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An inequality for local unitary Theta correspondence

Z. Gong[1] and L. Grenié[2]

Abstract. — Given a representation π of a local unitary group G and another local unitary group H, either the Theta correspondence provides a representation $\theta_H(\pi)$ of H or we set $\theta_H(\pi) = 0$. If G is fixed and H varies in a Witt tower, a natural question is: for which H is $\theta_H(\pi) \neq 0$? For given dimension m there are exactly two isometry classes of unitary spaces that we denote $H_m^{\pm}$. For $\varepsilon \in \{0,1\}$ let us denote $m_\varepsilon^{\pm}(\pi)$ the minimal m of the same parity of ε such that $\theta_{H_m^{\pm}}(\pi) \neq 0$, then we prove that $m_\varepsilon^+(\pi) + m_\varepsilon^-(\pi) \geqslant 2n+2$ where n is the dimension of π.

Résumé. — Étant donnée une représentation π d'un groupe unitaire local G et un autre groupe unitaire local H, on sait que soit la correspondance Theta fournit une représentation $\theta_H(\pi)$ de H soit on pose $\theta_H(\pi) = 0$. Si on fixe G et on laisse H varier dans une tour de Witt, une question naturelle est : pour quels H a-t-on $\theta_H(\pi) \neq 0$? Pour chaque dimension m il y a exactement deux classes d'équivalence d'espaces unitaires que nous dénotons $H_m^{\pm}$. Pour $\varepsilon \in \{0;1\}$, dnotons $m_\varepsilon^{\pm}(\pi)$ le plus petit m de la parité de ε tel que $\theta_{H_m^{\pm}}(\pi) \neq 0$, alors nous montrons que $m_\varepsilon^+(\pi) + m_\varepsilon^-(\pi) \geqslant 2n+2$ où n est la dimension de π.

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1. Introduction

The Theta correspondence is a powerful tool for the study of automorphic and local representations. It has been studied and used in the global and in the local case by various authors, see for instance [Har07], [HKS96], [How], [Kud86], [KR05], [MVW87], [Ral84], [Wal90]. We will restrict ourselves to the local case: we suppose that the base field is a p-adic field with $p \neq 2$. The Theta correspondence builds a duality between the representations of two reductive groups forming a dual pair inside a given symplectic (or metaplectic) group. The theory will be explained in greater detail in section 2. We will be interested in the so-called unitary case where both groups are unitary. To an irreducible representation π of the first group G corresponds at most one representation of the second group H that we denote $\theta(\pi) = \theta(G, H, \pi)$ where $\theta(\pi) = 0$ if there is no representation of H corresponding to π (in the unitary case, θ depends on the choice of a auxiliary character χ, we will thus write θ_χ instead of θ in that case). One can fix a representation π of an unitary group $G = \mathrm{U}(W)$ and vary the second group $H = \mathrm{U}(V)$, where W and V are Hermitian spaces and G and H are their respective unitary groups. One way to vary the space V is to start from a given irreducible space V_0 and to add hyperbolic planes $V_{1,1}$. One obtains a so-called Witt tower of spaces $V_r = V_0 \oplus (V_{1,1})^r$ and groups $H_r = H(V_r)$. We have (up to isometry) four such towers depending on the parity of r and on the sign of the Hasse invariant (see below for its definition). We denote them, with a slight notation shift, $V^\pm_{2r+m_0}$ where $m_0 = 0$ or 1, the dimension of $V^\pm_{2r+m_0}$ is $2r + m_0$ and $\pm$ is the sign of the Hasse invariant. It is now well known that if $\theta_\chi(G, H(V^\pm_{2r+m_0}), \pi) \neq 0$ then $\theta_\chi(G, H(V^\pm_{2r+2+m_0}), \pi) \neq 0$. We can thus consider, for a given m_0, the two integers $m^\pm_\chi(\pi)$ which are the minimal $m = 2r + m_0$ such that $\theta_\chi(G, H(V^\pm_m), \pi) \neq 0$.

We prove here a part of a conjecture of Harris, Kudla and Sweet (see Conjecture 2.7), namely

Theorem 3.10. — *Let π be an irreducible admissible representation of $G(W)$ where $\dim W = n$. Then*

$$m^+_\chi(\pi) + m^-_\chi(\pi) \geqslant 2n + 2 \ .$$

The conjecture (the Conservation Relation, see Conjecture 2.7) asserts that the inequality is in fact an equality.

In some important cases, Theorem 3.10, combined with the results of [HKS96] on local zeta integrals, suffices to prove stronger results. In particular, it is known, thanks to [HKS96], that

$$m = \inf(m_\chi^+(\pi), m_\chi^-(\pi)) \leqslant n.$$

When $m = n$ Harris and Kudla use this inequality and Theorem 3.10 to prove the *Dichotomy Conjecture* of [HKS96] ([Har07][Theorem 2.1.7]), which determines whether $m = m_\chi^+(\pi)$ or $m = m_\chi^-(\pi)$ in terms of local root numbers.

The (still-conjectural) Conservation Relation, the Dichotomy Conjecture (now proved), and Kudla's Persistence Principle (Proposition 2.6) go a long way toward providing a complete explicit determination of the local theta correspondence. Resolving the remaining ambiguities will require a better understanding of the poles of local zeta integrals. A key step in the present paper, as in [KR05], is to prove simplicity of these poles for unramified representations. This implies the Conservation Relation when π is the trivial representation, and a doubling argument that goes back to Kudla and Rallis, together with a cocycle calculation, then implies Theorem 3.10.

The inequality proved in Theorem 3.10 is applied in a global situation in [Har07] to study special values of L-functions.

While we were writing this manuscript, Harris brought to our attention that a proof in his article [Har07] was incomplete. Since the arguments are related to the ones explained here, we have added that proof as an appendix to this paper.

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2. Notations

This section recalls the local Theta correspondence as in [Kud96] and cites some of the results of [HKS96].

We fix once and for all a non archimedean local field F of residual characteristic different from 2.

The mapping Δ will always be a diagonal embedding, usually from G to $G \times G$ except in one point where it will be precised.

2.1. Heisenberg group

Let W be a vector space with a symplectic form $\langle .,. \rangle$ on which the group $\mathrm{GL}(W)$ will act on the right – accordingly, if f and g are two endomorphisms of W, we will denote $f \circ g$ the endomorphism such that $(f \circ g)(w) = g(f(w))$. We will denote, as usual,

$$\mathrm{Sp}(W) = \{g \in \mathrm{GL}(W) | \forall (x,y) \in W^2, \langle xg, yg \rangle = \langle x, y \rangle\}$$

its isometry group.

DEFINITION 2.1. — *The* Heisenberg group *of W if the group $H(W) = W \ltimes F$ with product*

$$(w_1, t_1)(w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \frac{1}{2}\langle w_1, w_2 \rangle) \ .$$

The centre of $H(W)$ is $\{(0,t) | t \in F\}$ and $\mathrm{Sp}(W)$ acts on $H(W)$ via its action on W:

$$(w,t)^g = (wg, t) \ .$$

We recall

THEOREM 2.2 (STONE–VON NEUMANN). — *Let ψ be a non trivial unitary character of F. There exists, up to isomorphism, one smooth irreducible representation (ρ_ψ, S) of $H(W)$ such that*

$$\rho_\psi\big((0,t)\big) = \psi(t) \cdot \mathrm{id}_S \ .$$

If we fix such a representation (ρ_ψ, S), then for any $g \in \mathrm{Sp}(g)$, the representation $h \longmapsto \rho_\psi^g(h) = \rho_\psi(h^g)$ is a representation of $H(W)$ with the same central character, which means that it must be isomorphic to ρ_ψ. Hence there is an isomorphism $A(g) \in \mathrm{GL}(S)$, unique up to a scalar, such that

$$\forall h \in H, \quad A(g)^{-1} \rho_\psi(h) A(g) = \rho_\psi^g(h). \tag{2.1}$$

The group

$$\mathrm{Mp}(W) = \{(g, A(g)) | \text{ equation (1) holds}\}$$

is independent of the choice of ψ and is a central extension of $\mathrm{Sp}(W)$ by $\mathbf{C}^\times$:

$$0 \longrightarrow \mathbf{C}^\times \longrightarrow \mathrm{Mp}(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1 \ .$$

The group $\mathrm{Mp}(W)$ has a natural representation, called the Weil representation, ω_ψ on S given by

$$\begin{array}{rcl} \omega_\psi : \mathrm{Mp}(W) & \longrightarrow & \mathrm{End}(S) \\ (g, A(g)) & \longmapsto & A(g) \end{array}$$

2.2. The Schrödinger model of the Weil representation

The natural mapping $(g, A(g)) \mapsto A(g)$ defines a representation of $\mathrm{Mp}(W)$ which has several models. We are interested in the so-called Schrödinger model.

Let Y be a Lagrangian of W, i.e. a maximal isotropic subspace of W and $W = X \oplus Y$ a complete polarisation of W. We consider Y as a degenerate symplectic space and see $H(Y) = Y \ltimes F$ as a maximal abelian subgroup of $H(W)$. We consider the extension ψ_Y of the character ψ from F to $H(Y)$ defined by $\psi_Y(y,t) = \psi(t)$. Let

$$S_Y = \mathrm{Ind}_{H(Y)}^{H(W)} \psi_Y \ .$$

We recall that S_Y is the space of the functions $f : H(W) \longrightarrow \mathbf{C}$ such that

$$\forall h \in H, \forall h_1 \in H(Y), f(h_1 h) = \psi_Y(h_1) f(h)$$

and such that there exists a compact open subgroup L of W satisfying

$$\forall h \in H, \forall l \in L, f\big(h(l,0)\big) = f(h) \ .$$

We fix an isomorphism of S_Y with the space $\mathcal{S}(X)$ of Schwartz functions on X by

$$\begin{array}{rcllcl} S_Y & \longrightarrow & \mathcal{S}(X) & & & \\ f & \longmapsto & \varphi : & X & \rightarrow & \mathbf{C} \\ & & & x & \mapsto & \varphi(x) = f(x,0). \end{array}$$

The group $H(W)$ acts on S_Y by right translation while it acts on $\varphi \in \mathcal{S}(X)$ by

$$\big(\rho(x+y,t)\varphi\big)(x_0) = \psi\left(t + \langle x_0, y\rangle + \frac{1}{2}\langle x, y\rangle\right)\varphi(x_0 + x)$$

where $x + y \in W$ is with $x \in X$ and $y \in Y$. Then (see [MVW87]) $(\rho, \mathcal{S}(X))$ is a model for the Weil representation.

We specify the operator ω_ψ as follows. We identify an element $w \in W$ with the row vector $(x, y) \in X \oplus Y$. An element $g \in \mathrm{Sp}(W)$ will be of the form $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a \in \mathrm{End}(X)$, $b \in \mathrm{Hom}(X,Y)$, $c \in \mathrm{Hom}(Y,X)$ and $d \in \mathrm{End}(Y)$. Let $P_Y = \{g \in \mathrm{Sp}(W) | c = 0\}$ be the maximal parabolic subgroup of $\mathrm{Sp}(W)$ that stabilises Y and $N_Y = \{g \in P_Y | d = \mathrm{id}_Y\}$ its unipotent radical. We have a Levi subgroup $M_Y = \{g \in P_Y | b = 0\}$ of P_Y and $P_Y = M_Y N_Y$.

We define the following natural mappings:

$$\begin{array}{rcl} m : \mathrm{GL}(X) & \longrightarrow & M_Y \\ a & \longmapsto & m(a) = \begin{pmatrix} a & 0 \\ 0 & a^\vee \end{pmatrix} \\ n : \mathrm{Her}(X,Y) & \longrightarrow & N_Y \\ b & \longmapsto & n(b) = \begin{pmatrix} \mathrm{id}_X & b \\ 0 & \mathrm{id}_Y \end{pmatrix} \end{array}$$

where $a^\vee$ is the inverse of the dual of a and $\mathrm{Her}(X,Y)$ is the subset of those $b \in \mathrm{Hom}(X,Y)$ which are Hermitian (in both cases we identify the dual of $X \oplus Y$ with $Y \oplus X$ using $\langle .,. \rangle$).

Proposition 2.3 ([Kud96, Proposition 2.3, p.8). — *Let* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(g)$. *The operator* $r(g)$ *of* $\mathcal{S}(X)$ *defined by*

$$r(g)(\varphi)(x) = \int_{\mathrm{Ker} c \backslash Y} \psi\left(\frac{1}{2}\langle xa, xb\rangle - \langle xb, yc\rangle + \frac{1}{2}\langle yc, yd\rangle\right) \varphi(xa+yc) \mathrm{d}\mu_g(y)$$

is proportional to $A(g)$ *and moreover is unitary for a unique Haar measure* $\mathrm{d}\mu_g(y)$ *on* $\mathrm{Ker}\ c^{\backslash Y}$.

2.3. Dual reductive pairs

Definition 2.4. — *A* dual reductive pair (G, G') *in* $\mathrm{Sp}(W)$ *is a pair of subgroups of* $\mathrm{Sp}(W)$ *such that both* G *and* G' *are reductive and*

$$\mathrm{Cent}_{\mathrm{Sp}(W)}(G) = G' \quad \textit{and} \quad \mathrm{Cent}_{\mathrm{Sp}(W)}(G') = G\ .$$

If (G, G') is a dual reductive pair in $\mathrm{Sp}(W)$, we denote $\widetilde{G}$ and $\widetilde{G}'$ the pullbacks of the subgroups in $\mathrm{Mp}(W)$. As seen in [MVW87], there exists a natural morphism

$$j : \widetilde{G} \times \widetilde{G}' \longrightarrow \mathrm{Mp}(W)$$

such that the restriction of j to $\mathbf{C}^\times \times \mathbf{C}^\times$ is the product.

