ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

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Tome XX, nº 2 (2011), p. 247-293.

<http://afst.cedram.org/item?id=AFST_2011_6_20_2_247_0>

Annales de la Faculté des Sciences de Toulouse Vol. XX, n° 2, 2011
pp. 247–293

A differential Puiseux theorem in generalized series fields of finite rank

Mickaël Matusinski[(1)]

Abstract. — We study differential equations $F(y, \dots, y^{(n)}) = 0$ where F is a formal series in $y, y', \dots, y^{(n)}$ with coefficients in some field of *generalized power series* $\mathbb{K}_r$ with finite rank $r \in \mathbb{N}^*$. Our purpose is to express the support Supp y_0, i.e. the set of exponents, of the elements $y_0 \in \mathbb{K}_r$ that are solutions, in terms of the supports of the coefficients of the equation, namely Supp F.

Résumé. — Nous étudions des équations différentielles $F(y, \dots, y^{(n)}) = 0$ où F est une séries formelle en $y, y', \dots, y^{(n)}$, à coefficients dans un *corps de séries généralisées* $\mathbb{K}_r$ de rang fini $r \in \mathbb{N}^*$. Notre objet est d'exprimer le support – c'est-à-dire l'ensemble Supp y_0 des exposants – des éléments $y_0 \in \mathbb{K}_r$ solutions, en fonction des supports des coefficients de l'équation, dont l'union est notée Supp F.

1. Introduction – About differential Puiseux theorems

In his PhD thesis [vdH97, Theorem 12.2], van der Hoeven proves that any well-ordered transseries solution to an algebraic differential equation with grid-based transseries coefficients is itself grid-based. Another version of this result can be found in [vdH06, Corollary 8.38]. Without entering into the details of definitions, transseries are formal series built from $\mathbb{R}$, x, field operations, composition with exponential and logarithmic functions, and an infinite summation process analogous to the construction of generalized

(*) Reçu le 07/06/2010, accepté le 03/12/2010

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series. Grid-based transseries are those whose support is included in a grid, that is to say the translation of a lattice. They were introduced by J. Ecalle in his proof of Dulac conjecture [É92]. Well-ordered transseries are those built with well-ordered supports.

This result for transseries addresses two difficulties concerning formal resolution of ordinary differential equations. On the one hand, it shows that what we may call the rank of any well-ordered transseries solution f (i.e. the Archimedean rank of the elements of the support of f) is finite, following the fact that it is finite for the grid-based coefficients of the equation [vdH06, Theorem 4.15: the rank of any grid-based transseries is bounded by the number of elements of a transbasis for it]. On the other hand, the support of such a finite rank well-ordered transseries solution is actually grid-based. This is a Puiseux type result [KP02, for instance] in the context of differential equations, which generalizes several recent results. Grigoriev and Singer [GS91, Corollary 3.1] considered polynomial differential equations $P(y,\dots,y^{(n)})=0$ with $P\in\mathbb{Q}[[x]][y,\dots,y^{(n)}]$ (i.e. the coefficients are formal power series in x with rational coefficients), and solutions that are power series with *real* exponents. Then they showed that these exponents belong to a finitely generated $\mathbb{Z}$-module in $\mathbb{R}$. Independently, J.Cano [Can93, Theorem 1] studies equations $f(x,y,y'\dots,y^{(n)})=0$, where $f\in\mathbb{R}[[x]][[y,\dots,y^{(n)}]]$, i.e. f is a power series in the $n+1$ variables $y,\dots,y^{(n)}$ and in x. He shows that any series with *rational* exponents greater than n which is a solution, is in fact a Puiseux series. In their proof of a desingularisation theorem, F.Cano, Moussu and Rolin used a generalization [CMR05, Appendix] of J. Cano's result to the case of real exponents: *given a differential equation $f(x,y,xy',xy'')=0$, where $f\in\mathbb{R}[[x]][[y,xy',xy'']]$, then the support of a solution is included in a lattice (i.e. a finitely generated sub-semigroup of $\mathbb{R}_{\geqslant 0}$).*

The main purpose of this paper is to prove a generalization (Theorem 1.2) of this second Puiseux type part of van der Hoeven's result (and so, of the other cited results) to the context of *generalized series fields* (which have well-ordered supports and do not carry any log-exp structure a priori) and for *differential equations defined by formal series* (not only by polynomials: see the equation (1.2)). Given a totally ordered abelian group Γ and a field $\mathcal{C}$, a generalized series with exponents in Γ and coefficients in $\mathcal{C}$ is a formal sum $a=\sum_{\gamma\in\Gamma}a_\gamma t^\gamma$, where t is an abstract variable, the coefficients $a_\gamma=a(\gamma)$ belong to $\mathcal{C}$ and its support $\operatorname{Supp} a=\{\gamma\in\Gamma\mid a_\gamma\neq 0\}$ is well-ordered in Γ (for classical definitions and properties in well-ordering theory, see [Bou70]). The series with empty support is denoted by 0. The set of such series endowed with component-wise sum and convolution product is a field [Hah07], which we denote by $\mathbb{K}$. We endow it also with the classical

valuation $v : \mathbb{K} \to \Gamma \cup \{\infty\}$, $v(a) = \min(\text{Supp } a)$ for $a \neq 0$ and $v(0) = \infty$, with the usual conventions.

In particular, we consider a group of exponents Γ of *finite rank* $r \in \mathbb{N}^*$ (i.e. with a finite number of Archimedean classes) which is also a real vector space, and $\mathbb{R}$ as field of coefficients (see Section 2.1). We denote by $\mathbb{K}_r$ the corresponding series field. We also suppose that this field is endowed with a derivation which behaves like the one in Hardy fields (see Definition 2.2). For any $I \in \mathbb{N}^{n+1}$, we denote $y^{(I)} = y^{i_0}(y')^{i_1} \cdots (y^{(n)})^{i_n}$. Let the following non trivial differential series

$$F(y, \ldots, y^{(n)}) = \sum_{I \in \mathbb{N}^{n+1}} c_I y^{(I)} \in \mathbb{K}_r[[y, \ldots, y^{(n)}]] \setminus \{0\} \tag{1.1}$$

of arbitrary fixed order $n \in \mathbb{N}$ of differentiation, with coefficients in $\mathbb{K}_r$ and with *well-ordered support* $\text{Supp } F = \cup_{I \in \mathbb{N}^{n+1}} \text{Supp } c_I$, be given. Then we consider the corresponding differential equation:

$$F(y) = F(y, y', \ldots, y^{(n)}) = 0. \tag{1.2}$$

By a solution of this equation, we mean a generalized series $y_0 \in \mathbb{K}_r$ such that $F(y_0)$ is well-defined (Definition 2.12) and $F(y_0) = 0$. Note that the condition $v(y_0^{(i)}) > \underline{0}$ is sufficient for $F(y_0)$ to be well-defined (Proposition 3.6). In the case where F is a differential polynomial, $F(y_0)$ is well-defined for any $y_0 \in \mathbb{K}_r$.

Example 1.1. —

1. There are natural settings in which such differential fields of series arise: algebraic or analytic differential equations. For example, following [CMR05], take $X : t_k' = F_k(t_1, \ldots, t_r),\ k \in \{1, \ldots, r\}$ to be an analytic vector field over a real analytic r-dimensional manifold m. Consider $\gamma : t \mapsto \gamma(t)$, $t \geqslant 0$, an integral curve of X having a unique ω-limit point p and assume that γ is sub-analytically non-oscillating. It is shown in Section 2 of [CMR05] that one can associate a Hardy field K_γ to γ which has at most rank r. So, in the case of maximal rank r, we consider as formal counterpart of K_γ the field $\mathbb{K}_r$ endowed with the derivation determined by X (see Example 2.8).

2. As cited before, one can take any finite rank differential subfield of the field of well-ordered transseries closed under real powers. For instance, take the field of generalized series $\mathbb{R}((\Gamma))$ with $\Gamma = \{\log(x), x, \exp(x)\}^{\mathbb{R}}$, the group of words in $t_1 = \exp(-x)$, $t_2 = 1/x$ and $t_3 = 1/\log(x)$ with real exponents, the usual comparison relations and derivation. Thus $t_1' = -t_1$, $t_2' = -t_2^2$ and $t_3' = -t_2 t_3^2$.

The following Puiseux type result is obtained as an immediate corollary of our Theorem 2.14.

THEOREM 1.2. — *Given a differential equation $F(y) = 0$ as in (1.2), the support of any solution $y_0 \in \mathbb{K}_r$ such that $v(y_0^{(i)}) > \underline{0}$ for any $i = 0, \dots, n$, is obtained by finitely many elementary transformations from the supports of F and t_k'/t_k, $k = 1, \dots, r$.*

The reader will find the necessary precisions in Definition 2.2 and Notation 2.1. We emphasize that the t_k'/t_k's do not depend on the equation (1.2), but only on the differential field $\mathbb{K}_r$. Our notion of "being deduced by finitely many elementary transformations" generalizes those of "being grid-based" and "belonging to a lattice" to the setting of well-ordered supports (Remark 2.15). Moreover, beyond this Puiseux type result, our Theorem 2.14 shows a dichotomy result identical to the one for pseudo-Cauchy sequences in valuation theory [Kuh00, Section 0.1]. In the case of non resolution of the equation, there is a stabilization of the valuation. This situation is also analogous to the one of monomialization of sub-analytic differential equations in [MR06]. The assumption that $v(y_0^{(i)}) > \underline{0}$ for any $i = 0, \dots, n$, means that y_0 and all its derivatives are infinitesimal. In other words, we can say that the solution is supposed to "tend to $(0, \dots, 0) \in \mathbb{R}^{n+1}$", this point being possibly a singular point for the equation (1.2). We intend to obtain the same result for arbitrary generalized series solutions, i.e. possibly non infinitesimal. In the Section 4.3, we show that such is the case for polynomial differential equations: see Theorem 4.17.

Our article is organized into four parts. In Section 2, we state the main Theorem 2.14. Since we are working with formal equations as in (1.2) rather than with polynomial ones, there may be some restrictions on the operations we can make. In Section 3, we check whether the differential series $F(y, \dots, y^{(n)})$ we consider can be evaluated at some series $y_0 \in \mathbb{K}_r$. In doing so, we also characterize the support of the obtained generalized series $F(y_0, \dots, y_0^{(n)})$. Section 4 deals with three transformations of differential series. Two of them, namely *additive conjugations* and *multiplicative conjugations*, are the ones used already in [vdH97] and [vdH06]. The third ones, namely changes of derivations, are not needed in van der Hoeven's work since he uses shiftings corresponding to the logarithmic-exponential structure. A first application of these transformations, in Section 4.3, is the reduction of the case of polynomial differential equations with arbitrary generalized series solutions, to our main Theorem 2.14. Then we use them to prove this Theorem 2.14 in Section 5. This proof, as is the case for the cited results,

uses an inductive method, evaluating the differential series $F(y, \dots, y^{(n)})$ at some longer and longer initial parts of a solution. But all the proofs of the above cited differential results, rely on an adaptation of the classical Newton polygonal method to the differential case: the Newton-Fine polygonal method [Fin89]. Instead of such a polygonal representation, we use a valuative approach and express directly the relation between the exponents of the solution and those of the coefficients of the equation.

Acknowledgment. — The major part of this work was done while the author was a doctoral fellow of J.-P. Rolin at the University of Bourgogne [Mat07]. Another version of this article was written while the author was a visiting postdoctoral fellow at the University of Saskatchewan, and was partially supported by Salma Kuhlmann's NSERC Discovery Grant. The author thanks gratefully Salma Kuhlmann for providing many good advises and interesting comments on this work. He thanks also the referee from the *Annales de la Faculté des Sciences de Toulouse* for his careful reading and interesting comments.

2. The main result

2.1. The Hahn field of rank r

For any positive integer r, the *Hahn group of rank* r is the product of r copies of $\mathbb{R}$ ordered lexicographically, say $\overrightarrow{\mathbb{R}}^r$. By Hahn's embedding theorem in [Hah07], any totally ordered group Γ with finite rank (i.e. with a finite number of Archimedean classes) embeds in such a Hahn group. From now on, we fix a positive integer r, and $\Gamma = \overrightarrow{\mathbb{R}}^r$, which is called the *group of exponents* of the generalized series. We write $\underline{0}$ its neutral element $(0, \dots, 0)$. Note that we consider the whole Hahn group instead of one of its subgroups, because the resolution of the differential equations leads to adjoint new real exponents to the support of the solutions with respect to the supports of the coefficients of the equation (see Definition 2.13 and Theorem 2.14 below).

We denote by $\mathbb{K}_r$ the corresponding field of generalized series *with real coefficients* and by $\mathbb{K}_r^{\prec}$ (respectively $\mathbb{K}_r^0$, $\mathbb{K}_r^{\succ}$, $\mathbb{K}_r^{\preccurlyeq}$) the subset of $\mathbb{K}_r$ given by $\{a \in \mathbb{K}_r \mid v(a) > 0 \ (respectively \ = 0, \ < 0, \ \geqslant 0)\}$. The symbols $\prec, \preceq$ and $\succ$ denote the usual dominance relations for functions ($t^\alpha \prec t^\beta \Leftrightarrow \alpha > \beta$). With this notation, $\mathbb{K}_r^{\preccurlyeq}$ is the valuation ring of $\mathbb{K}$ and $\mathbb{K}_r^{\prec}$ its maximal ideal. Given $\alpha, \beta \in \Gamma \setminus \{0\}$, we write $\alpha \gg \beta$ if $|\alpha|/k > k|\beta| > 0$ for any $k \in \mathbb{N}$. The corresponding symbol for the series is $\succ\succ$. We require the coefficients of the series to be real so that they will be compatible with the real exponents of the monomials, applying the Leibniz rule (HD0) below. We will also use the

notion of *leading term* of a series a, namely $\delta(a) = a_{v(a)}t^{v(a)}$ for any non zero $a \in \mathbb{K}_r$, and the classical *equivalence relations*

$$a \asymp b \Leftrightarrow v(a) = v(b) \quad \text{and} \quad a \sim b \Leftrightarrow v(a-b) > \min\{v(a), v(b)\}.$$

From now on, we also use another notation for elements of $\mathbb{K}_r$. For any $i \in \{1, \dots, r\}$, we set $t_k = t^{e_k}$, where $e_k = (0, \dots, 1, \dots, 0)$, with 1 in the k^{th} position. So any element $a \in \mathbb{K}_r$ can be written

$$\begin{aligned} a &= \sum_{\alpha \in \Gamma} a_\alpha t^\alpha \\ &= \sum_{\alpha_1 \in \mathbb{R}} t_1^{\alpha_1} \left(\sum_{\alpha_2 \in I_{\alpha_1}} t_2^{\alpha_2} \left(\cdots \left(\sum_{\alpha_r \in I_{\alpha_1, \dots, \alpha_{r-1}}} a_\alpha t_r^{\alpha_r} \right) \cdots \right) \right), \end{aligned}$$

where the $I_{\alpha_1,\dots,\alpha_k}$'s are subsets of $\mathbb{R}$. Thus, any element a of $\mathbb{K}_r^{\prec}$ can be written

$$\begin{aligned} a &= \textstyle\sum_{\alpha_r > 0} a_{(0,\dots,0,\alpha_r)} t_r^{\alpha_r} + \sum_{\alpha_{r-1} > 0} t_{r-1}^{\alpha_{r-1}} \left(\sum_{\alpha_r \in I_r} a_{(0,\dots,0,\alpha_{r-1},\alpha_r)} t_r^{\alpha_r} \right) \\ &\quad + \cdots \\ &= a_r + \cdots + a_1, \end{aligned}$$

with $v(a_i) \gg v(a_j)$ for any $i, j \in \{1, \dots, r\}$, $i < j$, whenever $a_j \neq 0$.

Notation 2.1. — For any $k \in \{1, \dots, r\}$, we denote by $\mathbb{K}_{r,k}$ the additive subgroup of $\mathbb{K}_r^{\prec}$, of elements

$$a_k = \textstyle\sum_{\alpha_k > 0} t_k^{\alpha_k} \left(\sum_{\alpha_{k+1} \in I_{\alpha_k}} t_{k+1}^{\alpha_{k+1}} \left(\cdots \left(\sum_{\alpha_r \in I_{\alpha_k, \dots, \alpha_{r-1}}} a_{(0,\dots,0,\alpha_k,\dots,\alpha_r)} t_r^{\alpha_r} \right) \cdots \right) \right),$$

Note that $\mathbb{K}_r^{\prec} = \mathbb{K}_{r,r} \oplus \cdots \oplus \mathbb{K}_{r,1}$ with $\underline{0} < v(\mathbb{K}_{r,r}^*) \ll \cdots \ll v(\mathbb{K}_{r,1}^*)$.

2.2. Endowing the series with a Hardy type derivation

We suppose from now on that $\mathbb{K}_r$ is endowed with a derivation D_0 as follows:

Definition 2.2. — We say that a map $D_0 : \mathbb{K}_r \to \mathbb{K}_r$, $a \mapsto a'$ with $t_k' \neq 0$ for any $k = 1, \dots, r$, is a *Hardy type derivation* if the following conditions are satisfied:

(HD0) $\forall \alpha \in \Gamma, D_0(t^\alpha) = (t^\alpha)' = (t_1^{\alpha_1} \cdots t_r^{\alpha_r})' = t^\alpha(\alpha_1 t_1'/t_1 + \cdots + \alpha_r t_r'/t_r)$;

(HD1) for any $a = \sum_{\alpha\in\Gamma} a_\alpha t^\alpha$ in $\mathbb{K}_r$, $D_0(a) = a' = \sum a_\alpha (t^\alpha)'$.
For any $a, b \in \mathbb{K}_r$ with $v(a), v(b) \neq 0$,

(HD2) L'Hospital's rule : $v(a) \leqslant v(b) \Leftrightarrow v(a') \leqslant v(b')$;

(HD3) Compatibility with the logarithmic derivatives :
$|v(a)| \geqslant |v(b)| > 0 \Rightarrow v(a'/a) \leqslant v(b'/b)$. Moreover, $v(a) \gg v(b) \Leftrightarrow v(a'/a) < v(b'/b)$.

