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Invariant Spin Structures on Riemann Surfaces

Sadok Kallel[1], Denis Sjerve[2]

Abstract. — We investigate the action of the automorphism group of a closed Riemann surface of genus at least two on its set of theta characteristics (or spin structures). We give a characterization of those surfaces admitting a non-trivial automorphism fixing either all of the spin structures or just one. The case of hyperelliptic curves and of the Klein quartic are discussed in detail.

Résumé. — Dans ce travail, nous étudions l'action du groupe d'automorphismes conformes d'une surface de Riemann de genre supérieur à deux sur ses structures spin. Nous caractérisons de telles surfaces qui admettent un automorphisme non-trivial fixant soit toutes les structures spin à la fois, soit seulement une. Les cas des courbes hyperelliptiques et de la quartique de Klein sont analysés en détail.

1. Introduction

Spin structures on Riemann surfaces or "theta characteristics" are classical objects of great use and interest in all of geometry, topology and physics. Let C be a closed Riemann surface and write $U(C)$ for its unit tangent bundle. A spin structure on C is a cohomology class in $H^1(U(C);\mathbb{Z}_2)$ whose

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restriction to each fiber is a generator of $H^1(S^1;\mathbb{Z}_2)$. There is of course a great number of ways to see what a spin structure is, a few of which we will encounter in the course of this paper.

Every Riemann surface C has 2^{2g} distinct spin structures where g is the genus of C. As described in [4], $Spin(C)$ has a natural structure of affine space over $\mathbb{Z}_2$ with $H^1(X,\mathbb{Z}_2)$ as its group of translations. The group of conformal automorphisms of C; $\mathrm{Aut}(C)$, acts on $H^1(U(C);\mathbb{Z}_2)$ and hence on $Spin(C)$ by pullback. This is an affine action.

In this paper we study invariant spin structures on compact Riemann surfaces, of genus at least two, under the action of $\mathrm{Aut}(C)$. Motivation for this work comes from a question by Jack Morava to the authors and from the paper [4] in which Atiyah proves that if $f : C \to C$ is an automorphism of a compact Riemann surface C then there necessarily exists a spin structure ξ that is invariant under $f : C \to C$.

This raises natural questions.

1. Is there an automorphism $f : C \to C$ that leaves only one spin structure invariant?

2. Is there a non-trivial automorphism $f : C \to C$ that leaves every spin structure invariant?

3. How many spin structures are invariant under a given $f : C \to C$?

4. How large can the isotropy subgroup of a given spin structure in $\mathrm{Aut}(C)$ be?

The first question admits the following answer.

THEOREM 1.1. — *Suppose $f : C \to C$ is an automorphism of order n, where n is odd. Then f leaves only one spin structure invariant if, and only if, the associated orbit surface $C/\mathbb{Z}_n$ has genus zero.*

We reach this theorem by reducing the problem to linear algebra using interesting constructions of Johnson [9]. This reduction takes the following form.

Let $A \in SL_{2g}(\mathbb{Z})$ denote the $2g \times 2g$ matrix representing the induced isomorphism $f_* : H_1(C;\mathbb{Z}) \to H_1(C;\mathbb{Z})$, with respect to some basis, and let $\bar{A}$ is its mod 2 reduction. Let $X = (x_1, \ldots, x_{2g})^T$ be a column vector representing an element of $H^1(C;\mathbb{Z})$, and let $\bar{X}$ be its mod 2 reduction.

Then we observe in section 2 that *spin structures left invariant by an automorphism $f : C \to C$ are in 1-1 correspondence with solutions $\bar{X}$ of the matrix equation* $(\bar{A}^T - I)\bar{X}^T = 0$ (Corollary 2.9). In particular we see that an orientation preserving diffeomorphism fixes all spin structures if and only if f acts trivially on $H_1(C; \mathbb{Z}_2)$. This is one form of the main theorem A of Sipe [15].

In section 2.1 we combine classical results of Nielsen and Serre to obtain an answer to question (2)

THEOREM 1.2. — *Let C be a compact Riemann surface of genus at least two. A non-trivial automorphism $f : C \to C$ leaves every spin structure invariant if, and only if, C is hyperelliptic and f is the hyperelliptic involution.*

Consequences of our calculations in section 2 also give answers to our remaining questions (3) and (4) respectively.

COROLLARY 1.3. —

(i) If k is the dimension of the eigenspace of $\bar{A}$ associated to the eigenvalue 1, then the number of f-invariant spin structures is 2^k.

(ii) The Klein quartic curve $\mathcal{K}$ of genus 3 and maximal group of automorphisms has a unique invariant spin structure under the entire group.

Corollary 1.3 (i) gives a more refined version of the theorem of Atiyah on the existence of an invariant spin-structure [4].

Corollary 1.3 (ii) is not new (see [7]). Our proof however seems novel and is simply obtained by inputting computations of [14] in our matrix equations. Moreover, in section 4 we give a very explicit and elementary description of the divisor associated to this unique invariant spin structure which we couldn't find in the literature (see theorem 4.2).

Finally, in section 3 we give a complete count of invariant spin structures for automorphisms of hyperelliptic surfaces based on a combinatorial definition of spin structures due to Mumford [12]. See propositions 3.3 and 3.4. The genus two case is then completely dissected.

Interesting results on invariant *r-roots* can be found in [15] (a "square root" being a spin structure). A growing number of references on invariant spin structures have to do with mapping class groups and moduli spaces of spin curves. A nice discussion of this theme is in [10].

Added in revision. — This work was posted on the archives in 2006[1]. In 2007 an independent proof of Theorem 1.2 appeared in work of Indranil Biswas and coauthors[2]. We were both unaware of each other's work and interestingly our methods are entirely different. This theorem has later been given an extension to irreducible smooth projective curves defined over an algebraically closed field of characteristic prime to two[3].

2. Spin Structures and Automorphisms

We assume C is smooth, closed, connected and orientable, of genus $g \geqslant 1$. Let $S^1 \xrightarrow{i} U(C) \xrightarrow{\pi} C$ be the unit tangent bundle. We adopt Johnson's definition of a spin structure [9]; namely this is a cohomology class $\xi \in H^1(U(C);\mathbb{Z}_2)$ that restricts to a generator of $H^1(S^1;\mathbb{Z}_2) \approx \mathbb{Z}_2$ for every fiber S^1.