We consider the pullback $(j^*(\omega_\psi), S)$ of ω_ψ to $\widetilde{G} \times \widetilde{G}'$. We note that the central character for both $\widetilde{G}$ and $\widetilde{G}'$ is the identity:

$$\omega_\psi(j(z_1, z_2)) = z_1 z_2 \cdot \mathrm{id}_S\ .$$

Let π be an irreducible admissible representation of $\widetilde{G}$ such that the central character of π is the identity. If

$$\mathcal{N}(\pi) = \bigcap_{\lambda \in \mathrm{Hom}_{\widetilde{G}}(S,\pi)} \mathrm{Ker}\,\lambda$$

then $S(\pi) = S/\mathcal{N}(\pi)$ is the largest quotient of S on which $\widetilde{G}$ acts by π. The action of $\widetilde{G}'$ on S commutes with the action of $\widetilde{G}$ so that $\widetilde{G}'$ acts on $S(\pi)$ and thus $S(\pi)$ is a representation of $\widetilde{G} \times \widetilde{G}'$. There exists (see [MVW87]) a smooth representation $\Theta_\psi(\pi)$ of G', unique up to isomorphism, such that

$$S(\pi) \simeq \pi \otimes \Theta_\psi(\pi)\ .$$

The principal result of the theory is the following

Theorem 2.5 (Howe duality principle). — *Let F be a non archimedean local field with residual characteristic different from 2 and let π be an irreducible admissible representation of $\widetilde{G}$.*

i) If $\Theta_\psi(\pi) \neq 0$, then it is an admissible representation of $\widetilde{G}'$ of finite length.

ii) If $\Theta_\psi(\pi) \neq 0$, there exists a unique $\widetilde{G}'$-submodule $\Theta^0_\psi(\pi)$ such that the quotient

$$\theta_\psi(\pi) = \Theta_\psi(\pi)/\Theta^0_\psi(\pi)$$

is irreducible. If $\Theta_\psi(\pi) = 0$, we let $\theta_\psi(\pi) = 0$.

iii) If two irreducible admissible representations π_1 and π_2 of $\widetilde{G}$ are such that $\theta_\psi(\pi_1) \simeq \theta_\psi(\pi_2) \neq 0$ then $\pi_1 \simeq \pi_2$.

2.4. The unitary case

Let E/F be a quadratic extension and $\epsilon_{E/F}$ the corresponding quadratic character of $F^\times$.

We fix a quadratic space W of dimension n with skew-Hermitian form

$$\langle .,. \rangle : W \times W \longrightarrow E$$

(linear in the second argument). We will denote $G(W)$ its isometry group.

Let V be a quadratic space of dimension m with Hermitian form

$$(.|.) : V \times V \longrightarrow E$$

(linear in the second argument). We will denote

$$G(V) = \{g \in \mathrm{GL}(V) | \forall v, w \in V, (gv|gw) = (v|w)\}$$

the isometry group of V. The space V will vary in the remaining of the paper.

Let $\mathbb{W} = \mathrm{R}_{E/F}(V \otimes_E W)$ with symplectic form

$$\begin{array}{rccl} \langle\langle .,. \rangle\rangle : & \mathbb{W} \otimes \mathbb{W} & \longrightarrow & F \\ & (v_1 \otimes w_1, v_2 \otimes w_2) & \longmapsto & \langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle \\ & & & = \dfrac{1}{2} \mathrm{Tr}_{E/F} \left((v_1, v_2) \langle w_1, w_2 \rangle \right). \end{array}$$

The pair $(G(V), G(W))$ is a dual reductive pair in $\mathrm{Sp}(\mathbb{W})$. We have a natural inclusion

$$\begin{array}{rcl} \imath : G(V) \times G(W) & \longrightarrow & \mathrm{Sp}(\mathbb{W}) \\ (g,h) & \longmapsto & \imath(g,h) = g \otimes h. \end{array}$$

For any pair of characters $\chi = (\chi_m, \chi_n)$ of $E^\times$ such that

$$\chi_n \,|_{F^\times} = \epsilon_{E/F}^n \,, \quad \chi_m \,|_{F^\times} = \epsilon_{E/F}^m \,,$$

one can define, see [Kud94, Proposition 4.8, p.396], a homomorphism

$$\tilde{\imath}_\chi : G(V) \times G(W) \longrightarrow \mathrm{Mp}(\mathbb{W})$$

lifting $\imath$ (the homomorphism $\tilde{\imath}_\chi$ *does* depend on χ). Since the context will usually make clear which of χ_m and χ_n is considered, we will often use χ instead of χ_m or χ_n. Moreover we define $\imath_{V,\chi}$ (resp. $\imath_{W,\chi}$) the restriction of $\imath_\chi$ to $G(V) \times 1$ (resp. $1 \times G(W)$).

We will denote ω_ψ the Weil representation of $\mathrm{Mp}(\mathbb{W})$ and ω_χ its pullback through $\tilde{\imath}_\chi$. As before, if π is an irreducible admissible representation of $G(V)$, we get a representation $\Theta_\chi(\pi, V)$ of $G(W)$ such that

$$S(\pi) \simeq \pi \otimes \Theta_\chi(\pi, V)$$

and if $\Theta_\chi(\pi, V) \neq 0$, we say that π appears in the local Theta correspondence for the pair $(G(V), G(W))$. This condition depends on χ_m but not on χ_n. As above we define $\theta_\chi(\pi, V)$ to be the unique irreducible quotient of $\Theta_\chi(\pi, V)$ (or 0 if $\Theta_\chi(\pi, V) = 0$).

Witt towers. For a fixed dimension m, there are two equivalence classes of Hermitian spaces of dimension m over E. These two classes are distinguished by their Hasse invariant

$$\epsilon(V) = \epsilon_{E/F}\big((-1)^{\frac{m(m-1)}{2}} \det V\big) \ .$$

We thus get two families of spaces $V_m^{\pm}$ where the sign is the sign of the Hasse invariant. As Hermitian spaces we have $V_{m+2}^{\pm} \simeq V_m^{\pm} \oplus V_{1,1}$, where $V_{1,1}$ is an hyperbolic plane and the direct sum is orthogonal. We thus get four so-called Witt towers

$$V_{2r}^{+} = V_0^{+} \oplus (V_{1,1})^r \ , \ V_{2r+2}^{-} = V_2^{-} \oplus (V_{1,1})^r \ ,$$

$$V_{2r+1}^{+} = V_1^{+} \oplus (V_{1,1})^r \ , \ V_{2r+1}^{-} = V_1^{-} \oplus (V_{1,1})^r$$

where V_0^{+} is the null vector space, V_2^{-} is an anisotropic 2-dimensional Hermitian space and $V_1^{\pm}$ are one dimensional anisotropic Hermitian spaces. In each case the integer r is the Witt index of the corresponding Hermitian space[1].

We have

Proposition 2.6 [HKS96],[Kud96]. — *Consider a Witt tower $\{V_m^{\epsilon}\}$ with $\epsilon = \pm$.*

i) (Persistence) If $\theta_\chi(\pi, V_m^{\epsilon}) \neq 0$ then $\theta_\chi(\pi, V_{m+2}^{\epsilon}) \neq 0$.

ii) (Stable range) We have $\theta_\chi(\pi, V_m^{\epsilon}) \neq 0$ if the Weil index r_0 of V_m is such that $r_0 \geqslant n$.

We fix $m_0 \in \{0,1\}$ and a character χ of $E^\times$ such that $\chi_{|F^\times} = \epsilon_{E/F}^{m_0}$ and we consider the two towers $V_m^{\pm}$ with m of the parity of m_0 (if $m_0 = 0$ we disregard V_0^{-} which does not exist). Let $m_\chi^{\pm}(\pi)$ be the smallest m such that

$$\theta_\chi(\pi, V_m^{\pm}) \neq 0 \ .$$

Based on several examples, we have

Conjecture 2.7 **(Conservation relation**, [HKS96, Speculations 7.5 and 7.6], [KR05, Conjecture 3.6]). — *If π is an irreducible admissible representation of $G(W)$, then*

$$m_\chi^{+}(\pi) + m_\chi^{-}(\pi) = 2n + 2 \ .$$

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(1) We recall that the Witt index of a quadratic space is the dimension of a maximal totally isotropic subspace

2.5. Degenerate principal series

Let W_+ and W_- be two copies of W with respectively the same form as W and its opposite. We keep the pair of characters $\chi = (\chi_m, \chi_n)$. We fix for the space $W_+ \oplus W_-$ the complete polarisation $X \oplus Y$ where $X = \{(w, -w) | w \in W\}$ and $Y = \{(w, w) | w \in W\} = \Delta(W)$ (recall that Δ is the diagonal embedding of W in $W_+ \oplus W_-$). We let then

$$\begin{array}{rclcrcl} \mathbb{W}_+ & = & \mathrm{R}_{E/F}(V \otimes_E W_+) & \qquad & \mathbb{W}_- & = & \mathrm{R}_{E/F}(V \otimes_E W_-) \\ \mathbb{X} & = & \mathrm{R}_{E/F}(V \otimes_E X) & & \mathbb{Y} & = & \mathrm{R}_{E/F}(V \otimes_E Y) \ . \end{array}$$

and we consider the representation $\omega_{V,W_+ \oplus W_-,\chi}$ of $G(V) \times G(W_+ \oplus W_-)$ induced by the Weil representation of $\mathbb{W}_+ \oplus \mathbb{W}_-$ on $S = \mathcal{S}(\mathbb{X}) \simeq \mathcal{S}(V^n)$. Let $R_n(V, \chi)$ be the maximal quotient of S on which $G(V)$ acts by the character χ_m. The space $R_n(V, \chi)$ can be seen as a representation of $G(W) \times G(W)$ via the natural embedding

$$i : G(W) \times G(W) = G(W_+) \times G(W_-) \hookrightarrow G(W_+ \oplus W_-) \ .$$

From now on, we will denote $G = G_n = G(W)$ and $\tilde{G} = \tilde{G}_n = G(W_+ \oplus W_-)$ so that $i : G \times G \hookrightarrow \tilde{G}$.

We then have

Proposition 2.8 ([HKS96], Proposition 3.1 and discussion before). — *If π be an irreducible admissible representation of $G(W)$, then*

$$\Theta_\chi(\pi, V) \neq 0 \iff \mathrm{Hom}_{G\times G}(R_n(V, \chi), \pi \otimes (\chi_m \cdot \pi^\vee)) \neq 0 \ .$$

Let P_Y be the parabolic subgroup of $\tilde{G}$ stabilising Y. We will denote M_Y its maximal Levi subgroup and N_Y its unipotent radical. As for the symplectic case, M_Y and N_Y are parametrised respectively by $\mathrm{GL}(X)$ and $\mathrm{Her}(X, Y)$.

For $s \in \mathbf{C}$ and χ a character of $E^\times$, let

$$I_n(s, \chi) = \mathrm{Ind}_{P_Y}^{\tilde{G}} \chi |\,.\,|^s$$

be the degenerate principal series (the induction is unitary and the elements of $I_n(s, \chi)$ are locally constant functions $\Phi(g, s)$).

We can identify $R_n(V, \chi)$ as a subspace of some $I_n(s, \chi)$ by sending an element $\varphi \in \mathcal{S}(X)$ to the function $g \longmapsto \omega_\chi(g)\varphi(0)$ – (we recall that we denote $\omega_\chi = \omega_\psi \circ \tilde{\imath}_{V,\chi}$). The spaces $R_n(V_m^\pm, \chi)$ allows us to decompose the various $I_n(s, \chi)$ as explained by the following proposition.