For any $k \in \{1, \dots, r\}$, we denote $\delta(t'_k) = T_k t^{\tau^{(k)}}$, $d_k := t^{\tau^{(k)}}/t_k$.

Remark 2.3. — Such a Hardy type derivation D_0 is a derivation in the usual sense, i.e. for any $a, b \in \mathbb{K}$, $(a+b)' = a' + b'$ and $(ab)' = a'b + ab'$. We note also that such a derivation is uniquely determined by its restriction to the t_k's. Indeed, for any non zero $a \in \mathbb{K}_r$, $a' = \sum a_\alpha (t^\alpha)' = [\sum (a_\alpha \alpha_1) t^\alpha] t'_1/t_1 \ + \cdots + [\sum (a_\alpha \alpha_r) t^\alpha] t'_r/t_r$.

Hypothesis (HD0) is an extension to real exponents of the usual Leibniz rule that holds for rational exponents for any derivation. Likewise, Hypothesis (HD1) is an extension of the linearity property for derivations. Thus D_0 is a *strong derivation* in the sense of van der Hoeven [vdH06, Section 5.1].

The valuation v is a *differential valuation* in the sense of Rosenlicht [Ros80]. Moreover, (HD3) is an additional property that holds in Hardy fields [Ros83, Propositions 3 and 4] as well as in pre-*H*-fields [AvdD05, Lemma 3.5]. Otherwise we are in the case that Rosenlicht calls a *Hardy type valuation* [Ros81]. To obtain a structure of *H-field* [AvdD05], it suffices to endow $\mathbb{K}_r$ with the ordering associated to the notion of *leading coefficient*, and to require that for any k, $t'_k > 0$, i.e. $T_k > 0$.

In our context, the hypotheses (HD2) and (HD3) can be substituted by their reduction to the t_i, $i = 1, \dots, r$:

(HD2') for all $k = 1, \dots, r-1$, $v(t'_k) \gg v(t'_{k+1})$;

(HD3') for all $k = 1, \dots, r-1$, $v(t'_k/t_k) < v(t'_{k+1}/t_{k+1})$.

Corollary 2.4. — *Supposing that (HD0) and (HD1) hold, we have:*

$$(\text{HD2}') \wedge (\text{HD3}') \Leftrightarrow (\text{HD2}) \wedge (\text{HD3})$$

Proof. — By (HD1), to prove l'Hospital's rule, it suffices to check it for monomials. Thus, consider t^α, t^β with $\alpha \geqslant \beta$. By (HD0), $(t^\alpha)' = t^\alpha(\alpha_k t'_i/t_i +$

$\cdots + \alpha_r t_r'/t_r)$ and $(t^\beta)' = t^\beta(\beta_j t_j'/t_j + \cdots + \alpha_r t_r'/t_r)$ where $\alpha = (0,\dots,0, \alpha_i,\dots,\alpha_r)$ and $\beta = (0,\dots,0,\beta_j,\dots,\beta_r)$. Using (HD2') and (HD3'), it is a routine to verify that $v((t^\alpha)') = \alpha + v(d_i) \geqslant v((t^\beta)') = \beta + v(d_j)$.

Consider now $a, b \in \mathbb{K}_r$ such that $|v(a)| = \alpha \geqslant |v(b)| = \beta > 0$. Suppose that $\alpha = (0,\dots,0,\alpha_i,\cdots,\alpha_r)$ and $\beta = (0,\dots,0,\beta_j,\dots,\beta_r)$. We have that $v(a') = \alpha + v(d_i)$ and $v(b') = \beta + v(d_j)$. So, $v(a'/a) = d_i$ and $v(b'/b) = d_j$. The Property (HD3) therefore follows from (HD3'). □

Corollary 2.5. — *The field of constants of D_0 is $\mathbb{R} \subset \mathbb{K}_r$.*

Proof. — If there was a non constant series a with $a' = 0$, then by (HD2), this would imply that there exists a non constant monomial m with derivative 0. Then, taking any other monomial $\tilde{m}$ with non zero derivative, we would have either $v(\tilde{m}) < v(m)$ or $-v(\tilde{m}) < v(m)$, but $v(m') = \infty < v((\tilde{m}^\pm)^{l'})$. This contradicts (HD2). □

In [KM10], we develop these ideas in greater details for more general series fields. To conclude this section, we give some key properties of the values $v(t_k') = \tau^{(k)}$.

Notation 2.6. — *For $k = 1,\dots,r$, whenever $v(d_k) \neq \underline{0}$, we write*

$$v(d_k) = \theta^{(k)} = (0,\dots,0,\theta_{\tilde{k}}^{(k)},\dots,\theta_r^{(k)}) \text{ where } \theta_{\tilde{k}}^{(k)} \neq 0.$$

Proposition 2.7. — *For any $k > l$, there exists $m \in \{l+1,\dots,r\}$ such that:*

$$v(d_k/d_l) = (0,\dots,0,\theta_m^{(k)} - \theta_m^{(l)},\dots,\theta_r^{(k)} - \theta_r^{(l)}) \text{ with } \theta_m^{(k)} - \theta_m^{(l)} > 0.$$

Proof. — Let $k \in \{1,...,r\}$ be given. We show that:

- for any $i = 1 \cdots k-1$, $\tau_i^{(k+1)} = \tau_i^{(k)}$;
- $\tau_k^{(k+1)} = \tau_k^{(k)} - 1$;
- $(0,\dots,0,\tau_{k+1}^{(k+1)} - 1,\dots,\tau_r^{(k+1)}) > (0,\dots,0,\tau_{k+1}^{(k)},\dots,\tau_r^{(k)})$.

From Definition 2.2, $v(t_k') > v(t_{k+1}')$ and $v(d_k) < v(d_{k+1})$. So

$$(\tau_1^{(k+1)} - \tau_1^{(k)},\dots,\tau_k^{(k+1)} - \tau_k^{(k)}, \tau_{k+1}^{(k+1)} - \tau_{k+1}^{(k)},\dots,\tau_r^{(k+1)} - \tau_r^{(k)}) < \underline{0}$$

and

$$(\tau_1^{(k+1)} - \tau_1^{(k)},\dots,\tau_k^{(k+1)} - \tau_k^{(k)} + 1, \tau_{k+1}^{(k+1)} - 1 - \tau_{k+1}^{(k)},\dots,\tau_r^{(k+1)} - \tau_r^{(k)}) > \underline{0}.$$

Thus, for any $i = 1 \cdots k-1$, $\tau_i^{(k+1)} = \tau_i^{(k)}$ and $\tau_k^{(k)} \geqslant \tau_k^{(k+1)} \geqslant \tau_k^{(k)} - 1$. But, for all $\alpha_k, \beta_{k+1} \in \mathbb{R}^*$, $(t_k^{\alpha_k})' = \alpha_k t_k^{\alpha_k} t_k'/t_k$ and $(t_{k+1}^{\beta_{k+1}})' = \beta_{k+1} t_{k+1}^{\beta_{k+1}} t_{k+1}'/t_{k+1}$. Moreover, for any $\alpha_k > 0$, $v(t_k^{\alpha_k}) > v(t_{k+1}^{\beta_{k+1}})$ which implies that $v((t_k^{\alpha_k})') > v((t_{k+1}^{\beta_{k+1}})')$. So for any $\alpha_k > 0$ and $\beta_{k+1} \in \mathbb{R}^*$, we obtain that

$$(0, \ldots, 0, \alpha_k, -\beta_{k+1}, 0, \ldots, 0) > v(d_{k+1}/d_k) = (0, \ldots, 0, \tau_k^{(k+1)} - \tau_k^{(k)} + 1, \tau_{k+1}^{(k+1)} - 1 - \tau_{k+1}^{(k)}, \ldots, \tau_r^{(k+1)} - \tau_r^{(k)}),$$

and so

$$\tau_k^{(k+1)} - \tau_k^{(k)} + 1 \leqslant 0 \Leftrightarrow \tau_k^{(k+1)} \leqslant \tau_k^{(k)} - 1.$$

Thus $\tau_k^{(k+1)} = \tau_k^{(k)} - 1$. We conclude recalling that $v(d_k) < v(d_{k+1})$. □

Example 2.8. — Consider a formal vector field $X : t_k' = F_k(t_1, \ldots, t_r)$, $k \in \{1, \ldots, r\}$. For all $i \in \{1, \ldots, r\}$, denote $\tau^{(k)} = (\tau_1^{(k)}, \ldots, \tau_r^{(k)})$ to be the least multi-exponent for the lexicographical ordering among those appearing in the monomials $t_1^{\tau_1^{(k)}} \cdots t_r^{\tau_r^{(k)}}$ of $F_k(t_1, \ldots, t_r)$. Proposition 2.7 gives that *the corresponding field of series $\mathbb{K}_r$ can be endowed with a Hardy type derivation if and only if the matrix $(\tau_j^{(k)})_{1 \leqslant j,k \leqslant r}$ is such that:*

- *for any $j = 1, \cdots, k-1$, $\tau_j^{(k+1)} = \tau_j^{(k)}$;*
- *$\tau_k^{(k+1)} = \tau_k^{(k)} - 1$;*
- *$(0, \ldots, 0, \tau_{k+1}^{(k+1)} - 1, \ldots, \tau_r^{(k+1)}) >_{lex} (0, \ldots, 0, \tau_{k+1}^{(k)}, \ldots, \tau_r^{(k)})$.*

It may be interesting to classify, up to a change of coordinates, the vector fields which verify such a property.

Corollary 2.9. — *With the Notation 2.6, the following dichotomy holds:*

- *either there exists $k \in \{1, \ldots, r\}$ such that $\tilde{k} = k$. Then such k is unique. We denote it by k_0. Moreover, for any $k < k_0$, $\tilde{k} > k$, and for any $k \geqslant k_0$, $\tilde{k} = k_0$ and $\theta^{(k)}$, $\theta^{(k_0)}$ have same leading exponent, i.e. $\theta_{\tilde{k}}^{(k)} = \theta_{k_0}^{(k_0)}$;*
- *or for any $k \in \{1, \ldots, r-1\}$, $\tilde{k} > k$, and $\theta^{(r)} = \underline{0}$. By convention, we set $k_0 = r$.*

COROLLARY 2.10. — *$v((t^\alpha)') = \underline{0}$ if and only if $\alpha = -v(d_{k_0})$.*

COROLLARY 2.11. — *Consider $k \in \{1, \dots, r\}$. If $v(d_k) = \underline{0}$, then $d_k^{(i)} = 0$ for any $i \in \mathbb{N}^*$. If not, then we have:*

1. *if $k \geqslant k_0$, then $v(d_k^{(i)}) = v(d_k) + iv(d_{k_0})$ for any $i \in \mathbb{N}$;*
2. *if $k < k_0$, then:*
   (a) *if $\tilde{k} < k_0$, then $v(d_k^{(i)}) = v(d_k) + iv(d_{\tilde{k}})$ for any $i \in \mathbb{N}^*$;*
   (b) *if $\tilde{k} > k_0$, then $v(d_k^{(i)}) = v(d_k) + v(d_{\tilde{k}}) + (i-1)v(d_{k_0})$ for any $i \in \mathbb{N}^*$;*
   (c) *if $\tilde{k} = k_0$, then:*
      – *either $\exists j \in \mathbb{N}^*$, $\theta_{k_0}^{(k)} = -j\theta_{k_0}^{(k_0)}$. Then, $v(d_k^{(i)}) = v(d_k) + iv(d_{k_0})$ for any $i = 1, \dots, j$, and $v(d_k^{(i)}) = v(d_k) + v(d_{\hat{k}}) + (i-1)v(d_{k_0})$ for some $\hat{k} > k_0$ and any $i > j$;*
      – *or $v(d_k^{(i)}) = v(d_k) + iv(d_{k_0})$ for any $i \in \mathbb{N}^*$.*

2.3. The main Theorem

To formulate our main theorem, we need the following definitions:

DEFINITION 2.12. — Given a differential series $F(y, \dots, y^{(n)})$ as in (1.1) and a non zero generalized series $y_0 \in \mathbb{K}_r$, we say that:

– the series y_0 is *compatible* with F if the family $(c_I y_0^{(I)})_{I \in \mathbb{N}^{n+1}}$ is strongly summable (see Definition 3.4). This implies that for any initial part p of y_0, the evaluation of F at p, is well-defined: $F(p, \dots, p^{(n)}) \in \mathbb{K}_r$. A *solution* of the corresponding differential equation (1.2) is a compatible y_0 such that $F(y_0, \dots, y_0^{(n)}) = 0 \in \mathbb{K}_r$;

– the series y_0, supposed to be compatible with F, *stabilizes* on F with initial part p_0 if there exists a proper initial part p_0 of y_0 such that, for any longer initial part p of y_0 (in particular for $p = y_0$), we have $v(F(p, \dots, p^{(n)})) = v(F(p_0, \dots, p_0^{(n)})) \neq v(0)$.

Note that, in the case where F is a *differential polynomial*, any $y_0 \in \mathbb{K}_r$ is compatible with it.

This notion of stabilization is identical to that of monomialization of sub-analytic differential equations proved in [MR06].

Definition 2.13. — Given two well-ordered subsets X_1 and X_2 of $\Gamma_{\geqslant \underline{0}}$, elements $\alpha > \underline{0}$ and β of Γ, we call *elementary transformations*:

- the sum of two sets: $X_1 + X_2 = \{\xi_1 + \xi_2 \mid \xi_1 \in X_1, \xi_2 \in X_2\}$;
- the generation of the additive semi-group: $\langle X_1 \rangle = \{k_1\xi_1 + \cdots + k_q\xi_q \mid k_i \in \mathbb{N},\ \xi_i \in X_1,\ q \in \mathbb{N}\}$;
- the addition of a new generator: $X_1 + \mathbb{N}\alpha$;
- the negative translation by β: $(X_1)_{\geqslant \beta} - \beta = \{\alpha \in X_1 \mid \alpha \geqslant \beta\} - \beta$.

Theorem 2.14. — *Given a differential series $F(y, \ldots, y^{(n)})$ as in (1.1) and a series $y_0 \in \mathbb{K}_r$ with $v(y_0^{(i)}) > \underline{0}$ for any $i = 0, \ldots, n$ (so, in particular, y_0 is compatible with F: see Proposition 3.6), there exists a well-ordered subset $\mathcal{R}$ of $\Gamma_{>0}$ obtained from* Supp F *and* Supp t_k'/t_k, $k = 1, \ldots, r$, *by a finite number of elementary transformations such that:*

- *either* Supp $y_0 \subseteq \mathcal{R}$;
- *or the series y_0 stabilizes on F with initial part p_0 and* Supp $p_0 \subseteq \mathcal{R}$. *In this case, y_0 can not be a solution of the corresponding differential equation $F(y) = 0$.*

As an immediate corollary, we obtain Theorem 1.2.

It can happen that y_0 does not stabilize on F while also $F(y_0) \neq 0$ (for instance, consider x near 0 in $\mathbb{R}_{>0}$, the equation $y - \sum_{k\in\mathbb{N}} x^k + \exp(-1/x) = 0$ and the series $y_0 = \sum_{k\in\mathbb{N}} x^k$ with the usual valuation).

Remark 2.15. — We can deduce from the theorem more particular results in the case that the supports are assumed to be grid-based or included in a lattice of $\Gamma_{\geqslant \underline{0}}$. For instance, suppose that the supports of the equation and of t_k'/t_k, $k = 1, \ldots, r$, are included in some lattice. It suffices to show that applying each elementary transformation in Definition 2.13 to some lattice again produces a lattice. For the three first transformations, it is a consequence of Proposition 2.1 and Exercise 2.1 in [vdH06]. For the fourth one, namely the negative translation, we prove the following claim:

Claim. — *Let $l \in \mathbb{N}^*$, $\lambda_k \in \Gamma_{\geqslant 0}$ for $k \in \{1, \ldots, l\}$, and $\Lambda = \langle \lambda_1, \ldots, \lambda_r \rangle$ be the corresponding lattice. Then, for any $\beta \in \Gamma$, $\Lambda_{\geqslant \beta} - \beta$, the negative translation by β of Λ, is included in some lattice $\langle \nu_1, \ldots, \nu_m \rangle$.*

Proof. — We proceed by induction on l, the number of generators of Λ. If $l = 1$, we denote $k_0 := \min\{k \in \mathbb{N} \,|\, k\lambda_1 \geqslant \beta\}$. Any element α of $\Lambda_{\geqslant\beta} - \beta = \langle\lambda_1\rangle_{\geqslant\beta} - \beta$ can be written $k\lambda_1 - \beta$ with $k \in \mathbb{N}$ and $k\lambda_1 \geqslant \beta$. Thus $\alpha = (k - k_0)\lambda_1 + (k_0\lambda_1 - \alpha)$. Then we set $\nu_1 = \lambda_1$ and $\nu_2 = k_0\lambda_1 - \beta$.

If $l \geqslant 2$, we suppose that the lemma holds for lattices generated by at most $l - 1$ elements. We consider a lattice $\Lambda = \langle\lambda_1, \dots \lambda_l\rangle$. Let k_0 be the least natural number k such that $k\lambda_l \geqslant \beta$. Any element α of $\langle\Lambda\rangle_{\geqslant\beta} - \beta$ can be written as $\sum_{i=1}^{l-1} k_i\lambda_i + k_l\lambda_l - \beta$. If $k_l \geqslant k_0$, then

$$\alpha = \textstyle\sum_{i=1}^{l-1} k_i\lambda_i + (k_l - k_0)\lambda_l + k_0\lambda_l - \beta.$$

So α belongs to $\langle\lambda_1, \dots, \lambda_{l-1}, \lambda_l, k_0\lambda_l - \beta\rangle$. If $k_l < k_0$, then α belongs to a set $\langle\lambda_1, \dots, \lambda_{l-1}\rangle_{\geqslant(\beta - k_l\lambda_l)} - (\beta - k_l\lambda_l)$ to which we apply the induction hypothesis. It follows that α belongs to some lattice $\tilde{\Lambda}_{k_l}$. Then

$$\langle\Lambda\rangle_{\geqslant\beta} - \beta \subset (\langle\lambda_1, \dots, \lambda_{l-1}, \lambda_l, k_0\lambda_l - \beta\rangle + \textstyle\sum_{k_l < k_0} \tilde{\Lambda}_{k_l}).$$

□

3. Differential series

3.1. Introducing new derivations

From D_0 we build r other derivations corresponding to the r Archimedean classes of the value group of $\mathbb{K}_r$.