There are short exact sequences

$$0 \longrightarrow H_1(S^1;\mathbb{Z}_2) \xrightarrow{i_*} H_1(U(C);\mathbb{Z}_2) \xrightarrow{\pi_*} H_1(C;\mathbb{Z}_2) \longrightarrow 0$$

$$0 \longleftarrow H^1(S^1;\mathbb{Z}_2) \xleftarrow{i^*} H^1(U(C);\mathbb{Z}_2) \xleftarrow{\pi^*} H^1(C;\mathbb{Z}_2) \longleftarrow 0$$

from which one deduces that the set $Spin(C)$ of spin structures on C is in $1-1$ correspondence with $H^1(C;\mathbb{Z}_2) \approx \mathbb{Z}_2^{2g}$, and therefore there are 2^{2g} spin structures on C. This correspondence is not a group isomorphism and $Spin(C)$ is in fact the non-trivial coset of $H^1(C;\mathbb{Z}_2)$ in $H^1(U(C);\mathbb{Z}_2)$.

Suppose ω is a smooth simple closed curve on C. Let $\overline{\omega}$ denote the lift of ω to $U(C)$ given by the unit tangent vector at each point of ω. There are actually 2 lifts, but they are homotopic in $U(C)$ by the homotopy that rotates one tangential direction to the other.

We will let $z \in H_1(U(C);\mathbb{Z}_2)$ denote the class represented by the tangential framing on $\partial(D^2)$, where D^2 is any closed 2-disc in C. As is nicely explained by Johnson, "Intuitively a spin structure $\zeta \in Spin(C)$ is a function assigning a number mod 2 to each framed curve of C, subject to the usual homological conditions and also that the boundary of a disc in C tangentially framed receives one".

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(1) http://front.math.ucdavis.edu/0610.5568v1
(2) *On theta characteristics of a compact Riemann surface*, Bull. Sci. math. 131 (2007) 493-499.
(3) http://front.math.ucdavis.edu/0804.1599

Lemma 2.1 *(Johnson [9]). — Suppose the homology class* $u \in H_1(C;\mathbb{Z}_2)$ *is represented by* $\omega_1 + \omega_2 + \cdots + \omega_r$*, where* $\{\omega_1, \omega_2, \ldots, \omega_r\}$ *is a set of non-intersecting smooth simple closed curves. Then* $\overline{\omega}_1 + \overline{\omega}_2 + \cdots + \overline{\omega}_r + rz$ *depends only on the class* u*, and not on the particular representation.*

Definition 2.2. — $\tilde{u} = \overline{\omega}_1 + \overline{\omega}_2 + \cdots + \overline{\omega}_r + rz.$

This canonical lifting from $H_1(C;\mathbb{Z}_2)$ to $H_1(U(C);\mathbb{Z}_2)$ fails to be a homomorphism. This is made explicit by the following result in [9].

Lemma 2.3. — *If* $a, b \in H_1(C;\mathbb{Z}_2)$ *then* $\widetilde{a+b} = \tilde{a} + \tilde{b} \; + \langle a, b\rangle z$*, where* $\langle a, b\rangle$ *is the intersection pairing.*

Let $e_1, \ldots, e_{2g}$ denote a basis of $H_1(C;\mathbb{Z}) \approx \mathbb{Z}^{2g}$. We do not assume that this basis is symplectic. The following is clear.

Lemma 2.4. — *A basis for* $H_1(U(C);\mathbb{Z})$ *is given by* $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{2g}, z$.

Let the dual basis of $H^1(U(C);\mathbb{Z})$ be denoted $\zeta_1, \ldots, \zeta_{2g}, \eta$. Then the mod 2 reduction $\bar{\eta} \in H^1(U(C);\mathbb{Z}_2)$ is a particular spin structure on C. It follows that the set of spin structures is given by

$$Spin(C) = \left\{ \sum_{i=1}^{2g} x_i \bar{\zeta}_i + \bar{\eta} \;\middle|\; \text{all } x_i \in \mathbb{Z}_2 \right\}.$$

We can now determine the action of an automorphism $f : C \to C$ on $Spin(C)$. Write $f_*(e_i) = \sum_{j=1}^{2g} a_{ji} e_j, \; i = 1, \ldots, 2g$, where the $a_{i,j}$ are the entries of $A \in SL_{2g}(\mathbb{Z})$.

Definition 2.5. — $v_i = \sum_{1 \leqslant j_1 < j_2 \leqslant 2g} a_{j_1 i} a_{j_2 i} \langle e_{j_1}, e_{j_2}\rangle$,

$V = [v_1, v_2, \ldots, v_{2g}]$.

We use the notation V_f or V_A if we want to emphasize that the vector V comes from f or A. Note that if f is an orientation preserving diffeomorphism of C inducing f_* (resp. f^*) on H_1 (resp. H^1) then $\widetilde{f_*(a)} = f_*(\widetilde{a})$ (and same for f^*). This is because $f_*(\overline{a}) = \overline{f_*(a)}$ since f_* acts by its differential on the tangent space and because $f_*(z) = z$. The following computation is an easy corollary of this fact

Lemma 2.6. — *With* $f_*(e_i) = \sum_{j=1}^{2g} a_{ji} e_j$ *for* $i = 1, \ldots, 2g$, *we have*

$$\begin{aligned} f_*(\tilde{e_i}) &= \widetilde{f_*(e_i)} = \sum_{j=1}^{2g} a_{ji}\tilde{e_j} + v_i z \ , \ \textit{and} \ \ f_*(z) = z \\ f^*(\zeta_i) &= \sum_{k=1}^{2g} a_{ik}\zeta_k \ \textit{and} \ f^*(\eta) = \sum_{k=1}^{2g} v_k\zeta_k + \eta. \end{aligned}$$

Thus the matrices for $H_1(U(C);\mathbb{Z}) \xrightarrow{f_*} H_1(U(C);\mathbb{Z})$ and $H^1(U(C);\mathbb{Z}) \xrightarrow{f^*} H^1(U(C);\mathbb{Z})$ are $\begin{bmatrix} A & 0 \\ V & 1 \end{bmatrix}$ and $\begin{bmatrix} A^T & V^T \\ 0 & 1 \end{bmatrix}$ respectively. After reducing mod-2, lemma 2.6 immediately yields