Proposition 2.9 ([KS97, Theorem 1.2, p.257]). — *Let* $V_m^{\pm}$ *be an* m*-dimensional unitary space and Hasse invariant* $\pm$. *Let* $s_0 = \frac{m-n}{2}$ *and* χ *a character of* $E^\times$ *such that* $\chi|_{F^\times} = \epsilon_{E/F}^m$.

i) If $m \leqslant n$, *i.e. if* $s_0 \leqslant 0$, *then the modules* $R_n(V_m^{\pm}, \chi)$ *are irreducible and* $R_n(V_m^+, \chi) \oplus R_n(V_m^-, \chi)$ *is the maximal completely reducible submodule of* $I_n(s_0, \chi)$.

ii) If $m = n$, *i.e. if* $s_0 = 0$, *then* $I_n(0, \chi) = R_n(V_n^+, \chi) \oplus R_n(V_n^-, \chi)$.

iii) If $n < m < 2n$, *i.e. if* $0 < s_0 < \frac{n}{2}$, *then* $I_n(s_0, \chi) = R_n(V_m^+, \chi) + R_n(V_m^-, \chi)$ *and* $R_n(V_m^+, \chi) \cap R_n(V_m^-, \chi)$ *is the unique irreducible submodule of* $I_n(s_0, \chi)$.

iv) If $m = 2n$, *i.e. if* $s_0 = \frac{n}{2}$, *then* $I_n(s_0, \chi) = R_n(V_{2n}^+, \chi)$, $R_n(V_{2n}^-, \chi)$ *is of codimension* 1 *and is the unique irreducible submodule of* $I_n(s_0, \chi)$.

v) If $m > 2n$, *i.e. if* $s_0 > \frac{n}{2}$, *then* $I_n(s_0, \chi) = R_n(V_m^{\pm}, \chi)$ *is irreducible.*

In all other cases $I_n(s, \chi)$ *is irreducible.*

To refine the aforementioned decompositions we begin with the Bruhat decomposition of $\tilde{G}$:

$$\tilde{G} = \coprod_{j=0}^{n} P_Y \omega_j P_Y, \quad \text{with } \omega_j = \begin{pmatrix} I_{n-j} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_j \\ 0 & 0 & I_{n-j} & 0 \\ 0 & -I_j & 0 & 0 \end{pmatrix}$$

and let us introduce, as in [Kud96, p.19] and [Rao93] the mapping

$$\begin{array}{rccc} x: & \tilde{G} & \longrightarrow & E^\times / \mathrm{N}_{E/F} E^\times \\ & p_1 \omega_j^{-1} p_2 & \longmapsto & \det(p_1 p_2|_Y) \mathrm{mod}\, \mathrm{N}_{E/F} E^\times \end{array}$$

Whenever $\chi|_{F^\times} = \mathbf{1}$ we can introduce the character $\chi_{\tilde{G}}$ of $\tilde{G}$

$$\chi_{\tilde{G}}(g) = \chi(x(g)) \ .$$

We extend the definition of R_n as follows:

$$R_n(V_0^+, \chi) = R_n(0, \chi) = \mathbf{C} \cdot \chi_{\tilde{G}}$$

and $R_n(V_0^+, \chi)$ is a submodule of dimension 1 o f $I_n(-\frac{n}{2}, \chi)$ (we are, at least formally, in the case *i)* of Proposition 2.9). As a last step, we define the intertwining operators

$$M_n(s, \chi) : I_n(s, \chi) \longrightarrow I_n(-s, \chi)$$

by the integral

$$M_n(s,\chi)(\Phi) = \int_{N_Y} \Phi(w_n u g, s)\mathrm{d}u = \int_{\mathrm{Her}(X,Y)} \Phi(w_n n(b) g, s)\mathrm{d}b\ ,$$

which is convergent for $\mathrm{Re}\, s > \frac{n}{2}$ and by meromorphic continuation for $s \in \mathbf{C}$. The Haar measure $\mathrm{d}b$ is chosen self-dual with respect to the Fourier transform

$$\hat{\phi}(y) = \int \phi(b)\psi(\mathrm{Tr}(by))\mathrm{d}b\ .$$

We normalise $M_n(s,\chi)$ using

$$a(s,\chi) = \prod_{j=0}^{n-1} L_F\left(2s + j - (n-1), \chi\epsilon_{E/F}^{j}\right)$$

and then $M_n^*(s,\chi) = \frac{1}{a(s,\chi)} M_n(s,\chi)$ is holomorphic and non zero (see [KS97, Proposition 3.2]).

Proposition 2.10 [KS97]. — *Let $V_m^{\pm}$ be the m-dimensional unitary space of dimension m and Hasse invariant $\pm$. Let $s_0 = \frac{m-n}{2}$ and χ a character of $E^{\times}$ such that $\chi|_{F^{\times}} = \epsilon_{E/F}^{m}$.*

i) If $m = 0$, i.e. if $s_0 = -\frac{n}{2}$, then $\mathrm{Ker}(M_n^*(-\frac{n}{2},\chi)) = R_n(V_0^+,\chi)$ and $\mathrm{Im}(M_n^*(-\frac{n}{2},\chi)) = R_n(V_{2n}^-,\chi)$.

ii) If $1 \leqslant m < n$, i.e. if $-\frac{n}{2} < s_0 < 0$, then $\mathrm{Ker}(M_n^*(s_0,\chi)) = R_n(V_m^+,\chi) \oplus R_n(V_m^-,\chi)$ and $\mathrm{Im}(M_n^*(s_0,\chi)) = R_n(V_{2n-m}^+,\chi) \cap R_n(V_{2n-m}^-,\chi)$.

iii) If $n \leqslant m < 2n$, i.e. if $0 \leqslant s_0 < \frac{n}{2}$, then $\mathrm{Ker}(M_n^*(s_0,\chi)) = R_n(V_m^+,\chi) \cap R_n(V_m^-,\chi)$, $M_n^*(s_0,\chi)(R_n(V_m^{\pm},\chi)) = R_n(V_{2n-m}^{\pm},\chi)$ thus we have $\mathrm{Im}(M_n^*(s_0,\chi)) = R_n(V_{2n-m}^+,\chi) \oplus R_n(V_{2n-m}^-,\chi)$.

iv) If $m = 2n$, i.e. if $s_0 = \frac{n}{2}$, then $\mathrm{Ker}(M_n^*(\frac{n}{2},\chi)) = R_n(V_{2n}^-,\chi)$ and $\mathrm{Im}(M_n^*(\frac{n}{2},\chi)) = M_n^*(\frac{n}{2},\chi)(R_n(V_{2n}^+),\chi) = R_n(V_0^+,\chi)$.

2.6. Local Zeta integral

The last element we will use is the local Zeta integral of a representation. We fix π an irreducible admissible representation of $G(W)$.

Definition 2.11. — *A* matrix coefficient *of π is a linear combination of functions of the form*

$$\phi(g) = \langle \pi(g)\xi, \xi^\vee \rangle$$

where ξ and $\xi^\vee$ are vectors of the space of π and $\pi^\vee$ respectively.

Moreover if $\xi_\circ$ and $\xi_\circ^\vee$ are preassigned spherical vectors of π and $\pi^\vee$, we let

$$\phi^\circ(g) = \langle \pi(g)\xi_\circ, \xi_\circ^\vee \rangle \ .$$

We parametrise the space of matrix coefficients with the space of $\pi \otimes \pi^\vee$ through the obvious projection. If $s \in \mathbf{C}$ with $\operatorname{Re} s$ large enough, $\xi \in \pi$, $\xi^\vee \in \pi^\vee$, $\Phi \in I_n(s, \chi)$, let

$$Z(s, \chi, \pi, \xi \otimes \xi^\vee, \Phi) = \int_G \langle \pi(g)\xi, \xi^\vee \rangle \Phi(i(g, I_n), s) \mathrm{d}g$$

and extend it linearly to the space of matrix coefficients of π. We fix a maximal compact subgroup K of $\tilde{G}$.

Definition 2.12. — *A* standard section Φ *is a mapping from $\mathbf{C}$ to the set of functions from $\tilde{G}$ to $\mathbf{C}$ such that $\forall s \in \mathbf{C}$, $\Phi(g, s) = \Phi(s)(g) \in I_n(s, \chi)$ and, moreover, $\Phi(s)|_K$ is independent of s.*

It is rather obvious that any element $\Phi(g, s) \in I_n(s, \chi)$ can be inserted in a (unique) standard section. The Zeta integral above defines, for $\operatorname{Re} s$ sufficiently large, an intertwining operator

$$Z(s, \chi, \pi) \in \operatorname{Hom}_{G \times G}\big(I_n(s, \chi), \pi \otimes (\chi \cdot \pi^\vee)\big) \ .$$

If Φ is a standard section, this operator can be meromorphically extended for all $s \in \mathbf{C}$ to an operator

$$Z^*(s, \chi, \pi) \in \operatorname{Hom}_{G \times G}\big(I_n(s, \chi), \pi \otimes (\chi \cdot \pi^\vee)\big) \ .$$

3. Our results

3.1. Decomposition of the degenerate principal series

Let $\Omega(W_+ \oplus W_-)$ be the Grassmannian of the Lagrangians of $W_+ \oplus W_-$. We can identify

$$P_Y \backslash G(W_+ \oplus W_-) \simeq \Omega(W_+ \oplus W_-)$$

using the map $P_Y \cdot g \longmapsto Yg$. There is a right action of $i(G(W) \times G(W))$ on $\Omega(W_+ \oplus W_-)$ which orbits are parametrised by the elements of the decomposition

$$G(W_+ \oplus W_-) = \coprod_{r=0}^{r_0} P_Y \delta_r i(G(W) \times G(W))$$

where r_0 is the Witt index of W. The aforementioned orbits are of the form

$$\Omega_r = P_Y \backslash P_Y \delta_r i(G(W) \times G(W)) \ .$$

The orbit Ω_r is made of the Lagrangians Z such that $\dim Z \cap W_+ = \dim Z \cap W_- = r$. The only open orbit is that of Y, which is Ω_0, while the only closed one is that of Ω_{r_0} and the closure of the orbit Ω_r is

$$\overline{\Omega}_r = \coprod_{j \geqslant r} \Omega_j \ .$$

We consider the filtration

$$I_n(s,\chi) = I_n^{(r_0)}(s,\chi) \supset \cdots \supset I_n^{(1)}(s,\chi) \supset I_n^{(0)}(s,\chi) \ ,$$

where

$$I_n^{(r)}(s,\chi) = \{\Phi \in I_n(s,\chi) | \Phi|_{\overline{\Omega}_{r+1}} = 0\} \ .$$

Let

$$Q_n^{(r)}(s,\chi) = I_n^{(r)}(s,\chi)/I_n^{(r-1)}(s,\chi)$$

be the successive quotients of the filtration. All $I_n^{(r)}(s,\chi)$ and $Q_n^{(r)}(s,\chi)$ are $G \times G$-stable.

Let T_W be the Witt tower containing W. For any $W' \in T_W$ of dimension $n' = n - 2r \leqslant n$, let $G_{n'} = G(W')$. We identify W' with a subspace of W isomorphic to W'. There is a Witt decomposition

$$W = U' \oplus W' \oplus U$$

where U and U' are dual isotropic subspaces of dimension r. Let P_r be the parabolic subgroup of G stabilising U. The Levi subgroup of P_r is isomorphic to $\mathrm{GL}(U) \times G_{n'}$ so that, if we denote M_r its Levi component and N_r its unipotent radical, we have isomorphisms

$$\begin{aligned} M_r &\simeq \mathrm{GL}(U) \times G_{n'} \\ P_r &\simeq (\mathrm{GL}(U) \times G_{n'}) \ltimes N_r. \end{aligned} \tag{3.2}$$

Note in particular for $r = 0$ that $U = U' = \{0\}$, $W' = W$ and $P_0 = G_n = G$.

Let

$$\mathrm{St}_r = i^{-1}(\delta_r^{-1} P_Y \delta_r \cap i(G \times G))$$

be the stabiliser of $P_Y \delta_r$ in $i^{-1}(P_Y)\backslash G \times G$.

Lemma 3.1. — *For a convenient choice of δ_r (specified in Equation (3.3) below), we have*

$$\mathrm{St}_r = (\mathrm{GL}(U) \times \mathrm{GL}(U) \times \Delta(G_{n'})) \ltimes (N_r \times N_r) \subset P_r \times P_r \ .$$

Moreover

$$Q_n^{(r)}(s,\chi) \simeq \mathrm{Ind}_{P_r \times P_r}^{G \times G} \Big(\chi|\,.\,|^{s+\frac{r}{2}} \otimes \chi|\,.\,|^{s+\frac{r}{2}} \otimes \big(\mathcal{S}(G_{n'}) \cdot (\mathbf{1} \otimes \chi)\big)\Big)$$

where the action of $G_{n'} \times G_{n'}$ on the space $\mathcal{S}(G_{n'}) \cdot (\mathbf{1} \otimes \chi)$ is given by $(g_1, g_2)\varphi(g) = \chi(\det g_2)\varphi(g_2^{-1} g g_1)$.

Proof. — We let $G' = G_{n'}$.

Recall the Witt decomposition

$$W = U' \oplus W' \oplus U$$

and consider the Lagrangian

$$Z = U \times \{0\} \oplus \Delta(W') \oplus \{0\} \times U$$

in $W_+ \oplus W_-$. Since the action of $\tilde{G}$ on $\Omega(W_+ \oplus W_-)$ is transitive, there exists $\delta_r \in \tilde{G}$ such that $Z = Y\delta_r$. Since any linear map from Y to Z can be extended to an element of $\tilde{G}$, we can furthermore require that

$$\begin{aligned} \forall v \in U', \delta_r|_{\Delta(U')}(v,v) &= (0, vd) \in \{0\} \times U \\ \delta_r|_{\Delta(W')} &= \mathrm{id}_{\Delta(W')} \\ \forall u \in U,\ \delta_r|_{\Delta(U)}(u,u) &= (u,0) \in U \times \{0\} \end{aligned} \tag{3.3}$$

where $d : U' \longrightarrow U$ is any isomorphism. Note in particular that $\delta_0 = \mathrm{id}_G$. Following [Kud96, Proof of Proposition 2.1, p.68], we find that there is a bijection between the orbit Ω_r of Z and the set

$$\{(Z_+, Z_-, \lambda)\}$$

where $Z_\pm$ is an isotropic subspace of $W_\pm$ of dimension r and

$$\lambda : Z_+^\perp / Z_+ \longrightarrow Z_-^\perp / Z_-$$

is an isometry[2]. The action of $(g_+, g_-) \in G \times G$ on this set is given by

$$(g_+, g_-)(Z_+, Z_-, \lambda) = (Z_+ g_+, Z_- g_-, g_+^{-1} \circ \lambda \circ g_-) .$$

The stabiliser of (Z_+, Z_-, λ) is

$$\{(g_+, g_-) \in G \times G | g_\pm \text{ stabilises } Z_\pm \text{ and } g_+^{-1} \circ \lambda \circ g_- = \lambda\} .$$

In our situation and with our choice of δ_r, we have $Z_+ = Z_- = U$, $Z_+^\perp / Z_+ = W'$ and $\lambda = \mathrm{id}_{W'}$. Hence, denoting $\mathrm{pr}_{W'}$ the projection on W' parallel to $U' \oplus U$,

$$\begin{aligned} \mathrm{St}_r &= \left\{(g_+, g_-) \in P_r \times P_r \middle| g_+|_{W'+U} \circ \mathrm{pr}_{W'} = g_-|_{W'+U} \circ \mathrm{pr}_{W'} \right\} \\ &= (\mathrm{GL}(U) \times \mathrm{GL}(U) \times \Delta(G')) \ltimes (N_r \times N_r) . \end{aligned}$$

For further reference, an element of P_r has the form

$$\begin{pmatrix} a & b & c \\ 0 & e & b^* \\ 0 & 0 & a^\vee \end{pmatrix}$$

where b^* depends on b, a and e and where c satisfies an equation depending on a, b and e. We thus have

$$g_\pm = \begin{pmatrix} a_\pm & b_\pm & c_\pm \\ 0 & e_\pm & b_\pm^* \\ 0 & 0 & a_\pm^\vee \end{pmatrix} \tag{3.4}$$

and the condition $g_+|_{W'+U} \circ \mathrm{pr}_{W'} = g_-|_{W'+U} \circ \mathrm{pr}_{W'}$ is simply $e_+ = e_-$.