Definition 3.1. — For any $k \in \{1, \dots, r\}$, we set $D_k(y) = y'/d_k$ where d_k is as in Definition 2.2.

Proposition 3.2. — *For any $k \in \{1, \dots, r\}$, D_k is a Hardy type derivation on $\mathbb{K}_r$ such that, for any monomial $t^\mu = t_k^{\mu_k} \cdots t_r^{\mu_r}$ in $\mathbb{K}_{r,k}$, we have*

$$D_k^i(t_k^{\mu_k} \cdots t_r^{\mu_r}) \sim \mu_k^i t_k^{\mu_k} \cdots t_r^{\mu_r}.$$

Proof. — The derivations D_k, $k = 1, \dots, r$ inherit the properties of D_0 (see Definition 2.2). Moreover, by (HD0) for D_0, we deduce that for any $\mu = (0, \dots, 0, \mu_k, \dots, \mu_r)$,

$$(t^\mu)' = t^\mu(\mu_k t_k'/t_k + \cdots + \mu_r t_r'/t_r) \sim t^\mu \mu_k t_k'/t_k,$$

and so $D_k(t^\mu) \sim \mu_k t^\mu$. □

For our proof of the Theorem 2.14, we will work with differential equations $F(y) = 0$ as in (1.2), but using as derivation some of these D_k, $k = 1, \ldots, r$, instead of the initial derivation D_0. For any $k \in \{1, \ldots, r\}$, for any $I \in \mathbb{N}^{n+1}$, we denote $y^{(I)_k} = y^{i_0}(D_k y)^{i_1} \cdots (D_k^n y)^{i_n}$. Let the following non trivial differential series

$$F(y) = F(y, \ldots, D_k^n y) = \sum_{I \in \mathbb{N}^{n+1}} c_I y^{(I)_k} \in \mathbb{K}_r[[y, D_k y, \ldots, D_k^n y]] \setminus \{0\}. \tag{3.3}$$

of arbitrary fixed order $n \in \mathbb{N}$ of differentiation, with coefficients in $\mathbb{K}_r$ and with *well-ordered support* $\mathrm{Supp}\ F = \cup_{I \in \mathbb{N}^{n+1}} \mathrm{Supp}\ c_I$, be given. Then the corresponding differential equation is:

$$F(y, \ldots, D_k^n y) = 0 \tag{3.4}$$

The problem of checking whether a change of derivation from an equation as in (1.2) to an equation as in (3.4) is well-defined, is the object of the Section 4.2.

PROPOSITION 3.3. — *Let a non zero series $y_0 \in \mathbb{K}_r$ and $k \in \{1, \ldots, r\}$ be given. If $v(y_0) > \underline{0}$, then $v(D_k^i(y_0)) > \underline{0}$ for all $i \in \mathbb{N}$.*

Proof. — We denote $v(y_0) = \mu = (0, \ldots, 0, \mu_l, \ldots, \mu_r)$ for some $l \in \{1, \ldots, r\}$ with $\mu_l \neq 0$. For the case $l = k$, the result follows from Proposition 3.2.

For the case $k < l$, we show by induction on i that for any $i \in \{0, \ldots, n\}$,

$$v(D_k^i y_0) = (0, \ldots, 0, \beta_{k_i}, \ldots, \beta_r)$$

for some $k_i > k$ and $\beta_{k_i} > 0$. If $i = 0$,

$$v(y_0) = (0, \ldots, 0, \mu_l, \ldots, \mu_r)$$

with $l > k$ and $\mu_l > 0$. Subsequently, we suppose that

$$v(D_k^i y_0) = (0, \ldots, 0, \beta_{k_i}, \ldots, \beta_r)$$

with $k_i > k$ and $\beta_{k_i} > 0$. But

$$D_k^{i+1} y_0 = D_k(D_k^i y_0) = (d_{k_i}/d_k) D_{k_i}(D_k^i y_0) \sim \beta_{k_i} D_k^i y_0 d_{k_i}/d_k.$$

So $v(D_k^{i+1}y_0) = v(D_k^i y_0) + v(d_{k_i}/d_k)$ and from Lemma 2.7, $v(d_{k_i}/d_k) > \underline{0}$ with the expected property.

For the case $k > l$, we show that $i \in \{0, \dots, n\}$, $D_k^i y_0 \sim \mu_l^i y_0 (d_l/d_k)^i$. We suppose that $D_k^i y_0 \sim \mu_l^i y_0 (d_l/d_k)^i$ for some $i \in \{0, \dots, n\}$. So

$$v(D_k^i y_0) = (0, \dots, 0, \mu_l, \mu_{l+1} + i(\theta_{l+1}^{(l)} - \theta_{l+1}^{(l)}), \dots),$$

which implies that

$$D_k^{i+1} y_0 = D_k(D_k^i y_0) \sim D_k(\mu_l^i y_0 (d_l/d_k)^i) \sim D_l(\mu_l^i y_0 (d_l/d_k)^i) d_l/d_k \sim \mu_l^{i+1} y_0 (d_l/d_k)^{i+1}.$$

So $v(D_k^i y_0)$ and $v(y_0)$ have the same sign. □

3.2. Dealing with formal equations rather than polynomial ones

There is an additional difficulty in dealing with a formal differential equation $F(y) = 0$ as in (1.2) rather than only a polynomial one: we have to verify when the evaluation $F(y_0, \dots, y_0^{(n)})$ of the differential series F at some series y_0 is well-defined. We state without proof the following easy generalization of a classical property [Fuc63, Ch.VIII,Sec.5,Lemma].

Definition 3.4. — Given an index set I, a family $\mathcal{F} = (a_i)_{i \in I} \in \mathbb{K}_r^I$ is said to be *strongly summable* if:

- Supp $\mathcal{F} := \bigcup_{i \in I}$ Supp a_i is a well-ordered subset of Γ;
- for all $\alpha \in$ Supp $\mathcal{F}$, the set $\{i \in I \mid \alpha \in \text{Supp } a_i\}$ is finite.

Lemma 3.5. — *Given a strongly summable family $(a_i)_{i \in I} \in \mathbb{K}_r^I$, then $\sum_{i \in I} a_i$ is well defined and, if we set $a_i = \sum_{\alpha \in \text{Supp } a_i} a_{i,\alpha} t^\alpha$ for all $i \in I$, then $\sum_{i \in I} a_i = \sum_{\alpha \in \Gamma} (\sum_{i \in I} a_{i,\alpha}) t^\alpha$.*

Proposition 3.6. — *Given a differential series $F(y, \dots, y^{(n)})$ and a generalized series $y_0 \in \mathbb{K}_r^\prec$ with $v(y_0^{(i)}) > \underline{0}$ for all $i \in \{0, \dots, n\}$, then y_0 is compatible with F (see Definition 2.12).*

Proof. — We proceed by induction on n, considering an arbitrary $y_0 \in \mathbb{K}_r^\prec$ with $v(y_0^{(i)}) > \underline{0}$ for all $i \in \{0, \dots, n\}$. If $n = 0$, we show that the family $(c_i y_0^i)_{i \in \mathbb{N}}$ is strongly summable. On the one hand, $\bigcup_{i \in \mathbb{N}}$ Supp $(c_i y_0^i) \subset$

$\bigcup_{i\in\mathbb{N}} \text{Supp } c_i + \langle \text{Supp } y_0\rangle$. By hypothesis $\bigcup_{i\in\mathbb{N}} \text{Supp } c_i = \text{Supp } F$ and Supp y_0 are well-ordered. Moreover Supp $y_0 \subset \Gamma_{>0}$, so $\langle \text{Supp } y_0\rangle$ is also well-ordered. On the other hand, for any $\alpha \in \Gamma$ and $m \in \mathbb{N}$, the set of $(\gamma^{(m)}, j_1\alpha_1 + \cdots + j_l\alpha_l) \in \text{Supp } c_m \times \langle \text{Supp } y_0\rangle$ such that $j_1 + \cdots + j_l = m$ and $\gamma^{(m)} + j_1\alpha_1 + \cdots + j_l\alpha_l = \alpha$ is clearly finite.

If $n > 0$, it suffices to consider a differential series $F(y, \ldots, y^{(n)})$ as an element of $\mathbb{K}_r[[y, \ldots, y^{(n-1)}]][[y^{(n)}]]$ and then apply the induction hypothesis and the preceding lemma. □

Corollary 3.7. — *Given a differential series $F(y, \ldots, D_k^n y)$ as in (3.3), any series $y_0 \in \mathbb{K}_r^{\prec}$ is compatible with it.*

This last result is an immediate consequence of the Proposition 3.3. The next one follows from the Corollary 2.9:

Corollary 3.8. — *We consider k_0 as defined in the Corollary 2.9. Given a generalized series $y_0 \in \mathbb{K}_r$, $v(y_0^{(i)}) > \underline{0}$ for all $i \in \{0, \ldots, n\}$ if and only if:*

$$v(y_0) > \max\{\underline{0}, -nv(d_{k_0})\}. \tag{3.5}$$

Remark 3.9. —

- *Note that in the case where $v(d_{k_0}) < \underline{0}$, if $y_0 \in \mathbb{K}_r$ is such that $v(y_0) > -nv(d_{k_0})$ as in the preceding corollary, then its analysis as in the Notation 2.1, spells $y_0 = y_{0,k} + \cdots + y_{0,r}$ with $k \geqslant k_0$.*
- *This condition (3.5) generalizes the condition $\rho_i > n$ used in [Can93] for the rational exponents ρ_i, $i \in \mathbb{N}$ of the series solution considered.*

3.3. Controlling the supports

The purpose of this section is to understand the support of the evaluation of a differential series F as in (1.1) or (3.3) at some series $y_0 = \sum_{\mu\in\text{Supp } y_0} m_\mu t^\mu \in \mathbb{K}_r$.

Notation 3.10. — For any multi-indexes $I = (i_0, \ldots, i_n), J = (j_0, \ldots, j_n) \in \mathbb{N}^{n+1}$, we set $|I| = i_0 + \cdots + i_n$, $\|I\| = 1i_1 + 2i_2 + \cdots + ni_n$ and $I! = i_0!\, i_1! \cdots i_n!$;
• $I + J$, $I - J$, denote respectively the termwise addition, subtraction.
• We will use the classical partial ordering on $\mathbb{N}^{n+1}$:

$$I \leqslant J \Leftrightarrow i_l \leqslant j_l \text{ for any } l = 0, \ldots, n.$$

• $F^{(I)} = \partial^{|I|} F / [\partial y^{i_0} \cdots (\partial y^{(n)})^{i_n}]$;
• For any initial segment S of Supp y_0, we denote by p_S, or simply by p if the context is clear, the *initial part of* y_0 *with support* S, i.e. $p = p_S = \sum_{\mu \in S} m_\mu t^\mu$.
• For any $k = 0, \dots, r$, $p^{(I)_k} = p^{i_0} D_k p^{i_1} \cdots D_k^n p^{i_n}$. (Recall that D_0 is the original derivation.) If D_k is known from the context, we denote simply $p^{(I)_k} = p^{(I)}$;
• $f_p^{(I)} = F^{(I)}(p)$ and $f_p = F(p)$;
• $v_p^{(I)} = v(F^{(I)}(p))$ and $v_p = v(F(p))$;
• For any proper initial segment $S \subsetneq$ Supp y_0 and any successor segment $\tilde{S}$ to S in Supp y_0 (*that is to say any initial segment of* Supp $y_0 \backslash S$), we call $\tilde{p} = p_{\tilde{S}}$ a *successor part* to $p = p_S$ in y_0. By the *Taylor expansion formula*, we have:

$$f_{p+\tilde{p}} = \sum_{I \in \mathbb{N}^{n+1}} \frac{f_p^{(I)}}{I!} \tilde{p}^{(I)}. \tag{3.6}$$

Now we introduce the following well-ordered subsets of $\Gamma_{>0}$ useful for the description of the supports of the differential series.

Definition 3.11. — For all $k \in \{1, \dots, r\}$, we define:

$$\mathcal{T}_k = \sum_{i=1}^{n} \sum_{l=k}^{r} \big\langle \text{Supp } D_k^i t_l / t_l \big\rangle.$$

Lemma 3.12. — *For any* $k \in \{1 \dots, r\}$, $\mathcal{T}_k$ *is a well-ordered subset of* $\Gamma_{>0}$, *obtained from* Supp t_k'/t_k, $k = 1, \dots, r$, *by finitely many elementary transformations.*

Proof. — It follows from Definition 2.2 and the proof of Proposition 3.3, that for all $k \geqslant l \in \{1, \dots, r\}$ and $i \in \{1, \dots, n\}$, $v(D_k^i t_l/t_l) > \underline{0}$. So the $\mathcal{T}_k$'s are well-ordered subsets of $\Gamma_{>0}$.

From Remark 2.3, we deduce that, for any series $a \in \mathbb{K}_r$ and any $k = 1, \dots, r$:

$$\begin{aligned} D_k(a) &= a'/d_k \\ &= [\textstyle\sum_\alpha (a_\alpha \alpha_1) t^\alpha] t_1'/(t_1 d_k) + \cdots + [\sum_\alpha (a_\alpha \alpha_r) t^\alpha] t_r'/(t_r d_k). \end{aligned}$$

Thus, Supp $D_k(a) \subset (\sum_{l=1}^r$ Supp $a +$ Supp $t_l'/t_l) - d_k$, and therefore is obtained from Supp a and Supp t_l'/t_l, $l = 1, \dots, r$, by finitely many elementary transformations. □

PROPOSITION 3.13. —

1. *Let $k \geqslant l \in \{1, \dots, r\}$, a differential series $F(y, D_l y, \dots, D_l^n y)$, and $y_{0,k} \in \mathbb{K}_{r,k}$ be given. Then:*

$$\text{Supp } (F(y_{0,k}, \dots, D_l^n y_{0,k})) \subset \text{Supp } F + \mathcal{T}_l + \langle \text{Supp } y_{0,k} \rangle.$$

2. *Consider a differential series $F(y, \dots, y^{(n)})$, and a compatible series $y_{0,k} \in \mathbb{K}_{r,k}$ for some $k \in \{1, \dots, r\}$. Then, with the notations of Corollary 2.9:*
   (a) *if $k < k_0$, then we have:*

$$\begin{gathered}\text{Supp } (F(y_{0,k}, \dots, y_{0,k}^{(n)})) \subset \\ \text{Supp } F + \mathcal{T}_k + \langle \textstyle\bigcup_{i=0}^n \text{Supp } y_{0,k} + iv(d_k) \rangle;\end{gathered}$$

   (b) *if $k \geqslant k_0$, then we have:*

$$\begin{gathered}\text{Supp } (F(y_{0,k}, \dots, y_{0,k}^{(n)})) \subset \\ \text{Supp } F + \mathcal{T}_{k_0} + \langle \textstyle\bigcup_{i=0}^n \text{Supp } y_{0,k} + iv(d_{k_0}) \rangle\end{gathered}$$

   and

$$\begin{gathered}\text{Supp } (F(y_{0,k}, \dots, y_{0,k}^{(n)})) \subset \\ (\textstyle\bigcup_I \text{Supp } c_I + \|I\| v(d_{k_0}) + \mathcal{T}_{k_0} + \langle \text{Supp } y_{0,k} \rangle.\end{gathered}$$

Proof. — We treat the two cases at a time, by taking $l \in \{0, \dots, r\}$. For any $I \in \mathbb{N}^{n+1}$, we have $\text{Supp } (c_I y_{0,k}^{(I)}) \subset \text{Supp } c_I + \text{Supp } y_{0,k}^{i_0} + \dots + \text{Supp } (D_l^n y_{0,k})^{i_n}$. But, for any element $\mu = (0, \dots, 0, \mu_k, \dots, \mu_r)$ with $\mu_k \neq 0$ of Γ, we have:

$$D_l(t^\mu) = t^\mu(\mu_k D_l t_k/t_k \ + \cdots + \mu_r D_l t_r/t_r).$$

By induction, one can deduce the following intermediate result (for a detailed proof, see [Mat07, Lemma 4.2.12]):

LEMMA 3.14. — *Let $k \in \{1, \dots, r\}$, $l \in \{0, \dots, r\}$, and $i \in \{1, \dots, n\}$. Let $y_{0,k} \in \mathbb{K}_{r,k}$. Then*

$$\begin{gathered}\text{Supp } D_l^i y_{0,k} \subset \text{Supp } y_{0,k} + \\ \textstyle\bigcup \text{Supp } [(D_l t_k/t_k)^{j_1^{(k)}} \cdots (D_l^i t_k/t_k)^{j_i^{(k)}} \cdots (D_l t_r/t_r)^{j_1^{(r)}} \cdots (D_l^i t_r/t_r)^{j_i^{(r)}}]\end{gathered}$$

where the union is taken over

$$j_1^{(k)} + \cdots + ij_i^{(k)} + \cdots + j_1^{(r)} + \ldots + ij_i^{(r)} = i.$$

In the case where $k \geqslant l \in \{1, \ldots, r\}$, we deduce that:

$$\text{Supp } D_l^i y_{0,k} \subset \text{Supp } y_{0,k} + \mathcal{T}_k.$$

Now, we note that, for any series $a \in \mathbb{K}$ and any $j \in \mathbb{N}^*$, $\text{Supp}\, a^j \subset \langle \text{Supp}\, a \rangle$. So, for any $I \in \mathbb{N}^{n+1}$:

$$\text{Supp } (c_I y_{0,k}^{(I)_l}) \subset \text{Supp } c_I + \langle \text{Supp } y_{0,k} \rangle + \mathcal{T}_k.$$

which implies the case (1).

For the case (2), we show by induction on i that:

Case (a) $\forall i = 1, \ldots, n\ \forall j \geqslant k$, $\text{Supp}(t_j^{(i)})/t_j\ \subset \mathcal{T}_k + i\theta^{(k)}$;

Case (b) $\forall i = 1, \ldots, n\ \forall j \geqslant k \geqslant k_0$, $\text{Supp}(t_j^{(i)})/t_j\ \subset \mathcal{T}_{k_0} + i\theta^{(k_0)}$.

