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Maximal subextensions of plurisubharmonic functions

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Dedicated to Professor Nguyen Thanh Van
on the occasion of his retirement

Abstract. — In our earlier [CKZ], we proved that any plurisubharmonic function on a bounded hyperconvex domain in $\mathbb{C}^n$ with zero boundary values in a quite general sense, admits a plurisubharmonic subextension to a larger hyperconvex domain. Here we study important properties of its maximal subextension and give informations on its Monge-Ampère measure. More generally, given a quasi-plurisubharmonic function φ on a given quasi-hyperconvex domain $D \subset X$ of a compact Kähler manifold (X, ω), with well defined Monge-Ampère measure such that $\int_D (\omega + dd^c \varphi)^n \leqslant \int_X \omega^n$, we prove that φ admits a global quasi-plurisubharmonic subextension $\tilde{\varphi}$ to the whole manifold X. If moreover $(\omega + dd^c \varphi)^n$ puts no mass on pluripolar sets of D, the maximal subextension is shown to have a well defined global Monge-Ampère measure on X. Moreover we give a good control on the weigthed energy of the subextension in terms of the weigthed energy of the original function. Finally we provide an exemple in $\mathbb{P}^2$ which shows that in general the maximal subextension do not have a well defined Monge-Ampère measure on $\mathbb{P}^2$ if the original function concentrates some mass in an analytic disc.

Résumé. — Dans notre travail précédent paper [CKZ], nous avions démontré que toute fonction plurisousharmonique sur un ouvert hyperconvexe borné de $\mathbb{C}^n$ ayant des valeurs au bord nulles en un sens assez général possède une sous-extension plurisousharmonic dans un dmaine hyperconvex plus grand. D'une façon plus générale, étant donnée une fonction quasi-plurisousharmonique φ sur un domaine quasi-hyperconvexe

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$D \subset X$ d'une variété kählerienne compacte (X, ω), de valeurs au bord nulle en un sens généralisé et ayant une mesure de Monge-Ampère bien définie sur D et vérifiant $\int_D (\omega + dd^c\varphi)^n \leqslant \int_X \omega^n$, nous démontrons que φ admet une sousextension $\tilde{\varphi}$ à la variété X toute entière. Si de plus $(\omega + dd^c\varphi)^n$ ne charge pas les ensembles pluripolaires de D, la sousextension maximale possède une mesure de Monge-Ampère globale bien définie sur X dont nous étudions la mesure de Monge-Ampère. De plus nous donnons un contrôle précis en terme d'energie de Monge-Ampère pondérée de la sousextension maximale en fonction de l'energie pondérée de la donnée φ. Enfin nous donnons un exemple dans $\mathbb{P}^2$ qui montre qu'en général la sousextension maximale n'a pas une mesure de Monge-Ampère globale bien définie si la mesure de Monge-Ampère de la fonction donné concentre de la masse sur un disque analytique.

1. Introduction

This is the sequel to our earlier paper [CKZ]. There we proved that given a plurisubharmonic function φ from the class $\mathcal{F}(\Omega)$ (see the next section for definitions) in a hyperconvex domain $\Omega \Subset \mathbb{C}^n$ one can find its maximal subextension φ which is plurisubharmonic in $\mathbb{C}^n$ and which has logarithmic growth at infinity. If, in addition, the Monge-Ampère measure of φ vanishes on pluripolar sets then the Monge-Ampère of φ is a well defined positive measure on $\mathbb{C}^n$ in the sense that it is the weak limit of the sequence of positive measures $(dd^c\varphi^j)^n$ for any sequence of continuous plurisubharmonic functions $\varphi^j \downarrow \varphi$ having the same rate of growth at infinity as φ. In Subsection 4.3 of this article we complete this picture studying in more detail the Monge-Ampère measures of maximal subextensions $\tilde{\varphi}$. If the sublevel sets of those subextensions are bounded then such a measure can be split into μ_1, dominated by $(dd^c\varphi)^n$ and essentially supported on the contact set where $\varphi = \tilde{\varphi}$, and μ_2 living on the set $\partial\{\tilde{\varphi} < 0\}$. In general the maximal global subextension of a function from the class $\mathcal{F}(\Omega)$ may not have well defined Monge-Ampère measure. It is the case for generic multipole Green function as we show in the last section.