The description of the stabiliser allows us to describe the induced representations. If $\tilde{g} \in \mathrm{St}_r$, then $p(\tilde{g}) = \delta_r i(\tilde{g}) \delta_r^{-1} = n \cdot m(a_r(\tilde{g})) \in P_Y$. Let $\xi_{s,r}$ be the character of St_r defined by $\xi_{s,r}(\tilde{g}) = \chi(a_r(\tilde{g})) |\det a_r(\tilde{g})|^{s+\frac{r}{2}}$. Consider the morphism of $G \times G$-modules

$$\begin{array}{ccc} Q_n^{(r)}(s, \chi) & \longrightarrow & \mathrm{Ind}_{\mathrm{St}_r}^{G \times G}(\xi_{s,r}) \\ \overline{f} & \longmapsto & \phi_{\overline{f}}(g_1, g_2) = \int_{N_r'} f(\delta_r n(u) i(g_1, g_2)) \mathrm{d}u \end{array}$$

where $f \in I_n^{(r)}(s, \chi)$ is a representative of $\overline{f}$. This morphism is an isomorphism (see [HKS96, Equation (4.9), p.963]). Let $\tilde{g} = (g_+, g_-)$ be an element of St_r decomposed as in (3.4). Then $\det(a_r(\tilde{g})) = \det a_+ \det a_- \det e_+$ (where we recall that $e_+ = e_-$). Since $e_+ \in G'$, $|\det e_+| = 1$ hence

$$\begin{aligned} Q_n^{(r)}(s, \chi) &\simeq \mathrm{Ind}_{\mathrm{St}_r}^{G \times G}(\chi|\,.\,|^{s+\frac{r}{2}} \otimes \chi|\,.\,|^{s+\frac{r}{2}} \otimes \chi) \\ &\simeq \mathrm{Ind}_{P_r \times P_r}^{G \times G}\left(\mathrm{Ind}_{\mathrm{St}_r}^{P_r \times P_r}(\chi|\,.\,|^{s+\frac{r}{2}} \otimes \chi|\,.\,|^{s+\frac{r}{2}} \otimes \chi)\right) . \end{aligned}$$

(2) in [Kud96] it is an anti-isometry but, since W_- has the opposite form of W_+, here λ is an isometry.

The induction from St_r to $P_r \times P_r$ is an induction from $\Delta(G')$ to $G' \times G'$. Moreover, if $f \in \mathrm{Ind}_{\Delta(G')}^{G'\times G'} \chi$ then $f(h_1, h_2) = \chi(h_2) f(h_2^{-1} h_1, 1)$. Hence

$$\mathrm{Ind}_{\Delta(G')}^{G'\times G'} \chi \simeq \mathcal{S}(G') \cdot (\mathbf{1} \otimes \chi)$$

where the action of $G' \times G'$ on $\mathcal{S}(G') \cdot (\mathbf{1} \otimes \chi)$ is given by

$$\rho(g_1, g_2)\varphi(g) = \chi(\det g_2)\varphi(g_2^{-1} g g_1) \ .$$

Hence

$$\begin{aligned}\mathrm{Ind}_{\mathrm{St}_r}^{P_r\times P_r}(\chi|\,.\,|^{s+\frac{r}{2}} \otimes \chi|\,.\,|^{s+\frac{r}{2}} \otimes \chi) &\simeq \chi|\,.\,|^{s+\frac{r}{2}} \otimes \chi|\,.\,|^{s+\frac{r}{2}} \otimes \mathrm{Ind}_{\Delta(G')}^{G'\times G'} \chi \\ &\simeq \chi|\,.\,|^{s+\frac{r}{2}} \otimes \chi|\,.\,|^{s+\frac{r}{2}} \otimes \big(\mathcal{S}(G') \cdot (\mathbf{1} \otimes \chi)\big) \ .\end{aligned}$$

The result follows. □

3.2. Simplicity of poles

We prove in our case the result of [KR05, section 5]. We follow the same method. We denote χ_0 the trivial character of $F^\times$.

Proposition 3.2. — *Let $\mathfrak{z}_s \in \mathcal{H}(G\,/\!/\,K) \otimes \mathbf{C}[q^s, q^{-s}]$ be the element defined by*

$$\mathfrak{z}_s = \prod_{i=1}^{r_0} (1 - q^{-s-\frac{1}{2}} t_i)(1 - q^{-s-\frac{1}{2}} t_i^{-1}) \ .$$

where we recall that $\mathcal{H}(G\,/\!/\,K) \simeq \mathbf{C}[t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}]^{W_G}$. For an unramified representation π of G, let $\pi(\mathfrak{z}_s)$ be the scalar by which $\mathfrak{z}_s$ acts on the unramified vector in π. Then for all matrix coefficients ϕ of π and all standard sections $\Phi(s) \in I_n(s)$, the function

$$\pi(\mathfrak{z}_s) \cdot Z(s, \chi_0, \pi, \phi, \Phi)$$

is an entire function of s.

Proof of Proposition 3.2. — We divide the proof into four steps.

3.2.1. *Step 1*

By linearity of Z, we can limit ourselves to the case where ϕ is of the form

$$\phi(g) = \langle \pi(g)\pi(g_1)\xi_\circ, \pi^\vee(g_2)\xi_\circ^\vee \rangle$$

where $\xi_\circ$ and $\xi_\circ^\vee$ are spherical vectors in π and $\pi^\vee$ and g_1, $g_2 \in G$. Then we have

$$\begin{aligned} Z(s,\chi_0,\pi,\phi,\Phi) &= \int_G \langle \pi(g)\pi(g_1)\xi_\circ, \pi^\vee(g_2)\xi_\circ^\vee\rangle \Phi_s(i(g,I_n))\mathrm{d}g \qquad (3.5)\\ &= \int_G \langle \pi(g)\xi_\circ, \xi_\circ^\vee\rangle \Phi_s(i(g_2 g g_1^{-1}, I_n))\mathrm{d}g \\ &= |\det g_2|^{s+r_0-\frac{1}{2}} \int_G \phi^\circ(g)\Phi_s(i(g,I_n)i(g_1^{-1},g_2^{-1}))\mathrm{d}g \end{aligned}$$

since $|\det g_2| = 1$ and ϕ° is bi-K invariant, for all k_1, $k_2 \in K$,

$$\begin{aligned} &= \int_G \phi^\circ(g)\Phi_s(i(k_2^{-1} g k_1, I_n)i(g_1^{-1},g_2^{-1}))\mathrm{d}g \\ &= \int_G \phi^\circ(g)\Phi_s(i(g,I_n)i(k_1,k_2)i(g_1^{-1},g_2^{-1}))\mathrm{d}g \end{aligned}$$

and thus

$$= \int_G \phi^\circ(g)\Psi_s(i(g,I_n))\mathrm{d}g$$

where, for any $h \in H = G_{2n}$,

$$\Psi_s(h) := \int_{K\times K} \Phi_s(hi(k_1,k_2)i(g_1^{-1},g_2^{-1}))\mathrm{d}k_1\mathrm{d}k_2. \qquad (3.6)$$

Note that Ψ_s is $K \times K$-invariant section of $I_n(s)$ which is not necessarily standard.

3.2.2. *Step 2*

We consider the algebra

$$\mathcal{A} = \mathbf{C}[X,X^{-1}] \otimes \mathcal{H}(G /\!/ K) \simeq \mathbf{C}[X,X^{-1}] \otimes \mathbf{C}[t_1,\ldots,t_n,t_1^{-1},\ldots,t_n^{-1}]^{W_G}\ ,$$

where $\mathcal{H}(G /\!/ K)$ is the K-spherical Hecke algebra of G and the element $\mathfrak{z} \in \mathcal{A}$ defined as:

$$\mathfrak{z} = \prod_{i=1}^{r_0}(1 - Xq^{-\frac{1}{2}}t_i)(1 - Xq^{-\frac{1}{2}}t_i^{-1})\ .$$

We let $G \times G$ act on $I_n(s)$ through i. We extend this action to $\mathcal{H}(G /\!/ K) \times \mathcal{H}(G /\!/ K)$ and we let any $\phi \in \mathcal{H}(G /\!/ K)$ act as $(\phi,1) \in \mathcal{H}(G /\!/ K) \times \mathcal{H}(G /\!/ K)$. We define the action of $\mathcal{A}$ on the space $I_n(s)^{K\times 1}$ of $K \times 1$-fixed vectors of $I_n(s)$ by the aforementioned action of $\mathcal{H}(G /\!/ K)$ and by $X \cdot \varphi = q^{-s}\varphi$ for any $\varphi \in I_n(S)$. Note that action of $1 \times G$ commutes with the action of $\mathcal{A}$.

Proposition 3.3. — *For any standard section Φ_s with associated section Ψ_s defined by (3.6), we have*

$$\Psi_s * \mathfrak{z} \in I_n^{(0)}(s)^{K\times K}.$$

Proof of Proposition 3.3. — We want to show the the image of $\Psi_s * \mathfrak{z}$ in each $Q_n^{(r)}(s) = Q_n^{(r)}(s,\chi_0)$ is 0 for $0 < r \leqslant r_0$. As an illustration, we will do the first step separately in the case of a split Hermitian space (in particular $n = 2r_0$). Consider the projection induced by restriction to the closed orbit:

$$\begin{array}{rcl} \mathrm{pr}_{r_0} : I_n(s) = I_n^{(r_0)}(s) & \longrightarrow & Q_n^{(r_0)}(s) \simeq \mathrm{Ind}_{P_{r_0}}^G\left(|\,.\,|^{s+\frac{r_0}{2}}\right) \otimes \mathrm{Ind}_{P_{r_0}}^G\left(|\,.\,|^{s+\frac{r_0}{2}}\right) \\ \Phi_s & \longmapsto & \big((g_1,g_2) \mapsto \Phi_s(i(g_1,g_2))\big)\,. \end{array}$$

If we let $\mathfrak{z}$ act only on the first term of the tensor product on the right side, we have

$$\mathrm{pr}_{r_0}(\Psi_s * \mathfrak{z}) = \mathrm{pr}_{r_0}(\Psi_s) * \mathfrak{z}\ .$$

On the other hand, we have

$$\mathrm{Ind}_{P_{r_0}}^G\left(|\,.\,|^{s+\frac{r_0}{2}}\right) \subset \mathrm{Ind}_B^G(\lambda)$$

where B is the standard Borel subgroup of G and λ is the unramified principal series representation with Satake parameter

$$(q^{s+r_0-\frac{1}{2}}, q^{s+r_0-\frac{3}{2}}, \ldots, q^{s+\frac{1}{2}})\ .$$

The element $\mathfrak{z}$ acts on the K-fixed vector of this representation by the scalar

$$\prod_{i=1}^{r_0}(1-q^{-s-\frac{1}{2}}q^{s+r_0+\frac{1}{2}-i})(1-q^{-s-\frac{1}{2}}q^{-s-r_0-\frac{1}{2}+i}) = 0\ .$$

This means that $\mathrm{pr}_{r_0}(\Psi_s * \mathfrak{z}) = 0$ i.e. that $\Psi_s * \mathfrak{z} \in I_n^{(r_0-1)}(s)$.