Indeed, for $i = 1$, we have $t_j' = d_k D_k(t_j) = d_{k_0} D_{k_0}(t_j)$. Suppose that the property holds for some $i = 1, \ldots, n-1$. Thus, in the case (a), $t_j^{(i)} = d_k^i t_j \sum_m m$ for some series $\sum_m m$ with support in

$$\sum_{s=1}^{i} \sum_{p=k}^{r} \langle \text{Supp } D_k^s t_p/t_p \rangle,$$

which is a subset of $\mathcal{T}_k$ (see Definition 3.11). Therefore:

$$\begin{array}{rcl} t_j^{(i+1)} & = & d_k D_k(t_j^{(i)}) \\ & = & i d_k^i D_k(d_k) t_j \sum_m m + d_k^{i+1} D_k(t_j) \sum_m m + d_k^{i+1} t_j \sum_m D_k(m) \\ & = & i d_k^{i+1} t_j [\theta_{\tilde{k}}^{(k)} D_k(t_{\tilde{k}})/t_{\tilde{k}}\ + \cdots + \theta_r^{(k)} D_k(t_r)/t_r] \sum_m m + \\ & & d_k^{i+1} t_j D_k(t_j)/t_j \sum_m m + d_k^{i+1} t_j \sum_m D_k(m). \end{array}$$

Then it suffices to remark that for any $k \in \{1, \ldots, r\}$, for any monomial m with support in $\mathcal{T}_k$, $D_k(m)$ has also its support in

$$\sum_{s=1}^{i+1} \sum_{p=k}^{r} \langle \text{Supp } D_k^s t_p/t_p \rangle,$$

also included in $\mathcal{T}_k$.

In the case (b), $t_j^{(i)} = d_{k_0}^i t_j \sum_m m$ for some series $\sum_m m$ with support in

$$\sum_{s=1}^{i}\sum_{p=k_0}^{r}\langle \text{Supp } D_{k_0}^s t_p/t_p\rangle,$$

which is a subset of $\mathcal{T}_{k_0}$. Therefore:

$$\begin{aligned} t_j^{(i+1)} &= d_{k_0}D_{k_0}(t_j^{(i)}) \\ &= id_{k_0}^i D_{k_0}(d_{k_0})t_j\textstyle\sum_m m + d_{k_0}^{i+1}D_{k_0}(t_j)\sum_m m + d_{k_0}^{i+1}t_j\sum_m D_{k_0}(m) \\ &= id_{k_0}^{i+1}t_j[\theta_{k_0}^{(k_0)}D_{k_0}(t_{k_0})/t_{k_0} \ +\cdots+\theta_r^{(k_0)}D_{k_0}(t_r)/t_r]\textstyle\sum_m m+ \\ &\quad d_{k_0}^{i+1}t_jD_{k_0}(t_j)/t_j\textstyle\sum_m m + d_{k_0}^{i+1}t_j\sum_m D_{k_0}(m). \end{aligned}$$

The conclusion follows as for the preceding case.

Now, from the Lemma 3.14, we deduce that for any $I\in\mathbb{N}^{n+1}$:

Case (a) $\text{Supp}(c_I y_{0,k}^{(I)})\subset \text{Supp } c_I + \|I\|\theta^{(k)} + \text{Supp } y_{0,k}^{|I|} + \mathcal{T}_k$;

Case (b) $\text{Supp}(c_I y_{0,k}^{(I)})\subset \text{Supp } c_I + \|I\|\theta^{(k_0)} + \text{Supp } y_{0,k}^{|I|} + \mathcal{T}_{k_0}$.

□

Note that, in the case (2)(a) ($k<k_0$), since $\tilde{k}>k$ (see Corollary 2.9), $v(y_{0,k})+i\theta^{(k)}$ has same sign as $v(y_{0,k})$ for any $i\in\mathbb{N}$. The following result provides some criteria of compatibility.

Corollary 3.15. — *Let a differential series $F(y,\dots,y^{(n)})$ as in (1.1) and a generalized series $y_{0,k}\in\mathbb{K}_{r,k}$ be given. The series $y_{0,k}$ is compatible with F if:*

either $k<k_0$

or $k\geqslant k_0$ *and* $v(y_{0,k})>\max\{\underline{0},-n\theta^{(k_0)}\}$

or $k\geqslant k_0$, $\theta^{(k_0)}<\underline{0}$, $\underline{0}<v(y_{0,k})\leqslant -n\theta^{(k_0)}$, *and for all but finitely many* $I\in\mathbb{N}^{n+1}$, $c_I = d_{k_0}^{\|I\|}a_I$ *for some strongly summable family* $(a_I)_I$, $a_I\in\mathbb{K}_r$.

3.4. The Weierstrass order of an equation

The following notion is a key one in our proof of Theorem 2.14 in Section 5. It plays a role comparable to the one of Newton degree in the Newton polygon method [vdH97, Section 2.3.5], [vdH06, Section 8.3.3]. We will need also to control its evolution when applying the transformations of the equations described in the next section.

Definition 3.16 **(Weierstrass order).** — A differential series $F(y)$ as in (1.1) or (3.3) has Weierstrass order $w \in \mathbb{N}$ if:

- for any $I \in \mathbb{N}^{n+1}$, $v(c_I) \geqslant \underline{0}$;
- there exists $I \in \mathbb{N}^{n+1}$ with $|I| = w$ and $v(c_I) = \underline{0}$;
- for any $I \in \mathbb{N}^{n+1}$ with $|I| < w$, $v(c_I) > \underline{0}$.

Given a differential series F, if we divide it by $t^{\min(\text{Supp } F)}$ ($\min(\text{Supp } F$ exists since the support of F is well-ordered), we obtain a series with Weierstrass order equal to the minimum of $|I|$ for multi-indexes $I \in \mathbb{N}^{n+1}$ such that $v(c_I) = \min(\text{Supp } F)$. Then, note that the support of the factored series is equal to Supp $F - \min(\text{Supp } F)$, which is the application of an elementary transformation to Supp F.

So, *without loss of generality, we will suppose from now that the differential series we consider has such a Weierstrass order.*

4. Transformations of differential series

4.1. Changes of variable: additive and multiplicative conjugations

In [Mat07], we used these transformations respectively under the name "shiftings" and "blow-ups". Then we became aware of, and now we resume the terminology "additive and multiplicative conjugations" for the same kind of transformations, which was introduced by van der Hoeven in [vdH97, Sections 5.2.2 and 5.2.3] and in [vdH06, Sections 8.2.1 and 8.2.2]. They play a role in particular in the proof of his Theorem 12.2 cited before.

Definition 4.1. — Given a differential series $F(y)$ as in (1.1) or (3.3), and some series $a \in \mathbb{K}_r$, we call additive conjugation by a the change of variable $y = a + \tilde{y}$ and we denote the differential series thus obtained:

$$F_a(\tilde{y}, \dots, D_k^n \tilde{y}) = F(a + \tilde{y}, \dots, D_k^n a + D_k^n \tilde{y}) \quad (k \in \{0, \dots, r\})$$

Proposition 4.2. —

1. *Let $k \in \{1, \dots, r\}$, a differential series $F(y, \dots, D_k^n y)$ as in (3.3), and $a \in \mathbb{K}_r^{\prec}$ with $a = a_k + \cdots + a_1$, be given. The differential series F_a derived from (3.3) by additive conjugation by a has a well-ordered support. Moreover:*

$$\text{Supp } F_a \subset \text{Supp } F + \mathcal{T}_k + \langle \text{Supp } a \rangle.$$

2. *Let a differential series $F(y,\dots,y^{(n)})$ as in (1.1) and a series $a\in\mathbb{K}_r^{\prec}$ with $v(a^{(i)})>\underline{0}$ for all $i=0,\dots,n$, be given. The differential series F_a derived from (1.1) by additive conjugation by a, has a well-ordered support. Moreover, if $a=a_l\in\mathbb{K}_{r,l}$ for some $l=1,\dots,r$, then we consider the following cases:*

   (a) *if $l<k_0$, then we have:*

   $$\text{Supp } F_a\subset \text{Supp } F+\mathcal{T}_l+\langle\textstyle\bigcup_{i=0}^n \text{Supp } a+iv(d_l)\rangle;$$

   (b) *if $l\geqslant k_0$, then we have:*

   $$\text{Supp } F_a\subset \text{Supp } F+\mathcal{T}_{k_0}+\langle\textstyle\bigcup_{i=0}^n \text{Supp } a+iv(d_{k_0})\rangle$$

   and

   $$\text{Supp } F_a\subset (\textstyle\bigcup_I \text{Supp } c_I+\|I\|v(d_{k_0}))+\mathcal{T}_{k_0}+\langle\text{Supp } a\rangle.$$

Proof. — By the additive conjugation $y=\tilde{y}+a$, we have in the two cases of the proposition $D_k^i y=D_k^i\tilde{y}+D_k^i a$ for any $i\in\{1,\dots,n\}$, $k\in\{0,\dots,r\}$. By the Taylor expansion formula 3.6, we have:

$$F_a(\tilde{y},\dots,D_k^n\tilde{y})=F(\tilde{y}+a,\dots,D_k^n\tilde{y}+D^n a)=\textstyle\sum_{J\in\mathbb{N}^{n+1}}(f_a^{(J)}/J!)\tilde{y}^{(J)}.$$

Note that $F^{(J)}=\sum_{I\geqslant J}c_I y^{(I-J)}$, which implies that Supp $F^{(J)}\subset$ Supp F for any J. By hypothesis, a is compatible with any differential series $f^{(J)}$, $J\in\mathbb{N}^{n+1}$: $f_a^{(J)}=F^{(J)}(a,\dots,D_k^n a)$ is a well-defined element of $\mathbb{K}_r$. Moreover, from Proposition 3.13, we deduce that for any $J\in\mathbb{N}^{n+1}$:

1. if $k\neq 0$, then Supp $f_a^{(J)}\subset$ Supp $F+\mathcal{T}_k+\langle\text{Supp } a\rangle$;
2. if $k=0$, then:
   (a) if $l<k_0$, then: Supp $f_a^{(J)}\subset$ Supp $F^{(J)}+\mathcal{T}_k+\langle\bigcup_{i=0}^n \text{Supp } a+iv(d_k)\rangle$
   (b) if $l\geqslant k_0$, then: Supp $f_a^{(J)}\subset$ Supp $F^{(J)}+\mathcal{T}_{k_0}+\langle\bigcup_{i=0}^n \text{Supp } a+iv(d_{k_0})\rangle$ and also Supp $f_a^{(J)}\subset(\bigcup_{c_I\neq 0,\ I\geqslant J}\text{Supp } c_I+\|I-J\|v(d_{k_0}))+\mathcal{T}_{k_0}+\langle\text{Supp } a\rangle$.

So, we deduce for Supp $F_a=\cup_J$Supp $f_a^{(J)}$, the desired properties. □

Remark 4.3. — Consider a differential series $F(y)$ as in (1.1) or (3.3), and a series $y_0\in\mathbb{K}_r^{\prec}$ such that $v(y_0^{(i)})>\underline{0}$ for any $i=0,\dots,n$. If we denote $F=\sum_{I\in\mathbb{N}^{n+1}}c_I y^{(I)}$ and $F_a=\sum_{I\in\mathbb{N}^{n+1}}c'_I\tilde{y}^{(I)}$, we have $c_I=F^{(I)}(0)/I!$ and $c'_I=f_a^{(I)}/I!$. So, $v(c_I)=\underline{0}$ if and only if $v(c'_I)=\underline{0}$, and for any such I's, $\delta(c_I)=\delta(c'_I)$. Therefore, *the differential series F and F_a have same Weierstrass order.*

In the sequel, we will use *anti-lexicographical ordering* for multi-indexes: for any $I = (i_0, \dots, i_n)$ and any $J = (j_0, \dots, j_n) \in \mathbb{N}^{n+1}$,

$$I <_{antilex} J \Leftrightarrow \text{Supp } (I - J) \neq \emptyset \text{ and } i_k < j_k \text{ where } k = \max(\text{Supp } (I - J)).$$

Definition 4.4. — Let $m = t^\lambda$, $\lambda \in \Gamma$, be a monic monomial. We call multiplicative conjugation by m the change of variable $y = mz$.

Proposition 4.5. —

1. *Let a differential series $F(y, \dots, D_k^n y)$ as in (3.3) and a monic monomial $m = t^\lambda$ with $\lambda = (0, \dots, 0, \lambda_l, \dots, \lambda_r)$, $\lambda_k > 0$, for some $l = k, \dots, r$, be given. Performing the multiplicative conjugation $y = mz$, we obtain a differential series*

$$\hat{F}(z, \dots, D_k^n z) = \sum_{J \in \mathbb{N}^{n+1}} \hat{c}_J z^{(J)_k}$$

with well-ordered support such that:

$$\text{Supp } \hat{F} \subset \text{Supp } F + \mathbb{N}\lambda + \mathcal{T}_k.$$

2. *Let a differential series $F(y, \dots, y^{(n)})$ as in (1.1) and a monic monomial $m = t^\lambda$ with $\lambda = (0, \dots, 0, \lambda_l, \dots, \lambda_r)$, $\lambda \neq 0$ be given. Performing the multiplicative conjugation $y = mz$, we obtain a differential series*

$$\hat{F}(z, \dots, z^{(n)}) = \sum_{J \in \mathbb{N}^{n+1}} \hat{c}_J z^{(J)}$$

such that:

(a) if $l < k_0$, then we have:

$$\text{Supp } \hat{F} \subset \text{Supp } F + \mathcal{T}_l + \sum_{i=0}^n \mathbb{N}(\lambda + iv(d_l));$$

(b) if $l \geqslant k_0$, then we have:

$$\text{Supp } \hat{F} \subset \text{Supp } F + \mathcal{T}_{k_0} + \sum_{i=0}^n \mathbb{N}(\lambda + iv(d_{k_0}))$$

and

$$\text{Supp } \hat{F} \subset (\bigcup_{c_I \neq 0} \bigcup_{0 \leqslant j \leqslant \|I\|} \text{Supp } c_I + |I|\lambda + jv(d_{k_0})) + \mathcal{T}_{k_0}.$$

If m is strongly compatible with F, then $\hat{F}$ has well-ordered support.

Proof. — We treat the two cases at a time, taking $k = 0, \dots, r$. For any $j = 0, \dots, n$, $D_k^j y = \sum_{i=0}^j C_j^i D_k^{j-i} m D_k^i z$. So, for any $I \in \mathbb{N}^{n+1}$,

$$y^{(I)} = (mz)^{i_0}(D_k mz + mD_k z)^{i_1} \cdots (\sum_{i=0}^{n} C_n^i D_k^{n-i} m D_k^i z)^{i_n} = \sum l_{J,K} m^{(K)} z^{(J)}$$

where $l_{J,K} \in \mathbb{N}^*$ and the sum is taken over $J, K \in \mathbb{N}^{n+1}$ such that $|J| = |K| = |I|$, $\|K\| + \|J\| = \|I\|$ and $J, K \leqslant_{antilex} I$. We deduce that:

LEMMA 4.6. — *For any $J \in \mathbb{N}^{n+1}$, $\hat{c}_J = \sum k_{I,K} c_I m^{(K)}$ where $k_{I,K} \in \mathbb{N}^*$ and the sum is taken over the $I, K \in \mathbb{N}^{n+1}$ such that $|I| = |J| = |K|$, $\|I\| = \|K\| + \|J\|$ and $J, K \leqslant_{antilex} I$.*

Therefore, we have:

$$\operatorname{Supp} \hat{c}_J = \bigcup_{I,J,K} \operatorname{Supp} c_I + \operatorname{Supp} m^{(K)}.$$

Then the result is obtained as for the proof of the Proposition 3.13. □

Remark 4.7. — In the case where $l = k \in \{1, \ldots, r\}$ and $\lambda_k > 0$, we have for any multi-index K, $m^{(K)} \sim \lambda_l^{\|K\|} m^{|K|}$. Then, by Lemma 4.6, we obtain that

$$\hat{c}_J = \sum k_{I,J} c_I \lambda_l^{\|K\|} m^{|K|} = \sum k_{I,J} \lambda_l^{\|K\|} c_I m^{|I|}.$$

It means that a term $D_k^i y$ generates by multiplicative conjugation an analogous term $mD_k^i z$, plus terms with order of derivation in z lower than i. Fix $l \in \mathbb{N}$ and consider the terms $c_I y^{(I)}$ of F with $|I| = l$ (suppose that there exists at least one). We denote $v_0 = \min\{v(c_I) \mid |I| = k\}$, and

$$\mathcal{A}_l = \{I \in \mathbb{N}^{n+1} \mid v(c_I) = v_0,\ |I| = l\}.$$

Then the terms $c_I y^{(I)}$ for $I \in \mathcal{A}_l$ provide by multiplicative conjugation at least one term $\hat{c}_{I_0} z^{(I_0)}$ with $v(\hat{c}_{I_0}) = v_0 + lv(M)$: the one for $I_0 = \max_{anti-lex}(\mathcal{A}_l)$ (its leading term can not be canceled by any other term).

In particular, for a differential series with Weierstrass order w, we consider $\mathcal{A}_w$ the set of multi-indexes I for which $v(c_I) = \underline{0}$. Then there exists a coefficient $\hat{c}_{I_0}$ with $|I_0| = w$ and with valuation $wv(M)$. Since for any I with $|I| > w$, we have $v(\hat{c}_I) \geqslant (w+1)v(M)$ which is bigger than $w.v(M)$, then, denoting

$$v_{min} = \min\{v(\hat{c}_I) \mid I \in \mathbb{N}^{n+1}\},$$

we have $v_{min} \leqslant w.v(M)$. Thus *if we divide the new series $\hat{F}$ by $t^{v_{min}}$, we obtain a series with Weierstrass order $\hat{w}$ at most equal to w the Weierstrass order of the initial series F.*

4.2. Changes of derivation

Given a differential series $F(y)$ as in (1.1) or (3.3), denoted equally with D_k for some $k \in \{0, \dots, r-1\}$, can we transform it into some $\tilde{F}_{k,l}(y, \dots, D_l^n y)$ for some $l \in \{k+1, \dots, r\}$? To do this, on the one hand we need to express the transformation, i.e. find formal formulas connecting F and $\tilde{F}_{k,l}$. This is the purpose of Proposition 4.8. On the other hand, since it may happen that $\tilde{F}_{k,l}$ does not have a well-ordered support, we must check when such a transformation is well defined (Proposition 4.11). Finally we determine the support of such a well-defined $\tilde{F}_{k,l}$ (Proposition 4.14).

