$ with $j \neq k$,*

$$L_{jk}(\phi) = \omega_{jk} \wedge \phi = -L_{kj}(\phi)$$

Some canonical mutually orthogonal distributions W_i ($i = 0, 1, 2, \dots, s$) are induced on T_p^*X by the forms ω_{jk} (see Section 2). Therefore other natural operators, called V_i ($i = 0, 1, 2, \dots, s$) arise from wedging with the local volume forms of these distributions.

One then uses all these operators, and their (pointwise) adjoints, to build a natural bundle of real Lie algebras $\mathcal{L}_{\mathbb{R}}^s$ (and its complexified bundle $\mathcal{L}_{\mathbb{C}}^s$) acting on forms. The global sections of these bundles (which are flat with respect to the naturally induced flat connections) act naturally on cohomology. We use this construction in the rational case to build an action of a rational form of $\mathbf{su}(s+1, s+1)$ on rational Hodge classes of Abelian varieties with rational period matrix.

Coming to a more detailed description of the contents of the present paper, in Section 2 we give the definiton of *almost* $s-$Kähler structure, which is a weaker version of the defintion of $s-$Kähler structure. Then we provide a first geometric description of an (almost) $s-$Kähler manifold X: we discuss the existence of an (almost) complex structure, the natural distributions on the cotangent space, the group of local structure-preserving tranformations and the orientability properties.

Section 3 is devoted to the definition of the bundles of Lie algebras $\mathcal{L}_{\mathbb{R}}^s$ and $\mathcal{L}_{\mathbb{C}}^s$ on X which constitute the main object to be studied in this paper. We also point out two other real forms ($\mathbf{s}\mathcal{L}^s, \mathbf{u}\mathcal{L}^s$), defined in terms of geometric generators, of the bundle $\mathcal{L}_{\mathbb{C}}^s$. We then define Lef^s as the real sub-bundle of $\mathcal{L}_{\mathbb{C}}^s$ which is the direct (real) generalization of the classical $\mathbf{sl}(2, \mathbb{C})$ Lefschetz bundle of Kähler geometry.

The sections from 4 to 7 are a detailed study of the fibres of above mentioned Lie algebra bundles: in Section 4 the Lefschetz bundle Lef^s is studied in detail, by showing some fundamental relations among its generators; the fibres of the bundle turn out to be isomorphic to the orthogonal algebra $\mathbf{so}(s+1, s+1, \mathbb{R})$ and Serre generators are presented in terms of simple brackets of geometric generators (Theorem 4.5).

In sections 5 and 6 we describe completely the main complex bundle $\mathcal{L}_{\mathbb{C}}^s$: we use the Hodge decomposition on ${\bigwedge^*}_{\mathbb{C}} T_p^*X$ with respect to the (almost) complex structure and Clifford algebra techniques to show that the fibres of $\mathcal{L}_{\mathbb{C}}^s$ are isomorphic to $\mathbf{sl}(2s+2, \mathbb{C})$; furthermore, we characterize $\mathcal{L}_{\mathbb{C}}^s$ as the bundle of all quadratic elements of trace zero (compatible with the almost complex structure) of a Clifford algebra bundle (Theorem 6.2). In the second part of Section 6 we describe completely also the reals forms $\mathcal{L}_{\mathbb{R}}^s$ and $\mathbf{s}\mathcal{L}^s$. In particular we show that the fibres of $\mathcal{L}_{\mathbb{R}}^s$ are unitary Lie algebras isomorphic to $\mathbf{su}(s+1, s+1)$. We observe that the complete description of $\mathcal{L}_{\mathbb{R}}^s$ fully answers to the question (first rised in [G1] and then more precisely in the rank two case in [GG1]) on the nature of the algebraic bundles generated by the real canonical operators associated to an s-Kähler structure.

In Section 7 $\mathcal{L}^s_{\mathbb{R}}$ and $\mathbf{u}\mathcal{L}^s$ are shown to be the real bundles of operators which preserve two natural non degenerate hermitean inner products on $\bigwedge^*{}_{\mathbb{C}}T^*_pX$ (Theorems 7.4 and 7.6) . A superHermitean variant of one of these inner products was introduced in [GG2] to study rank three WSD structures.

In Section 8 we show how one can pass from an action on forms to an action on cohomology when the manifold is s-Kähler . In Section 9 we put many nonequivalent s-Kähler structures on complex tori. When the period matrix is rational we build in particular rational s-Kähler structures. We then find a representation of a rational form of $\mathbf{su}(s+1, s+1)$ on rational Hodge classed; this representation is then shown to be irreducible in Theorem 10.2 of Section 10.

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2. Introduction to the geometric setting

This section is again introductory in nature, but with a stronger emphasis on the geometric aspects of the theory. First, as will have been clear already to the reader, one can isolate the pointwise aspects of the definition of an s-Kähler manifold. The notion of almost s-Kähler manifold given below is actually a hybrid between pointwise and local properties, which was introduced in [G1] (in the nondegenerate case):

Definition 2.1. — *A smooth manifold X of dimension $r(s+1)+c$ together with a Riemannian metric $\mathbf{g}$ and 2-forms $\omega_1, ..., \omega_s$ is* almost s-Kähler *(of rank r) if for each point of X there exist an open neighborhood $\mathcal{U}$ of p and a system of coordinates x_i, y_i^j, z_k, $i = 1, ..., r$, $j = 1, ..., s$, $k = 1, .., c$ on $\mathcal{U}$ such that:*

1) $\forall j \;\; \omega_j = \sum_{i=1}^r dx_i \wedge dy_i^j$,

2′) $\mathbf{g}_p = \left(\sum_{i=1}^r dx_i \otimes dx_i + \sum_{i,j} dy_i^j \otimes dy_i^j + \sum_{k=1}^c dz_k \otimes dz_k\right)_p$.

The forms $(dx_1)_p, \ldots, (dx_s)_p, (dy_1^1)_p, \ldots, (dy_s^r)_p, (dz_1)_p, \ldots, (dz_c)_p$ are an adapted coframe *at p. The forms $\omega_1, ..., \omega_s$ are the* structure forms.

Definition 2.2. — *Associated to an almost s-Kähler structure of rank r on a smooth manifold X there are smooth two-forms ω_{jk} for any pair of* *distinct indices* $j,k \in \{0\dots s\}$. *We have* $\omega_1 = \omega_{01}, ..., \omega_s = \omega_{0s}$ *and for any point* $p \in X$ *and any adapted coframe* $v_{01}, \dots, v_{0s}, v_{11}, \dots, v_{rs}, u_1, ..., u_c$ *at* p

$$\omega_{jk} = \sum_{i=1}^{r} v_{ij} \wedge v_{ik}$$

The forms ω_{jk} *with* $j,k \in \{1,...,s\}$ *are called* dualizing forms.

Remark 2.3. — It is easily verified that the dualizing forms do not depend on the particular adapted coframe used to define them (see for example [G1]).

Remark 2.4. — Since condition 2) of Definition 1.1 implies condition 2′) of Definition 2.1, s-Kähler implies almost s-Kähler .

Clearly there are some redundancies in the definition given above: for example, one has always $\omega_{jk} = -\omega_{kj}$. Observe also that an almost 1-Kähler manifold is simply an almost Kähler manifold, and for this reason in this paper we consider only the case $s \geqslant 2$ which is moreover the range where our constructions do exist. Recall also that an almost 2-Kähler manifold in which the structure forms are closed is a Weakly Self Dual manifold (see [G2], Definition 2.6 and [G3], where one of the authors built families of compact WSD manifolds of toric type by a quotient construction). In this respect, in particular a 2-Kähler manifold is WSD. Notice that the term "WSD" has been used by various authors (e.g. Apostolov et al.) in a very different context.

The almost s-Kähler structure on a manifold X splits its cotangent space as $T_p^*X = W_0 \oplus W_1 \oplus \cdots \oplus W_s$ where the W_j are $s+1$ mutually orthogonal canonical distributions defined as:

$$W_j = \{\phi \in T_p^*X \mid \phi \wedge \omega_{jk} = 0 \ \text{ for } k \text{ in } 0,..,\hat{j},..,s\}$$

The almost s-Kähler structure also determines canonical pairwise linear identifications among the spaces W_j, so that one can also write $T_p^*X \cong W_0 \otimes_{\mathbb{R}} \mathbb{R}^{s+1}$ or more simply

$$T_p^*X \cong W \otimes_{\mathbb{R}} \mathbb{R}^{s+1}$$

where $W = W_0 \cong W_1 \cong \cdots \cong W_s$.

Let us now come back to the canonical operators L_{jk} mentioned in the Introduction.

We now choose an orientaion of W_0 at a fixed point $p \in X$, and a (non-canonical) orthonormal basis γ_1, γ_2 compatible with this orientation;

this together with the standard identifications of the W_j determines an orientation and an orthonormal basis for T_p^*X, which we write as $\{v_{ij} = \gamma_i \otimes e_j \mid i = 1,2,\ j = 0,..,s\}$. We remark that the v_{ij} are an *adapted coframe* for the almost s-Kähler structure as defined above, and therefore we have the explicit expressions:

$$\omega_{jk} = v_{1j} \wedge v_{1k} + v_{2j} \wedge v_{2k}$$

A different choice of the γ_1, γ_2 would be related to the previous one by an element in $\mathbf{O}(2,\mathbb{R})$. The Lie algebra of the group $\mathbf{O}(2,\mathbb{R})$ expressing the change from one adapted basis to another is generated point by point by the operator J, which is determined and determines a (pointwise, local or global if possible) orientation of the distribution W_0:

Definition 2.5. — *The operator $J \in End_{\mathbb{R}}(\bigwedge^* T_p^*(X))$ associated to the standard basis v_{ij} is defined as*

$$J(v_{1j}) = v_{2j}, \qquad J(v_{2j}) = -v_{1j} \qquad \text{for } j \in \{0,1,\ldots,s\}$$

and $J(v \wedge w) = J(v) \wedge w + v \wedge J(w)$ for $v, w \in \Lambda^* T_p^* X$

Remark 2.6. — As J commutes with itself, and it is determined at every point $p \in X$ by an orientation of $(W_0)_p \subset T_pX$, it is always well defined locally. Of course, J admits a global determination if and only if W_0 admits a global orientation. This happens for example if X is orientable and s is even.

Whenever we will need a local volume form on X, we will use the one induced by a local choice of J which we will call Ω_p over the point $p \in X$.

From the above considerations one gets the following fundamental remark:

Remark 2.7. — An (almost) s-Kähler manifold of rank 2 is in particular an (almost) complex manifold of complex dimension s+1, when there is a global determination of J. This happens in particular when X is orientable and s is even.

