ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE Mathématiques

KARIM BOULABIAR, JAMEL JABEUR

Arens regularity of lattice-ordered rings

Tome XIX, nº S1 (2010), p. 25-36.

<http://afst.cedram.org/item?id=AFST_2010_6_19_S1_25_0>

Annales de la Faculté des Sciences de Toulouse Vol. XIX, n° Spécial, 2010
pp. 25–36

Arens regularity of lattice-ordered rings

Karim Boulabiar[(1)], Jamel Jabeur[(2)]

Abstract. — This work discusses the problem of Arens regularity of a lattice-ordered ring. In this prospect, a counterexample is furnished to show that without extra conditions, a lattice-ordered ring need not be Arens regular. However, as shown in this paper, it turns out that any f-ring in the sense of Birkhoff and Pierce is Arens regular. This result is then used and extended to the more general setting of almost f-rings introduced again by Birkhoff.

Résumé. — Ce travail aborde le problème de l'Arens régularité des anneaux réticulés. A cet égard, un contre-exemple est fourni pour montrer que, sans conditions supplémentaires, un anneau réticulé peut ne pas être Arens régulier. Néanmoins, comme il est démontré dans ce papier, il s'avère qu'un f-anneau au sens de Birkhoff et Pierce est Arens régulier. Ce résultat est ensuite employé et généralisé aux presque f-anneaux, introduits encore par Birkhoff.

1. Introduction

The classical books [3, 4] and the fundamental article [5] are adopted as the unique sources of unexplained terminology and notation in this paper. In order to avoid unnecessary repetition, we assume throughout that *all rings under consideration are associative and have multiplicative identities.* As usual, the symbol $\mathbb{R}$ is used to indicate the field of all real numbers.

(1) Département de Mathématiques, Faculté des Sciences de Tunis, Université Tunis-El Manar, Campus Universitaire, Tunis-2092, Tunisie.
karim.boulabiar@ipest.rnu.tn
(2) Département du Cycle Préparatoire, IPEST, Université du 7 Novembre à Carthage BP 51, 2070-La Marsa, Tunisia
jamel.jabeur@ipest.rnu.tn

A normed algebra is said to be Arens regular if the so-called first and second Arens multiplications coincide in its normed biconjugate. One of the famous and still unsettled problems in functional analysis is to characterize Arens regular normed algebras. However, neither in this introduction nor anywhere else in this work do we intend to touch the vast literature on the problem of Arens regularity of normed algebras. In fact, the main purpose of this paper is to discuss the corresponding problem in the completely different setting of lattice-ordered rings. A few more details seem to be in order.

Let $\mathcal{R}$ stand for a lattice-ordered ring. In the preliminaries section, we introduce the first and the second Arens multiplications in the complete lattice-ordered group $\mathrm{Hom}_b\left(\mathrm{Hom}_b\left(\mathcal{R},\mathbb{R}\right),\mathbb{R}\right)$. Then we define $\mathcal{R}$ to be Arens regular if these multiplications coincide in $\mathrm{Hom}_b\left(\mathrm{Hom}_b\left(\mathcal{R},\mathbb{R}\right),\mathbb{R}\right)$. In this regard, a counterexample is provided to show that without additional assumptions a lattice-ordered ring need not be Arens regular. In the third section of this note, we prove that any f-ring in the sense of Birkhoff and Pierce in [5] is Arens regular. This result is used and extended in the last section of this work. Namely, we prove that any almost f-ring as defined by Birkhoff in [4] is again Arens regular.

2. Preliminaries and a counterexample

The first paragraph of this section deals with the order biconjugate of an abelian lattice-ordered group (also called an ℓ-group). Let $\mathcal{G}$ be an abelian ℓ-group and let $\mathcal{G}^+$ denote the set of all positive elements in $\mathcal{G}$. In particular, if $u \in \mathcal{G}$, then $u^+ = u \vee 0 \in \mathcal{G}^+$ and $u^- = (-u) \vee 0 \in \mathcal{G}^+$. A map $\mathfrak{f} : \mathcal{G} \to \mathbb{R}$ is called a *group homomorphism* on $\mathcal{G}$ if

$$\mathfrak{f}\left(u+v\right) = \mathfrak{f}\left(u\right) + \mathfrak{f}\left(v\right) \quad \text{for all } u, v \in \mathcal{G}.$$

The group homomorphism $\mathfrak{f}$ on $\mathcal{G}$ is said to be *increasing* if

$$\mathfrak{f}\left(u\right) \leqslant \mathfrak{f}\left(v\right) \quad \text{for all } u, v \in \mathcal{G} \text{ with } u \leqslant v.$$

Of course, the group homomorphism $\mathfrak{f}$ on $\mathcal{G}$ is increasing if and only if $\mathfrak{f}\left(u\right) \geqslant 0$ for all $u \in \mathcal{G}^+$. The group homomorphism $\mathfrak{f}$ on $\mathcal{G}$ is said to be *bounded* if for every $u \in \mathcal{G}^+$ there exists $\mu \in (0, \infty)$ such that $\mathfrak{f}\left(v\right) \leqslant \mu$ whenever $0 \leqslant v \leqslant u$. This is equivalent to the condition that $\mathfrak{f}$ sends bounded sets to bounded sets (where we call a set *bounded* if it possesses an upper bound and a lower bound). The set $\mathrm{Hom}_b\left(\mathcal{G},\mathbb{R}\right)$ of all bounded group homomorphisms on $\mathcal{G}$ is a complete (and hence archimedean) abelian ℓ-group with respect to the pointwise addition and the ordering defined by

$$u \in \mathcal{G} \text{ and } u \geqslant 0 \text{ imply } \mathfrak{f}\left(u\right) \geqslant 0 \text{ in } \mathbb{R}$$