More generally, if we restrict the orbit of a section to Ω_r, we obtain a map

$$\mathrm{pr}_r : I_n(s) \longrightarrow \mathrm{Ind}_{P_r\times P_r}^{G\times G}\left(|\,.\,|^{s+\frac{r}{2}} \otimes |\,.\,|^{s+\frac{r}{2}} \otimes C(G_{n-2r})\right) =: B_r(s)$$

where $C(G_{n-2r})$ is the space of smooth functions on G_{n-2r}. There is a non-degenerate pairing between $Q_n^{(r)}(s)$ and $B_r(-s-r)$ given by

$$\langle f_1, f_2\rangle = \int_{P_r\times P_r\backslash G\times G} \langle f_1(g_1,g_2), f_2(g_1,g_2)\rangle_{G_{n-r}} \mathrm{d}\mu(g_1)\mathrm{d}\mu(g_2)\ ,$$

where the internal pairing is the integration over G_{n-r} and the external integral is the invariant functional for functions which transform on the left according to the square of the modulus character. A straightforward density argument shows that $\phi \in Q_n^{(r)}(s)$ is 0 if and only if it pairs to zero against all elements of the subspace $Q_n^{(r)}(-s-r) \subset B_r(-s-r)$. In addition if $\phi \in Q_n^{(r)}(s)^{K\times K}$ we can limit ourselves to the elements of $Q_n^{(r)}(-s-r)^{K\times K}$. Let $f_s \in Q_n^{(r)}(-s-r)^{K\times K}$ and $\mathfrak{z}_s = \mathfrak{z}\big|_{X=q^{-s}}$. We have

$$\langle \mathrm{pr}_r(\Psi_s * \mathfrak{z}), f_2\rangle = \langle \mathrm{pr}_r(\Psi_s) * \mathfrak{z}_s, f_s\rangle = \langle \mathrm{pr}_r(\Psi_s), f_s * \mathfrak{z}_s^\vee\rangle \ .$$

Lemma 3.4. — *For any $f_s \in Q_n^{(r)}(-s-r)^{K\times K}$ we have*

$$f_s * \mathfrak{z}_s^\vee = 0\ .$$

Proof of Lemma 3.4. — Since f_s is an element of a parabolic induction and is fixed by a maximal compact, it is determined by its value at the identity element I_n. It is not difficult to see that $f_s(I_n) \in \mathcal{S}(G)^{K_{n-r}\times K_{n-r}}$ where $K_{n-r} = G_{n-r} \cap K$. Let τ be an irreducible admissible representation of G_{n-r}. The action of $\mathcal{S}(G_{n-r})$ on τ determines a $G_{n-r}\times G_{n-r}$-equivariant map

$$\mu_\tau : \mathcal{S}(G_{n-r}) \longrightarrow \mathrm{Hom}^{\mathrm{smooth}}(\tau,\tau) \simeq \tau^\vee \otimes \tau$$

where $\mathrm{Hom}^{\mathrm{smooth}}$ is the space of vector-space homomorphisms fixed by a compact open subgroup of $G_{n-r}\times G_{n-r}$. The two factors of $G_{n-r} \times G_{n-r}$ act respectively by pre- and post-multiplication on the elements of $\mathrm{Hom}^{\mathrm{smooth}}(\tau,\tau)$ so that each has finite dimensional image. A function $\varphi \in \mathcal{S}(G_{n-r})^{K_{n-r}\times K_{n-r}}$ is nonzero if and only if there exists an irreducible admissible representation τ such that $\tau(\varphi) \neq 0$, i.e. such that $\mu_\tau(\varphi) \neq 0$.

Consider $f_s * \mathfrak{z}_s^\vee$. Let τ be, as above, an irreducible admissible representation of G_{n-r}. The map μ_τ induces

$$\begin{aligned}\mathrm{Ind}(\mu_\tau) : \mathrm{Ind}_{P_r\times P_r}^{G\times G}\big(|\,.\,|^{-s-\frac{r}{2}} \otimes |\,.\,|^{-s-\frac{r}{2}} \otimes \mathcal{S}(G_{n-r})\big)&\\ \longrightarrow \mathrm{Ind}_{P_r\times P_r}^{G\times G}\big(|\,.\,|^{-s-\frac{r}{2}} \otimes |\,.\,|^{-s-\frac{r}{2}} \otimes \tau^\vee \otimes \tau\big)&\end{aligned}$$

which satisfies $\mathrm{Ind}(\mu_\tau)(f_s)(I_n) = \mu_\tau(f_s(I_n))$. The latter induced representation is isomorphic to

$$\mathrm{Ind}_{P_r}^G\big(|\,.\,|^{-s-\frac{r}{2}} \otimes \tau^\vee\big) \otimes \mathrm{Ind}_{P_r}^G\big(|\,.\,|^{-s-\frac{r}{2}} \otimes \tau\big)$$

which can be embedded in

$$\mathrm{Ind}_B^G\lambda_1 \otimes \mathrm{Ind}_B^G\lambda_2$$

where the Satake parameters are

$$\begin{aligned}\lambda_1 &= (q^{-s-\frac{1}{2}}, q^{-s-\frac{3}{2}}, \dots, q^{-s+\frac{1}{2}-r}, q^{-\nu_1}, \dots, q^{-\nu_{n-r}})\\ \lambda_2 &= (q^{-s-\frac{1}{2}}, q^{-s-\frac{3}{2}}, \dots, q^{-s+\frac{1}{2}-r}, q^{\nu_1}, \dots, q^{\nu_{n-r}})\end{aligned}$$

(where $(q^{\nu_1}, \dots, q^{\nu_{n-r}})$ is the Satake parameter of τ). The operator $\mathfrak{z}_s^\vee$ acts on the unique line of $K \times K$-invariant vectors of this representation by the scalar

$$\prod_{i=1}^{r}(1 - q^{-s}q^{-\frac{1}{2}}q^{s-\frac{1}{2}+i})(1 - q^{-s}q^{-\frac{1}{2}}q^{-s+\frac{1}{2}-i}) \cdot (\text{factor}) = 0 \ .$$

But $\mathrm{Ind}(\mu_\tau)(f_s)$ is a $K \times K$-invariant vector in this representation so that $\mathrm{Ind}(\mu_\tau)(f_s) * \mathfrak{z}_s = 0$ and

$$\begin{aligned} \mu_\tau(f_s * \mathfrak{z}_s^\vee(I_n)) &= \mathrm{Ind}(\mu_\tau)(f_s * \mathfrak{z}_s^\vee)(I_n) \\ &= (\mathrm{Ind}(\mu_\tau)(f_s * \mathfrak{z}_s^\vee))(I_n) \\ &= 0 \ . \end{aligned}$$

Since this is true for all τ, we have $f_s * \mathfrak{z}_s^\vee(I_n) = 0$ and thus $f_s * \mathfrak{z}_s^\vee = 0$. □

We have $\mathrm{pr}_r(\Psi_s * \mathfrak{z}) = 0$ for all $r > 0$, which means that the support of $\Psi_s * \mathfrak{z}$ is included in Ω_0, which concludes the proof of Proposition 3.3. □

3.2.3. *Step 3*

Consider the isomorphism

$$\mathrm{pr}_0 : I_n(s) \longrightarrow Q_n^{(0)}(G) \simeq \mathcal{S}(G) \ .$$

Proposition 3.3 shows that, for a fixed s, we have $\mathrm{pr}_0(\Psi_s * \mathfrak{z}) \in \mathcal{S}(G)^{K \times K}$. Its support could vary with s. The following proposition shows that the support of $\mathrm{pr}_0(\Psi_s * \mathfrak{z})$ is bounded uniformly in s.

Lemma 3.5. — *We have*

$$\mathrm{pr}_0(\Psi_s * \mathfrak{z}) \in \mathbf{C}[q^s, q^{-s}] \otimes \mathcal{S}(G)^{K \times K} = \mathbf{C}[q^s, q^{-s}] \otimes \mathcal{H}(G \,/\!/\, K) \ .$$

Proof of Lemma 3.5. — Using the Cartan decomposition, write

$$\mathrm{pr}_0(\Psi_s * \mathfrak{z}) = \sum_{\lambda \in \Lambda} c_\lambda(s) L_\lambda \ ,$$

where L_λ is the characteristic function of the double coset $K g_\lambda K$ and Λ is the usual semigroup.

Lemma 3.6. — *We have*

$$c_\lambda(s) \in \mathbf{C}[q^s, q^{-s}]$$

and thus is an entire function of s.

Proof. — We have

$$c_\lambda(s)\cdot\|L_\lambda\|^2=\int_G(\Psi_s*\mathfrak{z})(i(g,I_n))\cdot L_\lambda(g)\mathrm{d}g. \tag{3.7}$$

The integral on the right is a (finite) linear combination, with coefficients in $\mathbf{C}[q^s,q^{-s}]$ of integrals of the form

$$\begin{aligned}&\int_G\int_G(\Psi_s*\mathfrak{z})(i(g,I_n)i(g_0,I_n))\cdot L_\mu(g_0)\mathrm{d}g_0\cdot L_\lambda(g)\mathrm{d}g \qquad (3.8)\\ =\;&\int_G\int_G(\Psi_s*\mathfrak{z})(i(g_0,I_n))\cdot L_\mu(g^{-1}g_0)\cdot L_\lambda(g)\mathrm{d}g_0\mathrm{d}g\\ =\;&\int_G\int_G(\Psi_s*\mathfrak{z})(i(g_0,I_n))\cdot\varphi(g_0)\mathrm{d}g_0\end{aligned}$$

where φ is a function depending on λ and μ. Since this function is a (finite) linear combination of characteristic functions of cosets gK, the integral in the last line of (3.8) is a (finite) linear combination with coefficients in $\mathbf{C}[q^s,q^{-s}]$ of integrals of the form

$$\int_K\int_{K\times K}\Phi_s\big(i(gk,I_n)i(k_1,k_2)i(g_1^{-1},g_2^{-1})\big)\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k\ .$$

But Φ_s is standard, hence it is right-invariant under a fixed compact open subgroup H, uniformly in s. This means that the set of g necessary to obtain the full integral (3.7) is finite and fixed. The elements g_1 and g_2 are fixed by the matrix coefficient ϕ we are considering and thus the integral (3.7) is a (finite) linear combination of $q^{\ell s}$ with $\ell\in\mathbf{Z}$. □

Let then Λ_1 be the set of $\lambda\in\Lambda$ such that $c_\lambda\neq 0$ and for $\lambda\in\Lambda$ let

$$D_\lambda=\{s\in\mathbf{C}:c_\lambda(s)=0\}\ .$$

If $\lambda\in\Lambda_1$ then D_λ is a numerable subset of $\mathbf{C}$. Hence $\bigcup_{\lambda\in\Lambda_1}D_\lambda$ is numerable and thus different from $\mathbf{C}$. Let $s_0\in\mathbf{C}$ be such that $\forall\lambda\in\Lambda_1$, $c_\lambda(s_0)\neq 0$. Since

$$\mathrm{pr}_0(\Psi_{s_0}*\mathfrak{z})=\sum_{\lambda\in\Lambda_1}c_\lambda(s_0)\cdot L_\lambda$$

has compact support, Λ_1 is finite and thus for all $s\in\mathbf{C}$, $\mathrm{pr}_0(\Psi_s*\mathfrak{z})$ has support in $\cup_{\lambda\in\Lambda_1}L_\lambda$. □

3.2.4. *Step 4*

Going back to the Zeta integral in (3.5), we define

$$Z^*(s,\chi_0,\pi,\phi,\Phi)=\int_G\phi^\circ(g)(\Psi_s*\mathfrak{z})(i(g,I_n))\mathrm{d}g.$$

This integral is equal to the scalar by which $\mathrm{pr}_0(\Psi_s * \mathfrak{z})$ acts on $\xi_\circ$ and is thus an entire function of s because it is an element of $\mathbf{C}[q^s, q^{-s}]$. On the other hand, if $\mathrm{Re}\,(s)$ is large enough we can unfold

$$\begin{aligned} Z^*(s,\chi_0,\pi,\phi,\Phi) &= \pi(\mathfrak{z}_s)\int_G \phi^\circ(g)\Psi_s(i(g,I_n))\mathrm{d}g \\ &= \pi(\mathfrak{z}_s)Z(s,\chi_0,\pi,\phi,\Phi) \end{aligned}$$

where $\pi(\mathfrak{z}_s)$ is the scalar by which $\mathfrak{z}_s = \mathfrak{z}\big|_{X=q^{-s}}$ acts on the spherical vector of π. Since $Z^*(s,\chi_0,\pi,\phi,\Phi)$ is an entire function of s, this completes the proof or Proposition 3.2. □

3.3. The conjecture holds for the trivial representation in the even dimensional tower

Definition 3.7 ([HKS96, Definition 4.6, p.963]). — *For $s_0 \in \mathbf{C}$, χ a character and π and irreducible admissible representation of G, we say that* π occurs in the boundary at the point $s = s_0$ *if*

$$\mathrm{Hom}_{G\times G}(Q_n^{(r)}(s_0,\chi), \pi\otimes(\chi\cdot\pi^\vee)) \neq 0$$

for some $r > 0$.

Proposition 3.8. — *Let $\pi = \mathbf{1}$ the trivial representation of G, ϖ_E an uniformiser of E and $q_E = |\varpi_E|$. We will denote $X^u(E^\times)$ the set of unramified characters of $E^\times$. Let*

$$X(\mathbf{1}) \neq \left\{(s,\chi)\in\mathbf{C}\times X^u(E^\times)\,\middle|\,\chi(\varpi_E) = (-1)^k, s = \frac{n}{2} - r - \frac{ki\pi}{\log q_E}, 1\leqslant r\leqslant r_0\right\}$$

with $1 \leqslant r \leqslant r_0$ and $k \in \mathbf{Z}$.