For this reason, rank two (almost) s-Kähler manifolds can be seen as a chapter in (almost) complex geometry. This allows on one hand to "import" the techniques of complex geometry, and on the other hand allows one to apply the results of rank two almost s-Kähler geometry to the complex world.

The following direct construction of almost s-Kähler manifolds is used in [G3] and [GG2] for the applications of the theory to Mirror Symmetry and mathematical physics. Let M be a smooth Riemannian manifold with metric $\mathbf{h}$, and let

$$X = \underbrace{T^*M \otimes_M \cdots \otimes_M T^*M}_{s \text{ times}}$$

The manifold X is naturally almost s-Kähler of rank $dim(M)$. The metric $\mathbf{h}$ induces naturally a Riemannian metric $\mathbf{g}$ on X and the differential forms ω_{jk} come in two sets, with different constructions: the ones in which j or k is equal to zero and the other ones. If π_j is the natural projection from X to the j-th copy of T^*M, then $\omega_{0j} = \pi_j^* \omega_{st}$ where ω_{st} is the standard symplectic form on the cotangent bundle T^*M. Observe then that using the Levi-Civita connection associated to the metric (induced by $\mathbf{g}$ on the cotangent bundle of M) we have a natural identification at any point $Q = (p, \phi_1, ..., \phi_s) \in X$

$$T_Q X \cong T_p M \oplus \underbrace{T_p^* M \oplus \cdots \oplus T_p^* M}_{s \text{ times}}$$

Let us call $W_{jk} \subset T_Q X$ the direct sum of the j^{th} and of the k^{th} summands among the copies of $T_p^* M$ in the identification above:

$$T_p^* M \oplus T_p^* M \cong W_{jk} \subset T_Q X$$

Using the metric, we can define $\omega_{ij} \in \bigwedge^2 T_Q^* X$ simply by defining a natural element in $\bigwedge^2 W_{jk}^*$. To do so, it is enough to observe that the identity (bundle) map from $T_p M$ to itself is an element $Id \in T_p^* M \otimes T_p M$. This space is naturally isomorphic (using the metric $\mathbf{h}$) to $T_p M \otimes T_p M$ and this last space maps naturally to

$$\bigwedge^2 (T_p M \oplus T_p M) \cong \bigwedge^2 W_{jk}^* \subset \bigwedge^2 T_Q^* X$$

where the last inclusion is again induced by the use of the metric. For more details see [G1] (Example 2.3), [G3] and [GG2].

When the structure is s-Kähler , one has that all the structure forms are covariant constant with respect to the Levi-Civita connection associated to the metric. This allows one to perform many of the same constructions that one usually performs in the Kähler case. In particular, one recovers (the analog of) the Hodge identities, and the adjoints of the canonical operators L_{jk} operate on cohomology (see Theorem 8.1 and Corollary 8.2). This is the context in the case of Abelian varieties, which in our opinion will provide many interesting applications of the constructions to be detailed in the present paper.

For a general rank of the structure $r \geqslant 1$, many of the above considerations generalize; for example the group of pointwise transformations which preserve the structure is $\mathbf{O}(r)$. As we have seen above, in the $r = 2$ case we obtain $\mathbf{O}(2)$ whose algebra is generated by J, while the $r = 3$ case (in which comes into play $\mathbf{O}(3)$) was discussed in detail in [GG2]. Clearly however, not everything generalizes to arbitrary rank: for example, a rank three s-Kähler manifold may be of (real) dimension 9, which is odd and therefore it is impossible to have an almost complex structure on such a manifold. Still in case $r = 3$, one has natural operators also in odd degree, and therefore the natural algebras which come out of the geometry are Lie superalgebras, instead of Lie algebras (see [GG2]).

3. Construction of the natural algebras

In this section we fix a point p in an almost s-Kähler manifold X and we mostly work on tensor powers of T_pX.

As was mentioned in the previous sections, using the forms ω_{jk} of the almost s-Kähler structure, we can build corresponding operators on forms, much in the way as the L operator is built on Kähler manifolds:

Definition 1.2. — *For $\phi \in \Omega^*_{\mathbb{C}}X$ and $j, k \in \{0, \dots s\}$ with $j \neq k$,*

$$L_{jk}(\phi) \;=\; \omega_{jk} \wedge \phi \;=\; -\,L_{kj}(\phi)$$

The above operators restrict also to $\bigwedge^* T^*_pX$ for any $p \in X$ where, using the chosen (orthonormal) basis, one can define also corresponding (non canonical) wedge and contraction operators:

Definition 3.1. — *Let $i \in \{1, 2\}$, $j \in \{0, 1, \dots, s\}$ and $p \in X$. The operators E_{ij} and I_{ij} are respectively the wedge and the contraction operator with the form v_{ij} on $\bigwedge^* T^*_pX$ (defined using the given basis); we use the notation $\frac{\partial}{\partial v_{ij}}$ to indicate the element of T_pX dual to $v_{ij} \in T^*_pX$:*

$$E_{ij}(\phi) = v_{ij} \wedge \phi, \qquad I_{ij}(\phi) = \frac{\partial}{\partial v_{ij}} \rightharpoonup \phi$$

Proposition 3.2. — *The operators E_{ij}, I_{ij} satisfy the following relations:*

$$\forall i, j, k, l \qquad E_{ij}E_{kl} = -E_{kl}E_{ij}, \quad I_{ij}I_{kl} = -I_{kl}I_{ij}$$

$$\forall i,j \qquad E_{ij}I_{ij} + I_{ij}E_{ij} = Id$$
$$\forall (i,j) \neq (k,l) \qquad E_{ij}I_{kl} = -I_{kl}E_{ij}$$
$$\forall i,j \qquad E_{ij}^* = I_{ij}, \quad I_{ij}^* = E_{ij}$$

where $*$ is adjunction with respect to the metric.

Proof. — The proof is a simple direct verification, which we omit. □

It is then immediate to check that:

PROPOSITION 3.3. — *J can be expressed on the whole $\bigwedge^* T_p^*X$ as*

$$J = \sum_{j=0}^{s} (E_{2j}I_{1j} - E_{1j}I_{2j})$$

Remark 3.4. — From this expression and the previous proposition one obtains that $J^* = -J$, i.e. for every p the Lie algebra generated by J is a subalgebra of $\mathbf{o}(\bigwedge^* T_p^*X)$ isomorphic to $\mathbf{so}(2, \mathbb{R}) \cong \mathbb{R}$.

Using the (non canonical) operators E_{ij} we can obtain simple expressions for the pointwise action of the canonical wedge operators V_j associated to the volume forms of the distributions W_j:

DEFINITION 3.5. — *For $\phi \in \bigwedge^* T_p^*X$ and $j \in \{0, \ldots, s\}$,*

$$V_j(\phi) = E_{1j}E_{2j}(\phi)$$

Remember however that the operators V_j, being simply multiplication by the volume forms of the spaces W_j, depend on the choice of a pointwise orientation for these spaces, which is implied for example by the choice of a determination for the operator J. Notice that when s is even, and X is oriented, it is always possible to define J (and consequently V_j) globally on X. On the opposite extreme situation, if X is non-orientable, it is certainly not possible to orient globally any one of the distributions W_j (and a fortiori you cannot determine J globally).

The Riemannian metric induces a Riemannian metric on T_p^*X and on the space $\bigwedge^* T_p^*X$.

DEFINITION 3.6. — *For $j \neq k \in \{0, 1, \ldots, s\}$*

$$\Lambda_{jk} = L_{jk}^*, \qquad A_j = V_j^*$$

By construction the canonical operators L_{jk}, Λ_{jk} on $\bigwedge^* T_p^* X$ are the pointwise restrictions of corresponding global operators on smooth differential forms, which we indicate with the same symbols: for $j \neq k \in \{0, 1, \ldots, s\}$,

$$L_{jk}, \Lambda_{jk} : \Omega^*(X) \to \Omega^*(X)$$

In the study of Kähler geometry, a central role is played by the Lie algebra generated by Lefschetz operator and its adjoint. The direct generalization of that algebra to the setting of (almost) s-Kähler manifolds is the following:

DEFINITION 3.7. — *The smooth bundle of Lie algebras Lef^s is the real sub-bundle of Lie algebras of $End_{\mathbb{R}}(\Omega^*(X))$ generated locally by the operators*

$$\{L_{jk}, \Lambda_{jk} \mid \text{for} \quad j = 0, 1, \ldots, s\}$$

The V_j, A_j instead can be always determined locally via a local determination of the operator J even when s is odd. Summing up:

DEFINITION 3.8. — *The smooth bundle of Lie algebras $\mathcal{L}^s_{\mathbb{R}}$ is the real sub-bundle of Lie algebras of $End_{\mathbb{R}}(\Omega^*(X))$ generated locally by the operators*

$$\{L_{jk}, V_j, \Lambda_{jk}, A_j \mid \text{for} \quad j = 0, 1, \ldots, s\}$$

for any fixed determination of J. The $*$-Lie algebra $\mathcal{L}^s_{\mathbb{C}}$ is $\mathcal{L}^s_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. The $*$ operator on $\mathcal{L}^s_{\mathbb{C}}$ is induced by the adjoint with respect to the Hermitean metric induced by the Riemannian one via complexification.

As mentioned in the Introduction, in the present paper we will describe completely the structure of the fibers of the bundles $Lef^s, \mathcal{L}^s_{\mathbb{R}}, \mathcal{L}^s_{\mathbb{C}}$, and we will further describe two other real forms of $\mathcal{L}^s_{\mathbb{C}}$, which are especially significant from a geometric point of view. Here are their definitions:

DEFINITION 3.9. — *The real form $\mathbf{s}\mathcal{L}^s$ of the complex bundle of $*$-Lie algebras $\mathcal{L}^s_{\mathbb{C}}$ is generated (as a bundle of real Lie algebras) by the local operators:*

$$L_{jk}, \ \imath V_j, \ \Lambda_{jk}, \ \imath A_j$$

DEFINITION 3.10. — *The real form $\mathbf{u}\mathcal{L}^s$ of the complex bundle of $*$-Lie algebra $\mathcal{L}^s_{\mathbb{C}}$ is generated (as a bundle of real Lie algebras) by the local operators:*

$$\imath L_{jk}, \ \imath V_j, \ i\Lambda_{jk}, \ \imath A_j$$

4. Clifford algebras and a natural presentation of Lef^s as a $\mathbf{so}(s+1, s+1, \mathbb{R})$ bundle

In this section we will show that $\mathcal{L}^s_{\mathbb{R}}$ lies inside a (real) Clifford algebra bundle over the $(4s+4)$-dimensional real bundle $TX \oplus T^*X$; we will also point out that the natural bundle of Lie subalgebras $Lef^s \subset \mathcal{L}^s_{\mathbb{R}}$ is isomorphic to the constant bundle having as fibre the orthogonal algebras $\mathbf{so}(s+1, s+1, \mathbb{R})$. Notice that the above considerations do not apply to the $s = 1$ (Kähler) situation; Lef^s in that case is simply a constant $\mathbf{sl}(2, \mathbb{R})$ bundle, as it is well know classically. Notice also that this global trivialization of Lef^s does not depend on a determination of the (almost) complex structure J.