(see [3, Theorem 11.2.5]). For convenience, we use the notation $\mathcal{G}^{\star}$ instead of $\mathrm{Hom}_b(\mathcal{G},\mathbb{R})$. The complete abelian ℓ-group $\mathcal{G}^{\star}$ is called the *order conjugate* of $\mathcal{G}$. The order conjugate $(\mathcal{G}^{\star})^{\star}$ of $\mathcal{G}^{\star}$ is called the *order biconjugate* of $\mathcal{G}$ and denoted briefly by $\mathcal{G}^{\star\star}$. In other words, we have $\mathcal{G}^{\star\star} = \mathrm{Hom}_b(\mathrm{Hom}_b(\mathcal{R},\mathbb{R}),\mathbb{R})$.

The next lines are devoted to the first and the second Arens multiplications in the order biconjugate of a lattice-ordered ring. Recall that an abelian ℓ-group $\mathcal{R}$ which is simultaneously an associative ring with the property that $\mathcal{R}^{+}$ is closed under multiplication is called a *lattice-ordered ring* (briefly, an *ℓ-ring*). Throughout this note, $\mathcal{R}$ stands for an ℓ-ring with e as a multiplicative identity. Following constructions in [2] by Arens, a multiplication in the order biconjugate $\mathcal{R}^{\star\star}$ of $\mathcal{R}$ can be introduced in three steps, as follows. For every $f \in \mathcal{R}^{\star}$ and $u \in \mathcal{R}$, define $\mathfrak{f}u : \mathcal{R} \to \mathbb{R}$ by

$$(\mathfrak{f}u)(v) = \mathfrak{f}(uv) \quad \text{for all } v \in \mathcal{R}.$$

For every bounded subset $\mathcal{S}$ of $\mathcal{R}$ and every $u \in \mathcal{R}$, the set $u\mathcal{S} = \{uv : v \in \mathcal{S}\}$ is bounded. Therefore,

$$\mathfrak{f}u \in \mathcal{R}^{\star} \quad \text{for all } \mathfrak{f} \in \mathcal{R}^{\star} \text{ and } u \in \mathcal{R}.$$

Then, for $\mathfrak{F} \in \mathcal{R}^{\star\star}$ and $\mathfrak{f} \in \mathcal{R}^{\star}$, a map $\mathfrak{F}\mathfrak{f} : \mathcal{R} \to \mathbb{R}$ can be defined by

$$(\mathfrak{F}\mathfrak{f})(u) = \mathfrak{F}(\mathfrak{f}u) \quad \text{for all } u \in \mathcal{R}.$$

Also, it is not hard to see that

$$\mathfrak{F}\mathfrak{f} \in \mathcal{R}^{\star} \quad \text{for all } \mathfrak{F} \in \mathcal{R}^{\star\star} \text{ and } \mathfrak{f} \in \mathcal{R}^{\star}.$$

Finally, let $\mathfrak{F}, \mathfrak{G} \in \mathcal{R}^{\star\star}$ and $\mathfrak{F} \rtimes \mathfrak{G} : \mathcal{R}^{\star} \to \mathbb{R}$ be the map given by

$$(\mathfrak{F} \ltimes \mathfrak{G})(\mathfrak{f}) = \mathfrak{F}(\mathfrak{G}\mathfrak{f}) \quad \text{for all } \mathfrak{f} \in \mathcal{R}^{\star}. \tag{2.1}$$

Again, it is readily verified that

$$\mathfrak{F} \ltimes \mathfrak{G} \in \mathcal{R}^{\star\star} \quad \text{for all } \mathfrak{F}, \mathfrak{G} \in \mathcal{R}^{\star\star}.$$

The multiplication $\ltimes$ defined in (2.1) is called the *first Arens multiplication* in $\mathcal{R}^{\star\star}$. It is quite simple to check that the complete ℓ-group $\mathcal{R}^{\star\star}$ is an ℓ-ring with respect to the first Arens multiplication. Moreover, the group homomorphism $e^{\star\star}$ on $\mathcal{R}^{\star}$ defined by

$$e^{\star\star}(\mathfrak{f}) = \mathfrak{f}(e) \quad \text{for all } \mathfrak{f} \in \mathcal{R}^{\star}$$

is a multiplicative identity in $\mathcal{R}^{\star\star}$. Similarly, three steps are needed to introduce the second Arens multiplication in $\mathcal{R}^{\star\star}$. Indeed, let $\mathfrak{f} \in \mathcal{R}^{\star}$ and $u \in \mathcal{R}$. Define $u\mathfrak{f} \in \mathcal{R}^{\star}$ by

$$(u\mathfrak{f})(v) = \mathfrak{f}(vu) \quad \text{for all } v \in \mathcal{R}.$$