Now, a subextension of a plurisubharmonic function from a domain D in $\mathbb{C}^n$ to a function defined in the whole space and of logarithmic growth can be viewed upon as a subextension of an ω-plurisubharmonic function (with ω a multiple of the Fubini-Study form) from a subset of $\mathbb{C}P^n$ to the whole manifold. Here the domain D is special since there exists a potential for ω in D. If, for instance, $D \subset \mathbb{C}P^n$ contains an algebraic set of positive dimension then there are no strictly plurisubharmonic functions in D. Thus on a compact Kähler manifold X we face a more general problem of subextension of an ω-plurisubharmonic function in $D \subset X$ to an ω-plurisubharmonic function in X. In Section 3 we introduce classes of ω-plurisubharmonic functions on $D \subset X$ modelled on the classes defined by Cegrell and prove the subextension results which are generalizations of the ones on global subextensions in $\mathbb{C}^n$. We refer to [CKZ] for a historical account on subextension problems.

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Dédicace. — *C'est avec un grand plaisir que nous apportons cette contribution au volume spécial en l'honneur du Professeur Nguyen Thanh Van à l'occasion de sa retraite. Ses travaux de recherche notamment en Théorie du Pluripotentiel et ses applications à la théorie de l'approximation font partie de ceux nombreux qui ont contribué à la naissance de cette "belle théorie" dans les années* 1980.

2. Monge-Ampère measure of maximal subextensions

We assume the notational convention $d^c = \frac{i}{2\pi}(\bar\partial - \partial)$. Let us recall some definitions from ([Ce1], [Ce2]). Let $D \Subset \mathbb{C}^n$ be a hyperconvex domain. We denote by $\mathcal{E}_0(D)$ the set of negative and bounded plurisubharmonic functions φ on D which tend to zero at the boundary and satisfy $\int_D (dd^c\varphi)^n < +\infty$.

Let us denote by $\mathcal{F}(D)$ the set of all $\varphi \in PSH(D)$ such that there exists a sequence (φ_j) of plurisubharmonic functions in $\mathcal{E}_0(D)$ such that $\varphi_j \searrow \varphi$ and $\sup_j \int_D (dd^c\varphi_j)^n < +\infty$.

Before we consider the subextensions from a hyperconvex domain to $\mathbb{C}^n$, we first need a result on subextensions to just a larger hyperconvex set. Let $D \Subset \Omega \Subset \mathbb{C}^n$ be two bounded hyperconvex domains (open and connected) and and let $u \in \mathcal{F}(D)$ be a given function. Then u admits a subextension $\tilde u \in \mathcal{F}(\Omega)$ i.e. $\tilde u \leqslant u$ on D (see [CZ]). Therefore we can define the maximal subextension of u to Ω by

$$(\star) \qquad \tilde u = \sup\{v \in PSH(\Omega); v < 0,\ v|_D \leqslant u\}.$$

It follows from [Ce2] that $\tilde u \in \mathcal{F}(\Omega)$. The following theorem provides a description of the Monge-Ampère measure of the maximal subextension.

Theorem 2.1. — *Let $D \subset\subset \Omega$. For every $u \in \mathcal{F}(D), \tilde u \in \mathcal{F}(\Omega)$, $(dd^c\tilde u)^n \leqslant \chi_D (dd^c u)^n$ and $\int_{\{\tilde u < u\}} (dd^c\tilde u)^n = 0$.*

For the proof of the last equality we need the following elementary lemma.

Lemma 2.2. — *Suppose* (μ_j) *is a sequence of positive measures on* D *with uniformly bounded mass and that to every* $\epsilon > 0$ *there is a* $\delta > 0$ *such that to every* $E \subset D$ *with* $cap(E) < \delta$ *we have* $\mu_j(E) < \epsilon$ *for all* j*. If* $\lim \mu_j = \mu$ *and if* $f, g \in PSH(D)$ *then*

$$\int_{\{f<g\}} d\mu \leqslant \liminf_j \int_{\{f<g\}} d\mu_j .$$

To prove the lemma, one can use Bedford-Taylor capacity and the quasicontinuity of g (see [BT2]).