Then $\mathbf{1}$ appears in the boundary at s if and only if $(s,\chi) \in X(\mathbf{1})$. Moreover if $(s_0,\chi) \notin X(\mathbf{1})$, for any standard section Φ the operator $Z(s,\chi,\mathbf{1})$ is holomorphic at $s = s_0$ and

$$\mathrm{Hom}_{G\times G}(I_n(s_0,\chi), \mathbf{1}\otimes\chi) = \mathbf{C}\cdot Z(s,\chi,\mathbf{1})\ .$$

Proof. — We know from Lemma 3.1 that

$$\begin{aligned} &\mathrm{Hom}_{G\times G}(Q_n^{(r)}(s,\chi), \mathbf{1}\otimes\chi) \\ = \ &\mathrm{Hom}_{G\times G}\left(\mathrm{Ind}_{P_r\times P_r}^{G\times G}\left(\chi|.|^{s+\frac{r}{2}}\otimes\chi|.|^{s+\frac{r}{2}}\otimes\left(\mathcal{S}(G')\cdot(\mathbf{1}\otimes\chi)\right)\right),\ \mathbf{1}\otimes\chi\right) \\ \simeq \ &\mathrm{Hom}_{G\times G}\left(\mathbf{1}\otimes\chi^{-1},\ \mathrm{Ind}_{P_r\times P_r}^{G\times G}\left(\chi^{-1}|\,.\,|^{-s-\frac{r}{2}}\otimes\chi^{-1}|\,.\,|^{-s-\frac{r}{2}}\otimes\left(\mathrm{C}^\infty(G')\cdot(\mathbf{1}\otimes\chi^{-1})\right)\right)\right) \\ \simeq \ &\mathrm{Hom}_{M_r\times M_r}\left(\mathbf{1}\otimes\chi^{-1}, \chi^{-1}|\,.\,|^{-s-\frac{r}{2}+\frac{n-r}{2}}\otimes\chi^{-1}|\,.\,|^{-s-\frac{r}{2}+\frac{n-r}{2}}\otimes\left(\mathrm{C}^\infty(G')\cdot(\mathbf{1}\otimes\chi^{-1})\right)\right) \end{aligned}$$

because the Jacquet module for $\mathbf{1}\otimes\chi^{-1}$ is $\mathbf{1}\otimes\chi^{-1}$ (as a representation of M_r).
Now if g corresponds to (a,g') in Equation (3.2) then $\det g = \det a\overline{\det a^{-1}}$ $\det g'$ so that $\chi(\det g) = \chi(\det a)^2\chi(\det g')$ but $\dim \mathrm{Hom}_{G'\times G'}(\mathbf{1}\otimes\chi^{-1}, \mathrm{C}^\infty(G')\cdot(\mathbf{1}\otimes\chi^{-1})) = 1$ (see [HKS96, end of section 4, p.964] for general π). Thus

$$\simeq \mathrm{Hom}_{\mathrm{GL}(U)\times\mathrm{GL}(U)}\big(\mathbf{1}\otimes\chi^{-2}, \chi^{-1}|\,.\,|^{-s+\frac{n}{2}-r}\otimes\chi^{-1}|\,.\,|^{-s+\frac{n}{2}-r}\big)$$

It follows that π occurs in the boundary at s if and only if χ is unramified, $\chi(\varpi_E) = (-1)^k$ and $(s-\frac{n}{2}+r)\log q_E + ki\pi = 0$, as required.

Suppose $(s_0,\chi)\notin X(\mathbf{1})$, i.e. $\mathbf{1}$ does not appear in the boundary. Let k be the maximum order of the pole of the Z integral in $s = s_0$ (as Φ varies). Thus

$$Z(s,\chi,\mathbf{1},\Phi) = \frac{\tau_{-k}(s,\chi,\mathbf{1},\Phi)}{(s-s_0)^k} + \cdots + \tau_0(s,\chi,\mathbf{1},\Phi) + \cdots$$

where the τ_i are holomorphic functions of s in a neighbourhood of s_0 and τ_{-k} is non-zero. The leading term τ_{-k} is itself an intertwining operator. If we had $k>0$, that is, if the Z integral had a pole in $s=s_0$, the restriction of τ_{-k} to $I_n^{(0)}(s_0,\chi)$ would be zero because the Z integral is convergent on

$$I_n^{(0)}(s_0,\chi) = Q_n^{(0)}(s,\chi) \simeq \mathcal{S}(G)\cdot(\mathbf{1}\otimes\chi)$$

thus convergent for every standard section $\Phi(s)$ such that $\Phi\in I_n^{(0)}(s,\chi)$. This means that we would have a non-zero intertwining operator in $\mathrm{Hom}_{G\times G}(Q_n^{(r)}(s,\chi),\mathbf{1}\otimes\chi)$ for some $r>0$, which is impossible by hypothesis. Thus $k\leqslant 0$, i.e. the integral is entire for any $\Phi\in I_n(s_0,\chi)$. Moreover, $Z(s_0,\chi,\mathbf{1})$ is a non-zero intertwining operator between $I_n^{(0)}(s_0,\chi)$ and $\mathbf{1}\otimes\chi$, which means that $\mathrm{Hom}_{G\times G}(I_n^{(0)}(s_0,\chi),\mathbf{1}\otimes\chi)$ is non zero, thus has dimension 1, and that $Z(s_0,\chi,\mathbf{1})$ is its basis.

Let $\lambda\in\mathrm{Hom}_{G\times G}(I_n(s_0,\chi),\mathbf{1}\otimes\chi)$. Its restriction $\bar\lambda$ to $I_n^{(0)}(s_0,\chi)$ is a multiple of $Z(s_0,\chi,\mathbf{1})$. Since $\mathbf{1}$ is supposed not to appear in the boundary, if $\lambda\neq 0$, then $\bar\lambda\neq 0$, i.e. $\bar\lambda = cZ(s_0,\chi,\mathbf{1})$ for some $c\neq 0$. Since $\lambda - cZ(s_0,\chi,\mathbf{1})$ is zero on $I_n^{(0)}(s_0,\chi)$, it must be zero everywhere, i.e. $\lambda = cZ(s_0,\chi,\mathbf{1})$. □

Theorem 3.9. — *Let m be an even integer and χ_0 the trivial character of $E^\times$, then*

$$\forall m\leqslant 2n,\quad \mathrm{Hom}_{G\times G}(R_n(V_m^-,\chi_0),\mathbf{1}) = 0\ ,$$

so that by (ii) of Proposition 2.6

$$\mathrm{Hom}_{G\times G}(R_n(V^-_{2n+2},\chi_0),\mathbf{1})\neq 0$$

and thus $m^-_{\chi_0}(\mathbf{1})=2n+2$. *Since* $m^+_{\chi_0}(\mathbf{1})=0$, *we have*

$$m^+_{\chi_0}(\mathbf{1})+m^-_{\chi_0}(\mathbf{1})=2n+2\ .$$

Proof. — By (i) of Proposition 2.6, it suffices to prove that

$$\mathrm{Hom}_{G\times G}(R_n(V^-_{2n},\chi_0),\mathbf{1})=0\ .$$

From Proposition 3.8 we know that

$$\mathrm{Hom}_{G\times G}\left(I_n\left(-\frac{n}{2},\chi_0\right),\mathbf{1}\right)$$

is non zero and is generated by

$$Z\left(-\frac{n}{2},\chi_0,\mathbf{1}\right)$$

which is holomorphic at $-\frac{n}{2}$. The element of $I_n(-\frac{n}{2},\chi_0)$ equal to 1 on K is $\chi_{0,\tilde{G}}$. As seen in [Li92, Theorem 3.1, p.186] and [LR05, Proposition 3, p.333] we have

$$Z\left(-\frac{n}{2},\chi_0,\mathbf{1},\phi^\circ,\chi_{0,\tilde{G}}\right)\neq 0$$

and thus $Z(-\frac{n}{2},\chi_0,\mathbf{1})(\chi_{0,\tilde{G}})\neq 0$. Let

$$\phi\in\mathrm{Hom}_{G\times G}(R_n(V^-_{2n},\chi_0),\mathbf{1})$$

and

$$\tilde{\phi}=\phi\circ M^*_n\left(-\frac{n}{2},\chi_0\right)\in\mathrm{Hom}_{G\times G}\left(I_n\left(-\frac{n}{2},\chi_0\right),\mathbf{1}\right)\ .$$

We have $\chi_{0,\tilde{G}}\in R_n(V^+_0,\chi_{0,\tilde{G}})=\ker M^*_n(-\frac{n}{2},\chi_0)$ so that $\tilde{\phi}(\chi_{0,\tilde{G}})=0$. This means that $\tilde{\phi}=0$ because it is a multiple of $Z\left(-\frac{n}{2},\chi_0,\mathbf{1}\right)$. We know from Proposition 2.10 that the mapping

$$M^*_n\left(-\frac{n}{2},\chi_0\right):I_n\left(-\frac{n}{2},\chi_0\right)\longrightarrow R_n(V^-_{2n},\chi_0)$$

is surjective so that $\phi=0$. □

3.4. Half of the conjecture

THEOREM 3.10. — *Let π be an irreducible admissible representation of $G(W)$, then*

$$m_\chi^+(\pi) + m_\chi^-(\pi) \geqslant 2n + 2\ .$$

Proof. — Fix $m_0 \in \{0, 1\}$, a character χ of $E^\times$ such that $\chi|_{F^\times} = \epsilon_{E/F}^{m_0}$ and suppose we have two Hermitian spaces V_a^+ and V_b^- such that

$$\theta_\chi(\pi, V_a^+) \neq 0 \quad \text{and} \quad \theta_\chi(\pi, V_b^-) \neq 0\ ,$$

with $\dim V_a^+ = a$, $\dim V_b^- = b$, a and b of the parity of m_0, $\epsilon(V_a^+) = 1$ and $\epsilon(V_b^-) = -1$. Let $V_{b,-}^-$ be the same space as V_b^- with opposite form and

$$\mathbb{W}_a = V_a^+ \otimes W, \quad \mathbb{W}_b = V_b^- \otimes W, \quad \mathbb{W}_{b,-} = V_{b,-}^- \otimes W.$$

We denote $\omega_{a,\chi}$ (resp. $\omega_{b,\chi}$, $\omega_{b,-,\chi}$) the representations of G induced by the representations $\omega_{a,\psi}$ (resp. $\omega_{b,\psi}$, $\omega_{b,-,\psi}$) of $\mathrm{Mp}(\mathbb{W}_a)$ (resp. $\mathrm{Mp}(\mathbb{W}_b)$, $\mathrm{Mp}(\mathbb{W}_{b,-})$). By hypothesis on V_a^+ and V_b^- we have two non-zero (and thus surjective) elements

$$\lambda \in \mathrm{Hom}_G(\omega_{a,\chi}, \pi), \quad \mu \in \mathrm{Hom}_G(\omega_{b,\chi}, \pi)\ .$$

Let $g_0 \in \mathrm{GL}_F(W)$ be an F-automorphism of W which is conjugate-linear as an E-morphism. Then $\mathrm{Ad}(g_0)$ is a MVW involution on G. Conjugating μ and π by $\mathrm{Ad}(g_0)$ we get a non-zero morphism

$$\mu^\vee \in \mathrm{Hom}_G(\omega_{b,\chi}^\vee, \pi^\vee)$$

and thus a surjective

$$\nu_0 = \lambda \otimes \mu^\vee \in \mathrm{Hom}_{G\times G}(\omega_{a,\chi} \otimes \omega_{b,\chi}^\vee, \pi \otimes \pi^\vee)\ .$$

We consider the projection of ν_0 on the trivial subquotient and see it as a G-homomorphism through the diagonal action of G. We get a non-zero element

$$\nu \in \mathrm{Hom}_G(\omega_{a,\chi} \otimes \omega_{b,\chi}^\vee, \mathbf{1})\ .$$

We have

$$\omega_{b,\psi}^\vee \simeq \omega_{b,\overline{\psi}} \simeq \omega_{b,-,\psi}.$$ [3]

On the other hand we can identify $\mathrm{Mp}(\mathbb{W}_b)$ and $\mathrm{Mp}(\mathbb{W}_{b,-})$ in which case we get the following

---

(3) The first isomorphism holds true because $\omega_{b,\psi}$ is unitary, the second because of the definition of $r(g)$ in 2.3

Lemma 3.11. — *We have*

$$\tilde{\imath}_{b,\chi} \simeq \tilde{\imath}_{b,-,\chi^{-1}} \ ,$$

where we added a subscript to $\tilde{\imath}$ *to remember which Hermitian space is involved.*

Proof. — The space V_b^- can be decomposed as an orthogonal direct sum of a split space and zero, one or two anisotropic lines. Since the splitting $\tilde{\imath}$ is additive, we consider separately the split and the anisotropic case.