Thus, if $\mathfrak{F} \in \mathcal{R}^{\star\star}$ and $\mathfrak{f} \in \mathcal{R}^{\star}$, then $\mathfrak{f}\mathfrak{F} \in \mathcal{R}^{\star}$ can be given by

$$(\mathfrak{f}\mathfrak{F})(u) = \mathfrak{F}(u\mathfrak{f}) \quad \text{for all } u \in \mathcal{R}.$$

Now, let $\mathfrak{F}, \mathfrak{G} \in \mathcal{R}^{\star\star}$ and define $\mathfrak{F} \rtimes \mathfrak{G} \in \mathcal{R}^{\star\star}$ by

$$(\mathfrak{F} \rtimes \mathfrak{G})(\mathfrak{f}) = \mathfrak{G}(\mathfrak{f}\mathfrak{F}) \quad \text{for all } \mathfrak{f} \in \mathcal{R}^{\star}. \tag{2.2}$$

The *second Arens multiplication* in $\mathcal{R}^{\star\star}$ is the multiplication $\rtimes$ defined in (2.2). Under the second Arens multiplication, $\mathcal{R}^{\star\star}$ is again a complete ℓ-ring with the same $e^{\star\star}$ as a multiplicative identity.

We are in position at this point to present the central definition of this note.

Definition 2.1. — *An ℓ-ring $\mathcal{R}$ is said to be Arens regular if the first Arens multiplication and the second Arens multiplication coincide in the order biconjugate $\mathcal{R}^{\star\star}$ of $\mathcal{R}$.*

Next, we provide an example to illustrate the fact that without additional conditions, the ℓ-ring $\mathcal{R}$ need not be Arens regular. The key idea of this example goes back to the problem recently discussed in the interesting note [8] by Buskes and Page. Notice first that if the ℓ-ring $\mathcal{R}$ is commutative, then $\mathcal{R}$ is Arens regular if and only if $\mathcal{R}^{\star\star}$ is commutative with respect to the first (or the second) Arens multiplication.

Example 2.2. — Let $C^*(\mathbb{Z})$ denote the set of all bounded real-valued functions on the set $\mathbb{Z}$ of all integers. Under the pointwise addition and ordering, $C^*(\mathbb{Z})$ is a compete abelian ℓ-group. Consider the sub ℓ-group $\mathcal{R}$ of $C^*(\mathbb{Z})$ such that $u \in \mathcal{R}$ if and only if the series $\sum u(n)$ is absolutely convergent.

First, we claim that the order conjugate $\mathcal{R}^{\star}$ of $\mathcal{R}$ can be identified as an ℓ-group with $C^*(\mathbb{Z})$. To this end, pick $n \in \mathbb{Z}$ and define $e_n \in \mathcal{R}$ by

$$e_n(n) = 1 \quad \text{and} \quad e_n(m) = 0 \quad \text{for all } m \in \mathbb{Z} \text{ with } m \neq n.$$

Let $\mathfrak{f} \in \mathcal{R}^{\star}$ and define a map $\widetilde{\mathfrak{f}} : \mathbb{Z} \to \mathbb{R}$ by $\widetilde{\mathfrak{f}}(n) = \mathfrak{f}(e_n)$ for all $n \in \mathbb{Z}$. It is readily verified that $\widetilde{\mathfrak{f}} \in C^*(\mathbb{Z})$ for all $\mathfrak{f} \in \mathcal{R}^{\star}$. Moreover, it is simple to check that the map $\varphi : \mathcal{R}^{\star} \to C^{*}(\mathbb{Z})$ defined by $\varphi(\mathfrak{f}) = \widetilde{\mathfrak{f}}$ for all $\mathfrak{f} \in \mathcal{R}^{\star}$ is an ℓ-group isomorphism, which gives us the desired fact. This identification is used below without further mention.

Now, it is easily seen that $\mathcal{R}$ is a commutative ℓ-ring with respect to the multiplication given by

$$(uv)(n) = \sum_{m=-\infty}^{\infty} u(m)\, v(n-m) \quad \text{for all } u, v \in \mathcal{R}.$$

Moreover, the ℓ-ring $\mathcal{R}$ has e_0 as a multiplicative identity. A direct computation yields that

$$(u\mathfrak{f})(n) = \sum_{m=-\infty}^{\infty} \mathfrak{f}(m)\, u(m-n) \quad \text{for all } u \in \mathcal{R}, \mathfrak{f} \in \mathcal{R}^{\star}, \text{ and } n \in \mathbb{Z}.$$

Put

$$\mathfrak{A} = \left\{ \mathfrak{f} \in \mathcal{R}^{\star} : \lim_{\infty} \mathfrak{f} = 0 \ \textit{and} \ \lim_{-\infty} \mathfrak{f} \text{ exist in } \mathbb{R} \right\}$$

and

$$\mathfrak{B} = \left\{ \mathfrak{f} \in \mathcal{R}^{\star} : \lim_{\infty} \mathfrak{f} \text{ exists in } \mathbb{R} \right\}.$$

Define a map $\mathfrak{F} : \mathfrak{A} \to \mathbb{R}$ by $\mathfrak{F}(\mathfrak{f}) = \lim_{-\infty} \mathfrak{f}$ for all $\mathfrak{f} \in \mathfrak{A}$ and a map $\mathfrak{G} : \mathfrak{B} \to \mathbb{R}$ by $\mathfrak{G}(\mathfrak{f}) = \lim_{\infty} \mathfrak{f}$ for all $\mathfrak{f} \in \mathfrak{B}$. Also, define a map $p : \mathcal{R}^{\star} \to \mathbb{R}$ by

$$p(\mathfrak{f}) = \sup \{ |\mathfrak{f}(n)| : n \in \mathbb{Z} \} \quad \text{for all } \mathfrak{f} \in \mathcal{R}^{\star}.$$