In the following we define some new operators, and in the meantime we introduce a unifying notation which concerns the L_{jk}, Λ_{jk}. These operators will be shown in Corollary 4.3 to be (global) sections of $\mathcal{L}^s_{\mathbb{R}}$.

Definition 4.1. — *For $j, k \in \{0, \ldots, s\}$*

$$L_{jk} = \sum_{i=1}^{2} E_{ij} E_{ik} \qquad L_{j\bar{k}} = \sum_{i=1}^{2} E_{ij} I_{ik}$$

$$L_{\bar{k}\bar{j}} = \Lambda_{jk} = \sum_{i=1}^{2} I_{ik} I_{ij} \qquad L_{\bar{j}k} = \sum_{i=1}^{2} I_{ij} E_{ik}$$

In accordance with the notation introduced in [G2] Section 7, we will use the shortcuts $L_{\alpha\beta}$ with $\alpha, \beta \in \{0, \ldots, s, \overline{0}, \ldots, \overline{s}\}$, with the convention that $\overline{\overline{\alpha}} = \alpha$.

Notice that with the above notation $L_{\alpha\alpha} = 0$ for any $\alpha \in \{0, \ldots, s, \overline{0}, \ldots, \overline{s}\}$.

Lemma 4.2. — *Given $\alpha, \beta, \gamma \in \{0, \ldots, s, \overline{0}, \ldots, \overline{s}\}$ with $\alpha \neq \beta, \alpha \neq \overline{\gamma}, \gamma \neq \overline{\beta}$:*

$$[L_{\alpha\beta}, L_{\overline{\beta}\gamma}] = L_{\alpha\gamma}$$

Given $\alpha \neq \beta \in \{0, \ldots, s, \overline{0}, \ldots, \overline{s}\}$:

$$[L_{\alpha\beta}, L_{\overline{\beta}\,\overline{\alpha}}] = L_{\alpha\overline{\alpha}} + L_{\beta\overline{\beta}}$$

Given $\alpha, \beta, \gamma, \delta \in \{0, \ldots, s, \overline{0}, \ldots, \overline{s}\}$ with $\{\alpha, \beta\} \cap \{\overline{\gamma}, \overline{\delta}\} = \emptyset$:

$$[L_{\alpha\beta}, L_{\gamma\delta}] = 0$$

Proof. — We prove the first relations with $\alpha = i, \beta = j, \gamma = \overline{k}$. The other cases are proved similarly. The third set of relations is straightforward due to the anticommutativity of the degree one operators which appear in the expressions of $L_{\alpha\beta}, L_{\gamma\delta}$.

For the first set of relations, a direct computation which is based on the fundamental relations 3.2 among the operators E_{ij} and I_{rs} proves:

$$[L_{ij}, L_{\overline{j}\,\overline{k}}] = \sum_r E_{ri}E_{rj} \sum_s I_{sj}I_{sk} - \sum_s I_{sj}I_{sk} \sum_r E_{ri}E_{rj} =$$

$$= \sum_r E_{ri}E_{rj} \sum_s I_{sj}I_{sk} - \sum_{s\neq r} E_{ri}E_{rj}I_{sj}I_{sk} - \sum_{s\ (s=r)} I_{sj}I_{sk}E_{si}E_{sj} =$$

$$= \sum_r E_{ri}E_{rj} \sum_s I_{sj}I_{sk} - \sum_{s\neq r} E_{ri}E_{rj}I_{sj}I_{sk} + \sum_{s\ (s=r)} E_{si}I_{sj}E_{sj}I_{sk} =$$

$$= \sum_r E_{ri}E_{rj} \sum_s I_{sj}I_{sk} - \sum_{s\neq r} E_{ri}E_{rj}I_{sj}I_{sk} + \sum_s E_{si}I_{sk} - \sum_s E_{si}E_{sj}I_{sj}I_{sk} =$$

$$= \sum_s E_{si}I_{sk} = L_{i\,\overline{k}}$$

□

Corollary 4.3. — *Given any choice of indices $j \neq k$, the elements $L_{\overline{j}\,k}, L_{j\,\overline{k}}$ belong to $\Gamma(X, Lef^s) \subset \Gamma(X, \mathcal{L}^s_{\mathbb{R}} \cap \mathbf{s}\mathcal{L}^s)$. Furthermore, for every $j = 0, 1, 2, \ldots, s$, the elements $L_{j\,\overline{j}}$ belong to $\Gamma(X, Lef^s) \subset \Gamma(X, \mathcal{L}^s_{\mathbb{R}} \cap \mathbf{s}\mathcal{L}^s)$.*

Proof.— For any fixed $p \in X$, the values of the elements L_{jk} and $L_{\overline{j}\,\overline{k}}$ at p are (maybe up to a sign) among the generators of the fibre of $\mathcal{L}^s_{\mathbb{R}} \cap \mathbf{s}\mathcal{L}^s$ at p. To show that $L_{j\,\overline{k}}$ is a section of $\mathcal{L}^s_{\mathbb{R}} \cap \mathbf{s}\mathcal{L}^s$ we notice that, since $s \geqslant 2$, we can find an index $i \in \{0, 1, 2, \ldots, s\}$ which is different from both j and k. Then we can use the lemma above and construct $L_{j\,\overline{k}}$ as:

$$[L_{ji}, L_{\overline{i}\,\overline{k}}] = L_{j\,\overline{k}}$$

The element $L_{\overline{j}\,k}$ is equal to $-L^*_{j\,\overline{k}}$ and therefore also is a section of $\mathcal{L}^s_{\mathbb{R}} \cap \mathbf{s}\mathcal{L}^s$. As for the last assertion, it follows from the first one and the fact that (according to the above lemma) $[L_{ij}, L_{\overline{j}\,\overline{i}}] = L_{i\,\overline{i}} + L_{j\,\overline{j}}$ and $[L_{i\,\overline{j}}, L_{j\,\overline{i}}] = L_{i\,\overline{i}} - L_{j\,\overline{j}}$. □

The operators defined below give rise to a set of Serre generators for $\Gamma(X, Lef^s)$, as shown in the following Theorem.

Definition 4.4. — *Let us define:*

$e_1 = L_{1\overline{0}},\ e_2 = L_{2\overline{1}},\ e_3 = L_{3\overline{2}},\ \ldots.., e_{s-1} = L_{s-1\overline{s-2}},\ e_s = L_{s\overline{s-1}},\ e_{s+1} = L_{\overline{s-1}\overline{s}}$

Moreover, for every $i = 1, 2, \ldots, s+1$, let f_i be the adjoint of e_i.

Theorem 4.5. — *The global operators* e_i, f_j *and* $h_i = [e_i, f_i]$ *restrict to a set of Serre generators of* Lef^s_p *for any* $p \in X$, *and* Lef^s *is (canonically) a trivial Lie algebra bundle with fibre isomorphic to* $\mathbf{so}(s+1, s+1, \mathbb{R})$.

Proof. — From the previous corollary, the global operators e_i , f_j and $h_i = [e_i, f_i]$ are sections of Lef^s. It is immediate, using Lemma 4.2, to check that these elements are also enough to produce a set of linear generators of Lef^s_p. We are left with the verification of the Serre relations for a root system of type $\mathbf{D}_{s+1}$. We consider a basis of simple roots $\alpha_1, \alpha_2, \ldots, \alpha_{s-2}, \alpha_{s-1}, \alpha_s$, α_{s+1} indexed according to the labelled Dynkin diagram of type $\mathbf{D}_{s+1}$, where the bifurcation node is associated to α_{s-1} and $\alpha_1, ..., \alpha_s$ are consecutive roots forming a diagram of type $\mathbf{A}_{s-1}$. The Serre relations are actually all consequence of Lemma 4.2. For instance, $[h_1, e_1] = 2e_1$ follows from the observation that $h_1 = [L_{1,\overline{0}}, L_{0,\overline{1}}] = L_{1,\overline{1}} - L_{0,\overline{0}}$ and then

$$[h_1, e_1] = [L_{1,\overline{1}} - L_{0,\overline{0}}, L_{1,\overline{0}}] = L_{1,\overline{0}} - [L_{0,\overline{0}}, L_{1,\overline{0}}] = 2L_{1,\overline{0}} = 2e_1$$

Two other examples:

$$[h_{s+1}, e_s] = [[L_{\overline{s-1}\overline{s}}, L_{s(s-1)}], L_{s\overline{s-1}}] = -[L_{s-1\overline{s-1}} + L_{s\overline{s}}, L_{s\overline{s-1}}] = L_{s\overline{s-1}} - L_{s\overline{s-1}} = 0$$

$$[h_{s+1}, e_{s-1}] = -[L_{s-1\overline{s-1}} + L_{s,\overline{s}}, L_{s-1\overline{s-2}}] = -[L_{s-1\overline{s-1}}, L_{s-1\overline{s-2}}] - 0$$

$$= -L_{s-1\overline{s-2}} = -e_{s-1}$$

□

Remark 4.6. — We notice that Theorem 4.5 is in accordance with [GG1] and [GG2] where the specialization of these computations to the case of WSD manifolds of rank two and three led us to the description of a natural subalgebra isomorphic to $\mathbf{sl}(4, \mathbb{R}) \cong \mathbf{so}(2, 2, \mathbb{R})$.

An alternative interpretation of the relations in Lemma 4.2 and of the appearance of $\mathbf{D}_{s+1}$ is thruough the use of two different Clifford Algebras, which will play a prominent role in the rest of this paper. For the first one, generalizing to arbitrary s the $s = 2$ case considered in [GG1], we define:

Definition 4.7. — *For* $p \in X$, *the Clifford algebra* $\mathcal{C}_p$ *is*

$$\mathcal{C}_p = Cl(T_pX \oplus T^*_pX, q)$$

with the quadratic form q *induced by the metric*

$$\begin{array}{ll} \forall i, j, h, k & < v_{ij}, v_{hk} >= 0 \\ \forall i, j, h, k & < \frac{\partial}{\partial v_{ij}}, \frac{\partial}{\partial v_{hk}} >= 0 \\ \forall (i, j) \neq (h, k) & < v_{ij}, \frac{\partial}{\partial v_{hk}} >= 0 \\ \forall i, j & < v_{ij}, \frac{\partial}{\partial v_{ij}} >= -\frac{1}{2} \end{array}$$

Remark 4.8. — The Clifford algebras $\mathcal{C}_p$ for varying p define a Clifford bundle $\mathcal{C}$ on X, as the definition of $\mathcal{C}_p$ is independent on the choice of a basis. Indeed, the quadratic form used to define it is simply induced by $-\frac{1}{2}$ times the natural bilinear pairing $T_pX \otimes T_p^*X \to \mathbb{R}$.