Lemma 2.7. — *Suppose* $\xi = \sum_{i=1}^{2g} x_i\bar{\zeta_i} + \bar{\eta}$ *is a spin structure. Then*

$$f^*(\xi) = \sum_{k=1}^{2g}\left(\sum_{i=1}^{2g} a_{ik}x_i + v_k\right)\bar{\zeta_k} + \bar{\eta}.$$

Let X denote the column vector $(x_1, \ldots, x_{2g})^T$. Then

Corollary 2.8. — $\xi = \sum_{i=1}^{2g} x_i\bar{\zeta_i} + \bar{\eta}$ *is an invariant spin structure for* $f : C \to C$ *if, and only if,*

$$(\bar{A}^T - I)\bar{X} = \bar{V}^T \tag{2.1}$$

If we suppose that ξ_1, ξ_2 are invariant spin structures associated to column vectors X_1, X_2, then each vector satisfies equation (2.1), and therefore $X = X_1 - X_2$ satisfies $(\bar{A}^T - I)\bar{X}^T = 0$. Since we know by a result of Atiyah [4] that for a given $f \in \mathrm{Aut}(C)$ there must exists at least an invariant spin structure ζ, we readily deduce that

Corollary 2.9. — *Spin structures left invariant by an automorphism* $f : C \to C$ *are in 1-1 correspondence with solutions* $\bar{X}$ *of the matrix equation* $(\bar{A}^T - I)\bar{X}^T = 0$.

2.1. Proofs of Main Statements

We first need information on the similarity class of the matrix $A \in SL_{2g}(\mathbb{Z})$.

DEFINITION 2.10. — *Let $\Phi_d(x)$ be the cyclotomic polynomial generated by the primitive d^{th} roots of unity and let C_d denote the $\phi(d) \times \phi(d)$ companion matrix of $\Phi_d(x)$, that is*

$$C_d = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & 1 \\ -a_0 & -a_1 & \cdot & \cdot & \cdots & -a_{\phi(d)-1} \end{bmatrix}$$

where $\phi(d)$ denotes Euler's totient function and $\Phi_d(x) = a_0 + a_1 x + \cdots + a_{\phi(d)-1} x^{\phi(d)-1} + x^{\phi(d)}$.

Suppose $f : C \to C$ is an automorphism of order n. If $g \geqslant 2$ then A also has order n. It follows that over the rationals $\mathbb{Q}$, A is similar to a direct sum of companion matrices. In fact, according to [16], $\exists$ unique distinct divisors $1 \leqslant d_1 < d_2 < \cdots < d_r \leqslant n$ of n and unique positive integers $e_1, \ldots, e_r$ such that

- A is similar to $e_1 C_{d_1} \oplus e_2 C_{d_2} \oplus \cdots \oplus e_r C_{d_r}$ as a matrix in $GL_{2g}(\mathbb{Q})$.
- $n = LCM(d_1, \cdots, d_r)$ and $2g = e_1\phi(d_1) + e_2\phi(d_2) + \cdots e_r\phi(d_r)$.

The minimal and characteristic polynomials of C_d are both $\Phi_d(x)$. It follows that the minimal and characteristic polynomials of A are

$$\mu_A(x) = \prod_{i=1}^{r} \Phi_{d_i}(x),\ \ \gamma_A(x) = \prod_{i=1}^{r} \Phi_{d_i}(x)^{e_i} \text{ respectively.}$$

From the factorization $x^{n-1} + x^{n-2} + \cdots + 1 = \prod_{d|n,d>1} \Phi_d(x)$ it follows that $n = \prod_{d|n,d>1} \Phi_d(1)$ and $det(I - A^T) = \gamma_A(1) = \prod_{i=1}^{r} \Phi_{d_i}^{e_i}(1)$. The following lemma is well known.

LEMMA 2.11. — $\Phi_d(1) = \begin{cases} 0 & \textit{if } d = 1 \\ p & \textit{if } d = p^k,\ p = \textit{a prime} \\ 1 & \textit{in all other cases} \end{cases}$

COROLLARY 2.12. — *$\bar{A}^T - I$ is invertible (as a matrix over $\mathbb{Z}_2$) if, and only if, $d_1 > 1$ and none of the d_i are powers of 2.*

From corollary 2.9 it follows that $f : C \to C$ has a unique invariant spin structure if, and only if, $\bar{A}^T - I \in SL_{2g}(\mathbb{Z}_2)$. This happens if, and only if, $det(A - I)$ is odd. Now theorem (1.1) of §1 will follow from this observation.

Theorem 2.13. — *Suppose $f : C \to C$ is an automorphism of order n, where n is odd. Then f leaves only one spin structure invariant if, and only if, the associated orbit surface $C/\mathbb{Z}_n$ has genus zero.*

Proof. — If V is a vector space and $L : V \to V$ is a linear map let $E(L, \lambda) = \{v \in V \mid L(v) = \lambda v\}$, the eigenspace of L associated to λ. Let $\mathcal{H}$ (resp. $\mathcal{H}^*$) denote the complex vector space of holomorphic (resp. anti-holomorphic) differentials on the curve C. Then $\dim \mathcal{H} = g$ the genus of C. The automorphism $f : C \to C$ induces linear automorphisms $L : \mathcal{H} \to \mathcal{H}$, $L^* : \mathcal{H}^* \to \mathcal{H}^*$. The subspaces $E(L, 1)$ and $E(L^*, 1)$ correspond to the $\mathbb{Z}_n$-invariant differentials, where $\mathbb{Z}_n$ is the cyclic subgroup generated by f, and thus they have dimension $h := genus(C/\mathbb{Z}_n)$. Now there is an isomorphism $H^1(C; \mathbb{Z}) \otimes \mathbb{C} \approx \mathcal{H} \oplus \mathcal{H}^*$ which is compatible with $f^* : H^1(C; \mathbb{Z}) \otimes \mathbb{C} \to H^1(C; \mathbb{Z}) \otimes \mathbb{C}$ on the one hand, and with $L \oplus L^* : \mathcal{H} \oplus \mathcal{H}^* \to \mathcal{H} \oplus \mathcal{H}^*$ on the other. From this it follows that the dimension of $E(f^*, 1) = E(A, 1)$ equals $2h$. From the rational canonical form $e_1 C_{d_1} \oplus e_2 C_{d_2} \oplus \cdots \oplus e_r C_{d_r}$ we see that the dimension of $E(A, 1)$ is 0 if $d_1 > 1$ and e_1 otherwise. The result now follows from corollary 2.12 and corollary 2.9. $\square$

Remark 2.14. — Note that if f has order n, and k is the dimension of the eigenspace of $\bar{A}$ associated to the eigenvalue 1, then necessarily $k \geqslant 2h$ where h is the genus of the orbit surface $C/\mathbb{Z}_n$. This is because $2h$ is the dimension of the eigenspace of A associated to the eigenvalue 1.