Proof (Of the theorem). — The first statement of the theorem was proved in [CH].

Observe that the function $\tilde{u}$ defined by $(\star)$ is plurisubharmonic if u is just any continuous function on D. Using the balayage procedure, it is easy to show that in that case we have $\int_{\{\tilde{u}<u\}} (dd^c\tilde{u})^n = 0$.

Assume now that $u \in \mathcal{F} \cap L^\infty(D)$ and take a sequence of continuous functions u_j on D decreasing to u. Then $\tilde{u}_j$ decreases to $\tilde{u}$ and the sequence $(\tilde{u}_j)$ is uniformly bounded on Ω since $\tilde{u} \leqslant \tilde{u}_j \leqslant 0$ on Ω. Therefore the Monge-Ampère measures $(dd^c\tilde{u}_j)^n$ are uniformly dominated by the Monge-Ampère capacity.

So if we put $\mu_j = (dd^c\tilde{u}_j)^n$ we can apply the lemma to conclude that for every $s \geqslant 0$:

$$\int_{\{\tilde{u}_s<u\}} (dd^c\tilde{u})^n \leqslant \liminf_j \int_{\{\tilde{u}_s<u\}} (dd^c\tilde{u}_j)^n \leqslant \liminf_j \int_{\{\tilde{u}_j<u_j\}} (dd^c\tilde{u}_j)^n = 0,$$

since by the remark at the beginning of this proof $\int_{\{\tilde{u}_j<u_j\}} (dd^c\tilde{u}_j)^n = 0$. To complete the proof in this case, we let s tend to $+\infty$.

If $u \in \mathcal{F}(D)$ only, consider $u_j = \max\{u, -j\}$. Then, for $t > 0$ fixed

$$(1 + \max\{u/t, -1\})\,(dd^c u_j)^n \to (1 + \max\{u/t, -1\})\,(dd^c u)^n, j \to +\infty.$$

Observe that the function $(1 + \max\{u/t, -1\})$ vanishes on $\{u \leqslant -t\}$ and is bounded from above by 1. Moreover for any $j > t$ we have $\{u > -t\} \subset \{u > -j\}$ and the sequence of measures $\mathbf{1}_{\{u>-j\}}(dd^c u_j)^n$ increases to the measure $\mathbf{1}_{\{u>-\infty\}}(dd^c u)^n$ (see [BGZ]). Therefore we obtain for $j > t$

$$(1 + \max\{u/t, -1\})\,(dd^c u_j)^n \leqslant \mathbf{1}_{\{u>-j\}}(dd^c u_j)^n \leqslant \mathbf{1}_{\{u>-\infty\}}(dd^c u)^n.$$

It follows that, for every fixed t, the sequence of measures

$$\mu_j := (1 + \max\{u/t, -1\})\,(dd^c u_j)^n$$

and therefore $(1 + \max\{u/t, -1\})\,(dd^c \tilde{u}_j)^n$ satisfy the requirements of the lemma, so we get for every fixed s and t:

$$\int_{\{\tilde{u}_s < u\}} (1 + \max\{u/t, -1\})\,(dd^c \tilde{u})^n \leqslant \liminf_j \int_{\{\tilde{u}_s < u\}} (1 + \max\{u/t, -1\})\,(dd^c \tilde{u}_j)^n$$

$$\leqslant \liminf_j \int_{\{\tilde{u}_s < u\}} (dd^c \tilde{u}_j)^n \leqslant \liminf_j \int_{\{\tilde{u}_j < u_j\}} (dd^c \tilde{u}_j)^n = 0.$$

We now let t tend to $+\infty$. Then since $1 + \max\{u/t, -1\} \nearrow \mathbf{1}_{\{u > -\infty\}}$ as $t \nearrow +\infty$, it follows from the previous inequalities that $\int_{\{\tilde{u}_s < u\}} (dd^c \tilde{u})^n = 0$.