PROPOSITION 4.9. — *The Clifford algebra $\mathcal{C}_p$ has a canonical representation ρ_p on $\bigwedge T_p^*X$, induced by the wedge and contraction operators E_{ij} and I_{ij} via the map*

$$\rho_p(v_{ij}) = E_{ij}, \qquad \rho_p\left(\frac{\partial}{\partial v_{ij}}\right) = I_{ij}$$

Proof. — The Clifford relations

$$\phi\psi + \psi\phi = -2 < \phi, \psi >$$

are precisely the content of Proposition 3.2. □

Abusing slightly the notation, we will identify $\mathcal{C}_p$ with its (faithful) image inside $End_{\mathbb{R}}\left(\bigwedge^* T_p^*X\right)$, and we will omit any reference to the map ρ_p when it will not be necessary. Actually, as the representation above is a real analogue of the Spinor representation, it is easy to check that the map ρ_p is an isomorphism of associative algebras. One then has:

DEFINITION 4.10. — *The linear subspace $\mathcal{C}_p^2$ of $\mathcal{C}_p$ is the image of the natural map $\bigwedge^2(T_pX \oplus T_p^*X) \to \mathcal{C}_p$. The linear subspace $\mathcal{C}_p^0$ of $\mathcal{C}_p$ is the subspace generated by* 1.

Recall (see for instance [LM]) that $\mathcal{C}_p^2$ is a Lie subalgebra of $\mathcal{C}_p$ (with the commutator bracket).

PROPOSITION 4.11. — *The bundle of Lie algebras $\mathcal{L}_{\mathbb{R}}^s$ is a sub-bundle of $\mathcal{C}^2$. Any local determination of the operator J is a (local) section of $\mathcal{C}^2$.*

Proof. — Let us fix $p \in X$. We consider the pointwise values of the operators $L_{\alpha\beta}$, the V_j and the A_j; they all lie inside $\mathcal{C}_p^2 \oplus \mathcal{C}_p^0$ by Proposition 3.2 and by the fact that the forms ω_{ij} restrict to elements of $\bigwedge^2 T_p^*X$. The space $< J >$ lies inside $\mathcal{C}_p^2 \oplus \mathcal{C}_p^0$ by Proposition 3.3. By definition the elements $\mathcal{C}_p^2$ are commutators, and therefore have trace zero in any representation, and hence also in the ρ_p. Moreover, again by inspection all the generators of the fibre in p of $\mathcal{L}_{\mathbb{R}}^s$ have trace zero once represented via ρ_p (they are nilpotent), and therefore they must lie inside $\mathcal{C}_p^2$. Both pointwise determinations of operator J are in the Lie algebra of the isometry group, and therefore they too have trace zero and hence sit inside $\mathcal{C}_p^2$. As $\mathcal{C}_p^2$ is closed under the commutator bracket of $\mathcal{C}_p$, and this commutator coincides with the composition bracket of operators, we have the conclusion. □

Remark 4.12. — For any $p \in X$, the Clifford algebra $\mathcal{C}_p$ is isomorphic to the standard Clifford Algebra $\mathbf{Cl}_{2s+2,2s+2}$, as the metric used to define it has signature $(2s+2, 2s+2)$. The previous proposition therefore shows that all the fibres of $\mathcal{L}^s_{\mathbb{R}}$ are Lie subalgebras of $\mathbf{Cl}^2_{2s+2,2s+2} \cong \mathbf{spin}_{2s+2,2s+2}$.

Remark 4.13. — For any fixed $p \in X$, giving degree 1 to the operators E_{ij} and degree -1 to the operators I_{ij}, we induce a $\mathbb{Z}$-degree on $\mathcal{C}_p$. This degree coincides with the degree of the operators induced from the grading on the forms from $\bigwedge^* T_p^* X$.

Similarly to Definition 4.7, for any $p \in X$ one could define a Clifford Algebra

$$\mathcal{C}l(\mathbb{R}^{s+1} \oplus (\mathbb{R}^{s+1})^*, q_{nat})$$

where q_{nat} is the quadratic form induced by ($-\frac{1}{2}$ times) the natural paring and

$$\mathbb{R}^{s+1} =< \tilde{E}_0, ..., \tilde{E}_s >, \quad (\mathbb{R}^{s+1})^* =< \tilde{I}_0, ..., \tilde{I}_s >$$

One has also a natural representation on $\bigwedge^* T_p X$ of the operators $[\tilde{E}_j, \tilde{E}_k]$, $[\tilde{E}_j, \tilde{I}_k]$, $[\tilde{I}_j, \tilde{I}_k]$ generating the degree two part of this Clifford Algebra, induced by the map which acts as follows:

$$\begin{array}{lcl} [\tilde{E}_j, \tilde{E}_k] & \to & 2L_{jk} \\ [\tilde{E}_j, \tilde{I}_k] & \to & 2L_{j\,\overline{k}} \\ [\tilde{I}_j, \tilde{I}_k] & \to & 2L_{\overline{j}\,\overline{k}} \end{array}$$

This gives directly the bundle Lef^s as a quotient of the $\mathbf{spin}_{s+1,s+1}$ Lie Algebra bundle of this Clifford bundle, proving again that its fibre is indeed $\mathbf{so}(s+1, s+1, \mathbb{R})$.

5. Quadratic invariants and Hodge decomposition

Fixing $p \in X$ and a determination J at p, the complex structure J acts on all the Clifford algebra $\mathcal{C}_p$ by adjunction with respect to the commutator bracket, and sends its quadratic part $\mathcal{C}_p^2$ to itself from Proposition 4.11.

DEFINITION 5.1. — *We call* quadratic invariants *the elements in* $\mathcal{C}_p^2$ *which commute with* J. *For varying* p, *we obtain a bundle of quadratic invariants.*

As usual, to decompose the representation $\bigwedge^* T^*X$ with respect to the weight induced by J, it is necessary to consider complexified forms (and algebras). The weight decomposition of the space $T_p^*X \otimes \mathbb{C}$ is obtained introducing a new basis for each $W_j \otimes \mathbb{C} =< v_{1j}, v_{2j} >_{\mathbb{C}}$:

$$w_j = \frac{1}{\sqrt{2}}(v_{1j} + \imath\, v_{2j}), \qquad \overline{w}_j = \frac{1}{\sqrt{2}}(v_{1j} - \imath v_{2j})$$

To describe explicitely the space of (complex) quadratic invariants in the Clifford algebra $\mathcal{C}_p$, let us introduce the following notation, which gives a basis of eigenvectors for the (adjoint) action of J:

Definition 5.2. —

$$E_{w_j} = \frac{1}{\sqrt{2}}(E_{1j} + \imath E_{2j}), \qquad E_{\overline{w}_j} = \frac{1}{\sqrt{2}}(E_{1j} - \imath E_{2j})$$

$$I_{w_j} = \frac{1}{\sqrt{2}}(I_{1j} - \imath I_{2j}), \qquad I_{\overline{w}_j} = \frac{1}{\sqrt{2}}(I_{1j} + \imath I_{2j})$$

Lemma 5.3. — *The adjoint action of the complex structure operator J on E_{w_j}, I_{w_j}, $E_{\overline{w}_j}$, $I_{\overline{w}_j}$ is:*

$$[J, E_{w_j}] = -\imath E_{w_j}, \quad [J, I_{w_j}] = \imath I_{w_j}$$

$$[J, E_{\overline{w}_j}] = \imath E_{\overline{w}_j}, \quad [J, I_{\overline{w}_j}] = -\imath I_{\overline{w}_j}$$

Proof. — It is enough to consider the corresponding J-weights of the $w_j, \overline{w}_j$. □

As $\mathcal{L}^s_{\mathbb{C}} \subset \mathcal{C}^2 \otimes \mathbb{C}$ from Proposition 4.11, in the following we show that $\mathcal{L}^s_{\mathbb{C}}$ lies inside the bundle of quadratic invariants. Immediately after we will give a basis for the space of quadratic invariants, thus providing a first upper bound for $\mathcal{L}^s_{\mathbb{C}}$ (which will be later shown to be off by only 1).

Proposition 5.4. — *The operator J commutes with all the elements in the fiber at p of $\mathcal{L}^s_{\mathbb{C}}$.*

Proof. — We prove the statement by a direct computation. Since

$$\omega_{jk} = v_{1j} \wedge v_{1k} + v_{2j} \wedge v_{2k} = \frac{1}{2}\left(w_j \wedge \overline{w}_k + \overline{w}_j \wedge w_k\right)$$

we have

$$L_{jk} = \frac{1}{2}\left([E_{w_j}, E_{\overline{w}_k}] - [E_{w_k}, E_{\overline{w}_j}]\right)$$

The vanishing $[J, L_{jk}] = 0$ then follows immediately from Lemma 5.3. Similarly, to show that $[J, V_k] = 0$ one uses

$$V_k = \frac{\imath}{2}[E_{w_k}, E_{\overline{w}_k}]$$

The corresponding commutation relations for the adjoint operators follow from the fact that $J^* = -J$, as noticed in Remark 3.4. □

The following proposition will show that, except for a toral part which will be discussed later, all the quadratic invariants of the Clifford bundle $\mathcal{C}$ lie inside $\mathbf{s}\mathcal{L}^s \subset \mathcal{L}^s_{\mathbb{C}}$. It will follow therefore that

$$4(s+1)^2 - 2(s+1) \leqslant dim_{\mathbb{R}}\, \mathbf{s}\mathcal{L}^s \leqslant dim_{\mathbb{C}}\, \mathcal{L}^s_{\mathbb{C}} \leqslant 4(s+1)^2$$

PROPOSITION 5.5. — *The following* $4(s+1)^2$ *operators are a linear basis for the quadratic* J*-invariants:*

(1) $[E_{w_i}, E_{\overline{w}_j}]$ *with* $i \neq j$	(2) $[I_{w_i}, I_{\overline{w}_j}]$ *with* $i \neq j$
(3) $[E_{w_i}, E_{\overline{w}_i}]$ *where* $i = 0, 1, \ldots, s$	(4) $[I_{w_i}, I_{\overline{w}_i}]$ *where* $i = 0, 1, \ldots, s$
(5) $[E_{w_i}, I_{w_j}]$ *with* $i \neq j$	(6) $[E_{\overline{w}_i}, I_{\overline{w}_j}]$ *with* $i \neq j$
(7) $[E_{w_i}, I_{w_i}]$ *where* $i = 0, 1, \ldots, s$	(8) $[E_{\overline{w}_i}, I_{\overline{w}_i}]$ *where* $i = 0, 1, \ldots, s$

The $4(s+1)^2 - 2(s+1)$ *operators of type* $(1), (2), (3), (4), (5), (6)$ *belong to the bundle of real algebras* $\mathbf{s}\mathcal{L}^s \subset \mathcal{L}^s_{\mathbb{C}}$.