Notice that if $\mathfrak{f}$ is positive in $\mathcal{R}^{\star}$ then $0 \leqslant \mathfrak{F}(\mathfrak{f}) \leqslant p(\mathfrak{f})$ and $0 \leqslant \mathfrak{G}(\mathfrak{f}) \leqslant p(\mathfrak{f})$. By a Hahn-Banach type theorem (see, for instance, Theorem 2.1 in [1]), both $\mathfrak{F}$ and $\mathfrak{G}$ extend to positive elements in $\mathcal{R}^{\star\star}$, denoted again by $\mathfrak{F}$ and $\mathfrak{G}$, respectively.

Finally, let $\mathfrak{h} \in \mathcal{R}^{\star}$ such that $\mathfrak{h}(n) = 1$ if $n \leqslant 0$ and $\mathfrak{h}(n) = 0$ if $n > 0$. For $u \in \mathcal{R}$, we have

$$(\mathfrak{h}u)(n) = \sum_{m=n}^{\infty} u(-m) \quad \text{for all } n \in \mathbb{Z}.$$

It follows that $\mathfrak{h}u \in \mathfrak{A} \cap \mathfrak{B}$, so

$$(\mathfrak{F}\mathfrak{h})(u) = \lim_{-\infty} \mathfrak{h}u = \sum_{n=-\infty}^{\infty} u(n) \quad \text{and} \quad (\mathfrak{G}\mathfrak{h})(u) = \lim_{\infty} \mathfrak{h}u = 0.$$

We derive quickly that

$$(\mathfrak{F} \ltimes \mathfrak{G})(\mathfrak{h}) = 0 \neq 1 = (\mathfrak{G} \ltimes \mathfrak{F})(\mathfrak{h}).$$

This means that under the first Arens multiplication, the ℓ-ring $\mathcal{R}^{\star\star}$ is not commutative. Since $\mathcal{R}$ is commutative, the remark made just before this example implies that $\mathcal{R}$ is not Arens regular.

To advance our discussion, it seems natural to impose an extra assumption on the ℓ-ring under consideration. This is exactly what we intend to do in the next sections.

3. Arens regularity of f-rings

Birkhoff and Pierce in [5, page 55] called the ℓ-ring $\mathcal{R}$ an *f-ring* if

$$(uw) \wedge v = (wu) \wedge v = 0 \quad \text{for all } u, v, w \in \mathcal{R}^+ \text{ with } u \wedge v = 0.$$

This additional hypothesis turns out to be sufficient for the ℓ-ring $\mathcal{R}$ to be Arens regular. First, a few well-known f-ring properties from [5] have to be revisited. For instance, squares in the f-ring $\mathcal{R}$ are nonnegative [5, Corollary 1, page 57]. In particular, the multiplicative identity e is nonnegative. On the other hand, an archimedean ℓ-ring $\mathcal{R}$ with a positive multiplicative identity e is an f-ring if and only if e is a weak order unit in $\mathcal{R}$ [5, Corollary 3, page 61]. Recall here that the positive element e is called a *weak order unit* in $\mathcal{R}$ if $u \in \mathcal{R}$ and $u \wedge e = 0$ imply $u = 0$ [5, page 49]. Below, the symbol $\mathbb{N}$ is used to indicate the set of all positive integers $\{1, 2, 3, ...\}$. All the ingredients are gathered now for the proof of the following theorem.

Theorem 3.1. — *Any f-ring is Arens regular.*

Proof. —First of all, from [6, Lemma 2.2] it follows directly that

$$0 \leqslant nu \leqslant u^2 + n\,(u \wedge ne) \quad \text{for all } u \in \mathcal{R}^+ \text{ and } n \in \mathbb{N}. \tag{3.3}$$

These inequalities play a key role in the sequel.

Now, we will see that $e^{\star\star}$ is a weak order unit in $\mathcal{R}^{\star\star}$. To show this, let $\mathfrak{F} \in \mathcal{R}^{\star\star}$ such that $\mathfrak{F} \wedge e^{\star\star} = 0$ and pick $0 \leqslant \mathfrak{f} \in \mathcal{R}^{\star}$. We claim that

$$0 = (\mathfrak{F} \wedge e^{\star\star})(\mathfrak{f}) = \inf\{\mathfrak{F}(\mathfrak{f} - \mathfrak{g}) + e^{\star\star}(\mathfrak{g}) : \mathfrak{g} \in \mathcal{R}^{\star} \text{ and } 0 \leqslant \mathfrak{g} \leqslant \mathfrak{f}\}$$