Next we give a proof of Theorem 1.2.

Proof. — Suppose C is hyperelliptic and f is the hyperelliptic involution J. Then the induced isomorphism $J_* : H_1(C; \mathbb{Z}) \to H_1(C; \mathbb{Z})$ is $-I$. Therefore $V_J = [0, 0, \ldots, 0]$ and equation (2.1) for an invariant spin structure becomes trivial.

Conversely, suppose $f : C \to C$ is a non-trivial automorphism leaving every spin structure invariant. We want to prove that C is necessarily hyperelliptic and that f is its hyperelliptic involution. Observe in that case that equation (2.1) is valid for all vectors X and therefore $\bar{A} = I$. By a theorem of Serre (see p. 293 of [8]) we see that A must have order 2, and therefore since $\mathrm{Aut}(C)$ embeds in $Sp(g, \mathbb{Z})$ for $g \geqslant 2$, the automorphism $f : C \to C$ is also of order 2. Now a theorem of Nielsen [13] states that f is determined up to conjugacy by its fixed point data. In the case of an automorphism of order 2 the fixed point data is just the number of fixed points, and therefore two involutions with the same number of fixed points are conjugate. Let the number of fixed points be r. It is known [8] that if

$f \neq id$, then $r \leqslant 2g+2$. Assume $r < 2g+2$ and write $2s = 2g+2-r > 0$ (r must be even because of the Riemann-Hurwitz formula). Then f must be conjugate to an automorphism as depicted in Figure 1.

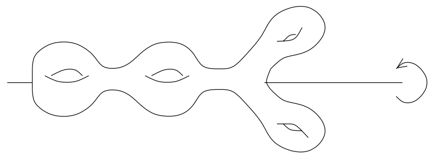


Figure 1. — An involution with $r < 2g+2$ fixed points

Therefore we can choose a basis so that $A = \begin{bmatrix} -I_r & 0 & 0 \\ 0 & 0 & -I_s \\ 0 & -I_s & 0 \end{bmatrix}$. But this contradicts the equation $\bar{A} = I$, and therefore $r = 2g+2$. Thus C is hyperelliptic and f is the hyperelliptic involution. $\square$

2.2. Mapping Class Group

Let $\Gamma_g(C)$ be the mapping class group of C; that is the group of isotopy classes of orientation preserving diffeomorphisms of C, and assume $g \geqslant 2$. It is well-known that $\Gamma_g(C)$ is a finitely generated group with generators the Dehn twists around representative loops of a symplectic basis of C. If $[f] \in \Gamma_g$, then the action of f on $Spin(C)$ (viewed as cohomology classes) is independent of the choice of this representative so that Γ_g acts on $Spin(C)$. Let $S_g \subset \Gamma_g$ be the subgroup of Γ_g that fixes all spin structures. The following is then immediate from our equation (2.1).

Corollary 2.15 *[15]. — The subgroup S_g is precisely the subgroup of elements that induce the identity on $H_1(C;\mathbb{Z}_2)$.*

In fact in [15], Sipe states her theorem for all n-roots and she identifies S_g with the subgroup of elements which induce the identity on $H_*(U(C);\mathbb{Z}_2)$. But an orientation preserving automorphism is the identity on $H_*(U(C);\mathbb{Z}_2)$ if and only if it is the identify on $H_*(C;\mathbb{Z}_2)$.

3. Spin Structures on Hyperelliptic Curves

We use a convenient description of the spin structures on a hyperelliptic curve in terms of divisors due to Mumford ([12] or [3], appendix B) to count invariant spin structures for subgroups of the automorphism group.

From the appendix we have that

$$Spin(C) = \{D \in Cl(C) \mid 2D = K\} \tag{3.1}$$

where $Cl(C)$ is the divisor class group of C. We will denote by $J_2(C)$ the subgroup of points of order two in $J(C)$; the "Jacobian" of line bundles of degree zero. See [3].

3.1. Spin Divisors on a Hyperelliptic Curve

We consider the hyperelliptic surface $y^2 = \prod_{i=1}^{2g+2}(x - e_i)$ having genus g and branch set $B = \{e_1, \ldots, e_{2g+2}\}$. The e_i are distinct points in the Riemann sphere. We will write $\pi : C \longrightarrow \mathbb{P}^1$ for the degree two covering map sending $(x, y) \mapsto x$, and we let p_i be the ramification points such that $\pi(p_i) = e_i$. Let D be any divisor of the form $2p_i$ or $x + y$ if $x, y \notin \{p_1, \ldots, p_{2g+2}\}$ (i.e. D is the divisor associated to the line bundle L obtained as the pullback of the unique line bundle over $\mathbb{P}^1$ of degree $+1$). Then the canonical divisor is given according to ([3, 12])

$$K_C = -2D + p_1 + \cdots + p_{2g+2} = (g-1)D \tag{3.2}$$

We can next determine the divisor of an element $L \in J_2(C)$. Let $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be the meromorphic function $\phi(z) = (z - e_i)/(z - e_j), i \neq j$, and consider the composite $C \xrightarrow{\pi} \mathbb{P}^1 \xrightarrow{\phi} \mathbb{P}^1$. The divisor of the composite is $(\phi \circ \pi) = 2p_i - 2p_j$ and hence by definition $p_i - p_j$ is the divisor of a line bundle of order two on C $(i \neq j)$. More generally one sees that if $L \in J_2(C)$, then it is represented by either one of the divisors

$$E = p_{i_1} + \cdots + p_{i_k} - p_{j_1} - \cdots - p_{j_k} = kD - p_{l_1} - \cdots - \cdots - p_{l_{2k}} \tag{3.3}$$

The above formula shows that any finite set $T \subset \{1, \ldots, 2g+2\}$ of even cardinality gives an element in $J_2(C)$. One can further see that any such T and its complement $T^c \subset B$ give rise to the same element. If we define E_g to be the quotient

$$E_g := \{T \mid T \subseteq B, |T| \text{ even}\}/\sim, \text{ where } T' \sim T \Longleftrightarrow T' = T \text{ or } T' = T^c,$$

then the map

$$\alpha : E_g \longrightarrow J_2(C) \quad , \quad T \mapsto \alpha_T = \sum_{i \in T} (p_i - p_{2g+2})$$

defines an isomorphism between two copies of $\mathbb{Z}_2^{2g}$ [6].