Proof. — In this proof, we fix $p \in X$ and all the bundles and operators will be considered at this point. To be J-invariant means simply to have weight zero, and the computation of the J-weight of the quadratic mononials follows immediately from those of $E_{w_j}, I_{w_j}, E_{\overline{w}_j}, I_{\overline{w}_j}$, which are respectively $-\imath, \imath, \imath, -\imath$.

It remains to be shown that the monomials of type $(1), (2), (3), (4), (5), (6)$ belong to $\mathbf{s}\mathcal{L}^s$. By adjunction it is enough to deal with types $(1), (3), (5)$. From the proof of Proposition 5.4 we know that the monomials of type (3) are (up to a scalar) the generators $\imath V_j$. The proof for types (1) and (5) is similar. Let us consider for instance a monomial of type (1) $[E_{w_i}, E_{\overline{w}_j}]$ with $i \neq j$. Since E_{w_i} and $E_{\overline{w}_j}$ anticommute, this is equal to $2E_{w_i}E_{\overline{w}_j}$. Then

$$2E_{w_i}E_{\overline{w}_j} = (E_{1i} + \imath E_{2i})(E_{1j} - \imath E_{2j}) = E_{1i}E_{1j} + E_{2i}E_{2j} + \imath(E_{2i}E_{1j} - E_{1i}E_{2j})$$

$$= L_{ij} + \imath(E_{2i}E_{1j} - E_{1i}E_{2j})$$

We have therefore to show that $\imath(E_{2i}E_{1j} - E_{1i}E_{2j})$ belongs to $\mathbf{s}\mathcal{L}^s$.

We recall that, by Corollary 4.3, the elements $L_{i\,\overline{j}}$ belong to $\mathbf{s}\mathcal{L}^s$ and notice that

$$[L_{i\,\overline{j}}, \imath V_j] = \imath(E_{1i}I_{1j} + E_{2i}I_{2j})E_{1j}E_{2j} - \imath E_{1j}E_{2j}(E_{1i}I_{1j} + E_{2i}I_{2j})$$

$$= \imath(E_{1i}E_{2j} - E_{1i}E_{1j}I_{1j}E_{2j} - E_{2i}E_{1j} + E_{2i}E_{1j}E_{2j}I_{2j} - E_{1j}E_{2j}E_{1i}I_{1j} - E_{1j}E_{2j}E_{2i}I_{2j})$$
$$= \imath(E_{1i}E_{2j} - E_{2i}E_{1j})$$

□

6. The fibres of the bundles $\mathcal{L}^s_{\mathbb{C}}$, $\mathcal{L}^s_{\mathbb{R}}$ and $\mathbf{s}\mathcal{L}^s$

In this section we fix once and for all a determination of J at the point p and consider the Hodge decomposition of $\bigwedge^*_{\mathbb{C}} T^*_p X$ with respect to the (almost) complex structure J. We will use this information to first study the complex algebra formed by the fibers at the point p of the bundle $\mathcal{L}^s_{\mathbb{C}}$. In the second part of this section we will concentrate on its real forms associated to the bundle $\mathcal{L}^s_{\mathbb{R}}$ and to the bundle $\mathbf{s}\mathcal{L}^s$ generated by the Lefschetz operators $L_{\alpha\beta}$ and by the $\imath V_j, \imath A_k$.

The Hodge (type) decomposition of forms on X with respect to the complex structure J can be described as usual explicitely as follows, using the J-homogeneous basis $w_j, \overline{w}_k$:

$$\bigwedge^{r,t} T^*_{\mathbb{C}} X_p = < w_{i_1} \wedge \cdots \wedge w_{i_r} \wedge \overline{w}_{j_1} \wedge \cdots \wedge \overline{w}_{j_t} \mid i_1, ..., j_t \in \{0, 1, 2, \ldots, s\} >_{\mathbb{C}}$$

Definition 6.1. — *At a given point $p \in X$, and with the chosen a determination of J at p, we indicate with $\mathcal{I}_\alpha$ the subspace (isotypical component) of forms of J-weight $\imath\alpha$ $(-s-1 \leqslant \alpha \leqslant s+1)$.*

Theorem 6.2. — *Let X be a (almost, pointwise) s-Kähler manifold or rank two.*

a) The Lie algebra bundle $\mathcal{L}^s_{\mathbb{C}}$ has fibre isomorphic to $\mathbf{sl}(2s+2, \mathbb{C})$.

b) At a given point $p \in X$, the direct sum of $\mathcal{L}^s_{\mathbb{C},p}$ with the space spanned by the operator J_p is the set of all quadratic invariants of $\mathcal{C}_p$.

c) At a given point $p \in X$, the restriction of $\mathcal{L}^s_{\mathbb{C}}$ to the $2s+2$ dimensional space $\mathcal{I}_{-s}$ of forms of J-weight $-s$ is faithful.

Proof. — We work at a fixed point p.

The isotypical component $\mathcal{I}_{-s}$ has dimension $2s+2$ and has a basis $\{b_i\}$ $(0 \leqslant i \leqslant 2s+2)$ given by the following monomials:

- $b_i = w_0 \wedge \ldots \wedge \widehat{w_i} \wedge \cdots \wedge w_s$, for $i \in \{0,1,2,\ldots,s\}$, where $\widehat{w_i}$ means that w_i is omitted and therefore the monomial has degree s.
- $b_{s+1+i} = w_0 \wedge \cdots \wedge w_s \wedge \overline{w}_i$, where $i \in \{0,1,2,\ldots,s\}$ and the monomial has degree $s+2$.

It is then immediate to check that, for instance:

$$[E_{w_0}, I_{w_1}](b_0) = [E_{w_0}, I_{w_1}](w_1 \wedge \cdots \wedge w_s) = 2w_0 \wedge \widehat{w_1} \wedge \cdots \wedge w_s = 2b_1$$

$$[E_{w_s}, E_{\overline{w}_0}](b_s) = [E_{w_s}, E_{\overline{w}_0}](w_0 \wedge \cdots \wedge w_{s-1}) = 2w_0 \cdots \wedge w_s \wedge \overline{w}_0 = 2b_{s+1}$$

Completely analogous computations show that, when we represent the action of $\mathcal{L}^s_{\mathbb{C},p}$ on the isotypical component $\mathcal{I}_{-s}$ using the above mentioned basis, all the elementary matrices e_{ij} (where $i \neq j$ and e_{ij} is the matrix with all the entries equal to 0 except for the entry (i,j) which is 1) are obtained using the quadratic invariants of type (1), (2), (3), (4), (5), (6) which in Proposition 5.5 were shown to lie in $\mathcal{L}^s_{\mathbb{C},p}$.

More precisely, we have the following identifications for the "positive" set of Serre generators $e_{j+1,j}$:

- $e_{j+2,j+1} = \frac{1}{2}[E_{w_j}, I_{w_{j+1}}]$ for $0 \leqslant j \leqslant s-1$;
- $e_{s+2,s+1} = \frac{1}{2}[E_{w_s}, E_{\overline{w}_0}]$;
- $e_{s+3+j,s+2+j} = \frac{1}{2}[E_{\overline{w}_{j+1}}, I_{\overline{w}_j}]$ for $0 \leqslant j \leqslant s-1$.

Therefore $\mathcal{L}^s_{\mathbb{C},p}$ acts as $\mathbf{sl}(2s+2,\mathbb{C})$ on $\mathcal{I}_{-s}$ (notice that, as the generators L_{ij}, $\Lambda_{ij} = L_{\bar{j}\,\bar{i}}$, V_i, A_i of $\mathcal{L}^s_{\mathbb{C},p}$ are nilpotent, they still have trace zero when restricted to $\mathcal{I}_{-s}$).

Summing up, the algebra $\mathcal{L}^s_{\mathbb{C},p}$ has a quotient isomorphic to the simple algebra $\mathbf{sl}(2s+2,\mathbb{C})$ and is embedded in the $4(s+1)^2$-dimensional space of the quadratic invariants; now, since the quadratic invariant J_p doesn't belong to $\mathcal{L}^s_{\mathbb{C},p}$ (in fact the trace of its restriction to $\mathcal{I}_{-s}$ is different from 0, since J_p acts on $\mathcal{I}_{-s}$ as multiplication by $-\imath s$), we conclude that $\mathcal{L}^s_{\mathbb{C}}$ has dimension $4(s+1)^2-1$. Therefore the restriction to $\mathcal{I}_{-s}$ provides us with an isomorphism of $\mathcal{L}^s_{\mathbb{C},p}$ with $\mathbf{sl}(2s+2,\mathbb{C})$. $\square$

We want now to characterize explicitely the matrices of $\mathcal{L}^s_{\mathbb{R},p}$. Notice that the basis that we use is not real, but we will show that the matrices are nevertheless in the standard form for $\mathbf{su}(s+1, s+1)$.

THEOREM 6.3. — *The algebra $\mathcal{L}^s_{\mathbb{R},p}$ is isomorphic to $\mathbf{su}(s+1, s+1)$. With respect to the basis $\{c_i\}$ of $\mathcal{I}_{-s}$ defined below, $\mathcal{L}^s_{\mathbb{R},p}$ is faithfully presented as the algebra of matrices*

$$\begin{pmatrix} D & H_2 \\ H_1 & -\overline{D}^t \end{pmatrix}$$

with D an $(s+1)\times(s+1)$ complex matrix and H_1, H_2 two $(s+1)\times(s+1)$ complex antihermitean matrices.

Proof. — We start by noticing that the operators of degree zero $L_{j\,\overline{k}}$ $(j \neq k)$ (which lie in $Lef^s_p \subset \mathcal{L}^s_{\mathbb{R},p}$) can be expressed in terms of the basis of quadratic invariant monomials as:

$$2L_{j\,\overline{k}} = [E_{w_j}, I_{w_k}] + [E_{\overline{w}_j}, I_{\overline{w}_k}]$$

It is convenient to use a basis $\{c_r\}$ of $\mathcal{I}_{-s}$ which differs from the basis $\{b_r\}$ provided in the proof of Theorem 6.2 only for some signs. This same basis will be used in the proof of Theorem 7.4 below.