To prove the second equality, note that

$$(\mathfrak{F} \wedge e^{\star\star})(\mathfrak{f}) \geqslant \inf\{\mathfrak{F}(\mathfrak{f} - \mathfrak{g}) + e^{\star\star}(\mathfrak{g}) : \mathfrak{g} \in \mathcal{R}^{\star} \text{ and } 0 \leqslant \mathfrak{g} \leqslant \mathfrak{f}\}$$

is trivial, upon taking $\mathfrak{g} = 0$ and $\mathfrak{g} = \mathfrak{f}$. To prove the inverse inequality, let $\mathfrak{G} \in \mathcal{R}^{\star\star}$ be any lower bound for $\mathfrak{F}$ and $e^{\star\star}$; we must show that

$$\mathfrak{G}(\mathfrak{f}) \leqslant \inf\{\mathfrak{F}(\mathfrak{f} - \mathfrak{g}) + e^{\star\star}(\mathfrak{g}) : \mathfrak{g} \in \mathcal{R}^{\star} \text{ and } 0 \leqslant \mathfrak{g} \leqslant \mathfrak{f}\}.$$

For this, suppose $\mathfrak{g} \in \mathcal{R}^{\star}$ with $0 \leqslant \mathfrak{g} \leqslant \mathfrak{f}$. Then $\mathfrak{G}(\mathfrak{f}-\mathfrak{g}) \leqslant \mathfrak{F}(\mathfrak{f}-\mathfrak{g})$ and $\mathfrak{G}(\mathfrak{g}) \leqslant e^{\star\star}(\mathfrak{g})$, whence

$$\mathfrak{G}(\mathfrak{f}) = \mathfrak{G}(\mathfrak{f}-\mathfrak{g}) + \mathfrak{G}(\mathfrak{g}) \leqslant \mathfrak{F}(\mathfrak{f}-\mathfrak{g}) + e^{\star\star}(\mathfrak{g}),$$

and the claim is proved (the idea for the proof of the claim can be found in [3, proof of Theorem 11.2.5]). Fix $\varepsilon \in (0,\infty)$ and choose $n \in \mathbb{N}$. There exists $\mathfrak{g}_n \in \mathcal{R}^{\star}$ such that $0 \leqslant \mathfrak{g}_n \leqslant \mathfrak{f}$ and

$$0 \leqslant \mathfrak{F}(\mathfrak{f}-\mathfrak{g}_n) + \mathfrak{g}_n(e) = \mathfrak{F}(\mathfrak{f}-\mathfrak{g}_n) + e^{\star\star}(\mathfrak{g}_n) \leqslant 2^{-n}\varepsilon.$$

Put $\mathfrak{h}_n = \inf\{\mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_n\}$ and observe that

$$\begin{aligned} 0 &\leqslant \mathfrak{F}(\mathfrak{f}-\mathfrak{h}_n) = \mathfrak{F}(\mathfrak{f}-\inf\{\mathfrak{g}_1, \mathfrak{g}_2, .., \mathfrak{g}_n\}) \\ &= \mathfrak{F}(\sup\{\mathfrak{f}-\mathfrak{g}_1, \mathfrak{f}-\mathfrak{g}_2, ..., \mathfrak{f}-\mathfrak{g}_n\}) \\ &\leqslant \mathfrak{F}(\mathfrak{f}-\mathfrak{g}_1) + \mathfrak{F}(\mathfrak{f}-\mathfrak{g}_2) + \cdots + \mathfrak{F}(\mathfrak{f}-\mathfrak{g}_n) \\ &\leqslant \varepsilon\left(2^{-1} + 2^{-2} + \cdots + 2^{-n}\right). \end{aligned}$$

Thus,

$$0 \leqslant \mathfrak{F}(\mathfrak{f}-\mathfrak{h}_n) \leqslant \varepsilon \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

On the other hand, let $\mathfrak{h} \in \mathcal{R}^{\star}$ be such that $0 \leqslant \mathfrak{h} \leqslant \mathfrak{h}_n$ for all $n \in \mathbb{N}$. Hence,

$$0 \leqslant \mathfrak{h}(e) \leqslant \mathfrak{h}_n(e) \leqslant \mathfrak{g}_n(e) \leqslant 2^{-n}\varepsilon \quad \text{for all } n \in \mathbb{N}.$$

It follows that $\mathfrak{h}(e) = 0$. According to (3.3), we have

$$0 \leqslant n\mathfrak{h}(u) \leqslant \mathfrak{h}\left(u^2\right) + n^2\mathfrak{h}(e) = \mathfrak{h}\left(u^2\right) \quad \text{for all } u \in \mathcal{R}^{+} \text{ and } n \in \mathbb{N}.$$

That is, $\mathfrak{h} = 0$, which yields that

$$\inf\{\mathfrak{h}_n : n \in \mathbb{N}\} = 0. \tag{3.5}$$

Lemma. — *Suppose $\mathcal{G}$ and $\mathcal{H}$ are abelian ℓ-groups with $\mathcal{H}$ complete, and $\mathfrak{h}_1, \mathfrak{h}_2, ...$ are elements of* $\mathrm{Hom}_b(\mathcal{G}, \mathcal{H})$ *admitting an upper bound. Then*

$$(\sup\{\mathfrak{h}_n : n \in \mathbb{N}\})(u) = \sup\{\mathfrak{h}_n(u) : n \in \mathbb{N}\} \quad \textit{for all } u \in \mathcal{G}^{+}.$$

Proof. —See the end of the proof of [3, Theorem 11.2.5].