THEOREM 3.1 *[12]. — Let $p_i \in C$ be as above. Then every theta characteristic is of the form*

$$E_T = kD + p_{i_1} + \cdots + p_{i_{g-1-2k}}$$

for $-1 \leqslant k \leqslant \frac{g-1}{2}$, $T := \{p_{i_1}, \cdots, p_{i_{g-1-2k}}\}$ with the i_α distinct. Moreover such a representation is unique if $k \geqslant 0$ and subject to a single relation $-D + p_{i_1} + \cdots + p_{i_{g+1}} = -D + p_{j_1} + \cdots + p_{j_{g+1}}$ when $k = -1$.

For a simple proof see also [3], exercises 26-32, p. 287-288.

Since to each branch point there corresponds a unique ramification point, we have

LEMMA 3.2. — *$Spin(C)$ corresponds to the set of all $T \subset B$ such that $|T| \equiv g + 1(2)$ modulo the equivalence relation $T \sim T^c$. This has a natural affine structure over E_g given by $\theta_S + \alpha_T = \theta_{T+S}$.*

3.2. Automorphisms

Let $f : C \longrightarrow C$ be an automorphism of a Riemann surface. Then f acts on $Spin(C)$ as in (3.1) by

$$f(\sum n_i P_i) = \sum n_i f^{-1}(P_i)$$

Indeed one way to see this is to replace the cotangent bundle by the tangent bundle, and since f is an automorphism, replace the pullback $f^* : T^*_{f(x)}C \longrightarrow T^*_x C$ by $f^{-1} : T_{f(x)}C \longrightarrow T_x C$.

For the rest of this section C will be hyperelliptic, B its branch set as in section 3.1 and J the hyperelliptic involution. This is a central element in $\mathrm{Aut}(C)$ and so we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}_2\{J\} \longrightarrow \mathrm{Aut}(C) \longrightarrow \overline{Aut}(C) \longrightarrow 0$$

where $\overline{Aut}(C)$ is necessarily a finite subgroup of $PSL_2(\mathbb{C})$ and hence is a cyclic, dihedral or polyhedral group. If $f : C \to C$ is an automorphism then we associate $\bar{f} \in \overline{Aut}(C)$. Here f acts on the ramification (i.e. Weierstrass) points and $\bar{f}$ on the branch set B. Since an element of $PSL_2(\mathbb{C})$ other than the identity fixes at most two points, we see that if $f \neq id, J$, then $\bar{f}$ acts on $B = \{e_1, \ldots, e_{2g+2}\}$ with at most two fixed points.

We will be then distinguishing <u>three cases</u>: when $\bar{f}$ acts fixed point freely, with one fixed point or with two.

We assume below that $f \neq id, J$, f has order n so that (wlog) $\bar{f}(e) = \zeta e$, $\zeta = e^{2\pi\imath/n}$. We use the notation $\langle e\rangle := \{e, \zeta e, \ldots, \zeta^{n-1}e\}$ to denote the orbit of $e \in \mathbb{P}^1(\mathbb{C}) = \mathbb{C}\cup\{\infty\}$ under the action of the cyclic group $\mathbb{Z}_n \subset PSL_2(\mathbb{C})$ generated by $\bar{f}$. Every orbit $\langle e\rangle$ has length n except possibly for two which are singletons: $\langle 0\rangle = \{0\}$ and $\langle\infty\rangle = \{\infty\}$.

A spin structure $[T]$ as in lemma 3.2 is invariant under the automorphism $f : C \to C$ if, and only if, $\zeta T = T$ or T^c. We now put this to good use.

Proposition 3.3. — *Assume $f \neq id$ has odd order n. Then the number of invariant spin structures under f is*

$$\begin{cases} 2^{r-2} & \textit{if } \bar{f} \textit{ acts freely}, \quad \textit{where } 2g+2 = nr \\ 2^{r-1} & \textit{if } \bar{f} \textit{ acts with one fixed point}, \quad \textit{where } 2g+2 = nr+1 \\ 2^{r} & \textit{if } \bar{f} \textit{ acts with two fixed points}, \quad \textit{where } 2g = nr \end{cases}$$

Proof. — Since n is odd an invariant spin structure is determined by a subset $T \subset B$ such that $\zeta T = T$ and $card(T) \equiv\ g+1\ (mod\ 2)$. Assume $\bar{f}$ acts freely on B so that B is the disjoint union

$$B = \langle e_1\rangle \sqcup \langle e_2\rangle \sqcup \cdots \sqcup \langle e_r\rangle,$$

where all the e_i are in $\mathbb{C}^* = \mathbb{C} - \{0\}$. By this notation we also mean that we have reordered the e_i so that $e_1, \ldots, e_r$ are representatives of the orbits $\langle e_i\rangle$ for $i = 1, \ldots, 2g+2$. There is no loss of generality in doing this. In that case and if some $\zeta^m e_i \in T$, then the entire orbit $\langle e_i\rangle \subset T$. Therefore T must be a disjoint union of the form

$$T = \langle e_{i_1}\rangle\sqcup\langle e_{i_2}\rangle\sqcup\cdots\sqcup\langle e_{i_k}\rangle, \text{ where } 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant r \text{ and } k = 0, \ldots, r.$$

Therefore $card(T) = kn \equiv k\ (mod\ 2)$. The number of such T is $\displaystyle\sum_{k=0}^{r}\binom{r}{k} = 2^r$, and the number satisfying $card(T) \equiv g+1\ (mod\ 2)$ is 2^{r-1}. This follows from the identity $\displaystyle\sum_{k=0}^{r}(-1)^k\binom{r}{k} = 0$. These subsets come in complementary pairs $\{T, T^c\}$, and therefore the number of invariant spin structures is 2^{r-2}.