PROPOSITION 4.8. — *Let $k, l \in \{0, \dots, r\}$, we denote $m = d_l/d_k$ (setting $d_0 = 1$). We set for any $i \in \{1, \dots, n\}$,*

$$D_k^i y = q_{1,i} D_l y + q_{2,i} D_l^2 y + \dots + q_{i,i} D_l^i y$$

where the multi-sequence $(q_{j,i})_{j=1,..,i;i=1,..,n}$ is defined by:

$$\begin{cases} q_{1,1} &= m, & & \\ q_{1,i+1} &= D_k q_{1,i}, & \forall i = 1, \cdots, n-1, & \\ q_{j+1,i+1} &= q_{j,i} m + D_k q_{j+1,i}, & \forall i = 1, \cdots, n-1, & \forall j = 1, \cdots, i-1, \\ q_{i+1,i+1} &= q_{i,i} m, & \forall i = 1, \cdots, n-1, & \end{cases}$$

Then we denote $F(y, \dots, D_k^n y) = \tilde{F}_{k,l}(y, \dots, D_l^n y)$ (note that $\tilde{F}_{k,l}$ may not have a well-ordered support). In particular, $q_{1,i} = D_l^{i-1} m$ and $q_{i,i} = m^i$ for any $i \in \{1, \dots, n\}$.

Proof. — We proceed by induction on i. If $i = 1$, we have $D_k y = y'/d_k = (d_l/d_k).(y'/d_l) = (d_l/d_k) D_l y$. Thus we set $q_{1,1} = m$. For the induction, we suppose that

$$D_k^i y = q_{1,i} D_l y + q_{2,i} D_l^2 y + \dots + q_{i,i} D_l^i y.$$

Therefore:

$$\begin{aligned} D_k^{i+1} y &= D_k q_{1,i} D_l y + q_{1,i} D_k D_l y + \dots + D_k q_{i,i} D_l^i y + q_{i,i} D_k D_l^i y \\ &= D_k q_{1,i} D_l y + q_{1,i} (d_l/d_k) D_l^2 y + \dots + D_k q_{i,i} D_l^i y + q_{i,i} (d_l/d_k) D_l^{i+1} y \\ &= D_k q_{1,i} D_l y + (q_{1,i} m + D_k q_{2,i}) D_l^2 y + \dots + (q_{i-1,i} m + D_k q_{i,i}) D_l^i y \\ &\quad + q_{i,i} m D_l^{i+1} y. \end{aligned}$$

The property holds at the step $i + 1$. □

LEMMA 4.9. — *For any $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, i\}$, $q_{j,i}$ is a differential polynomial in m such that $q_{j,i} = \sum q_{j,i,I} m^{(I)_k} \in \mathbb{K}_r$ where the sum is taken over $I \in \mathbb{N}^i$ with $|I| = j$ and $\|I\| = i - j$.*

Proof.— We proceed by induction on i. If $i = 1$, $q_{1,1} = m$ and we set $q_{1,1,(1)} = 1$. If $i = 2$, $q_{1,2} = D_k m$ and $q_{2,2} = m^2$. So $q_{1,2,(0,1)} = q_{2,2,(2,0)} = 1$.

For the induction, we suppose that $q_{1,i+1} = D_k^i m,\ q_{1,i+1,(0,..,1)} = 1$ and $q_{i+1,i+1} = m^{i+1},\ q_{i+1,i+1,(i+1,0,..)} = 1$. Then

$$q_{j+1,i+1} = mq_{j,i} + D_k q_{j+1,i} = \sum_{|I|=j,\|I\|=i-j} q_{j,i,I} m^{(I+(1,0,..,0))_k} + \sum_{|I|=j+1,\|I\|=i-j-1} q_{j,i,I} D_k(m^{(I)_k}).$$

In the right hand term, for the first sum, we set $J = I + (1, 0, .., 0)$. So $|J| = j + 1$ and $\|J\| = i + 1 - j - 1$. For the second one, we have

$$D_k(m^{(I)_k}) = D_k(m^{i_0}(D_k m)^{i_1} \dots (D_k^{i-1} m)^{i_{i-1}}) = \sum_{l=0}^{i-1} i_l m^{i_0} \dots (D_k^l m)^{i_l - 1}(D_k^{l+1} m)^{i_l+1} \dots (D_k^{i-1} m)^{i_{i-1}}.$$

We set $K = I + (0, .., -1, 1, .., 0)$ with -1 in l^{th} position. So $|K| = j + 1$ and $\|K\| = i + 1 - j - 1$. Then we obtain, for any $l = 1, \dots, i - 1$,

$$q_{j+1,i+1,I} = q_{j,i,I-(1,0,..,0)} + (i_l + 1) q_{j,i,I-(0,..,-1,1,..,0)}$$

and for all $I \in \mathbb{N}^{i+1}$ with $|I| = j+1$, $\|I\| = i+1-j-1$, $i_1 \geqslant 1$ and $i_{l+1} \geqslant 1$. □

As an example, we provide in the following table some polynomials $q_{j,i}$.

$i\backslash j$	1	2	3	4
1	m	-	-	-
2	$D_k m$	m^2	-	-
3	$D_k^2 m$	$3mD_k m$	m^3	-
4	$D_k^3 m$	$4mD_k^2 m + 3(D_k m)^2$	$6D_k m.m^2$	m^4
5	$D_k^4 m$	$5mD_k^3 m + 10D_k m D_k^2 m$	$10m^2 D_k^2 m + 15m(D_k m)^2$	$10m^3 D_k m$

Remark 4.10.— From Proposition 4.8 and Lemma 4.9, $D_k^i z = \sum_{j=1}^i q_{i,j} D_l^j z$ with $q_{i,j} = \sum q_{j,i,I} m^{(I)_k} \in \mathbb{K}_r^{\prec}$ where $m = d_l/d_k$, $q_{j,i,I} \in \mathbb{N}$ and the sum is taken over multi-indexes $I \in \mathbb{N}^i$ such that $|I| = j$ and $\|I\| = i - j$. In particular, the coefficient of $D_l^i z$ is $q_{i,i} = m^i$. So, given some differential monomials $a_I y^{(I)_k}$ with coefficients that have same valuation $v^{(0)}$, by the change of derivation they generate new differential monomials $b_J y^{(J)_l}$ with $|I| = |J|$ and with one of them (the one with $J = I_0$ where I_0 is the greatest multi-index for the anti-lexicographical ordering among these I's) with coefficient that has valuation $v^{(0)} + \|I_0\| v(m)$. But $v(m) = (0, \dots, 0, \mu_{l_1}, \dots, \mu_r)$ with $\mu_{l_1} > 0$ and $l_1 > k$ (see Proposition 2.7). So such a coefficient has a valuation of type

$$v^{(0)} + \|I\|(0, \dots, 0, \mu_{l_1}, \dots, \mu_r)$$

with $\mu_{l_1} > 0$ and $l_1 > k$. *This remark will be useful to control the evolution of the Weierstrass order of the differential series through a change of derivation.*

Proposition 4.11. —

1. *We consider a differential series $F(y, \dots, D_k^n y)$ as in (3.3). The changes of derivation*
$$F(y, \dots, D_k^n y) = \tilde{F}_{k,l}(y, \dots, D_l^n y)$$
are well defined for any $k < l$ in $\{1, \dots, r\}$.

2. *We consider a differential series $F(y, \dots, y^{(n)})$ as in (1.1), the integer $k_0 \in \{1, \dots, r\}$ defined in the Corollary 2.9, and some $l \in \{1, \dots, r\}$, with $l \leqslant k_0$ if $v(d_{k_0}) < \underline{0}$ (see Corollary 3.8).*

   (a) *If $v(d_l) \geqslant \underline{0}$, then, by the change of derivation*
   $$F(y, \dots, y^{(n)}) = \tilde{F}_{0,l}(y, \dots, D_l^n y),$$
   we obtain a differential series $\tilde{F}_{0,l}$ with well-ordered support.

   (b) *If $v(d_l) < \underline{0}$, then $l \leqslant k_0$. We apply successively to F the multiplicative conjugation $y = d_l^{-n} z$, and the change of derivation $F = \tilde{F}_{0,l}$. Then, $\tilde{F}_{0,l}$ has well-ordered support.*

Proof. — Given $k, l \in \{0, \dots, r\}$, by the Proposition 4.8, if for any $i \in \{0, \dots, n\}$, $v(D_k^i m) \geqslant \underline{0}$ (where $m = d_l/d_k$), then by the change of derivation $F(y, \dots, D_k^n y) = \tilde{F}_{k,l}(y, \dots, D_l^n y)$, we obtain a differential series $\tilde{F}_{k,l}$ which has clearly a well-ordered support. But, in the case (1), $m = d_l/d_k$ with $l > k$. So $v(m) = v(d_l) - v(d_k) > \underline{0}$, and therefore, $v(D_k^i m) > \underline{0}$ for all $i \in \{0, \dots, n\}$ (see Proposition 3.3).

In the case (2)(a), $m = d_l$. By the Corollary 2.11, $v(m^{(i)}) = v(d_l^{(i)}) \geqslant \underline{0}$ for any i.

For the case (2)(b), we denote $M = d_l^{-n}$. Combining the results of the Propositions 4.5 and 4.8, we show that:

Lemma 4.12. — *For any $i \in \mathbb{N}$, $y^{(i)}$ becomes a linear combination with positive integer coefficients, of terms*
$$M^{(i-k-j)} d_l^{(I)} D_l^k z = M^{(i-k-j)} d_l^{i_0} (d_l')^{i_1} \cdots (d_l^{(j)})^{i_j} D_l^k z$$
where $k = 0, \dots, i$, $j = 0, \dots, i-k$, $I \in \mathbb{N}^{j+1}$ with $|I| = k$ and $\|I\| = j$.

Proof. — Consider such a term $M^{(i-k-j)}d_l^{(I)}D_l^k z$. Then its derivative is:

$$\begin{aligned}(M^{(i-k-j)}d_l^{(I)}D_l^k z)' &= M^{(i+1-k-j)}d_l^{(I)}D_l^k z + M^{(i-k-j)}[i_0 d_l^{(I+(-1,1,0,\cdots))} \\ &\quad + \cdots + i_j d_l^{(I+(0,\ldots,0,-1))}d_l^{(j+1)}]D_l^k z + \\ &\quad M^{(i-k-j)}d_l^{(I)}d_l D_l^{k+1}z \\ &= M^{(i+1-k-j)}d_l^{(\tilde{I})}D_l^k z + M^{(i+1-k-(j+1))} \\ &\quad [i_0 d_l^{(\tilde{I}+(-1,1,0,\cdots,0))} + \cdots + i_j d_l^{(\tilde{I}+(0,\ldots,0,-1,1))}]D_l^k z \\ &\quad + M^{(i+1-(k+1)-j)}d_l^{(\tilde{I}+(1,0,\ldots,0))}D_l^{k+1}z,\end{aligned}$$

where $\tilde{I} \in \mathbb{N}^{j+2}$ is defined as the multi-index I expanded by the addition of a new component 0 at its end. All the new exponents have the desired properties. □

Recall that $M = d_l^{-n}$. We show that:

Lemma 4.13. — *For any $m \in \mathbb{N}^*$, $M^{(m)}$ is an integral linear combination of terms*

$$d_l^{-n-p}d_l^{(Q)} = d_l^{-n-p}(d_l')^{q_1}\cdots(d_l^{(m-p)})^{q_{m-p+1}}$$

where $p = 1, \ldots, m$, $Q \in \mathbb{N}^m$ with $|Q| = p$ and $\|Q\| = m$.

Proof. — By induction on m, we consider such a general term $d_l^{-n-p}d_l^{(Q)}$. Then its derivative is:

$$\begin{aligned}(d_l^{-n-p}d_l^{(Q)})' &= (-n-p)d_l^{-n-(p+1)}d_l^{(Q)} + d_l^{-n-p}[q_1 d_l^{(Q+(-1,1,0,\ldots))} + \cdots + \\ &\quad q_{m-p+1}d_l^{(Q+(0,\ldots,-1))}d_l^{(m-p+2)}] \\ &= (-n-p)d_l^{-n-(p+1)}d_l^{(\tilde{Q})} + d_l^{-n-p}[q_1 d_l^{(\tilde{Q}+(-1,1,0,\ldots,0))} + \cdots + \\ &\quad q_{m-p+1}d_l^{(\tilde{Q}+(0,\ldots,-1,1))}].\end{aligned}$$

where $\tilde{Q} \in \mathbb{N}^{m-p+2}$ is defined as the multi-index Q expanded by the addition of a new component 0 at its end. All the new exponents have the desired properties. □

Now, combining the results of the two preceding lemmas, as the general term generated by $y^{(i)}$ we obtain

$$M^{(i-k-j)}d_l^{(I)}D_l^k z = d_l^{-n-i+k+j}d_l^{i_0}(d_l')^{i_1+q_1}\cdots(d_l^{(m)})^{i_m+q_m}D_l^k z$$

where $m = \max\{j, i-k-j-p\}$ and the multi-indexes I or Q are completed by zeros when needed. But, by the Corollary 2.11, since $l \leqslant k_0$, $v(d_l^{(q)}) \geqslant (q+1)v(d_l)$ for any $q \in \mathbb{N}$. Thus we deduce that the valuation of the coefficient of this general term is bigger than the minimum of:

$$\begin{array}{rcl} v(M^{(i-k-j)}d_l^{(I)}) & \geqslant & v(d_l^{-n-i+k+j}d_l^{|I|+\|I\|+|Q|+\|Q\|}) \\ & \geqslant & (-n+k+j+p)v(d_l) \\ & \geqslant & (-n+i)v(d_l) \\ & \geqslant & \underline{0} \end{array}$$

since $|I|=k$, $\|I\|=j$, $|Q|=p$, $\|Q\|=i-k-j$, with $p=1,\dots,i-k-j$, $j=0,\dots,i-k$, $k=0,\dots,i$ and $i=0,\dots,n$. □

PROPOSITION 4.14. — *For any $k<l\in\{0,\dots,r\}$, with the same hypothesis as in the Proposition 4.11, we have:*

1. Supp $\tilde{F}_{k,l}\subset$ Supp $F+\mathcal{T}_k$

2. *(a) (i) if $l<k_0$, then* Supp $\tilde{F}_{0,l}\subset$ Supp $F+\mathbb{N}\ v(d_l)+\mathcal{T}_l$;
   (ii) if $l\geqslant k_0$, then Supp $\tilde{F}_{0,l}\subset$ Supp $F+\mathbb{N}\ v(d_{k_0})+\mathcal{T}_{k_0}$;
   (b) Supp $\tilde{F}_{0,l}\subset$ Supp $F-\mathbb{N}v(d_l)+\mathcal{T}_l$.

Proof. — (1) We denote $m=d_l/d_k$. From Lemma 2.7, we have

$$\text{Supp } m=\{v(m)\}=\{v(d_l)-v(d_k)\}=\{(0,\dots,0,\beta_j,\dots,\beta_r)\}$$

for some $j>k$ and some $\beta_j,\dots,\beta_r\in\mathbb{R}$ with $\beta_j>0$. The change of derivation reads $D_k^i y=\sum_{j=1}^{i}q_{i,j}D_l^j y$ for any $i\in\{0,\dots,n\}$, with

$$q_{i,j}=\sum_{I\in\mathbb{N}^i,|I|=j,\|I\|=i-j}q^{j,i,I}m^{(I)_k}.$$

From Lemma 3.13, for any $i\in\{0,\dots,n\}$, we have Supp $D_k^i m\subset v(m)+\mathcal{T}_k$. Then it suffices to note that $m=d_l/d_k\sim t_l'/t_l d_k=D_k t_l/t_l$ and so Supp $m\subset\mathcal{T}_k$.

(2) (a) If $v(d_l)\geqslant\underline{0}$, by the Propositions 3.13, 4.8 and the Lemma 4.9, we have, for any $i\in\{1,\dots,n\}$ and any $j\in\{1,\dots,i\}$:

$$\begin{array}{rlllll} \text{(i)} & \text{if } l<k_0, \text{ then} & \text{Supp } q_{i,j} & \subset & \text{Supp } d_l+\mathbb{N}v(d_l)+\mathcal{T}_l; \\ \text{(ii)} & \text{if } l\geqslant k_0, \text{ then} & \text{Supp } q_{i,j} & \subset & \text{Supp } d_l+\mathbb{N}v(d_{k_0})+\mathcal{T}_{k_0}. \end{array}$$

We have Supp $d_l=\{v(d_l)\}\subset$ Supp t_l'/t_l. But, $t_l'/t_l=d_{k_0}D_{k_0}t_l/t_l$. So, $v(d_l)\in v(d_{k_0})+\mathcal{T}_{k_0}$. We deduce that:

$$\begin{array}{rlllll} \text{(i)} & \text{if } l<k_0, \text{ then} & \text{Supp } q_{i,j} & \subset & \mathbb{N}v(d_l)+\mathcal{T}_l; \\ \text{(ii)} & \text{if } l\geqslant k_0, \text{ then} & \text{Supp } q_{i,j} & \subset & \mathbb{N}v(d_{k_0})+\mathcal{T}_{k_0}, \end{array}$$

which leads directly to the desired conclusion.

(2) (b) As in the proof of the Proposition 4.11, by combination of the multiplicative conjugation $y = d_l^{-n}$ and the change of derivation $F = \tilde{F}_{0,l}$, a term $y^{(i)}$ for some $i = 0, \dots, n$ generates an integral linear combination of terms

$$d_l^{-n-i+k+j} d_l^{i_0} (d_l')^{i_1+q_1} \cdots (d_l^{(m)})^{i_m+q_m} D_l^k z$$

where $|I| = k$, $\|I\| = j$, $|Q| = p$, $\|Q\| = i-k-j$, with $p = 0, \dots, i-k-j$, $j = 0, \dots, i-k$, $k = 0, \dots, i$.