Combining the dual of this lemma (with $u = e$) with equation 3.5 above, we get

$$\inf\{\mathfrak{h}_n(e) : n \in \mathbb{N}\} = 0$$

Consequently, a subsequence $\left(\mathfrak{h}_{\alpha(n)}\right)_{n\in\mathbb{N}}$ of $(\mathfrak{h}_n)_{n\in\mathbb{N}}$ can be found so that

$$0 \leqslant \mathfrak{h}_{\alpha(n)}(e) \leqslant n^{-4} \quad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

If $u \in \mathcal{R}^+$ and $n \in \mathbb{N}$, then (3.3), in which we replace n by n^2, and (3.6) lead directly to

$$0 \leqslant \mathfrak{h}_{\alpha(n)}(u) \leqslant n^{-2}\mathfrak{h}_{\alpha(n)}\left(u^2\right) + n^2\mathfrak{h}_{\alpha(n)}(e) \leqslant n^{-2}\left(\mathfrak{h}_1\left(u^2\right)+1\right)$$

(notice here that the sequence $(\mathfrak{h}_n)_{n\in\mathbb{N}}$ is decreasing in $\mathcal{R}^\star$). Hence, the series $\sum \mathfrak{h}_{\alpha(n)}(u)$ converges in $\mathbb{R}$. A map $\mathfrak{t} : \mathcal{R}^+ \to \mathcal{R}$ can thus be defined by

$$\mathfrak{t}(u) = \sum_{n\in\mathbb{N}} \mathfrak{h}_{\alpha(n)}(u) \quad \text{for all } u \in \mathcal{R}^+.$$

It is readily checked that

$$\mathfrak{t}(u+v) = \mathfrak{t}(u) + \mathfrak{t}(v) \quad \text{for all } u, v \in \mathcal{R}^+.$$

By [3, 1.1.7], $\mathfrak{t}$ extends uniquely to an increasing group homomorphism on $\mathcal{R}$, again denoted by $\mathfrak{t}$. Since

$$\sum_{k=1}^{n} \mathfrak{h}_{\alpha(k)} \leqslant \mathfrak{t} \text{ in } \mathcal{R}^\star \quad \text{for all } n \in \mathbb{N}$$

and $\mathfrak{F}$ is an increasing group homomorphism on $\mathcal{R}^\star$, we get

$$0 \leqslant n \inf\left\{\mathfrak{F}(\mathfrak{h}_k) : k \in \mathbb{N}\right\} \leqslant \sum_{k=1}^{n} \mathfrak{F}\left(\mathfrak{h}_{\alpha(k)}\right) \leqslant \mathfrak{F}(\mathfrak{t}).$$

But then,

$$\inf\left\{\mathfrak{F}(\mathfrak{h}_n) : n \in \mathbb{N}\right\} = 0.$$

So,

$$\sup\left\{\mathfrak{F}(\mathfrak{f} - \mathfrak{h}_n) : n \in \mathbb{N}\right\} = \mathfrak{F}(\mathfrak{f}).$$

This equality together with (3.4) yields that $0 \leqslant \mathfrak{F}(\mathfrak{f}) \leqslant \varepsilon$. We derive that $\mathfrak{F}(\mathfrak{f}) = 0$ because ε is arbitrary in $(0, \infty)$. This means that $\mathfrak{F} = 0$, so $e^{\star\star}$ is a weak order unit in $\mathcal{R}^{\star\star}$, as required.

In summary, the multiplicative identity $e^{\star\star}$ in the archimedean ℓ-ring with respect to the first Arens multiplication is simultaneously a weak order unit. This implies that under the first Arens multiplication $\ltimes$, the lattice-ordered ring $\mathcal{R}^{\star\star}$ is an f-ring with $e^{\star\star}$ as a multiplicative identity. Analogously, $\mathcal{R}^{\star\star}$ is an f-ring with $e^{\star\star}$ as a multiplicative identity under the second Arens multiplication $\rtimes$. From Theorem 2.2 in [9], it follows that there exists $\mathfrak{H} \in \mathcal{R}^{\star\star}$ such that

$$\mathfrak{F} \ltimes \mathfrak{G} = \mathfrak{H} \rtimes \mathfrak{F} \rtimes \mathfrak{G} \quad \text{for all } \mathfrak{F}, \mathfrak{G} \in \mathcal{R}^{\star\star}.$$

In particular,

$$\mathfrak{H} = \mathfrak{H} \rtimes e^{\star\star} \rtimes e^{\star\star} = e^{\star\star} \ltimes e^{\star\star} = e^{\star\star}.$$

Therefore,

$$\mathfrak{F} \ltimes \mathfrak{G} = e^{\star\star} \rtimes \mathfrak{F} \rtimes \mathfrak{G} = \mathfrak{F} \rtimes \mathfrak{G} \quad \text{for all } \mathfrak{F}, \mathfrak{G} \in \mathcal{R}^{\star\star}.$$

This completes the proof of the theorem. □

Observe that in the previous proof, we have shown in particular that if $\mathcal{R}$ is an f-ring then so is $\mathcal{R}^{\star\star}$ with respect to the first (and the second) Arens multiplication. It should be pointed out that this result has been obtained for archimedean f-algebras with separating order dual in an alternative way by Huijsmans and de Pagter [10, Corollary 4.5]. In the next section, Theorem 3.1 is used then generalized to the more general setting of the so-called almost f-rings.