Assume $\bar{f}$ acts with one fixed point so that

$$B = \langle \cdot \rangle \sqcup \langle e_1 \rangle \sqcup \langle e_2 \rangle \sqcup \cdots \sqcup \langle e_r \rangle, \quad e_i \in \mathbb{C}^*, \forall i$$

where $\langle \cdot \rangle$ is either $\langle 0 \rangle$ or $\langle \infty \rangle$. If $T \subset B$ satisfies $\zeta T = T$, then the possibilities for T are

$$\begin{array}{rcl} T & = & \langle e_{i_1} \rangle \sqcup \langle e_{i_2} \rangle \sqcup \cdots \sqcup \langle e_{i_k} \rangle, \text{ or} \\ T & = & \langle \cdot \rangle \sqcup \langle e_{i_1} \rangle \sqcup \langle e_{i_2} \rangle \sqcup \cdots \sqcup \langle e_{i_k} \rangle \end{array}$$

In both cases we must have $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant r$ and $k = 0, \ldots, r$. The number of such subsets T is 2^{r+1}, and the number satisfying $card(T) \equiv g + 1 \ (mod\ 2)$ is 2^r. These subsets come in complementary pairs, and therefore the number of invariant spin structures is 2^{r-1}.

Finally assume that $\bar{f}$ has two fixed points in B so that

$$B = \langle 0 \rangle \sqcup \langle \infty \rangle \sqcup \langle e_1 \rangle \sqcup \langle e_2 \rangle \sqcup \cdots \sqcup \langle e_r \rangle, \ e_i \in \mathbb{C}^*, \forall i$$

If $T \subset B$ satisfies $\zeta T = T$ then T can be any collection of orbits with the right cardinality so that by arguments similar to those above we also get 2^r invariant spin structures. $\square$

Proposition 3.4. — *Assume $f \neq id, J$ has even order n. Then the number of invariant spin structures is*

$$\begin{cases} 2^{r-1} & if\ \bar{f}\ acts\ freely\ on\ B\ and\ g\ is\ even\ \ (2g+2 = nr) \\ 2^r & if\ \bar{f}\ acts\ freely\ on\ B\ and\ g\ is\ odd\ \ (2g+2 = nr) \\ 2^r & if\ \bar{f}\ acts\ on\ B\ with\ two\ fixed\ points\ \ (2g = nr) \end{cases}$$

Note that when n is even, $\bar{f}$ cannot act on B with a single fixed point since $2g + 2 = nr + 1$.

Proof. — First assume $\bar{f}$ acts on B fixed point freely so that again B is the disjoint union $\langle e_1 \rangle \sqcup \langle e_2 \rangle \sqcup \cdots \sqcup \langle e_r \rangle$, where all the e_i are in $\mathbb{C}^* = \mathbb{C} - \{0\}$. Then there are 2 possibilities for an invariant spin structure $[T]$: either $\zeta T = T^c$ or $\zeta T = T$.

If $\zeta T = T^c$ occurs then necessarily T is the disjoint union of the sets

$$\{\zeta^{\epsilon_j} e_j, \zeta^{2+\epsilon_j} e_j, \zeta^{4+\epsilon_j} e_j, \ldots, \zeta^{n-1+\epsilon_j} e_j\}, \text{ where } j = 1, \ldots, r \text{ and } \epsilon_j = \pm 1 \ \forall j$$

This means that n is necessarily even (as is the case) and that ζ fixes no branch points. The number of such subsets is 2^r since it equals the number of choices of the ϵ_j. Every such subset T determines a spin structure $[T]$ because $card(T) = g+1$ in this case. These subsets come in equivalent pairs $\{T, T^c\}$, and therefore there are 2^{r-1} invariant spin structures in this case.

For the second possibility there are 2^r subsets $T \in \mathcal{S}_B$ such that $\zeta T = T$, namely

$$T = \langle e_{i_1}\rangle \sqcup \langle e_{i_2}\rangle \sqcup \cdots \sqcup \langle e_{i_k}\rangle, \text{ where } 1 \leqslant i_1 < \cdots < i_k \leqslant r,\ k = 0, 1, \ldots, r$$

Now $card(T) = kn \equiv 0\ (mod\ 2)$, and therefore $[T]$ is a spin structure if, and only if, g is odd. These spin structures come in complementary pairs, so we have 2^{r-1} invariant spin structures for this possibility.

In summary, when $\bar{f}$ acts fixed point freely there are 2^{r-1} invariant spin structures if g is even, and $2^{r-1} + 2^{r-1} = 2^r$ if g is odd.

The arguments for the other case are similar. $\square$

3.3. Genus Two Curves

As an exercise we count the invariant spin structures for all possible subgroups of $\mathrm{Aut}(C)$ in the case when C is of genus two; i.e. $C : y^2 = f(x)$ where f is a polynomial of degree 5 or 6. A complete list of all possible reduced automorphism groups that can occur together with their associated equations has been long known (and is attributed to O. Bolza). For those groups we can count precisely the number of fixed spin structures. It is given by the table below of which middle two columns we have taken from ([2], p.6):

$g = 2$	$\overline{Aut}$	representative f	# of fixed spin structures
(i)	$\mathbb{Z}_2$	$(x^2-1)(x^2-a)(x^2-b)$	4
(ii)	D_2	$(x^2-1)(x^2-a^2)(x^2-\frac{1}{a^2})$	2
(iii)	D_3	$x^6 - 2ax^3 + 1$	1
(iv)	D_6	$x^6 + 1$	1
(v)	$\mathfrak{S}_4$	$x(x^4-1)$	0
(vi)	$\mathbb{Z}_5$	$x^5 - 1$	1

Here the actual new statement is in the last column. By D_n we mean the dihedral group of order $2n$. The groups in the table must be finite subgroups of both $SO(3)$ (since the hyperelliptic involution is normal in $\mathrm{Aut}(C)$), and of the symmetric group $\mathfrak{S}_6$ (permutations of the six branch points). Note that 5 is the largest possible prime order in $\overline{Aut}$ according to ([8], p. 268). Note also that A_5, the isocahedral group, is not on the list since it has order 60 and hence an extension by the hyperelliptic involution will have order 120 which exceeds the Hurwitz bound on the order of $\mathrm{Aut}(C)$.