We rewrite such a term as:

$$d_l^{-n-i+k+j} d_l^{|I|+\|I\|+|Q|+\|Q\|} (d_l'/d_l^2)^{i_1+q_1} \cdots (d_l^{(m)}/d_l^{m+1})^{i_m+q_m} D_l^k z = d_l^{-n+p+k+j} (d_l'/d_l^2)^{i_1+q_1} \cdots (d_l^{(m)}/d_l^{m+1})^{i_m+q_m} D_l^k z.$$

Note that $0 \leqslant p+k+j \leqslant n$.

Now, it suffices to show that:

LEMMA 4.15. — *For any* $s \in \mathbb{N}^*$, $\operatorname{Supp} d_l^{(s)} \subset (s+1)v(d_l) + \mathcal{T}_l$.

Proof. — We proceed by induction on s. Since $l \leqslant k_0$, we have for some $\tilde{l} \geqslant l$, $d_l' = d_l(\theta_{\tilde{l}}^{(l)} t_{\tilde{l}}'/t_{\tilde{l}} + \cdots + \theta_r^{(l)} t_r'/t_r)$. So $d_l' = d_l^2(\theta_{\tilde{l}}^{(l)} D_l(t_{\tilde{l}})/t_{\tilde{l}} + \cdots + \theta_r^{(l)} D_l(t_r)/t_r)$ which has a support as desired.

Suppose that, for some $s \in \mathbb{N}^*$, we have $d_l^{(s)} = d_l^{s+1} \sum c_\gamma t^\gamma$ for some series $\sum c_\gamma t^\gamma$ with support in $\mathcal{T}_l$. Then:

$$\begin{aligned} d_l^{(s+1)} &= d_l^{s+1}[(s+1)(d_l'/d_l^1) \textstyle\sum c_\gamma t^\gamma + \sum c_\gamma (t^\gamma)'] \\ &= d_l^{s+2}[(s+1)(d_l'/d_l^2) \textstyle\sum c_\gamma t^\gamma + \sum c_\gamma D_l(t^\gamma)]. \end{aligned}$$

□

To conclude, note that d_l'/d_l^2 and $\sum c_\gamma D_l(t^\gamma)$ have support included in $\mathcal{T}_l$ (the first one by the result for $s = 1$ here above, the second one by the case (1) of the Proposition 3.13). □

4.3. Reducing to positive valuation solutions for polynomial equations

We consider a differential polynomial

$$P(y) = P(y, \dots, y^{(n)}) = \sum_{I \in S \subset \mathbb{N}^{n+1},\ S \text{ finite}} c_I y^{(I)} \in \mathbb{K}_r[y, \dots, y^{(n)}] \setminus \{0\}. \tag{4.7}$$

We remind that in this case, any generalized series $y_0 \in \mathbb{K}_r$ is compatible with (4.7) (see Definition 2.12: the family $(c_I y_0^{(I)})_I$, being finite, is strongly summable). The purpose of this section is to derive from Theorem 2.14 the same result, but for *any* generalized series $y_0 \in \mathbb{K}_r$.

We consider a generalized series

$$y_0 = m_0 t^{\mu_0} + m_1 t^{\mu_1} + \cdots \in \mathbb{K}_r$$

where $\delta(y_0) = m_0 t^{\mu_0}$, such that $v(y_0^{(i)}) \leqslant \underline{0}$ for some $i = 0, \ldots, n$, i.e. (see Corollary 3.8) such that:

$$\mu_0 \leqslant \alpha_0 := \max\{\underline{0}, -nv(d_{k_0})\}.$$

Lemma 4.16. —

1. *By the additive conjugation $y = \tilde{y} + m_0 t^{\mu_0}$ (see Definition 4.1), we obtain from $P(y)$ as in (4.7) a new differential polynomial*

$$P_{m_0 t^{\mu_0}}(\tilde{y}, \ldots, \tilde{y}^{(n)}) = \textstyle\sum_{I \in S} \tilde{c}_I \tilde{y}^{(I)} \in \mathbb{K}_r[\tilde{y}, \ldots, \tilde{y}^{(n)}] \setminus \{0\}$$

with associated series

$$\tilde{y}_0 = y_0 - m_0 t^{\mu_0} = m_1 t^{\mu_1} + \cdots \in \mathbb{K}_r.$$

2. *By the multiplicative conjugation $\tilde{y} = t^{\mu_0 - \alpha_0} \hat{y}$ (see Definition 4.4), we obtain from $P_{m_0 t^{\mu_0}}$ a new differential polynomial*

$$\hat{P}(\hat{y}, \ldots, \hat{y}^{(n)}) = \textstyle\sum_{J \in \hat{S} \text{ finite}} \hat{c}_J \hat{y}^{(J)} \in \mathbb{K}_r[\hat{y}, \ldots, \hat{y}^{(n)}] \setminus \{0\}$$

with associated series

$$\hat{y}_0 = \tilde{y}_0 / t^{\mu_0 - \alpha_0} = m_1 t^{\mu_1 - \mu_0 + \alpha_0} + \cdots \in \mathbb{K}_r.$$

which has valuation $v(\hat{y}_0) = \mu_1 - \mu_0 + \alpha_0 > \alpha_0$.

Proof. — (1) By the Taylor expansion formula (3.6), we have

$$\begin{array}{rcl} P_{m_0 t^{\mu_0}}(\tilde{y}, \ldots, \tilde{y}^{(n)}) & = & P(y + m_0 t^{\mu_0}, \ldots, y^{(n)} + m_0 (t^{\mu_0})^{(n)}) \\ & = & \sum_{I \in S} \left(P^{(I)}(m_0 t^{\mu_0}, \ldots, m_0 (t^{\mu_0})^{(n)}) / I! \right) \tilde{y}^{(I)}. \end{array}$$

Since $P^{(I)}$ is a differential polynomial for any I, we have $P^{(I)}(m_0 t^{\mu_0}, \ldots, m_0 (t^{\mu_0})^{(n)}) \in \mathbb{K}_r$. Then $P_{m_0 t^{\mu_0}}$ is well-defined and has well-ordered support (which is the finite union of the supports of $\tilde{c}_I$).

(2) By the Lemma 4.6, we have $\hat{c}_J = \sum k_{I,K} \tilde{c}_I (t^{\mu_0 - \alpha_0})^{(K)}$ where $k_{I,J} \in \mathbb{N}^*$ and the sum is taken over the $I, K \in \mathbb{N}^{n+1}$ such that $|I| = |J| = |K|$, $\|I\| = \|K\| + \|J\|$ and $J, K \leqslant_{antilex} I$. Therefore, we have a finite number of coefficients $\hat{c}_J$, any of which is itself a finite sum of generalized series $\tilde{c}_I (t^{\mu_0 - \alpha_0})^{(K)}$. Thus, $\hat{P}$ is a differential polynomial. □

Thanks to the preceding lemma, in the case of a differential polynomial together with an arbitrary generalized series $y_0 \in \mathbb{K}_r$, we can reduce to the hypothesis of the Theorem 2.14. So, assuming that the latter holds, we obtain:

THEOREM 4.17. — *We denote* $\alpha_0 := \max\{\underline{0}, -nv(d_{k_0})\}$. *Given a differential polynomial* $P(y)$ *as in (4.7) and a non zero series* $y_0 \in \mathbb{K}_r$, *there exists a well-ordered subset* $\mathcal{R}$ *of* $\Gamma_{\geqslant 0}$ *obtained from* Supp F *and* Supp t'_k/t_k, $k = 1, \ldots, r$, *by a finite number of elementary transformations such that:*

- *either* Supp $y_0 \subseteq v(y_0) - \alpha_0 + \mathcal{R}$;
- *or the series* y_0 *stabilizes on* F *with initial part* p_0 *and* Supp $p_0 \subseteq v(y_0) - \alpha_0 + \mathcal{R}$. *In this case,* y_0 *can not be a solution of the corresponding differential equation* $P(y) = 0$.

As a direct consequence, we obtain that:

THEOREM 4.18. — *Given a differential equation*

$$P(y, \ldots, y^{(n)}) = 0 \tag{4.8}$$

where P *is a differential polynomial as in (4.7), the subset* (Supp y_0) $- v(y_0) + \alpha_0$ *of* $\Gamma_{>\underline{0}}$, *is obtained by finitely many elementary transformations from the supports of* F *and* t'_k/t_k, $k = 1, \ldots, r$.

In the case of a general differential series $F(y, \ldots, y^{(n)})$ as in (1.1), the changes of variable defined in the Lemma 4.16 may not generate a differential series $\hat{F}$ with well-ordered support. Nevertheless, we conjecture that the Theorems 4.17 and 4.18 (or slighlty adapted versions of them) hold in this more general context.

5. Proof of the Theorem 2.14

5.1. The main lemma

The Theorem 2.14 is a consequence of the following lemma:

LEMMA 5.1. — *Given $k \in \{1, \dots, r\}$ and $w \in \mathbb{N}$, we consider a differential series $F(y, \dots, D_k^n y)$ as in (3.3) and a generalized series $y_0 = y_{0,k} + \cdots + y_{0,1} \in \mathbb{K}_r^{\prec}$ with $y_{0,l} \in \mathbb{K}_{r,l}$ for any $l \in \{1, \dots, k\}$. Then there exists a well-ordered subset $\mathcal{R}$ of $\Gamma_{>0}$ obtained from* Supp F, $\mathcal{T}_k, \dots, \mathcal{T}_r$ *by a finite number of elementary transformations such that:*

- *either the exponents of $y_{0,k}$ belong to $\mathcal{R}$;*
- *or the series y_0 stabilizes on F with initial part p_0 which is also a proper initial part of $y_{0,k}$. Moreover the exponents of p_0 belong to $\mathcal{R}$.*

Proof of Theorem 2.14. — Indeed, we consider a series $y_0 \in \mathbb{K}_r^{\prec}$ with $v(y_0^{(i)}) > \underline{0}$ for any $i = 0, \dots, n$, as in the statement of Theorem 2.14. We denote by $y_0 = y_{0,k} + \cdots + y_{0,1}$ its decomposition as in the Notation 2.1. Then, applying the point (2) of the Proposition 4.11, we consider two cases. If $v(d_k) \geqslant \underline{0}$, then, by the change of derivation

$$F(y, \dots, y^{(n)}) = \tilde{F}_{0,k}(y, \dots, D_k^n y),$$

the new differential series $\tilde{F}_{0,k}$ has well-ordered support. So it has some Weierstrass order $w \in \mathbb{N}$ (see Definition 3.16). Moreover, by the Proposition 4.14, the support of the new series $\tilde{F}_{0,k}$ is derived from the one of F and the set $\mathcal{T}_k$ by elementary transformations. Thus, considering this series $\tilde{F}_{0,k}$ with y_0, we are reduced to the hypothesis of the Lemma 5.1.

If $v(d_k) < \underline{0}$, as in the case (b) of the Proposition 4.11, we have $k \leqslant k_0$, and we can apply successively to F the multiplicative conjugation $y = d_k^{-n} z$, and the change of derivation $F = \tilde{F}_{0,k}$. Then, we obtain a differential series $\tilde{F}_{0,k}$ with well-ordered support. As above it has some Weierstrass order. Its support is also derived from Supp F and $\mathcal{T}_k$ by elementary transformations. The associated generalized series $\tilde{y}_0 = y_0 d_k^n$.

If $k < k_0$, by the Corollary 2.9, then $\tilde{k} > k$. It implies that

$$\begin{array}{rcl} \tilde{y}_0 & = & y_{0,k} d_k^n + \cdots + y_{0,1} d_k^n \\ & = & \tilde{y}_{0,k} + \cdots + \tilde{y}_{0,1} \in \mathbb{K}_r^{\prec} \end{array}$$

as required to apply the Lemma 5.1. Note that the support of y_0 is then equal to Supp $\tilde{y}_0 - nv(d_k)$. Therefore, it is included into Supp $\tilde{y}_0 + \mathbb{N}(-v(d_k))$, which is the application of an elementary transformation to Supp $\tilde{y}_0$.

If $k = k_0$, since $v(y_0^{(i)}) > \underline{0}$ for any $i = 0, \ldots, n$, then $v(y_0)$ verifies the Condition 3.5. So, $v(\tilde{y}_0)$ is also positive, and $\tilde{y}_0$ is of the form

$$\tilde{y}_0 = \tilde{y}_{0,l} + \cdots + \tilde{y}_{0,1} \in \mathbb{K}_r^{\prec}$$

with $l \geqslant k_0$. Then, if necessary, we apply in the differential series $\tilde{F}_{0,k_0}$ just obtained, the change of derivation from D_{k_0} to D_l. Then we can apply to the latter, together with the series $\tilde{y}_0$, the Lemma 5.1. As before, Supp y_0 is deduced from Supp $\tilde{y}_0$ by an elementary transformation.

In any case, we are reduced to the hypothesis of the Lemma 5.1. To simplify the notations, let us denote in any of the described cases, the obtained differential series with Weierstrass order w by $G(y, \ldots, D_k^n y)$, and the corresponding generalized series by $y_0 = y_{0,k} + \cdots + y_{0,1}$. Applying the Lemma 5.1, we obtain a dichotomy for $y_{0,k}$, the second case being exactly the second one in the statement of Theorem 2.14.

For the first one, we have Supp $y_{0,k} \subset \mathcal{R}_k$, where $\mathcal{R}_k$ is a well-ordered subset of $\Gamma_{>0}$ obtained from Supp G (and therefore from Supp F) and $\mathcal{T}_k, \ldots, \mathcal{T}_r$ by a finite number of elementary transformations. Then we perform the additive conjugation $y = \tilde{y} + y_{0,k}$ in the original differential series F, and we get a new one $F_{y_{0,k}}(\tilde{y}, \ldots, \tilde{y}^{(n)})$ together with a series $\tilde{y}_0 = y_{0,l} + \ldots + y_{0,1}$ for some $l < k$. Moreover, by the Proposition 4.2, the support of $F_{y_{0,k}}$ is obtained from the one of F and the sets $\mathcal{T}_k$ and Supp $y_{0,k}$ by finitely many elementary transformations. But Supp $y_{0,k} \subset \mathcal{R}_k$ is obtained itself from Supp F, $\mathcal{T}_k, \ldots, \mathcal{T}_r$ by a finite number of elementary transformations. Then, so Supp $F_{y_{0,k}}$ is. We can resume the preceding arguments with $F_{y_{0,k}}(\tilde{y}, \ldots, \tilde{y}^{(n)})$ and $\tilde{y}_0$.

Thus we prove gradually in the case where there is no stabilization that Supp $y_0 \subset \mathcal{R} = \sum_{l=1}^{k} \langle \mathcal{R}_l \rangle$, the latter being obtained from Supp F, $\mathcal{T}_k, \ldots, \mathcal{T}_r$ by finitely many elementary transformations as desired. □

Concerning the proof of our main lemma, we consider the set $\{1, \ldots, r\}$ endowed with the *reverse ordering* $1 > 2 > \cdots > r$, and we denote it $\{r, \ldots, 1\}$. Then we define the lexicographical product $\overrightarrow{\mathbb{N} \times \{r, \ldots, 1\}}$, remark that it is a well-ordered set, and denote (w, k) its elements. To prove Lemma 5.1, we proceed by transfinite induction on $(w, k) \in \overrightarrow{\mathbb{N} \times \{r, \ldots, 1\}}$.

First, if the differential series $F(y, \dots, D_k^n y)$ has Weierstrass order 0, then for any $k \in \{r, \dots, 1\}$ and any initial part p of y_0,

$$F(p, \dots, D_k^n p) = c_0 + \sum_{I \in \mathbb{N}^{n+1} \setminus \{0\}} c_I p^{(I)}.$$

But for any $I \in \mathbb{N}^{n+1}$ with $|I| \geqslant 1$, $v(y_0^{(I)_k}) > \underline{0}$ (Lemma 3.3) and by hypothesis $v(c_0) = \underline{0} \leqslant v(c_I)$. So $v(F(p, \dots, D_k^n p)) = v(c_0) = \underline{0}$, which means that y_0 stabilizes on F with initial part 0.

Second, we consider $(w, k) \in \overrightarrow{\mathbb{N}^* \times \{r, \dots, 1\}}$ and suppose that the Theorem 2.14 holds for any $(\tilde{w}, \tilde{k}) < (w, k)$. We treat separately the case $w = 1$.

5.2. The case $w = 1$

Notation 5.2. — Given a differential series $F(y, \dots, D_k^n y)$ as in (3.3) with Weierstrass order 1, we denote:

- $\mathcal{A}$ the set of all multi-index I with length 1 such that $v(c_I) = \underline{0}$;
- for all $I \in \mathbb{N}^{n+1}$, $c_{I,0}$ the leading coefficient of c_I.

Definition 5.3. — With the terminology of [Inc44], we call *indicial polynomial* associated to a Weierstrass order 1 series $F(y, \dots, D_k^n y)$ the polynomial $\pi(X) = \sum_{I \in \mathcal{A}} c_{I,0} X^{\|I\|}$. We denote $\{\rho_1, \dots, \rho_m\}$ the set of its real positive roots ($m \leqslant n$ since its degree is at most n).

Lemma 5.4. — *Given a Weierstrass order 1 differential series $F(y, \dots, D_k^n y)$ as in (3.3) and a series $y_0 \in \mathbb{K}_r^{\prec}$ with initial part*

$$y_{0,k} = \sum_{\mu \in \mathrm{Supp}\ y_{0,k}} m_\mu t^\mu \in \mathbb{K}_{r,k},$$

there are two cases:

1. *either for any proper initial part p of $y_{0,k}$ with successor term $m_\mu t^\mu$ with $\mu = (0, \dots, 0, \mu_k, \dots, \mu_r)$ ($\mu_k > 0$), we have $\mu = v_p$, or $\mu_k \in \{\rho_1, \dots, \rho_m\}$;*
2. *or there exists a proper initial part p_0 of $y_{0,k}$ with successor term $m_0 t^{\mu^{(0)}}$ such that the two following properties hold:*
   (a) *for any successor part $\tilde{p}$ of p_0 in y_0, we have*

$$v_{p_0+\tilde{p}} = v_{p_0 + m_0 t^{\mu^{(0)}}} = \min\{v_{p_0}, \mu^{(0)}\} \leqslant \mu^{(0)};$$

(b) for any proper initial part p of p_0 with successor term $m_\mu t^\mu$, $\mu = (0, \dots, 0, \mu_k, \dots, \mu_r) < \mu^{(0)}$, we have $\mu = v_p$ or $\mu_k \in \{\rho_1, \dots, \rho_m\}$.