4. Arens regularity of almost f-rings

Birkhoff in [4, page 405] called the ℓ-ring $\mathcal{R}$ an *almost f-ring* if $uv = 0$ whenever $u \wedge v = 0$ in $\mathcal{R}$. Any f-ring is an almost f-ring but not conversely (see Corollary 1 [5, page 57] and Example 16 in [5, page 62]). This observation leads to a quite natural question, namely, is any almost f-ring Arens regular? The main objective of this section is to answer affirmatively this question. Some preparation are needed.

Let $\mathcal{R}$ be an ℓ-ring the multiplicative identity of which is denoted by e. An element $u \in \mathcal{R}$ is said to be *majorizable* if there exists $v \in \mathcal{R}$ such that

$$n\,|u| \leqslant v \quad \text{for all } n \in \mathbb{N}.$$

The set of all majorizable elements in $\mathcal{R}$ is denoted by $\mathfrak{s}$. It is easily seen that $\mathfrak{s}$ is an ℓ-ideal in $\mathcal{R}$. That is to say, $\mathfrak{s}$ is a two-sided ideal in $\mathcal{R}$ such that $|u| \leqslant v$ in $\mathcal{R}$ and $v \in \mathfrak{s}$ imply $u \in \mathfrak{s}$. Hence, one can consider the quotient ℓ-ring $\mathcal{R}/\mathfrak{s}$ [3, Section 8.3]. The residue class in $\mathcal{R}/\mathfrak{s}$ of $u \in \mathcal{R}$ is denoted by $[u]$. Next, we show that if $\mathcal{R}$ is an almost f-ring, then $\mathcal{R}/\mathfrak{s}$ is an f-ring. To do this, we have to notice that if u is an element in the almost f-ring $\mathcal{R}$ then $|u|^2 = u^2$. This elementary property is useful for the proof of the following result.

LEMMA 4.1. — *Let $\mathcal{R}$ be an almost f-ring and $\mathfrak{s}$ be the ℓ-ideal of all majorizable elements in $\mathcal{R}$. Then the quotient ℓ-ring $\mathcal{R}/\mathfrak{s}$ is an f-ring.*

Proof. — First, let $u, v \in \mathcal{R}$ such that $u \wedge v \in \mathfrak{s}$. Since,

$$(u - (u \wedge v))(v - (u \wedge v)) = 0,$$

we get

$$|uv| \leqslant |u(u \wedge v)| + |v(u \wedge v)| + |u \wedge v|^2 .$$

It follows that $uv \in \mathfrak{s}$ as $u \wedge v \in \mathfrak{s}$ and $\mathfrak{s}$ is an ℓ-ideal in $\mathcal{R}$.

Secondly, let $u \in \mathcal{R}^+$ and assume that $u^m \in \mathfrak{s}$ for some $m \in \mathbb{N}$ with $m \geqslant 2$. Thus, there exists $v \in \mathcal{R}$ such that $nu^m \leqslant v$ for all $n \in \mathbb{N}$. Now, fix $n \in \mathbb{N}$ and observe that

$$(e - nu^{m-1})^-(e - nu^{m-1})^+ = 0.$$

It follows that

$$\begin{aligned}(e - nu^{m-1})(e - \left(e \wedge nu^{m-1}\right)) &= (e - nu^{m-1})(e - nu^{m-1})^+ \\ &= \left((e - nu^{m-1})^+\right)^2 \geqslant 0.\end{aligned}$$

This implies the second inequality below:

$$nu^{m-1}(e - (e \wedge nu^{m-1})) \leqslant nu^{m-1} + e \wedge nu^{m-1} - nu^{m-1}(e \wedge nu^{m-1}) \leqslant e.$$

This together with

$$u^{m-1} \wedge nu^{2m-2} \geqslant u^{m-1}(e \wedge nu^{m-1})$$

implies

$$n(u^{m-1} - \left(u^{m-1} \wedge nu^{2m-2}\right)) \leqslant n(u^{m-1} - u^{m-1}(e \wedge nu^{m-1})) \leqslant e.$$

From this follows the second inequality below:

$$nu^{m-1} \leqslant e + n\left(u^{m-1} \wedge nu^{2m-2}\right) \leqslant e + n^2 u^{2m-2} \leqslant e + vu^{m-2}.$$

This means that $u^{m-1} \in \mathfrak{s}$. An obvious induction yields quickly that $u \in \mathfrak{s}$. Let us extend this fact to the nonpositive case. Hence, let $u \in \mathcal{R}$ and assume that there exists $m \in \mathbb{N}$ such that $m \geqslant 2$ and $u^m \in \mathfrak{s}$. Observe that $|u|^{2m} = u^{2m} \in \mathfrak{s}$. By the positive case, we get $|u| \in \mathfrak{s}$ and thus $u \in \mathfrak{s}$ because $\mathfrak{s}$ is an ℓ-ideal.

In summary, we have proved that the quotient ℓ-ring $\mathcal{R}/\mathfrak{s}$ is an almost f-ring which contains no nonzero nilpotent elements. In view of Lemma 4 in [5, Section 9], $\mathcal{R}/\mathfrak{s}$ is an f-ring and we are done. □

Combining Theorem 3.1 and Lemma 4.1, we see that if $\mathcal{R}$ is almost f-ring then the quotient ℓ-ring $\mathcal{R}/\mathfrak{s}$ is Arens regular. This observation is applied next to prove that the almost f-ring $\mathcal{R}$ is again Arens regular, which is the last result of this paper.