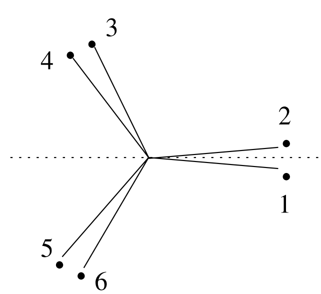
The spin structures in genus 2 are given by all equivalence classes of subsets of cardinality 3 in the branch set B (there are 10 of these corresponding to the even structures) together with all subsets of cardinality 5. For each case in the table above, the configuration of points of B in $\mathbb{P}^1$ is laid out according to the geometry of the associated group action. It is easier to write $B = \{1, 2, 3, 4, 5, 6\}$.

Proof. — (last column of table)

• (i) $\mathbb{Z}_2$ acts fixed point freely on B via the map $x \mapsto -x$ and hence according to proposition 3.4 it fixes $2^{r-1} = 4$ branch points (here $n = 2, r = 3$).

• (ii) This has to do with the geometry of a parallelogram with vertices at $a, \bar{a}, -\bar{a}, -a$ say ($|a| = 1$). Let's refer to these branch points by $1, 2, 3, 4$ respectively (so that "5" and "6" correspond to the remaining branch points 1 and -1). The two involutions are $\tau_1 = (12)(34)$ (reflection through the x-axis) and $\tau_2 = (13)(24)(56)$ (reflection through the y-axis). According to proposition 3.4, τ_1 and τ_2 fix 4 spin structures each but it is easy to see that they have only two such structures in common namely $\{5, 3, 4\}$ and $\{5, 1, 2\}$.

• (iii) This is the situation where the branch points are configured in the plane as in the figure below (each dot represents a branch point in $\mathbb{C} \subset \mathbb{P}^1$).



The polynomial in (iii) has roots $e^{\pm\alpha}, je^{\pm\alpha}, j^2e^{\pm\alpha}$ numbered as in the figure. There is only one spin structure invariant by the $\frac{2\pi}{3}$ rotation: namely $T = \{1,3,5\} \sim T' = \{2,4,6\}$. The involution with respect to the x-axis switches T and T' and hence D_3 has a unique invariant spin structure.

• (iv) $\mathbb{Z}_6$ acts cyclically on B without fixed points and hence according to proposition 3.4 it has a single fixed spin structure $T = \{1,3,5\} \sim T^c$.

• (v) This is the so-called "Bolza surface" with conformal automorphism group of order 48, which is the largest among all curves of genus 2. The branch points form an octahedron $\{1,2,3,4,N,S\}$ with north N and south S poles. The cylic group $\mathbb{Z}_4$ acts by rotating the equator $1 \mapsto 2 \mapsto 3 \mapsto 4$. According to proposition 3.4 this group fixes 2 spin structures which must be odd of the form $\{1,2,3,4,N\}$ or $\{1,2,3,4,S\}$. But there is an involution switching N and S and thus there can't be any invariant spin structure.

• (vi) corresponds to when $\overline{Aut}$ is of order $n = 5$ generated by $\bar{f}$ (in the notation of section 3.2) which acts by fixing the one branched point at ∞. According to proposition 3.3 $\bar{f}$ fixes $2^{r-1} = 1$ structures. $\square$

4. Spin Structures on Klein's Curve

In this section we show that the Klein's quartic curve $\mathcal{K}$ has a unique spin structure invariant under all automorphisms and give an explicit description of it.

Let $G = PSL(2, \mathbf{F}_7) = Aut(\mathcal{K})$. This is a simple group with presentation of the form

$$G = \langle R, S, T \mid R^2 = S^3 = T^7 = RST = 1,\ \text{etc}\rangle$$

Let $e_1, e_2, \ldots, e_6$ be a standard symplectic basis of $H^1(C;\mathbb{Z})$. That is a basis of the free abelian group $H^1(C;\mathbb{Z})$ chosen so that the intersection pairing is given as follows:

$$\langle e_i, e_j\rangle = \begin{cases} +1 & \text{if } j = i+3 \\ -1 & \text{if } j = i-3 \\ 0 & \text{in all other cases} \end{cases}$$

From the work of Rauch and Lewittes [14] we can find matrix representations for the induced maps R_*, S_*, T_* on $H_1(C;\mathbb{Z})$:

$$R_* = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$$S_* = \begin{bmatrix} -1 & 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 \end{bmatrix}$$

$$T_* = \begin{bmatrix} 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The corresponding V vectors are

$$V_R = [0,0,0,0,0,0], V_S = [-1,1,1,0,0,0], V_T = [0,0,0,-1,0,0].$$

Using equation (2.1), together with a Maple program, it is now direct to determine the sets of spin structures invariant under R, S, T respectively. The solution vectors $\bar{X}$ are:

1. $\bar{X} = [x_1, 0, x_3, x_4, x_5, x_4]$, where the x_j are arbitrary, for R-invariant spin structures.
2. $\bar{X} = [x_1 + x_2, x_1 + x_2, 1, x_1 + x_2, x_1, x_2]$, where the x_j are arbitrary, for S-invariant spin structures.
3. $\bar{X} = [0,0,1,0,0,0]$ for T-invariant spin structures.

It is now clear that Klein's curve admits only one invariant spin structure. This is the spin structure fixed by T and which by choosing the x_j appropriately we can show is also an invariant for R and S.

Remark 4.1. — The existence of a unique spin structure on Klein's curve is well-known to algebraic geometers [7]. It comes as follows. Let $X(p), p > 5$ prime, denote the modular curve defined as a compactification of the upper half-plane by the action of the principal congruence subgroup of level p. The group $G = \mathrm{PSL}(2, \mathbb{F}_{|})$ acts as a group of automorphisms on $X(p)$. Then [1, 6] the group $\mathrm{Pic}(X(p))^G$ of invariant line bundles is infinite with cyclic generator a $(2p-12)$-th root of the canonical bundle of degree $(p^2-1)/24$. When $p = 7$, $X(7) = K$ and the generator is a square root of the canonical bundle hence the spin structure in question.