Proof. — We consider a proper initial part p of $y_{0,k}$ with successor term $m_\mu t^\mu$, and some successor part $\tilde{p}$ of p (so $m_\mu t^\mu$ is the leading term of $\tilde{p}$). From the Taylor expansion formula (3.6):

$$f_{p+\tilde{p}} = \sum_{I \in \mathbb{N}^{n+1}} (f_p^{(I)}/I!)\tilde{p}^{(I)}.$$

But $\tilde{p} = m_\mu t^\mu(1+\epsilon)$ and so

$$\tilde{p}^{(I)} = m_\mu^{|I|} \mu_k^{\|I\|} t^{|I|\mu}(1+\epsilon_I)$$

for some $\epsilon, \epsilon_I \in \mathbb{K}_r^\prec$ (Proposition 3.2). Moreover, for any $I \in \mathcal{A}$, $\delta((f_p^{(I)}/I!)) = c_{I,0}$. So

$$\delta\left((f_p^{(I)}/I!)\tilde{p}^{(I)}\right) = m_\mu t^\mu \mu_k^{\|I\|} c_{I,0}.$$

For any $J \in \mathbb{N}^{n+1} \backslash \{0\}$, $J \notin \mathcal{A}$, we have

$$v((f_p^{(J)}/J!)\tilde{p}^{(J)}) = v_p^{(J)} + |J|\mu > \mu.$$

Thus we obtain $f_{p+\tilde{p}} = f_p + m_\mu t^\mu(\pi(\mu_k) + \epsilon)$ for some $\epsilon \in \mathbb{K}_r^\prec$. Then the dichotomy of the lemma follows from the ultrametric triangular inequality for the valuation v. For the second case, p_0 is the shortest initial part such that $\mu^{(0)} \neq v_{p_0}$ and $\mu_k^{(0)} \notin \{\rho_1, \dots, \rho_m\}$. Then for any successor part $\tilde{p}$ of p_0, we have

$$v_{p_0+\tilde{p}} = v_{p_0 + m_0 t^{\mu^{(0)}}} = \min\{v_{p_0}, \mu^{(0)}\}.$$

□

Returning to the proof of Lemma 5.1, in case $k = r$, we set

$$\mathcal{R} = \langle \text{Supp } F \rangle + \langle (0, \dots, 0, \rho_1), \dots, (0, \dots, 0, \rho_m) \rangle + \mathcal{T}_r.$$

Then, from Proposition 3.13, we remark that

$$v_p \in \text{Supp } F + \mathcal{T}_r + \langle \text{Supp } p \rangle$$

for any initial part p of $y_{0,r}$. So using the relations $\mu = v_p$ or $\mu_r \in \{\rho_1, \dots, \rho_m\}$, by a straightforward transfinite induction, we obtain that Supp $y_0 \subset \mathcal{R}$ in

case (1) of the preceding lemma, respectively Supp $p_0 \subset \mathcal{R}$ in case (2). The subcase (2)(a) means exactly that y_0 stabilizes on F with initial part $p_0 + m_0 t^{\mu^{(0)}}$.

In case $k \in \{r-1, \dots, 1\}$, we consider $y_0 = y_{0,k} + \cdots + y_{0,1}$ and a differential series $F(y, \dots, D_k^n y)$ with Weierstrass order 1. We suppose that the Lemma 5.1 holds for any $l \in \{r, \dots, k+1\}$. According to the dichotomy in the Lemma 5.4, we show by transfinite induction that Supp $y_{0,k}$ (respectively Supp p_0) is included in an additive sub-semigroup $\mathcal{R}_k$ of $\Gamma_{>0}$ of type

$$\mathcal{R}_k = \langle \text{Supp } F \rangle + \mathcal{T}_k + \cdots + \mathcal{T}_r + \mathcal{R}_{1,k} + \cdots + \mathcal{R}_{m,k}$$

where m is the number of positive roots of the indicial polynomial associated to F (see Definition 5.3), and the $\mathcal{R}_{i,k}$'s are additive sub-semigroups of $\Gamma_{>0}$ obtained from Supp F, $\mathcal{T}_k, \dots, \mathcal{T}_r$ by a finite number of elementary transformations.

Indeed, let us consider a proper initial part p of $y_{0,k}$ (respectively p_0) with successor term $m_\mu t^\mu$, $\mu = (0, \dots, 0, \mu_k, \dots, \mu_r)$. We suppose that S is included in an additive semi-group $\mathcal{R}$ that contains $\langle \text{Supp } F \rangle$, $\mathcal{T}_k, \dots, \mathcal{T}_r$. From Lemma 5.4, there are two cases. Either $\mu = v_p$. But from Proposition 3.13 we have

$$v_p \in \text{Supp } f_p \subset \text{Supp } F + \mathcal{T}_k + \langle \text{Supp } p \rangle$$

and $\langle \text{Supp } p \rangle \subset \mathcal{R}$ as well as Supp F and $\mathcal{T}_k$. So $\mu \in \mathcal{R}$.

Or $\mu_k \in \{\rho_1, \dots, \rho_m\}$. For instance, $\mu_k = \rho_h$ for some $h \in \{1, \dots, m\}$ fixed. We set $S_h = \{\mu \in \text{Supp } y_{0,k} \mid \mu_k = \rho_h\}$, $\sigma^{(h)} = \min S_h$,

$$S'_h = \{\mu \in \text{Supp } y_{0,k} \mid \mu < \sigma^{(h)}\}$$

and $S''_h = S'_h \cup \{\sigma^{(h)}\}$. So $S'_h \subset S \subset \mathcal{R}$ and $S''_h \subset \mathcal{R} + \mathbb{N}\sigma^{(h)}$.

Let us show that there exists an additive sub-semigroup $\mathcal{R}_{h,k}$ of $\Gamma_{>0}$ obtained from $\mathcal{R}$, Supp F, $\mathcal{T}_k, \dots, \mathcal{T}_r$ by a finite number of elementary transformations such that $S_h \subset \mathcal{R}_{h,k}$. We note that

$$\sigma^{(h)} = (0, \dots, 0, \rho_h, 0, \dots, 0, \sigma_{l,h}, \dots, \sigma_{r,h})$$

for some $l \in \{k+1, \dots, r\}$ with $\sigma_{l,h} \neq 0$. So

$$p_{S_h \setminus \{\sigma^{(h)}\}} / t^{\sigma^{(h)}} = z_{0,l} + \cdots + z_{0,k+1} \in \mathbb{K}_r^{\prec}$$

with initial part $z_{0,l} \in \mathbb{K}_{r,l}$. Then we reduce to a differential series $G(z, \ldots, D_l^n z)$ together with the series $z_{0,l} \in \mathbb{K}_{r,l}$ so as to apply the induction hypothesis.

The calculations we make are represented symbolically in cases $r = 2$ and $k = 1$ by the following picture, in which the black points represent elements of the support of $y_{0,k}$. The necessary changes of derivation are also mentioned.

(1) We perform the *additive conjugation* $y = \tilde{y} + p_{S_h''}$ in the differential series $F(y, \ldots, D_k^n y)$. From Proposition 4.2 and Remark 4.3, we obtain a new series

$$F_{p_{S_h''}}(\tilde{y}, \ldots, D_k^n \tilde{y}) = \textstyle\sum_{I \in \mathbb{N}^{n+1}} c_I' \tilde{y}^{(I)}$$

with Weierstrass order 1. We have $c_I' = f_{p_{S_h''}}^{(I)}/I!$ and in particular, for any $I \in \mathcal{A}$, $v(c_I') = v(c_I) = \underline{0}$. Moreover:

$$\text{Supp } F_{p_{S_h''}} \subset \text{Supp } F + \mathcal{T}_k + \langle S_h'' \rangle \subset \mathcal{R} + \mathbb{N}\sigma^{(h)}.$$

The associated series is $\tilde{y}_0 = y_0 - p_{S_h''}$ with initial part $\tilde{y}_{0,k} = y_{0,k} - p_{S_h''} \in \mathbb{K}_{r,k}$.

(2) We perform in $F_{p_{S_h''}}$ the *multiplicative conjugation* $y_1 = t^{\sigma^{(h)}} z$. From Proposition 4.5, we obtain a differential series

$$\hat{F}(z, \ldots, D_k^n z) = \textstyle\sum_{J \in \mathbb{N}^{n+1}} \hat{c}_J z^{(J)}$$

with

$$v(\hat{c}_J) \geqslant \min\{v(c_I') + v((t^{\sigma^{(h)}})^{(K)}) \ \mid \ |I| = |J| = |K|, \ \|I\| = \|K\| + \|J\|, \ J, K \leqslant_{antilex} I\}$$

and

$$(t^{\sigma^{(h)}})^{(K)} \sim \rho_h^{\|K\|} t^{|K|\sigma^{(h)}} = \rho_h^{\|K\|} t^{|I|\sigma^{(h)}}.$$

So

$$v(\hat{c}_J) \geqslant \min\{v(c_I') + |I|\sigma^{(h)} \ \mid \ |I| = |J|, \ J \leqslant_{antilex} I\}.$$

According to the Remark 4.7, we have $v(\hat{c}_{I_0}) = \sigma^{(h)}$ where I_0 is the greatest multi-index from $\mathcal{A}$ for anti-lexicographical ordering (in particular $|I_0| = 1$). Moreover, from Proposition 4.5,

$$\text{Supp } \hat{F} \subset \text{Supp } F_{p_{S''_h}} + \mathcal{T}_k + \mathbb{N}\sigma^{(h)} \subset \mathcal{R} + \mathbb{N}\sigma^{(h)}.$$

The associated series is $z_0 = \tilde{y}_0/t^{\sigma^{(h)}}$, which has $z_{0,l} \in \mathbb{K}_{r,l}$ as initial part.

(3) We perform the *change of derivation*

$$\hat{F}(z, \dots, D_k^n z) = \tilde{F}_{k,l}(z, \dots, D_l^n z) = \textstyle\sum_{L \in \mathbb{N}^{n+1}} \tilde{c}_L z^{(L)}$$

as in the Proposition 4.11. From the Remark 4.10, the terms $\hat{c}_J z^{(J)}$ with $v(\hat{c}_J) = \sigma^{(h)}$ provide at least one term $\tilde{c}_L z^{(L)}$ with $|L| = 1$ and

$$v(\tilde{c}_L) = (0, \dots, 0, \rho_h, \chi_{k+1}, \dots, \chi_r)$$

for some reals $\chi_{k+1}, \dots, \chi_r$ ($L = J_0$ being the greatest element for anti-lexicographical ordering among these J's is such a good candidate). Then we observe that

$$\min\{v(\tilde{c}_L), L \in \mathbb{N}^{n+1}, |L| \geqslant 1\} = v(\tilde{c}_{L_0}) = (0, \dots, 0, \rho_h, \chi_{k+1}^{(0)}, \dots, \chi_r^{(0)})$$

for some L_0 with $|L_0| = 1$. We set $v^{(0)} = v(\tilde{c}_{L_0})$ and there are two cases. Either $v(\tilde{c}_0) \leqslant v^{(0)}$. Then for any initial part $\tilde{p}$ of $z_{0,l}$, we have $v(\tilde{f}_{\tilde{p}}) = v(\tilde{c}_0)$. It means that z_0 stabilizes on $\tilde{F}_{k,l}$ with initial part 0, and equivalently y_0 stabilizes on F with initial part $p_{S''_h}$.
Or $v(\tilde{c}_0) > v^{(0)}$. Then we divide $\tilde{F}_{k,l}(z, \dots, D_l^n z)$ by $t^{v^{(0)}}$ and we obtain a differential series $G(z, \dots, D_l^n z)$ with Weierstrass order 1. Moreover

$$\text{Supp } G = \text{Supp } \tilde{F}_{k,l} - v^{(0)} \subset (\mathcal{R} + \mathbb{N}\sigma^{(h)} - v^{(0)})_{\geqslant \underline{0}}.$$

The associated series is

$$z_0 = \tilde{y}_0/t^{\sigma^{(h)}} = (y_0 - p_{S''_h})/t^{\sigma^{(h)}} \in \mathbb{K}_r^{\prec}$$

that has initial part $z_{0,l} \in \mathbb{K}_{r,l}$. For any initial part S' of Supp $z_{0,l}$, we denote by $q = q_{S'}$ the corresponding initial part of $z_{0,l}$ and $\tilde{v}_q = v(G(q))$.

From the induction hypothesis, there exists a well-ordered subset $\mathcal{R}'$ of $\Gamma_{>0}$ obtained from Supp G, $\mathcal{T}_l, \dots, \mathcal{T}_r$ by finitely many elementary transformations such that:

(a) : either Supp $z_{0,l} \subset \mathcal{R}'$;

(b) : or z_0 stabilizes on G with initial part $q_{S'_0}$ where S'_0 is a proper initial segment of Supp $z_{0,l}$ which is included in $\mathcal{R}'$.

We note that $\mathcal{R}'$ is obtained from Supp G, $\mathcal{T}_l, \ldots, \mathcal{T}_r$ by a finite number of elementary transformations and

$$\text{Supp } G \subset (\mathcal{R} + \mathbb{N}\sigma^{(h)} - v^{(0)})_{\geqslant \underline{0}}.$$

So, in case (ii), $S'_0 \subset \mathcal{R}'$ where $\mathcal{R}'$ is obtained from $\mathcal{R}$, Supp F, $\mathcal{T}_k, \ldots, \mathcal{T}_r$ by a finite number of elementary transformations. Moreover, we observe that any proper initial segment S' of Supp $z_{0,l}$ corresponds to a proper initial segment $S' + \sigma^{(h)}$ of Supp $y_{1,k}$, and so to a proper initial segment $S''_h \cup (S' + \sigma^{(h)})$ of Supp $y_{0,k}$. Thus, for any initial segment S' of Supp $z_{0,l}$,

$$\tilde{v}_q = v(G(q_{S'})) = v(\tilde{F}_{k,l}(q_{S'})) + v^{(0)}.$$

But

$$v(\tilde{F}_{k,l}(q_{S'})) = v(\hat{F}(q_{S'})) = v(F p_{S''_h}(p_{S'+\sigma^{(h)}}))$$

and

$$v(F_{p_{S''_h}}(p_{S'+\sigma^{(h)}})) = v(f_{p_{S''_h \cup (S'+\sigma^{(h)})}}) = v_{p_{S''_h \cup (S'+\sigma^{(h)})}}.$$

So $\tilde{v}_q = v_{p_{S''_h \cup (S'+\sigma^{(h)})}} + v^{(0)}$. It means that, in the case (b), y_0 stabilizes on F with initial segment p_{S_0} where $S_0 = S''_h \cup (S'_0 + \sigma^{(h)})$. It suffices then to set

$$\mathcal{R}_{h,k} = \langle \mathcal{R}' \rangle + \mathbb{N}\sigma^{(h)}.$$

In case (a), Supp $z_{0,l} \subset \mathcal{R}'$. So

$$\text{Supp } (t^{\sigma^{(h)}} z_{0,l}) \subset \mathcal{R}' + \sigma^{(h)}.$$

We set $\tilde{S}'_h = S'_h \cup$ Supp $(t^{\sigma^{(h)}} z_{0,l})$, $\tilde{\sigma}^{(h)}$ its successor element in Supp $y_{0,k}$ and $\tilde{S}''_h = \tilde{\sigma}^{(h)} \cup \{\tilde{\sigma}^{(h)}\}$. If we denote $\tilde{z}_0 = (y_0 - p_{\tilde{S}''_h})/t^{\tilde{\sigma}^{(h)}}$, then $\tilde{z}_0 \in \mathbb{K}_r^{\prec}$ with initial part $\tilde{z}_{0,l'} \in \mathbb{K}_{r,l'}$ for some $l' \in \{k+1, \ldots, l-1\}$. Therefore we repeat the previous process, starting with the additive conjugation $y = y_1 + p_{\tilde{S}''_h}$.

There are at most as many steps as

$$z_0 = p_{S_h \setminus \{\sigma^{(h)}\}}/t^{\sigma^{(h)}} = z_{0,l} + \cdots + z_{0,k+1}$$

has elements $z_{0,i} \in \mathbb{K}_{r,i}$ in its analysis. If we denote by j this number of steps, in the case where there is no stabilization, we obtain for any $i = 0 \cdots j-1$,

$$\text{Supp } (t^{\tilde{\sigma}^{(i)}_{(h)}} \tilde{z}^{(i)}_{0,l^{(i)}}) \subset \mathcal{R}^{(i+1)}$$

where $\mathcal{R}^{(i+1)}$ is a well-ordered subset of $\Gamma_{>0}$ obtained from $\mathcal{R}$, Supp F, and $\mathcal{T}_k, \dots, \mathcal{T}_r$ by finitely many elementary transformations. But

$$S_h = \bigcup_{i=0}^{j-1} \text{Supp } (t^{\tilde{\sigma}^{(i)}_{(h)}} \tilde{z}^{(i)}_{0,l^{(i)}}).$$

So $S_h \subset \bigcup_{k=1}^{j} \mathcal{R}^{(k)}$. Then we set

$$\mathcal{R}_{h,k} = \left\langle \bigcup_{k=1}^{j} \mathcal{R}^{(k)} \right\rangle$$

which is an additive sub-semigroup of $\Gamma_{>0}$ obtained from $\mathcal{R}$, Supp F, $\mathcal{T}_k, \dots, \mathcal{T}_r$ by finitely many elementary transformations.

5.3. The case $w > 1$

We consider a differential series $F(y, \dots, D^n_k y)$ as in (3.3) with Weierstrass order $w > 1$, together with a series $y_0 = y_{0,k} + \cdots + y_{0,1} \in \mathbb{K}^{\prec}_r$. To prove the Lemma 5.1, we proceed in three steps.