Corollary 4.2. — *Any almost f-ring is Arens regular.*

Proof. — Let $\mathcal{R}$ be an almost f-ring. Let $\mathfrak{f} \in \mathcal{R}^\star$ and choose $u, v \in \mathcal{R}$ with $[u] = [v]$ in $\mathcal{R}/\mathfrak{s}$. Hence, there exists $w \in \mathcal{R}^+$ such that $n\,|u - v| \leqslant w$ for all $n \in \mathbb{N}$. Thus, $n\,|\mathfrak{f}(u) - \mathfrak{f}(v)| \leqslant |\mathfrak{f}|(w)$ for all $n \in \mathbb{N}$, so $\mathfrak{f}(u) = \mathfrak{f}(v)$. Accordingly, a map $[\mathfrak{f}] : \mathcal{R}/\mathfrak{s} \to \mathbb{R}$ can be defined by

$$[\mathfrak{f}]([u]) = \mathfrak{f}(u) \quad \text{for all } u \in \mathcal{R}.$$

It is readily verified that $[\mathfrak{f}] \in (\mathcal{R}/\mathfrak{s})^\star$. Moreover, if $\mathfrak{f}, \mathfrak{g} \in \mathcal{R}^\star$ then $[\mathfrak{f}] = [\mathfrak{g}]$ implies $\mathfrak{f} = \mathfrak{g}$. Therefore, for each $\mathfrak{F} \in \mathcal{R}^{\star\star}$ a map $[\mathfrak{F}] : (\mathcal{R}/\mathfrak{s})^\star \to \mathbb{R}$ can be defined by

$$[\mathfrak{F}]([\mathfrak{f}]) = \mathfrak{F}(\mathfrak{f}) \quad \text{for all } \mathfrak{f} \in \mathcal{R}^\star.$$

As before, it is not hard to check that $[\mathfrak{F}] \in (\mathcal{R}/\mathfrak{s})^{\star\star}$. Also, one may check easily that if $[\mathfrak{F}] = [\mathfrak{G}]$ for $\mathfrak{F}, \mathfrak{G} \in \mathcal{R}^{\star\star}$, then $\mathfrak{F} = \mathfrak{G}$. Now, it is readily checked that

$$[\mathfrak{F} \ltimes \mathfrak{G}] = [\mathfrak{F}] \ltimes [\mathfrak{G}] \quad \text{and} \quad [\mathfrak{F} \rtimes \mathfrak{G}] = [\mathfrak{F}] \rtimes [\mathfrak{G}] \quad \text{for all } \mathfrak{F}, \mathfrak{G} \in \mathcal{R}^{\star\star}.$$

Since $\mathcal{R}/\mathfrak{s}$ is Arens regular, we have

$$[\mathfrak{F}] \ltimes [\mathfrak{G}] = [\mathfrak{F}] \rtimes [\mathfrak{G}] \quad \text{for all } \mathfrak{F}, \mathfrak{G} \in \mathcal{R}^{\star\star}.$$

Accordingly,

$$[\mathfrak{F} \ltimes \mathfrak{G}] = [\mathfrak{F} \rtimes \mathfrak{G}] \quad \text{for all } \mathfrak{F}, \mathfrak{G} \in \mathcal{R}^{\star\star}.$$

This yields that $\mathfrak{F} \ltimes \mathfrak{G} = \mathfrak{F} \rtimes \mathfrak{G}$ and the desired result follows. □

Notice finally that in the proof of Corollary 4.2, we have shown in particular that if $\mathcal{R}$ is an almost f-ring, then the order biconjugates $\mathcal{R}^{\star\star}$ and $(\mathcal{R}/\mathfrak{s})^{\star\star}$ of $\mathcal{R}$ and $\mathcal{R}/\mathfrak{s}$, respectively, are isomorphic as ℓ-rings.

Acknowledgement. — The authors would like to thank the referee for his helpful comments and suggestions.

Bibliography

[1] Aliprantis (C. D.) and Burkinshaw (O.). — Positive operators, Academic Press, Orlando, (1985).

[2] Arens (R.). — The adjoint of bilinear operation, Proc. Amer. Math. Soc., 2, p. 839-848 (1951).

[3] Bigard (A.), Keimel (K.) and Wolfenstein (S.). — Groupes et Anneaux Réticulés, Lecture Notes Math. 608, Springer Verlag, Berlin-Heidelberg-New York, (1977).

[4] Birkhoff (G.). — Lattice Theory, 3rd. edition Am. Math. Soc. Colloq. Publ. No. 25, Providence, Rhode Island, (1967).

[5] Birkhoff (G.) and Pierce (R. S.). — Lattice-ordered rings, An. Acad. Brasil. Ciènc. 28, p. 41-69 (1956).

[6] Boulabiar (K.). — Representation theorems for d-multiplications on Archimedean unital f-rings, Comm. Algebra, 32, p. 3955-3967 (2004).

[7] Buskes (G.) and Page (R.). — A positive note on a couterexample by Arens, Quaest. Math., 28, p. 117-121 (2005).

[8] Conrad (P.). — The additive group of an f-ring, Canad. J. Math., 26, p. 1157-1168 (1974).

[9] Huijsmans (C. B.) and de Pagter (B.). — The order bidual of lattice ordered algebras, J. Funct. Anal., 59, p. 41-64 (1984).