Namely, $\{c_r\}$ $(0 \leqslant r \leqslant 2s+2)$ is given by the following monomials:

- $c_r = b_r = w_0 \wedge \ldots \wedge \widehat{w_r} \wedge \cdots \wedge w_s$, for $r \in \{0, 1, 2, \ldots, s\}$;
- $c_{s+1+r} = w_0 \wedge \cdots \wedge w_r \wedge \overline{w}_r \wedge \cdots \wedge w_s$, for $r \in \{0, 1, 2, \ldots, s\}$.

This allows us to compute, for $i = 0, 1, 2, \ldots, s$:

$$L_{j\,\overline{k}}(c_i) = 0 \qquad if\ i \neq j$$

$$L_{j\,\overline{k}}(c_j) = \frac{1}{2}[E_{w_j}, I_{w_k}](c_j) = -I_{w_k}E_{w_j}(c_j) = -(-1)^{j+k}c_k$$

and

$$L_{j\,\overline{k}}(c_{s+1+i}) = 0 \qquad if\ i \neq k$$

$$L_{j\,\overline{k}}(c_{s+1+k}) = \frac{1}{2}[E_{\overline{w}_j}, I_{\overline{w}_k}](c_{s+1+k}) = E_{\overline{w}_j}I_{\overline{w}_k}(c_{s+1+k}) = (-1)^{j+k}c_{s+1+j}$$

This means that the matrices of the degree 0 subalgebra generated by the operators $L_{j\,\overline{k}}$ have real coefficients and, more precisely they are all the matrices with the following block-form:

$$\begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}$$

where A is a real $(s+1)\times(s+1)$ matrix with trace zero. This explicitely establishes an isomorphism between $< L_{j\overline{k}} >_{\mathbb{R}}$ and $\mathbf{sl}(s+1,\mathbb{R})$.

The computation of the matrices of the operators V_j is made easier by the use of the relation contained in the proof of Proposition 5.4:

$$V_j = \frac{\imath}{2}[E_{w_j}, E_{\overline{w}_j}]$$

We can now observe that, for $i = 0, 1, 2, \ldots, s$:

$$V_j(c_i) = 0 \qquad if\ i \neq j$$

$$V_j(c_j) = \frac{\imath}{2} 2 E_{w_j} E_{\overline{w}_j}(c_j) = \imath c_{s+1+j}$$

This, together with the observation that

$$V_j(c_{s+1+i}) = 0 \qquad \forall i = 0, \ldots, s$$

implies that the matrix of V_j has the following block-form:

$$\begin{pmatrix} 0 & 0 \\ \imath B & 0 \end{pmatrix}$$

where B is a real and symmetric (actually diagonal) $(s+1)\times(s+1)$ matrix.

It follows that all the matrices of the above form are in $\mathcal{L}^s_{\mathbb{R},p}$ since they provide an irreducible representation for the adjoint action of $< L_{j\overline{k}} >_{\mathbb{R}} \cong \mathbf{sl}(s+1,\mathbb{R})$: notice that the action of a matrix with upper diagonal A over one with lower left block $\imath B$ is as follows:

$$\imath B \rightarrow -\imath(BA\ +\ {}^t AB)$$

As for the operators L_{jk} $(j \neq k)$ of degree two, as it has been shown in Proposition 5.4:

$$2L_{jk} = [E_{w_j}, E_{\overline{w}_k}] - [E_{w_k}, E_{\overline{w}_j}]$$

Therefore,

$$L_{jk}(c_i) = 0 \qquad if\ i \neq j, k$$

and

$$L_{jk}(c_j) = -E_{\overline{w}_k} E_{w_j}(c_j) = (-1)^{j+k} c_{s+1+k}$$

$$L_{jk}(c_k) = E_{\overline{w}_j} E_{w_k}(c_k) = -(-1)^{j+k} c_{s+1+j}$$

This, together with the observation that

$$L_{jk}(c_{s+1+i}) = 0 \qquad \forall i = 0, \ldots, s$$

implies that the matrix of L_{jk} has the block-form:

$$\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$$

where C is a real and antisymmetric $(s+1)\times(s+1)$ matrix. Then all the matrices of the above form are in $\mathcal{L}^s_{\mathbb{R},p}$ since they provide an irreducible representation for the action of $< L_{j\,\overline{k}} >_{\mathbb{R}} \cong \mathbf{sl}(s+1,\mathbb{R})$ similarly as before.

In the same way, acting with $< L_{j\,\overline{k}} >_{\mathbb{R}} \cong \mathbf{sl}(s+1,\mathbb{R})$ on the adjoint operators Λ_{jk} and A_j, we can show that $\mathcal{L}^s_{\mathbb{R},p}$ contains all the matrices of the form

$$\begin{pmatrix} 0 & H \\ 0 & 0 \end{pmatrix}$$

where H is a complex antihermitean $(s+1)\times(s+1)$ matrix.

It is now immediate to check that the matrices constructed above generate the matrix algebra as in the claim. To conclude it is enough to use point c) of Theorem 6.2, which states that the restriction of $\mathcal{L}^s_{\mathbb{R},p}$ to $\mathcal{I}_{-s}$ is faithful. $\square$

Remark 6.4. — By inspection of the proof above one checks that the matrices of the generators are actually in $\mathbb{Q}[\imath]$, the structure constants of the algebra $\mathcal{L}^s_{\mathbb{R},p}$ turn out to be in $\mathbb{Q}$, and the algebra generated over $\mathbb{Q}$ is a rational form of $\mathbf{su}(s+1,s+1)$. This form is explicitely exhibited as the algebra of matrices described in the statement of the Theorem, with coefficients in $\mathbb{Q}[i]$.

THEOREM 6.5. — *The bundle of real Lie algebras* $\mathbf{s}\mathcal{L}^s$ *has fibre isomorphic to the Lie algebra* $\mathbf{sl}(2s+2,\mathbb{R})$.

Proof. — From the proofs of Theorem 6.2 and of Proposition 5.4 it follows that $\mathbf{s}\mathcal{L}^s$ is included in the real $\mathbf{sl}(2s+2,\mathbb{R})$ bundle generated by the quadratic monomials of types $(1),(2),\ldots,(5),(6)$. We conclude observing that this bundle is a real form of $\mathcal{L}^s_{\mathbb{C}}$, and therefore coincides with the real form $\mathbf{s}\mathcal{L}^s$. $\square$

Remark 6.6. — An explicit computation (based on the matrix representations of the algebras $\mathcal{L}^s_{\mathbb{R},p}$ and $\mathbf{s}\mathcal{L}^s_p$ with respect to the basis $\{c_r\}$ of $\mathcal{I}_{-s}$) shows that the orthogonal algebra $Lef^s_p \cong \mathbf{so}(s+1,s+1)$ coincides with the intersection $\mathcal{L}^s_{\mathbb{R},p} \cap \mathbf{s}\mathcal{L}^s_p$.

7. Invariant quadratic forms

In the previous section we described completely the complex bundles of Lie algebras $\mathcal{L}^s_{\mathbb{C}}$, $\mathcal{L}^s_{\mathbb{R}}$ and $\mathbf{s}\mathcal{L}^s$. Here, we will construct two natural quadratic forms, and we will show that the natural bundles of real Lie algebras $\mathbf{u}\mathcal{L}^s$ and $\mathcal{L}^s_{\mathbb{R}}$ are precisely the bundles of their respective invariant sections. This will show that the fibre of $\mathbf{u}\mathcal{L}^s$ is isomorphic at every point to $\mathbf{su}(s+1, s+1)$.

On the complex bundle of vector spaces $\bigwedge^*_{\mathbb{C}} T^*X$ there is a natural hermitean inner product $< \ , \ >$, obtained from the wedge operation on forms (cf. [GG2] where we used a superHermitean variant of this product for the rank 3 case), and defined below. Associated to this pairing, there is a natural notion of antihermitean operator. We will prove that the set of antihermitean operators inside $\mathcal{L}^s_{\mathbb{C}}$ is a real form for $\mathcal{L}^s_{\mathbb{C}}$, generated by operators naturally derived from the geometry and coinciding with $\mathbf{u}\mathcal{L}^s$.

Definition 7.1. — *For every $p \in X$ there is a natural non degenerate Hermitean inner product $< \ , \ >_p$ on $\bigwedge^*_{\mathbb{C}} T^*_pX$, defined using the natural (standard) Hermitean inner product $(\ , \ )_p$ associated to the metric $\mathbf{g}$ and the (pointwise) volume form Ω associated to the metric $\mathbf{g}$ and to the chosen determination of J at p:*

$$< \alpha, \beta >_p = \imath^{deg(\alpha)deg(\beta)} (\alpha \wedge \overline{\beta}, \Omega)_p$$

We indicate with $< \ , \ >$ the corresponding form with values in smooth functions.

Let us denote with $\overline{*}$ the (complex linear) operator obtained composing conjugation with the Hodge star associated to the metric.

Proposition 7.2. — *For every $p \in X$, the pairing $< \ , \ >_p$ satisfies the following properties:*

a) $< \alpha, \beta >_p = \imath^{(deg(\alpha)+2)deg(\beta))} (\alpha, \overline{*}\beta)_p$

b) $< \ , \ >_p$ *is preserved by the the operator J in derived sense, namely*

$$\forall \alpha\beta \ \ < J\alpha, \beta > + < \alpha, J\beta > = 0$$

c) $< \ , \ >_p$ *is preserved by the operator $\overline{*}$, namely*

$$\forall \alpha\beta \ \ < \overline{*}\alpha, \overline{*}\beta > = < \alpha, \beta >$$

d) The pure weight components $\mathcal{I}_k$ are mutually $< \ , \ >_p$-orthogonal and $< \ , \ >_p$ is nondegenerate when restricted to any one of them.

Proof. — The first three facts are standard. For the orthogonality statement in part d), we observe that, if $\alpha \in \mathcal{I}_h$ and $\beta \in \mathcal{I}_k$ with $deg(\alpha) + deg(\beta) = dim(X)$ then

$$< \alpha, \beta >_p \Omega = (\alpha \wedge \overline{\beta})_p$$

is a complex number times a form of J-weight zero, but from the right hand side it also must have J-weight equal to $(h-k)\imath$. Therefore if $h \neq k$, it must be zero.

Restricting to a single $\mathcal{I}_k$, notice that $\overline{*}$ sends this component to itself (as it commutes with J), and then if $\alpha \neq 0$ in $\mathcal{I}_k$, $< \alpha, \overline{*}\alpha >$ is a power of $\imath$ times (α, α) by point a), and is therefore nonzero. □

We want now to characterize the operators inside $\mathcal{L}^s_{\mathbb{C},p}$ which preserve the form $< \ , \ >_p$. First we observe that, since the dimension of $T^*_p X$ is even, $\overline{**}$ is equal to the identity on the forms of even degree while $\overline{**} = -I$ when restricted to the odd forms. Then, for fixed $p \in X$, using the expression $< \alpha, \beta >_p = (\alpha, \overline{*}\beta)_p$, we see that the "differential" condition for preservation of the form by the operator ϕ

$$\forall\alpha\forall\beta \quad < \phi(\alpha), \beta >_p + < \alpha, \phi(\beta) >_p = 0$$

is equivalent to $\phi^* = -\overline{*}\,\phi\,\overline{*}$ on the even forms and to $\phi^* = \overline{*}\,\phi\,\overline{*}$ on the odd forms.