First, since $w > 1$, there exists at least one multi-index $I \in \mathbb{N}^{n+1}$ with $|I| = w-1$ such that $F^{(I)}$ has Weierstrass order 1. Thus we apply the induction hypothesis to these differential series, and obtain the desired finiteness property for (at least) some proper initial part $p_{S_{w-1}}$ of $y_{0,k}$. We denote

$$\mu^{(w-1)} = (0, \dots, 0, \mu^{(w-1)}_k, \dots, \mu^{(w-1)}_r) = v(y_0 - p_{S_{w-1}})$$

and $S''_{w-1} = S_{w-1} \cup \{\mu^{(w-1)}\}$. Then we note that $(y_0 - p_{S''_{w-1}})/t^{\mu^{(w-1)}}$ has an initial part $z_{0,l} + \cdots + z_{0,k}$ for some $l \in \{r, \dots, k\}$.

In the case where $l \in \{r, \dots, k+1\}$, the second step is devoted to determine the support of $z_{0,l} + \cdots + z_{0,k+1}$. We reduce $F(y, \dots, D^n_k y)$ to another series $G(z, \dots, D^n_l z)$ with Weierstrass order $\tilde{w}$ together with a series associated z_0 with initial part $z_{0,l} \in \mathbb{K}_{r,l}$ such that $(\tilde{w}, l) < (w, k)$, by means of three successive transformations.

In the third step, we check the support of $z_{0,k}$ reducing to a differential series $\tilde{G}(\tilde{z}, \dots, D^n_k \tilde{z})$ of Weierstrass order lower than w.

First step. We need some new notations.

NOTATION 5.5. — We denote:

- $\mathcal{A} = \{I \in \mathbb{N}^{n+1} \ \mid \ |I| = w, v(c_I) = \underline{0}\}$;
- for any $I \in \mathcal{A},\ c_{I,0} = \delta(c_I)$;
- $\mathcal{A}_{w-1} = \{I \in \mathbb{N}^{n+1} \ \mid \ \exists J \in \mathbb{N}^{n+1},\ |J| = 1,\ I + J \in \mathcal{A}\}$. So we have $|I| = w - 1$ for all $I \in \mathcal{A}_{w-1}$.
- For any given $I \in \mathcal{A}_{w-1}$, we set:
  - $\mathcal{A}_I = \{J \in \mathbb{N}^{n+1} \ \mid \ |J| = 1,\ I + J \in \mathcal{A}\}$;
  - $\pi_I(X) = \sum_{J \in \mathcal{A}_I} c_{I+J,0} X^{\|J\|}$ (then we have $\deg \pi_I \leqslant n$);
  - $\{\rho_1^{(I)}, \ldots, \rho_{m_I}^{(I)}\}$ the set of the positive roots of π_I.

Then we apply the induction hypothesis and the Lemma 5.4 to the differential series $F^{(I)}(y, \ldots, D_k^n y)$, $I \in \mathcal{A}_{w-1}$ together with the series y_0. We obtain two cases:

- either there exists $I_0 \in \mathcal{A}_{w-1}$ such that we are in the first case of Lemma 5.1: there exists a well-ordered subset $\mathcal{R}_{I_0}$ of $\Gamma_{>0}$ obtained from Supp $F^{(I_0)}$, $\mathcal{T}_1, \ldots, \mathcal{T}_r$ by a finite number of elementary transformations such that Supp $y_{0,k} \subset \mathcal{R}_{I_0}$. It suffices then to observe that Supp $F^{(I_0)} \subset$ Supp F;
- or for any $I \in \mathcal{A}_{w-1}$, y_0 stabilizes on $F^{(I)}$ with an initial part $p_{S_I \cup \{\mu^{(I)}\}}$, S_I being an initial segment of Supp $y_{0,k}$ with successor element $\mu^{(I)}$. Moreover the value of stabilization is at most equal to $\mu^{(I)}$.

In this last case, we set $S_{w-1} = \bigcup_{I \in \mathcal{A}_{w-1}} S_0^{(I)}$, $\mu^{(w-1)}$ its successor element and $S''_{w-1} = S_{w-1} \cup \{\mu^{(w-1)}\}$. So there exists $I_0 \in \mathcal{A}_{w-1}$ such that $S_{w-1} = S_0^{(I)} \subset \mathcal{R}_{I_0}$. We denote

$$\mathcal{R}_1 = \left\langle \mathcal{R}_{I_0} \cup \{\mu^{(w-1)}\} \right\rangle = \langle \mathcal{R}_{I_0} \rangle + \mathbb{N}\mu^{(w-1)}.$$

So $S''_{w-1} \subset \mathcal{R}_1$. For any $I \in \mathcal{A}_{w-1}$, we denote $\hat{v}^{(I)}$ the value of stabilization of $v_p^{(I)}$, and so $\hat{v}^{(I)} = v^{(I)}_{p_{S''_{w-1}}} \leqslant \mu^{(w-1)}$. We set $\hat{v}_{w-1} = \min\{\hat{v}^{(I)},\ I \in \mathcal{A}_{w-1}\}$. Thus $\hat{v}_{w-1} \leqslant \mu^{(w-1)}$.

We examine now the valuations of the differential series $F^{(I)}(y, \ldots, D_k^n y)$ for $I \notin \mathcal{A}_{w-1}$, $|I| = w-1$. From the Taylor expansion formula (3.6), for any successor segment $\tilde{S}$ of S_{w-1},

$$f^{(I)}_{p_{S_{w-1}}+p_{\tilde{S}}} = f^{(I)}_{p_{S_{w-1}}} + \sum_{|J|\geqslant 1}(f^{(I+J)}_{p_{S_{w-1}}}/J!)p^{(J)}_{\tilde{S}}.$$

But for any multi-index J, $v(p^{(J)}_{\tilde{S}}) = |J|v(p_{\tilde{S}}) = |J|\mu^{(w-1)}$, and for any J with $|J| = 1$, $v(f^{(I+J)}_{p_{S_{w-1}}}) > \underline{0}$ since $I \notin \mathcal{A}_{w-1}$. Then there are two possibilities:

(i) : either $v^{(I)}_{p_{S''_{w-1}}} > \mu^{(w-1)}$. So, from the ultrametric triangular inequality, $v^{(I)}_{p_{S''_{w-1}}+p_{\tilde{S}}} > \mu^{(w-1)}$ for any successor segment $\tilde{S}$ of S''_{w-1};

(ii) : or $v^{(I)}_{p_{S''_{w-1}}} \leqslant \mu^{(w-1)}$ which implies that

$$v^{(I)}_{p_{S''_{w-1}}+p_{\tilde{S}}} = v^{(I)}_{p_{S''_{w-1}}} \leqslant \mu^{(w-1)}.$$

Then we denote by $\hat{v}^{(I)}$ this stabilized valuation, by $\mathcal{B}_{w-1}$ the set of multi-indexes I for whom that situation arises, and $\tilde{v}_{w-1} = \min\{\hat{v}^{(I)} \mid I \in \mathcal{A}_{w-1} \cup \mathcal{B}_{w-1}\}$. Thus $\tilde{v}_{w-1} \leqslant \mu^{(w-1)}$.

Second step. **(1)** We perform the *additive conjugation* $y = \tilde{y} + p_{S''_{w-1}}$. From Proposition 4.2, we obtain a differential series

$$F_{p_{S''_{w-1}}}(\tilde{y}, \dots, D^n_k\tilde{y}) = \sum_{I\in\mathbb{N}^{n+1}} c'_I\tilde{y}^{(I)}$$

where $c'_I = f^{(I)}_{p_{S_{w-1}}}/I!$. In particular, there exists $I_0 \in \mathbb{N}^{n+1}$ with $|I_0| = w-1$ such that

$$v(c'_{I_0}) = \tilde{v}_{w-1} \leqslant \mu^{(w-1)}.$$

From the Remark 4.3, $F_{p_{S''_{w-1}}}(\tilde{y}, \dots, D^n_k\tilde{y})$ has also Weierstrass order w. Moreover

$$\text{Supp } F_{p_{S''_{w-1}}} \subset \text{Supp } F + \langle S''_{w-1}\rangle + \mathcal{T}_k \subset \text{Supp } F + \mathcal{T}_k + \mathcal{R}_1.$$

The associated series is $\tilde{y}_0 = y_0 - p_{S''_{w-1}}$ with initial part $\tilde{y}_{0,k} = y_{0,k} - p_{S''_{w-1}}$.

(2) We perform the *multiplicative conjugation* $\tilde{y} = t^{\mu^{(w-1)}}z$ in $F_{p_{S''_{w-1}}}$. From the Proposition 4.5, we obtain a differential series

$$\hat{F}(z, \dots, D^n_k z) = \sum_{J\in\mathbb{N}^{n+1}} \hat{c}_J z^{(J)}$$

with

$$\text{Supp } \hat{F} \subset \text{Supp } F_{p_{S''_{w-1}}} + \mathbb{N}\mu^{(w-1)} + \mathcal{T}_k \subset \text{Supp } F + \mathcal{T}_k + \mathcal{R}_1$$

(by Proposition 4.5, since $\mathbb{N}\mu^{(w-1)} + \mathcal{R}_1 = \mathcal{R}_1$). Moreover

$$v(\hat{c}_J) \geqslant \min\{v(c'_I) + v((t^{\mu^{(w-1)}})^{(K)})\}$$

where the minimum is taken over multi-indexes $I \in \mathbb{N}^{n+1}$ such that $|I| = |J| = |K|$, $\|I\| = \|K\| + \|J\|$ and $J, K \leqslant_{antilex} I$. Furthermore

$$(t^{\mu^{(w-1)}})^{(K)} \sim (\mu_k^{(w-1)})^{\|K\|} t^{|I|\mu^{(w-1)}}.$$

In particular, by the Remark 4.7, $v(\hat{c}_{J_0}) = \tilde{v}_{w-1} + (w-1)\mu^{(w-1)}$ where J_0 is the greatest multi-index for the anti-lexicographical ordering among the J's such that $v(c'_J) = \tilde{v}_{w-1}$. So $v(\hat{c}_{J_0}) \leqslant w\mu^{(w-1)}$. Moreover, for any J with $|J| \geqslant w$,

$$v(\hat{c}_J) \geqslant |J|\mu^{(w-1)} \geqslant w\mu^{(w-1)}.$$

So there exists at least one multi-index I with $|I| < w$ such that

$$v(\hat{c}_I) = \min\{v(\hat{c}_J) \ \mid \ J \in \mathbb{N}^{n+1}\}.$$

We denote v_{min} this minimal valuation and I_0 some multi-index with $|I_0|$ as low as possible such that $v(\hat{c}_{I_0}) = v_{min}$. Then we observe that, dividing $\hat{F}(z, \ldots, D_k^n z)$ by $t^{v_{min}}$, we obtain a differential series $\tilde{F}(z, \ldots, D_k^n z)$ with Weierstrass order equal to $|I_0|$, and so lower than w. The associated series is $z_0 = \tilde{y}_0 / t^{\mu^{(w-1)}} \in \mathbb{K}_r^{\prec}$.

If $k \in \{r-1, \ldots, 1\}$, then z_0 has *a priori* an initial part

$$\tilde{y}_{0,k}/\mu^{(w-1)} = z_{0,l} + \cdots + z_{0,k}$$

for some $l \in \{r, \ldots, k+1\}$. We set

$$S_1 = \{\mu \in \text{Supp } y_{0,k} \ \mid \ \mu_k = \mu_k^{(w-1)}, \mu \geqslant \mu^{(w-1)}\}$$

(so S_1 is a successor segment of S_{w-1}). We remark that

$$S_1 = (\textstyle\bigcup_{i \in \{k+1, \ldots, l\}} \text{Supp } z_{0,i}) + \mu^{(w-1)}.$$

Now we determine Supp $z_{0,l}$.

(3) We perform the *change of derivation*

$$\hat{F}(z,\dots,D_k^n z) = \tilde{F}_{k,l}(z,\dots,D_l^n z)$$

as in Proposition 4.11. Then we show as before (proof of Lemma 5.1 in case $w=1$) that there are two cases. Either z_0 stabilizes on $\tilde{F}_{k,l}$ with initial part 0, which means that y_0 stabilizes on F with initial part $p_{S''_{w-1}}$.

Or dividing $\tilde{F}_{k,l}$ by $t^{v^{(0)}}$ where $v^{(0)}$ is the minimum of the valuation of the coefficients of $\tilde{F}_{k,l}$, we obtain a differential series $G(z,\dots,D_l^n z)$ with Weierstrass order $\tilde{w} \geqslant 1$. From Remarks 4.7 and 4.10, $\tilde{w} \leqslant w$. Moreover from Proposition 4.14,

$$\text{Supp } \tilde{F}_{k,l} \subset \text{Supp } \hat{F} + \mathcal{T}_k \subset \text{Supp } F + \mathcal{T}_k + \mathcal{R}_1.$$

Thus,

$$\text{Supp } G \subset (\text{Supp } F + \mathcal{T}_k + \mathcal{R}_1)_{\geqslant v^{(0)}} - v^{(0)}.$$

We apply the induction hypothesis to $G(z,\dots,D_l^n z)$ together with z_0 that has initial part $z_{0,l}$. There exists a well-ordered subset $\tilde{\mathcal{R}}_2$ of $\Gamma_{>0}$ obtained from Supp G, $\mathcal{T}_l,\dots,\mathcal{T}_r$ by a finite number of elementary transformations such that:

- either the exponents of $z_{0,l}$ belong to $\tilde{\mathcal{R}}_2$;
- or the series z_0 stabilizes on G with initial part q_0 which is also a proper initial part of $z_{0,l}$, and the support of q_0 is included in $\tilde{\mathcal{R}}_l$. Then it means that y_0 stabilizes on F with initial part $p_0 = p_{S''_{w-1}} + t^{\mu^{(w-1)}} q_0$ and that Supp $p_0 \subset \mathcal{R}_1 + (\tilde{\mathcal{R}}_2 + \mu^{(w-1)})$, which is obtained from Supp F, $\mathcal{T}_k,\dots,\mathcal{T}_r$ by a finite number of elementary transformations as desired.

If the preceding first case holds, we perform the additive conjugation $z = \tilde{z} + t^{\mu^{(w-1)}} z_{0,l}$ in the differential series $\hat{F}(z,\dots,D_k^n z)$ defined above. We obtain a differential series $\hat{\hat{F}}(z,\dots,D_k^n z)$ with coefficients that have also as minimal valuation v_{min} for some multi-index I_0 (the least multi-index for anti-lexicographical ordering among the coefficients having valuation v_{min}: see Remark 4.3). Moreover

$$\text{Supp } \hat{\hat{F}} \subset \text{Supp } \hat{F} + \left\langle \tilde{\mathcal{R}}_2 + \mu^{(w-1)} \right\rangle + \mathcal{T}_k$$

(see Proposition 4.2) which is obtained from Supp F, $\mathcal{T}_k, \dots, \mathcal{T}_r$ by a finite number of elementary transformations. The associated series is $\hat{z}_0 = z_0 - z_{0,l}$, with initial part z_{0,l_1} for some $l_1 \in \{l, \dots, k\}$. If $l_1 \neq k$, we resume the preceding arguments.

Thus, in the case where there is no stabilization, we prove gradually that for any $i \in \{l, \dots, k+1\}$, Supp $z_{0,i} \subset \mathcal{R}_2$ with $\mathcal{R}_2$ that is a well-ordered subset of $\Gamma_{>0}$ obtained from Supp F, $\mathcal{T}_k, \dots, \mathcal{T}_r$ by finitely many elementary transformations. Thus

$$S_1 = \text{Supp } (z_{0,l} + \cdots + z_{0,k+1}) + \mu^{(w-1)}$$

is included in $\mathcal{R}_2 + \mu^{(w-1)}$.

Third step. It remains to examine the support of $z_{0,k}$ so as to obtain the desired property for the set

$$\{\mu \in \text{Supp } y_{0,k} \ \mid \ \mu_k > \mu_k^{(w-1)}\}.$$

We return to the differential series $\tilde{F} = \hat{F}/t^{v_{\min}}$ defined in the second step, which has Weierstrass order $|I_0| < w$. Then we perform the additive conjugation

$$z = \tilde{z} + (z_{0,l} + \cdots + z_{0,k+1}).$$

By the Proposition 4.2 and the Remark 4.3, we obtain a new differential series $\tilde{G}(\tilde{z}, \dots, D_k^n \tilde{z})$ with the same Weierstrass order $|I_0| < w$, together with a series $\tilde{z}_0 = z_0 - (z_{0,l} + \cdots + z_{0,k+1})$ that has initial part $z_{0,k} \in \mathbb{K}_{r,k}$. Moreover,

$$\begin{gathered}\text{Supp } \tilde{G} \subset \text{Supp } \tilde{F} + \left\langle S_1 - \mu^{(w-1)} \right\rangle + \mathcal{T}_k \subset \\ (\text{Supp } F + \mathcal{T}_k + \mathcal{R}_1)_{\geqslant v_{\min}} - v_{\min} + \langle \mathcal{R}_2 \rangle + \mathcal{T}_k.\end{gathered}$$

Then we apply the induction hypothesis to $\tilde{G}(\tilde{z}, \dots, D_k^n \tilde{z})$ together with $\tilde{z}_0$. There exists a well-ordered subset $\mathcal{R}_3$ of $\Gamma_{>0}$ obtained from Supp $\tilde{G}$, $\mathcal{T}_k, \dots, \mathcal{T}_r$ by finitely many elementary transformations such that:

– either the exponents of $z_{0,k}$ belong to $\mathcal{R}_3$. So if we set

$$\mathcal{R} = \mathcal{R}_1 + [(\mathcal{R}_2 + \mathcal{R}_3) + \mu^{(w-1)}],$$

then Supp $y_{0,k} \subset \mathcal{R}$ which is obtained from Supp F, $\mathcal{T}_k, \dots, \mathcal{T}_r$ by a finite number of elementary transformations as desired;

– or the series $\tilde{z}_0$ stabilizes on $\tilde{G}$ with initial part q_0 which is also a proper initial part of $z_{0,k}$, and the support of q_0 is included in $\mathcal{R}_3$. In this case, it means that y_0 stabilizes on F with initial part $p_0 = p_{S''_{w-1}} + t^{\mu^{(w-1)}} q_0$ which is initial part of $y_{0,k}$. Moreover, Supp $p_0 \subset \mathcal{R}$. This concludes the proof of the Lemma 5.1, and therefore the one of the Theorem 2.14.

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