4.1. Spin Structures on Quartics

We would like to say more about what this particular spin structure on $\mathcal{K}$ is. Consider the points $a = [1:0:0], b = [0:1:0]$ and $c = [0:0:1]$ belonging to $\mathcal{K}$, where $\mathcal{K}$ is described as the locus $x^3y + y^3z + z^3x = 0$ in $\mathbb{P}^2$. Let K be the canonical divisor on $\mathcal{K}$ which is determined by the intersection of the curve with any hyperplane $\mathbb{P}^1 \subset \mathbb{P}^2$. Then

THEOREM 4.2. — *The divisor class $\theta := 2a + 2b + 2c - K$ is the unique spin structure on the Klein curve $\mathcal{K}$ fixed by the entire group $Aut(\mathcal{K})$.*

Before attempting the proof, we make general remarks about spin structures on a quartic curve C in $\mathbb{P}^2$. There are $2^6 = 64$ spin structures on C as pointed out, of which 28 are "odd" and 36 "even" (see appendix). The canonical divisor class on C is the locus of the intersection of a hyperplane $\mathbb{P}^1 \subset \mathbb{P}^2$ with C. Suppose C has a bitangent going through the points p, q, $p \neq q$. This is a line $\mathbb{P}^1$ intersecting the curve in p, q with multiplicity two, and hence $K = 2p + 2q$ (in the divisor class group). By definition, $p + q$ is a spin structure on C. Since there are 28 bitangents, this accounts for all odd spin structures.

The even spin structures on the other hand are described as follows. Following [11], a secant line $\overline{ab}$, $a \neq b \in C$ is a *strict tangent* of C if either:

(i) $\overline{ab}$ is tangent to C at a point r different from a, b, or

(ii) $\overline{ab}$ is a triple tangent at a or at b.

This is equivalent to the condition that the divisor class $K = a + b + 2r$ for a point $r \in C$. A triangle is now *strictly biscribed* if all sides are strict tangents (see figure 2). A plane cubic has no such triangles but a plane quartic has 288 of them! The way they arise as it turns out is in groups of 8 triangles, one for every even spin structure.

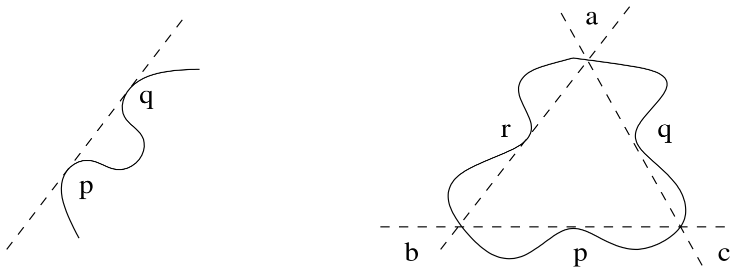


Figure 2. — A bitangent (odd theta) and a strictly biscribed triangle (even theta) fixed points

Let $\Delta(a, b, c)$ be a strictly biscribed triangle in a quartic C with double tangents at p, q, r as in figure 2. Then we can associate to Δ the even spin class

$$\theta(\Delta) = a + b + c + p + q + r - K$$

Using the relations $K = 2p + b + c = 2q + a + c = 2r + a + b$ we can indeed check that $2\theta(\Delta) = K$. It is known that for an even theta charateristic θ, there are 8 strictly biscribed triangles Δ such that $\theta(\Delta) = \Delta$ [6, 11].

Restricting to the case of the Klein curve $\mathcal{K}$, we know that $Aut(\mathcal{K}) = PSL(2, \mathbb{F}_7)$ fixes no bitangent and hence an invariant spin candidate must be even. Consider the points $a = [1 : 0 : 0], b = [0 : 1 : 0]$ and $c = [0 : 0 : 1] \in \mathcal{K}$. As discussed in [5] for instance, the transformation T of order 7 can be represented by

$$T : x \mapsto \gamma x \quad , \quad y \mapsto \gamma^4 y \quad , \quad z \mapsto \gamma^2 z$$

where $\gamma = e^{\frac{2\pi i}{7}}$. This transformation fixes the three points above and hence the triangle $xyz = 0$ they define. Note that a, b, c are inflexion points (i.e. points with triple tangents) and hence they form what Klein calls an inflexion triangle; that is the tangent at a meets $\mathcal{K}$ at b, and the tangent at b meets $\mathcal{K}$ at c and finally the tangent at c cuts $\mathcal{K}$ in a. This is also a biscribed triangle according to our definition with associated theta divisor

$$\theta := a + b + c + a + b + c - K = 2a + 2b + 2c - K$$

(since $r = a, p = b, q = c$).

Let's check that θ is fixed by the generators R and S of order two and three respectively. Now

$$S : x \mapsto y \quad , \quad y \mapsto z \quad , \quad z \mapsto x$$

so Δ and hence θ is invariant. To see that R acts trivially on θ it doesn't quite help to write the long expression for R given also in [5]. Instead consider the quotient $H = \mathcal{K}/\langle R\rangle$ by the cylic subgroup generated by R and write $\pi : \mathcal{K} \longrightarrow H$ the quotient map. Clearly H cannot be $\mathbb{P}^1$ since $\mathcal{K}$ is not hyperelliptic. The only other possibility by the Riemann-Hurwitz formula is that $genus(H) = 1$ and H is an elliptic curve. If $a \in \mathcal{K}$, then $\pi^{-1}(\pi(a)) = \{a, Ra\}$. But on H there is a meromorphic function f with a double pole at $\pi(a)$ and hence in the divisor class group and by definition $0 = (f \circ \pi) = 2a - 2R(a)$. This means that $2a + 2b + 2c = 2R(a) + 2R(b) + 2R(c)$ and hence $R(\theta) = \theta$ as desired.

The above discussion proves theorem 4.2.

Remark 4.3. — In [7], the action of $PSL(2, \mathbb{F}_7)$ on the set of even spin structures is described with orbit decomposition $36 = 1+7+7+21$. The one fixed spin generator is expressed explicitly in terms of the orbital divisors as $D_2 - D_3 - 7D_7$, where $\mathrm{Aut}(C)$ acts on $\mathcal{K}$ with orbits D_2, D_3, D_7 with stabilizers of orders $2, 3, 7$.

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