The next two theorems show that the bundle $\mathbf{u}\mathcal{L}^s$ (generated at any point by the value of the operators $\imath L_{j,k}$ $(j \neq k)$, $\imath V_i$ and their adjoints, see Definition 3.10) is precisely the bundle of Lie subalgebras given point by point by the operators which preserve the form $< \ , \ >$:

THEOREM 7.3. — *The Lie algebra bundle $\mathbf{u}\mathcal{L}^s$ preserves the form $<,>$.*

Proof. — As we observed before, the statement is equivalent to the fact that the condition $\phi^*(\alpha) = (-1)^{deg\ \alpha+1}\overline{*}\,\phi\,\overline{*}(\alpha)$ holds for all the sections ϕ of $\mathbf{u}\mathcal{L}^s$ and all the homogeneous elements $\alpha \in \bigwedge^*_{\mathbb{C}} T^*X$.

It is enough to check these equations for the generators of $\mathbf{u}\mathcal{L}^s$, at a fixed point $p \in X$, which as usual we omit from the notation for the operators when not strictly necessary.

Let Ψ be one of the generators $\imath L_{ij}$ or $\imath V_k$; this means that Ψ is the operator given by the wedge with an even form $\imath\psi$, where ψ is real. One has, given v homogeneous of degree h and w in $\bigwedge^*_{\mathbb{C}} T^*_p X$ with degree of the same parity (which is the only possibly non-vanishing case):

$$(\Psi(v), w)_p = (\imath\psi \wedge v \wedge \overline{\overline{*}w}, \Omega)_p = -(v \wedge \overline{\imath\psi \wedge \overline{*}w}, \Omega)_p =$$

$$= -(-1)^h (v \wedge \overline{\overline{**}(\imath\psi \wedge \overline{*}w)}, \Omega)_p = -(-1)^h (v \wedge \overline{\overline{*}(\overline{*}\Psi\overline{*})(w)}, \Omega)_p$$

On the other hand, $(\Psi(v), w)_p = (v, \Psi^*(w))_p = (v \wedge \overline{\overline{*}\Psi^*(w)}, \Omega)_p$. This implies $\overline{*}\Psi^* = -(-1)^h\overline{*}(\overline{*}\Psi\overline{*})$ which is equivalent to $\Psi^* = -(-1)^h\overline{*}\Psi\overline{*}$ that is the relation we wanted to check. The adjoint of this equation immediately proves the relation also for the generators $\imath L_{\overline{k},\overline{j}}$, $(j \neq k)$ and $\imath A_i$. □

Theorem 7.4. — *The Lie algebra bundle* $\mathbf{u}\mathcal{L}^s$ *is the full real Lie subalgebra bundle of* $\mathcal{L}^s_{\mathbb{C}}$ *of operators which preserve the form* $<\ ,\ >$*, and its fibre is isomorphic to* $\mathbf{su}(s+1, s+1)$.

Proof. — As usual, let us fix once and for all a point $p \in X$, which will be omitted from the notation when not strictly necessary.

In view of part *c*) of Theorem 6.2, to compute the signature of the form $<\ ,\ >_p$ we can restrict to $\mathcal{I}_{-s}$. We use here the basis $\{c_r\}$ $(0 \leqslant r \leqslant 2s+2)$ defined in Theorem 6.3. By construction, for every $r < j$, $< c_r, c_j >_p = 0$ unless $j = s+1+r$ and in this case we have that

$$< c_r, c_{s+1+r} >_p = \imath^{s(s+2)}(w_0 \wedge \ldots \wedge \widehat{w_r} \wedge \cdots \wedge w_s \wedge \overline{w_0 \wedge \cdots \wedge w_r \wedge \overline{w}_r \wedge \cdots \wedge w_s}, \Omega)_p$$

$$= \imath^{s(s+2)}(-1)^{s+1}(-1)^s \cdots (-1)^0 (w_0 \wedge \overline{w_0} \wedge \ldots \wedge w_r \wedge \overline{w_r} \wedge \cdots \wedge w_s \wedge \overline{w_s}, \Omega)_p$$

$$= \imath^{s(s+2)}(-1)^{s+1}(-1)^s \cdots (-1)^0 (-\imath)^{s+1} = \imath^{2s^2+1}$$

Thus we notice that $< c_r, c_{s+1+r} >_p$ does not depend on the index r, being equal to $\imath$ when s is even and to $-\imath$ when s is odd. If follows that

for every s the signature of $<\ ,\ >_p$ is $(s+1, s+1)$. From Theorem 7.3 and the above remark on the signature one deduces that the fibre of $\mathbf{u}\mathcal{L}^s$ at p can be identified with a subalgebra of $\mathbf{su}(s+1, s+1) \subseteq \mathbf{sl}(2s+2, \mathbb{C}) \cong \mathcal{L}^s_{\mathbb{C},p}$. Since, by construction, $\mathbf{u}\mathcal{L}^s$ is a real form of $\mathcal{L}^s_{\mathbb{C}}$, we can replace $\subseteq$ with $=$ in the inclusion above and the claim follows. □

The theorem above also provides us the key ingredient to build and invariant quadratic form for $\mathcal{L}^s_{\mathbb{R}}$, which is the most natural real Lie algebra bundle associated to the (almost) s-Kähler structure. Indeed, $\mathcal{L}^s_{\mathbb{R}}$ is generated at every point by the values of the operators L_{ij}, V_k and their pointwise adjoints. The main tool will be a new hermitean inner product $<<\ ,\ >>$ defined starting from $<\ ,>$ on the complex bundle of vector spaces $\bigwedge^*_{\mathbb{C}} T^*X$ which we now introduce:

Definition 7.5. — *For every $p \in X$ there is a natural non degenerate Hermitean inner product $<< \ , \ >>_p$ on $\bigwedge_{\mathbb{C}}^* T_p^* X$, defined, on homogeneous elements α, β, as:*

$$<< \alpha, \beta >>_p = \imath^{s+1+deg\ \beta} < \alpha, \beta >_p$$

We indicate with $<< \ , \ >>$ the corresponding form with values in smooth functions.

Theorem 7.6. — *The Lie algebra bundle $\mathcal{L}_{\mathbb{R}}^s$ is the full real Lie subalgebra bundle of $\mathcal{L}_{\mathbb{C}}^s$ of operators which preserve the form $<< \ , \ >>$, which has signature $(s+1, s+1)$.*

Proof. — Let Γ be any one of the generators $L_{jk}, V_j, \Lambda_{jk}, A_j$ of $\mathcal{L}_{\mathbb{R}}^s$. Since $\imath\Gamma$ preserves $< \ , \ >$, a direct computation reducing $<< \ , \ >>$ to $< \ , \ >$ shows that Γ preserves $<< \ , \ >>$. A dimensional argument concludes the proof of the first statement.

For the second part of the claim, it suffices to compute the signature of the form $<< \ , >>_p$ when restricetd to $\mathcal{I}_{-s}$. Using the basis $\{c_r\}$ of $\mathcal{I}_{-s}$ introduced in the proof of Theorem 7.4 we have:

$$<< c_r, c_{s+1+r} >>_p = \imath^{2s+3} < c_r, c_{s+1+r} >_p = \imath^{2s^2+2s+4} = \imath^{2s(s+1)} = 1$$

which shows that the total signature is $(s+1, s+1)$. □

Notice that the theorem shows again (in accordance with Theorem 6.3) that the fibre of $\mathcal{L}_{\mathbb{R}}^s$ is isomorphic to $\mathbf{su}(s+1, s+1)$.

8. Action on cohomology

Up to this point all the geometric and algebraic constructions applied to almost s-Kähler manifolds. From this section onwards we concentrate on s-Kähler manifolds, and on their global geometric properties.

On an s-Kähler manifold the (local) sections $L_{jk}, \Lambda_{jk}, V_i, A_i$ are all covariant constant with respect to the (Levi-Civita) connection induced by the metric, which therefore determines a flat connection on all the bundles of Lie algebras $\mathcal{L}_{\mathbb{C}}^s$, $\mathcal{L}_{\mathbb{R}}^s$, $\mathbf{s}\mathcal{L}^s$, $\mathbf{u}\mathcal{L}^s$, Lef^s.

We now show that we have a representation of the flat sections of the bundles of Lie algebras $\mathcal{L}_{\mathbb{C}}^s, \mathcal{L}_{\mathbb{R}}^s, \mathbf{s}\mathcal{L}^s, \mathbf{u}\mathcal{L}^s, Lef^s$ on the cohomology of an s-Kähler manifold, induced by the representation on the space of forms. This will be done showing that the Laplacian Δ_d commutes with the action of generators of these spaces of sections, as in Theorem 10.1 on page 46 of [G1].

THEOREM 8.1. — *Let* $(X, \omega_1, ..., \omega_s, \mathbf{g})$ *be a compact orientable s-Kähler manifold, and let* $\mathcal{U} \subset X$ *be an open set with a determination of* J. *We have that if* $\phi \in \{L_{jk}\} \cup \{V_j\}$, *and* d *is the de Rham differential:*

1) $[\phi, d] = 0$

2) If we define $d^c := [\phi, d^*]$, *we have that* $dd^c + d^c d = 0$*;*

3) $[\phi, \Delta_d] = [\phi^*, \Delta_d] = 0$, *where* Δ_d *is the d-Laplacian relative to the metric* $\mathbf{g}$ *and to the orientation.*

Proof. — We adapt the proof of Theorem 10.1 of [G1]. We omit the details of the computations, which are however standard.

1) This equation follows immediately from the fact that the forms ω_{jk} and the volume forms $Vol(W_j)$ of the distribtions W_j are covariant constant with respect to the Levi-Civita connection, and therefore closed.

2) If we write down the expression for d^c in standard s-Kähler coordinates centered at a point $p \in X$, we see that no derivative of the metric appears. Therefore, when we write down the expression for $dd^c + d^c d$, only the first derivatives of the metric are involved. We skip the details, as they are completely analogous to those of, for example, [GH, pages 111-115].
It follows, as in the classical case of Kähler manifolds, that to prove the equation it is enough to reduce to the case of a constant metric. When the metric is flat, however, the equation is easily seen to be equivalent (using 1)) to $[\phi, \Delta_d] = 0$, which with a flat metric follows immediately from the fact that the two-form corresponding to ϕ is constant in flat (orthonormal) coordinates.