We first consider the case in which V_b^- is split. We will need some additional notations (see [HKS96, n.10, p.950]). For any additive character η of F and $a \in F$ we will let η_a be the character such that $\eta_a(x) = \eta(ax)$, $\gamma_F(\eta) \in \mu_8$ is the Weil index of the quadratic character $x \longmapsto \eta(x^2)$ and $\gamma_F(a,\eta) = \frac{\gamma_F(\eta_a)}{\gamma_F(\eta)}$. Recall that (see [HKS96, n.11, p.950])

$$\gamma_F(ab,\eta) = (a,b)_F \gamma_F(a,\eta)\gamma_F(b,\eta) \ .$$

Let η be the character such that $\eta(x) = \psi(\frac{1}{2}x)$ (i.e. $\eta = \psi_{\frac{1}{2}}$). For $g \in G$, we denote $j(g)$ the integer such that $i(g, I_n) \in P_Y \delta_{j(g)} i(G \times G)$. Since V_b^- is split we have (see [HKS96, 1.15, p.953]),

$$\tilde{\imath}_{b,\chi}(g) = (\imath_b(g), \beta_{V_b^-,\chi}(g))$$

with

$$\beta_{V_b^-,\chi}(g) = \chi(x(g))\gamma_F(\eta\circ RV)^{-j(g)}$$

where

$$\gamma_F(\eta\circ RV) = (\Delta, \det V_b^-)_F \gamma_F(-\Delta,\eta)^b \gamma_F(-1,\eta)^{-b}.^4$$

Let

$$\begin{array}{rcccc} \varphi: & \mathrm{Sp}(\mathbb{W}_b) \times \mathbf{C}^1 \simeq \mathrm{Mp}(\mathbb{W}_b) & \longrightarrow & \mathrm{Sp}(\mathbb{W}_{b,-}) \times \mathbf{C}^1 \simeq \mathrm{Mp}(\mathbb{W}_{b,-}) \\ & (g \ , \ z) & \longmapsto & (g \ , \ \overline{z}) \end{array}$$

be the identification. Then $\overline{\chi(x(g))} = \chi^{-1}(x(g))$ and

$$\begin{aligned} \overline{\gamma_F(-\Delta,\eta)\gamma_F(-1,\eta)^{-1}} &= \overline{\left(\frac{\gamma_F(\eta_{-\Delta})}{\gamma_F(\eta_{-1})}\right)} = \frac{\gamma_F(\eta_\Delta)}{\gamma_F(\eta_1)} = \gamma_F(\Delta,\eta)\gamma_F(1,\eta)^{-1} \\ &= (\Delta,-1)_F \gamma_F(-\Delta,\eta)(-1,-1)_F \gamma_F(-1,\eta)^{-1} \\ &= (\Delta,-1)_F \gamma_F(-\Delta,\eta)\gamma_F(-1,\eta)^{-1} \end{aligned}$$

(4) for this single proof, we fix $\delta \in E^\times - F^\times$ such that $\Delta = \delta^2 \in F^\times$ and use it to identify the Hermitian and skew-Hermitian spaces

thus, since $\det V_{b,-}^- = (-1)^b \det V_b^-$, we have $\overline{\beta_{V_b^-,\chi}(g)} = \beta_{V_{b,-}^-,\chi^{-1}}(g)$ and

$$\varphi\circ\tilde{\imath}_{b,\chi} = \tilde{\imath}_{b,-,\chi^{-1}}$$

as claimed.

We now consider the case in which V_b^- is an anisotropic line. We identify V_b^- with E and if $(x,y) \in E^2$, we have $\langle x,y\rangle = \mathbf{a}\overline{x}y$ for some $\mathbf{a} \in F$. If $g \in G(V_b^-) = E^1$, we decompose $g = x + \delta y$ (with $x, y \in F$) and we have (see [Kud94, Proposition 4.8, p.396])

$$\begin{aligned}\beta_{V_b^-,\chi}(g) &= \chi(\delta(g-1))\gamma_F(2\mathbf{a}y(x-1),\eta)\gamma_F(\eta)(\Delta,-2y(1-x))_F \\ &= \chi(\delta(g-1))\gamma_F(\eta_{2\mathbf{a}y(x-1)})(\Delta,-2y(1-x))_F\end{aligned}$$

and

$$\beta_{V_{b,-}^-,\chi}(g) = \chi(\delta(g-1))\gamma_F(\eta_{-2\mathbf{a}y(x-1)})(\Delta,-2y(1-x))_F \ .$$

It is immediate that $\overline{\beta_{V_{b,-}^-,\chi^{-1}}(g)} = \beta_{V_b^-,\chi}(g)$ and

$$\varphi\circ\tilde{\imath}_{b,\chi} = \tilde{\imath}_{b,-,\chi^{-1}}$$

as claimed. □

Let

$$V_{a,b,-} = V_a^+ \oplus V_{b,-}^-, \quad \mathbb{W}_{a,b,-} = \mathbb{W}_a \oplus \mathbb{W}_{b,-}$$

and let, as before, χ_0 be the trivial character of $E^\times$. We denote, as above, $\omega_{a,b,-,\chi_0}$ the representation of G induced by the Weil representation $\omega_{a,b,-,\psi}$. Let

$$\tilde{\imath} : \mathrm{Mp}(\mathbb{W}_a) \times \mathrm{Mp}(\mathbb{W}_{b,-}) \longrightarrow \mathrm{Mp}(\mathbb{W}_{a,b,-})$$

be the natural map whose restriction to $\mathbf{C}^1$ is the product. Then

$$\tilde{\imath}^*\omega_{a,b,-,\psi} = \omega_{a,\psi} \otimes \omega_{b,-,\psi} \ .$$

According to [HKS96, Lemma 5.2, p.964],

$$\tilde{\imath}_{a,b,-,\chi_0} = \tilde{\imath}\circ(\tilde{\imath}_{a,\chi} \times \tilde{\imath}_{b,-,\chi^{-1}})\circ\Delta : G \longrightarrow \mathrm{Mp}(\mathbb{W}_{a,b,-}).$$

Thus as a representation of G we have

$$\omega_{a,\chi} \otimes \omega_{b,-,\chi^{-1}} \simeq \omega_{a,b,-,\chi_0} \ .$$

We thus have a non-zero element

$$\nu \in \mathrm{Hom}_G(\omega_{a,\chi} \otimes \omega_{b,\chi}^\vee, \mathbf{1}) \simeq \mathrm{Hom}_G(\omega_{a,b,-,\chi_0}, \mathbf{1}) \ .$$

We have $\dim V_{a,b,-} = a+b$ even. Let us compute $\epsilon(V_{a,b,-})$:

$$\begin{aligned}\epsilon(V_{a,b,-}) &= (-1)^{\frac{(a+b)(a+b-1)}{2}} \det V_{a,b,-}\\ &= (-1)^{\frac{a(a-1)+ab+ba+b(b-1)}{2}} \det V_a^+ \det V_{b,-}^-\\ &= (-1)^{\frac{a(a-1)+b(b-1)}{2}+ab} \det V_a^+ (-1)^b \det V_b^-\\ &= (-1)^{ab+b}(-1)^{\frac{a(a-1)}{2}} \det V_a^+ (-1)^{\frac{b(b-1)}{2}} \det V_b^-\\ &= (-1)^{ab+b}\epsilon(V_a^+)\epsilon(V_b^-)\ .\end{aligned}$$

Since both ab and b have the parity of m_0 we have $\epsilon(V_{a,b,-}) = \epsilon(V_a^+)\epsilon(V_b^-) = -1$. Thus, according to Theorem 3.9

$$a+b \geqslant 2n+2$$

as needed. □

3.5. Criterion

Definition 3.12. — *For a given $m \in \{0,...,2n\}$, let $m' = 2n-m$. The space $V_{m'}^{\pm}$ is said to be* complementary *to $V_m^{\pm}$ (the space V_{2n}^- has no complementary).*

Remark 3.13. — If $V_{m'}^{\pm}$ is complementary of $V_m^{\pm}$, then $s_0' = \frac{m'-n}{2} = \frac{2n-m-n}{2} = \frac{n-m}{2} = -s_0$.

Theorem 3.14. — *Fix $m_0 \in \{0,1\}$ and a character χ of $E^\times$ such that $\chi|_{F^\times} = \epsilon_{E/F}^{m_0}$. Suppose that*

$$\dim \mathrm{Hom}_{G\times G}(I_n(s_0,\chi), \pi \otimes (\chi\cdot\pi^\vee)) = 1$$

for all s_0 in

$$\begin{cases}\left\{-\frac{n}{2}, 1-\frac{n}{2}, ..., \frac{n}{2}-1, \frac{n}{2}\right\} & \text{if } m_0 = 0\\ \left\{\frac{1-n}{2}, \frac{3-n}{2}, ..., \frac{n-3}{2}, \frac{n-1}{2}\right\} & \text{if } m_0 = 1,\end{cases}$$

i.e. for all $s_0 \in \frac{m_0}{2} + \mathbf{Z}$ such that $|s_0| \leqslant \frac{n}{2}$. Then

$$m_\chi^+(\pi) + m_\chi^-(\pi) = 2n+2\ .$$

To prove the theorem, we will need the composition series for $I_n(s_0,\chi)$ in each case where it is reducible. Using [KS97], we give here those series explicitly with indication of the action of the operators $M^*(s_0,\chi)$. In the diagram we have implicitly $m' = 2n-m$. Note that V_0^- does not exist, but we define the space $R_n(V_0^-,\chi)$ as the zero-dimensional subspace in $R_n(V_0^+,\chi)$.

$$\begin{array}{ccccc}
0 & \subset & R_n(V_0^+,\chi) & \subset & I(-\frac{n}{2},\chi) \\
\| & & \| & & \\
R_n(V_0^-,\chi) & & M^*(\frac{n}{2},\chi)(R_n(V_{2n}^+,\chi)) & = & M^*(\frac{n}{2},\chi)(I(\frac{n}{2},\chi)) \\
\| & & \| & & \\
M^*(\frac{n}{2},\chi)(R_n(V_{2n}^+,\chi)) & & \operatorname{Ker} M^*(-\frac{n}{2},\chi) & & m=0,\ s_0=-\frac{n}{2}
\end{array}$$

$$\begin{array}{cccccc}
 & & M^*(-s_0,\chi)(R_n(V_{m'}^+,\chi)) & & & \\
 & & \| & & & \\
 & & R_n(V_m^+,\chi) & & & \\
 & \subset & & \subset & & \\
0 & & & & R_n(V_m^+,\chi)\oplus R_n(V_m^-,\chi) & \subset I_n(s_0,\chi) \\
 & \subset & & \subset & \| & \\
 & & R_n(V_m^-,\chi) & & \operatorname{Ker} M^*(s_0,\chi) & \\
 & & \| & & & \\
 & & M^*(-s_0,\chi)(R_n(V_{m'}^-,\chi)) & & 1\leqslant m<n,\ -\frac{n}{2}<s_0<0 &
\end{array}$$

$$\begin{array}{cccccc}
 & & M^*(0,\chi)(R_n(V_n^+,\chi)) & & & \\
 & & \| & & & \\
 & & R_n(V_n^+,\chi) & & & \\
 & \subset & & \subset & & \\
0 & & & & R_n(V_n^+,\chi)\oplus R_n(V_n^-,\chi) & = I(0,\chi) \\
\| & \subset & & \subset & & \\
\operatorname{Ker} M^*(0,\chi) & & R_n(V_n^-,\chi) & & & \\
 & & \| & & & \\
 & & M^*(0,\chi)(R_n(V_n^-,\chi)) & & m=n,\ s_0=0 &
\end{array}$$

$$\begin{array}{cccccc}
 & & R_n(V_m^+,\chi) & & & \\
 & \subset & & \subset & & \\
0\ \subset R_n(V_m^+,\chi)\cap R_n(V_m^-,\chi) & & & & R_n(V_m^+,\chi)+R_n(V_m^-,\chi) & = I_n(s_0,\chi) \\
\| & \subset & & \subset & & \\
\operatorname{Im} M^*(-s_0,\chi) & & R_n(V_m^-,\chi) & & & \\
\| & & & & & \\
\operatorname{Ker} M^*(s_0,\chi) & & & & n<m<2n,\ 0<s_0<\frac{n}{2} &
\end{array}$$

$$\begin{array}{ccccc}
0 & \subset & R_n(V_{2n}^-,\chi) & \subset & R_n(V_{2n}^+,\chi)=I_n\left(\frac{n}{2},\chi\right) \\
 & & \| & & \\
 & & \operatorname{Im} M^*(-\frac{n}{2},\chi) & & \\
 & & \| & & \\
 & & \operatorname{Ker} M^*(\frac{n}{2},\chi) & & m=2n,\ s_0=\frac{n}{2}
\end{array}$$

In each case an inclusion sign means that the quotient is non-zero and irreducible.

Proof. — Fix $m_0 \in \{0,1\}$ and a character χ of $E^\times$ such that $\chi|_{F^\times} = \epsilon_{E/F}^{m_0}$. For $0 \leqslant m' \leqslant 2n$, we put $m = 2n - m'$ and recall that $s_0 = \frac{m-n}{2}$.

The case $m_\chi^+(\pi) = 0$ is immediate because it implies $\pi = \mathbf{1}$ and Theorem 3.9 says that $m_\chi^-(\pi) = 2n+2$.

If $s_0 \geqslant 0$ we have $I_n(s_0,\chi) = R_n(V_m^+,\chi) + R_n(V_m^-,\chi)$ and thus, thanks to the hypothesis of the theorem, at least one of

$$\mathrm{Hom}_{G\times G}(R_n(V_m^\pm,\chi), \pi \otimes (\chi \cdot \pi^\vee))$$

is non zero. Thanks to Proposition 2.8 this in turn means that

$$\min(m_\chi^+(\pi), m_\chi^-(\pi)) \leqslant n+1$$

(the bound is $n+1$ and not n in case m and n have opposite parity). If $s_0 > \frac{n}{2}$ then $I_n(s_0,\chi)$ is irreducible and thus

$$R_n(V_m^\pm,\chi) = I_n(s_0,\chi)\ .$$

Since we have $m > 2n > \min(m_\chi^+(\pi), m_\chi^-(\pi))$, by the persistence principle (see Proposition 2.6, point (1.)) we have

$$\mathrm{Hom}_{G\times G}(R_n(V_m^\pm,\chi), \pi \otimes (\chi \cdot \pi^\vee)) \neq 0$$

for one and thus both signs $\pm$. This means $\max(m_\chi^+(\pi), m_\chi^-(\pi)) \leqslant 2n+2-m_0$.

Let $\epsilon = \pm$ be such that $m_\chi^\epsilon(\pi) = \min(m_\chi^+(\pi), m_\chi^-(\pi))$. We let m' be $m_\chi^\epsilon(\pi)$ (and choose m and s_0 accordingly). As observed above, the case $m' = 0$ has already been proved. If $m' = 1$, then from Theorem 3.10 we have $m_\chi^{-\epsilon}(\pi) \geqslant 2n+1$ and thus, thanks to the preceding bound, $m_\chi^{-\epsilon}(\pi) = 2n+1$ (observe that if $m' = 1$ then $m_0 = 1$).

We now suppose $2 \leqslant m' \leqslant n+1$, i.e. $-\frac{1}{2} \leqslant s_0 \leqslant \frac{n}{2} - 1$. By Theorem 3.10 we thus have $m_\chi^{-\epsilon}(\pi) \geqslant 2n+2-m' \geqslant n+1$. Since m' is the minimum of $m_\chi^\pm(\pi)$, we have

$$\mathrm{Hom}_{G\times G}(R_n(V_{m'-2}^+,\chi) \oplus R_n(V_{m'-2}^-,\chi), \pi \otimes (\chi \cdot \pi^\vee)) = 0 \qquad (3.9)$$

(here $R_n(V_0^-,\chi) = 0$ as defined above). This means that any element of $\mathrm{Hom}_{G\times G}(I_n(-s_0-1,\chi), \pi \otimes (\chi \cdot \pi^\vee))$ factors through

$$I_n(-s_0-1,\chi)/R_n(V_m^+,\chi) \oplus R_n(V_m^-,\chi) \simeq \mathrm{Im} M^*(-s_0-1,\chi)$$

and thus

$$\dim \mathrm{Hom}_{G\times G}(\mathrm{Im} M^*(-s_0-1,\chi), \pi\otimes(\chi\cdot\pi^\vee)) = 1 \ .$$

On the other hand, let

$$\mu \in \mathrm{Hom}_{G\times G}(I_n(s_0+1,\chi), \pi\otimes(\chi\cdot\pi^\vee))$$

with $\mu \neq 0$. Suppose

$$\mu|_{R_n(V^{-\epsilon}_{m+2})} = 0 \ .$$

Then, since $\mu \neq 0$ we have

$$\mu|_{R_n(V^{\epsilon}_{m+2})} = 0 \ ,$$

and thus

$$\mathrm{Hom}_{G\times G}(R_n(V^{-\epsilon}_{m+2})/R_n(V^{-\epsilon}_{m+2})\cap R_n(V^{\epsilon}_{m+2}), \pi\otimes(\chi\cdot\pi^\vee)) \neq 0 \ .$$

But $M^*(s_0+1)$ identifies

$$R_n(V^{-\epsilon}_{m+2})/R_n(V^{-\epsilon}_{m+2})\cap R_n(V^{\epsilon}_{m+2})$$

with $R_n(V^{-\epsilon}_{m'-2})$. This means that

$$\mathrm{Hom}_{G\times G}(R_n(V^{-\epsilon}_{m'-2}), \pi\otimes(\chi\cdot\pi^\vee)) \neq 0 \ .$$

From (3.9), we know that this is impossible. Hence μ must be non-zero on $R_n(V^{-\epsilon}_{m+2})$ thus

$$m^{-\epsilon}_\chi(\pi) \leqslant m+2 = 2n+2-m' \ .$$

We thus have $m^+_\chi(\pi) + m^-_\chi(\pi) = 2n+2$ as claimed. $\square$

Appendix

A. Completion of a proof

As announced in the introduction, we want to add a missing statement in the proof of [Har07, Theorem 3.4, p.128]. In the proof of the theorem, one should check that the spherical vector of the representation $I_n(s, \alpha^*)$ belongs to $R_n(V_m^+)$ for almost all places v. We prove it here in the following lemma.

Lemma A.1. — *We suppose E/F, V, W, m, n, G, H, $\mathbb{W}$, $\mathbb{X}$, $\mathbb{Y}$, χ and ψ are as above. We suppose in addition that E/F, χ and ψ are unramified. Then for any $s = \frac{m-n}{2}$ the spherical vector of $I_n(s, \chi)$ is in $R_n(V_m^+, \chi)$.*

Proof. — The spherical vector of $I_n(s, \chi)$ is the unique element Φ° such that $\Phi^\circ(K) = \{1\}$. Thus one only needs to check that there is an element in $\Phi \in R_n(V_m^+, \chi)$ such that $\Phi(K) = \{1\}$. Remember that

$$R_n(V_m^+, \chi) = \{g \longmapsto \omega_\chi(g)\varphi(0) : \varphi \in \mathcal{S}(\mathbb{X})\} .$$

We let V be any of the two spaces $V_m^\pm$. The action of G over the space $\mathcal{S}(\mathbb{X})$ can be summarised by (see [KS97, top of p.280]):

$$\begin{aligned} \omega_\chi(m(a))\varphi(x) &= \chi(\det a)|\det a|_E^{\frac{n}{2}}\varphi(x \cdot a) \\ \omega_\chi(n(b))\varphi(x) &= \psi(\mathrm{tr}((x,x)b))\varphi(x) \\ \omega_\chi(\delta_r)\varphi(x) &= \gamma^{-r}\int_{V^r} \psi\left(\mathrm{Tr}_{E/F}\mathrm{tr}(x'', z)\right)\varphi(x' + z)\mathrm{d}z \end{aligned}$$

with the following conventions for the last integral: V is decomposed as $V^{n-r} \oplus V^r$, $x = x' + x''$ according to this decomposition and the Haar measure $\mathrm{d}z$ is the r-power of the Haar measure of V which is self-dual for the Fourier transform defined by the pairing $\psi\circ\mathrm{Tr}_{E/F}(\,,\,)$ and γ is a quotient of Weil indexes of quadratic forms.

If $k \in P \cap K$, we obviously have $\omega_\chi(k)\varphi(0) = \varphi(0)$. An element $f \in I_n(0, \chi)$ is spherical if and only if $\forall k \in K$, $f(k) = f(I_n) \neq 0$. Thus the spherical vector of $I_n(0, \chi)$ will be in $R_n(V, \chi)$ if and only if $\omega_\chi(\delta_r)\varphi(0) = \varphi(0)$ for all r (and $\varphi(0) \neq 0$).

We now suppose that $V = V^+$; remember that the uniformiser ϖ of F is an uniformiser for E. We choose an orthonormal basis $(v_1, \ldots, v_n)$ of V.

We first compute the Haar measure of V. Let $V_\circ$ be the $\mathcal{O}_E$-module generated by $(v_1, ..., v_n)$ in V and φ° its characteristic function. After identification of V^* with V thanks to $\psi\circ\mathrm{Tr}_{E/F}(\,,\,)$, the Fourier transform of φ°

is

$$\widehat{\varphi^\circ}(y) = \int_V \psi\big(\mathrm{Tr}_{E/F}(x,y)\big)\varphi(x)\mathrm{d}x \ .$$

We readily see that $\widehat{\varphi^\circ} = \mu(\mathcal{O}_\circ)\varphi^\circ$ so that

$$\widehat{\widehat{\varphi^\circ}} = \mu(\mathcal{O}_\circ)^2\varphi^\circ$$

which means that the measure has to be normalised by $\mu(\mathcal{O}_\circ) = 1$.

We now compute γ in both cases for W: Hermitian or skew-Hermitian. Its precise definition, taken from [Kud94, Theorem 3.1, p.378, case 3_+], is as follows. Fix $\delta \in E^\times$ be such that $E = F(\delta)$ and $\Delta = \delta^2 \in F^\times$. Then

$$\gamma = (\det V, \Delta)_F \gamma_F(-\Delta, \eta)^m \gamma_F(-1,\eta)^{-m} \ .$$

Since E/F is unramified, Δ has valuation 0. Looking at [Rao93, Prop A.11, p.369] we readily see that $\gamma_F(-\Delta,\eta) = \gamma_F(-1,\eta) = 1$. One should note that the correct formula for $\gamma_F(a,\eta)$ in Proposition A.11 should be

$$\gamma_F(a,\eta) = \left(\frac{\bar{u}}{\overline{F}}\right)^{\alpha(\eta)} \cdot \left\{\left(\frac{\bar{u}}{\overline{F}}\right)\gamma_{\overline{F}}(\bar{\eta})\right\}^{\alpha(a)}$$

but that does not change anything for us because $\alpha(\eta) = 0$ anyway. Since $V = V^+$, we have $(\det V, \Delta)_F = 1$ and thus $\gamma = 1$. Observe that this remains true if W is skew-Hermitian (case 3_- of [Kud94]) because the definition of γ differs between the two cases by a scaling by δ for V and the product by $\chi(\delta)$; since δ has valuation 0 this does not change γ. □

This allows us to slightly reformulate [Har07, Theorem 3.2, p.125], since one hypothesis is now proved.

Th. 3.2 (Harris). —*Let $G = GU(W)$, a unitary group with signature (r,s) at infinity, and let π be a cuspidal automorphic representation of G. We assume $\pi \otimes \chi$ occurs in anti-holomorphic cohomology $\bar{H}^{rs}(Sh(W), E_\mu)$ where μ is the highest weight of a finite-dimensional representation of G. Let χ, α be algebraic Hecke characters of $\mathcal{K}^\times$ of type η_k and η_κ^{-1}, respectively. Let s_0 be an integer which is critical for the L-function $L^{mot,S}(s, \pi\otimes\chi, St, \alpha)$; i.e. s_0 satisfies the inequalities (3.3.8.1) of [Har97]:*

$$(**) \qquad \frac{n-\kappa}{2} \leqslant s_0 \leqslant min(q_{s+1}(\mu) + k - \kappa - \mathcal{Q}(\mu), p_s(\mu - k - \mathcal{P}(\mu)),$$

Define $m = 2s_0 - \kappa$. Let α^* denote the unitary character $\alpha/|\alpha|$ and assume

$$(3.2.1) \qquad \alpha^*|_{\mathbf{A}_{\mathbb{Q}}^\times} = \varepsilon_{\mathcal{K}}^m.$$

Suppose there is a positive-definite hermitian space V of dimension m and a finite set S of finite primes such that

(a) For every finite v in S, π_v does not occur in the boundary at s_0 for α_v^*, and π_v is ambiguous for m and α^*;
(b) For every finite v, $\Theta_{\alpha^*}(\pi_v \otimes \chi_v, V_v) \neq 0$;
(c) For every finite v outside S, all data (π_v, χ_v, α_v, and the additive character ψ_v) are unramified.

Then

(i) One can find a factorizable vector $\phi_f \in I_n(s, \alpha^*)_f$ such that for every finite v, $\phi_v \in R_n(V_v, \alpha^*)$ and ϕ_f takes values in $(2\pi i)^{(s_0+\kappa)n} L \cdot \mathbb{Q}^{ab}$ and two factorizable vectors $\varphi \in \pi \otimes \chi$, $\varphi' \in \alpha^* \cdot (\pi \otimes \chi)^\vee$ arithmetic over the field of definition $E(\pi)$ of π_f.

(ii) Suppose φ is as in (i). Then

$$L^{mot,S}(s_0, \pi \otimes \chi, St, \alpha) \sim_{E(\pi, \chi^{(2)} \cdot \alpha); \mathcal{K}} P(s_0, k, \kappa, \pi, \varphi, \chi, \alpha)$$

where $P(s_0, k, \kappa, \pi, \varphi, \chi, \alpha)$ is the period

$$(2\pi i)^{s_0 n - \frac{nw}{2} + k(r-s) + \kappa s} g(\varepsilon_{\mathcal{K}}^{[\frac{n}{2}]}) \cdot \pi^c P^{(s)}(\pi, *, \varphi) g(\alpha_0)^s p((\chi^{(2)} \cdot \alpha)^\vee, 1)^{r-s}$$

appearing in Theorem 3.5.13 of [Har97].

Proof. — With respect to the original theorem we just removed the existence of factorizable vectors in $\pi \otimes \chi$ and $\alpha^* \cdot (\pi \otimes \chi)^\vee$, the existence of ϕ_f and, accordingly, condition (a). The fact that there are factorizable vectors in $\pi \otimes \chi$ and $\alpha^* \cdot (\pi \otimes \chi)^\vee$ is well known. We know that for any v such that no data ramifies (neither the extension nor the characters), then the spherical vector ϕ_v° is in $R_n(V_{m,v}^+)$. However for all but finitely many v, we have $V_v \simeq V_{m,v}^+$. Denote S' the set of primes that are either infinite or such that some data ramify or such that $V_v \not\simeq V_{m,v}^+$. Then for $v \notin S'$, let $\phi_v = \phi_v^\circ$ the spherical vector. For any finite $v \in S'$, let ϕ_v be any element of $\mathrm{Soc}_{n,m}(s)$. Then $\phi_f = \otimes \phi_v \in I_n(s, \alpha^*)_f$ satisfies condition (a) of [Har07, Theorem 3.2]. Thus the hypotheses of Harris' Theorem are verified. □

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