3) The second equation is the adjoint of the first. The first one, once written down explicitely in terms of d and d^*, follows immediately from points 1) − 2). □

COROLLARY 8.2. — *Let* $(X, \omega_1, ..., \omega_s, \mathbf{g})$ *be a compact orientable s-Kähler manifold. Then there is a canonical representation of the Lie algebras of flat global sections of the bundles* $\mathcal{L}^s_{\mathbb{C}}, \mathcal{L}^s_{\mathbb{R}}, \mathbf{s}\mathcal{L}^s, \mathbf{u}\mathcal{L}^s, Lef^s$ *on* $H^*(X, \mathbf{C})$.

9. Complex tori and rational structures

The following example is a direct generalization of Example 2.7 of [G2], in the special case of rank equal to two. Let $\Gamma \subset \mathbb{C}^{s+1}$ be any (not necessarily maximal rank) lattice, and let

$$X = \mathbb{C}^{s+1}/\Gamma$$

As it is immediate to check, X has a natural structure of s-Kähler manifold of rank 2. Indeed, the (flat) metric is determined by the natural global coordinates induced by the projections, giving at every point an orthonormal (co)frame:

$$\left\{ y_i^j \ | i \in \{1, ..., 2\} \ , \ j \in \{0, ..., s\} \right\}.$$

Using these coordinates one can give directly the expressions for the forms:

$$\omega_{jk} = \sum_{i=1}^{r} dy_i^j \wedge dy_i^k$$

The metric being flat, condition 2) of Definition 1.1 is satisfied everywhere exactly (without the term $\mathbf{O}(2)$).

As the lattice Γ varies, we obtain different (and in general not isomorphic) s-Kähler structures all compatible with the complex structure (i.e. such that the J coming from the s-Kähler strucure coincides with the complex structure of the manifold).

Remark 9.1. — One can think of these s-Kähler structures as "decorations" or "enrichments" of the underlying Kähler structure. As such, one can use them to study the moduli problems for complex tori by first studying the moduli problems of related s-Kähler manifolds.

Definition 9.2. — *An s-Kähler structure on a manifold X is* integral *(resp.* rational*) if the standard operators L_{jk}, V_h and their adjoints act on integral (respectively rational) cohomology.*

In view of the following proposition and of the results of the next section, we are interested in studying rational structures on complex tori and Abelian varieties:

Proposition 9.3. — *Any rational s-Kähler structure compatible with the complex structure on an Abelian variety is such that the structure forms and the canonical volume forms belong to $Im(cl) \otimes \mathbb{Q} \subset H^{\cdot\cdot}_{Hodge}(X) \otimes \mathbb{Q}$*

Proof. — As all the forms of the statement lie in $H^{1,1}(X) \cap H^2(X, \mathbb{Q})$, the proposition follows from Lefschetz 1-1 theorem. □

In the remaining part of this section we will give a sufficient condition for the existence of rational s-Kähler structures on complex tori.

Let X be a complex torus as above. We indicate with $J = J_p$ the complex structure(s) on the tangent spaces T_pX for varying $p \in X$ and we fix a basis $\lambda_1, ..., \lambda_g, \mu_1, ..., \mu_g$ for Λ such that the matrix associated to $E = ImH$ (where H is the Hermitean form associated to the polarization) is "symplectic" with integral elementary divisors. Let Π be the period matrix, expressing the $\lambda_1, ..., \mu_g$ in terms of the $\lambda_1, ..., \lambda_g$. We indicate with B the righthand side of Π

$$\Pi = \begin{pmatrix} I & B \end{pmatrix}$$

We will show that if B is rational then X admits rational s-Kähler structures compatible with its complex structure.

PROPOSITION 9.4. — *A form $\phi \in \Omega^s(X, \mathbb{R})$ harmonic with respect to a translation invariant metric represents a rational cohomology class if and only if it assumes rational values on the lattice Λ, when considered as a multilinear form.*

Proof. — Let ϕ be a translation invariant (and hence harmonic) form of $\Omega^1(X, \mathbb{R})$ representing the cohomology class $[\phi] \in H^1_{DR}(X, \mathbb{R})$, and let $\lambda \in \Lambda$, representing the loop $[\lambda] \in \pi_1(X)$. The integration isormorphism

$$\int : \; H^1_{DR}(X, \mathbb{R}) \to Hom(\pi_1(X), \mathbb{R})$$

is compatible (via De Rham's theorem) with the finer $H^1(X, \mathbb{Z}) \cong Hom(\pi_1(X), \mathbb{Z})$, and can be read directly as

$$\int_{[\lambda]} [\phi] = \phi(\lambda)$$

Taking iterated exterior powers one obtains the statement of the proposition. □

We define

$$v_{1j} = \lambda^*_{j+1}, \quad v_{2j} = Jv_{1j} = -v_{1j} \circ J$$

with the dual taken with respect to the real basis $\lambda_1, ..., \lambda_g, J\lambda_1, ..., J\lambda_g$.

PROPOSITION 9.5. — *The $(g-1)$-Kähler structure determined by the above adapted coframe is rational on X, i.e. all the operators $L_{jk}, V_j, \Lambda_{jk}, A_j$ for varying and different j and k act on rational cohomology.*

Proof. — We start by proving that the structure forms ω_{jk} and the Vol_j have rational cohomolgy classes. From the previous proposition, it is enough to prove that for all r, if λ_p, μ_q are elements of this basis,

$$\{\lambda^*_r(\lambda_p), \lambda^*_r(\imath\lambda_p), \lambda^*_r(\mu_r), \lambda^*_r(\imath\mu_r)\} \subset \mathbb{Q}$$

The first family is made up of delta functions, and the second one is zero. The third and the fourth can be expressed as rational linear combinations of the real and imaginary parts of the entries of the period matrix B.

From this and the linear independence of the v_{ij} it follows that

$$H^*(X,\mathbb{Q}) = \bigwedge\nolimits_{\mathbb{Q}}^{*} < v_{ij} \mid i = 1,2 \;\; j = 0,...,g-1 >$$

We observe finally that the pointwise adjoints Λ_{ij}, A_j of the L_{jk}, V_j are covariant constant, and therefore commute with the Laplacian (see Theorem 8.1). Moreover, they send rational harmonic forms (which are represented by rational polynomials in the v_{ij} by the previous proposition) into polynomials in the v_{ij} which have again rational coefficients. $\square$

The construction above works when the period matrix is rational. We conjecture that it should be possible to perform a similar construction also when B has higher transcendence degree, and more precisely:

CONJECTURE 9.6. — *On any complex torus there exist rational s-Kähler structures compatible with the complex structure, with $g - trdeg(B) - 1 \leqslant s$ (and of course $s \leqslant g-1$).*

DEFINITION 9.7. — *Given a rational s-Kähler structure on the manifold X, we indicate with $\mathbf{L}_{jk}, \mathbf{V}_j, \mathbf{\Lambda}_{jk}, \mathbf{A}_j$ the classes of the corresponding global sections of operators of $\mathcal{L}^s_{\mathbb{R}}$ when acting on rational cohomology $H^*(X,\mathbb{Q})$. We indicate with $\mathbb{L}^s_{\mathbb{Q}}$ the Lie algebra over $\mathbb{Q}$ generated by the operators above inside $End_{\mathbb{Q}}(H^*(X,\mathbb{Q}))$.*

10. The irreducible module of rational Hodge classes

Recall from Theorem 6.3 that the algebra $\mathbb{L}^s_{\mathbb{R}} = \mathbb{L}^s_{\mathbb{Q}} \otimes \mathbb{R}$ is isomorphic to $\mathbf{su}(s+1, s+1)$.

Let X be an abelian variety with a rational period matrix and a fixed rational s-Kähler structure compatible with the complex structure (guaranteed to exist from the previous section). In this section we are going to show that the Hodge classes of X constitute an irreducible $\mathbb{L}^s_{\mathbb{Q}}$-module (see Theorem 10.2 below). Let us indicate with V the standard representation of $\mathbb{L}^s_{\mathbb{C}} = \mathbb{L}^s_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbf{sl}(2s+2,\mathbb{C})$ (i.e. the irreducible complex highest weight module of weight $(1,0,\ldots,0)$).

Proposition 10.1. — *For every $i = 0, 1, \ldots, 2s+2$ we have the following isomorphism of $\mathbb{L}^s_{\mathbb{C}}$ irreducible modules:*

$$\mathcal{I}_{-s-1+i} = \bigoplus_r H^{r,-s-i-1+r} \cong \bigwedge^i V$$

Proof. — When $i = 0$ or $i = 2s+2$ the spaces involved are 1-dimensional. When $1 \leqslant i \leqslant s+1$, the form

$$v_i = w_0 w_1 \cdots w_s \overline{w}_{s-i+1} \overline{w}_{s-i+2} \cdots \overline{w}_s$$

is a highest weight vector for the $\mathbb{L}^s_{\mathbb{Q}}$ representation $\mathcal{I}_{-s-1+i}$ and it generates an irreducible representation isomorphic to $\bigwedge^i V$ with weight $(0, \ldots, 0, 1, 0, \ldots, 0)$ (1 is in the i-th position).

To see this, it suffices to consider the identification of $\mathbb{L}^s_{\mathbb{Q}}$ with $\mathbf{sl}(2s+2, \mathbb{C})$ provided by the faithful representation $\mathcal{I}_{-s}$ and the set of "positive" Serre generrators described in the proof of Theorem 6.2. We can then conclude by a dimensional argument.

The case when $s+2 \leqslant i < 2s+2$ follows by considering the action of the $*$ operator. □

Theorem 10.2. — *Let X be an abelian variety with rational period matrix and a rational s-Kähler structure compatible with the complex structure. Then the algebra $\mathbb{L}^s_{\mathbb{Q}}$ acts irreducibly on the rational Hodge classes $Hodge^{\cdot,\cdot}(X) \otimes \mathbb{Q}$.*

Proof. — First of all, we observe that all the generators of $\mathbb{L}^s_{\mathbb{Q}}$ are J-invariant and rational, and therefore send rational Hodge classes to rational Hodge classes. Then, the previous proposition shows that if we complexify the representation on rational Hodge classes and the algebra we obtain the irreducible module $\mathcal{I}_0$ over $\mathbb{L}^s_{\mathbb{C}}$. Therefore, a fortiori, $Hodge^{\cdot,\cdot}(X) \otimes \mathbb{Q}$ is irreducible over $\mathbb{Q}$. □

Remark 10.3. — From Theorem 6.3 and the subsequent remark one immediately deduces an explicit matrix presentation of $\mathbb{L}^s_{\mathbb{Q}}$, via its faithful representation on the forms of $\mathcal{I}_{-s}$ at any given point.

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