To complete the proof, we let s tend to $+\infty$. □

Remark 2.3. — Independently the above theorem was proved in [P], Lemma 4:5.

Remark 2.4. — It follows that

$$\mathbf{1}_{\{\tilde{u} = -\infty\}} (dd^c \tilde{u})^n = \mathbf{1}_{\{u = -\infty\}} (dd^c u)^n.$$

Indeed, the inequality "$\leqslant$" follows from Theorem 2.1 and the other one from Demailly's inequality [D] (see also [ACCP], Lemma 4.1).

3. Potentials on Kähler domains

Here we want to establish some elementary facts in pluripotential theory on compact Kähler manifolds with boundary i.e. on domains in a compact Kähler manifold.

3.1. The comparison principle

The aim of this section is to give a semi global version of the comparison principle which contains the local one from pluripotential theory on bounded hyperconvex domains in $\mathbb{C}^n$ as well as the global one from the theory on compact Kähler manifolds (see [GZ2]).

Let X be a Kähler manifold of dimension n and ω the Kähler form on X. We want to consider bounded $\omega-$plurisubharmonic functions on Kähler domains in X with boundary. For any domain $D \subset X$, denote by $PSH(D, \omega)$ the set of $\omega-$plurisubharmonic functions on D.

By definition if φ is $\omega-$plurisubharmonic on D then locally in D the function $u := \varphi + p$ is a local plurisubharmonic function, where p is a local plurisubharmonic potential of the form ω i.e. $dd^c p = \omega$. Therefore by Bedford and Taylor [BT] the curvature current $\omega_\varphi := dd^c\varphi + \omega$ associated to φ is a globally defined closed positive current on D which can be witten locally as $\omega_\varphi = dd^c u$. Hence by Bedford and Taylor [BT], the wedge power ω_φ^p is a well defined closed positive current of bidegree (p,p) on D. More generally, if $\varphi_1, \cdots, \varphi_q$ are bounded $\omega-$plurisubharmonic functions on D, we can define inductively the wedge intersection product

$$T(\varphi_1, \cdots, \varphi_q) := \omega_{\varphi_1} \wedge \cdots \wedge \omega_{\varphi_q} \tag{3.1}$$

as a closed positive current of bidimension $(n-q, n-q)$ on D. Moreover these currents put no mass on pluripolar sets.

Actually all local results from pluripotential theory concerning bounded plurisubharmonic functions on domains in $\mathbb{C}^n$ are valid in the situation considered here. We will refer to these results as results from the "local theory".

Here we use ideas from the global case (see [GZ2]). Our starting point is the following "local version" of the comparison principle which follows from quasi-continuity of plurisubharmonic functions (see [BT2],[BT3]).

Proposition 3.1. — *Let T be a closed positive current of bidimension (p,p) $(1 \leqslant p \leqslant n)$ of type (3.1) and $\varphi, \psi \in PSH(D,\omega) \cap L^\infty(D)$. Then*

$$\mathbf{1}_{\{\varphi < \psi\}} (\omega + dd^c \sup\{\varphi, \psi\})^p \wedge T = \mathbf{1}_{\{\varphi < \psi\}} (\omega + dd^c \psi)^p \wedge T, \tag{3.2}$$

in the weak sense of Borel measures on D. In particular

$$\mathbf{1}_{\{\varphi \leqslant \psi\}} (\omega + dd^c \sup\{\varphi, \psi\})^p \wedge T \geqslant \mathbf{1}_{\{\varphi \leqslant \psi\}} (\omega + dd^c \psi)^p \wedge T, \tag{3.3}$$

in the weak sense of Borel measures on D.

To perform a useful integration by parts formula, we need to consider special domains.

Definition 3.2. — *We will say that a domain $D \subset X$ is quasi-hyperconvex if D admits a continuous negative $\omega-$plurisubharmonic exhaustion function $\rho : D \longmapsto [-1, 0[$.*

Observe that any domain $D \subset X$ with smooth boundary given by $D := \{r < 0\}$, where r is smooth in a neighbourhood of $\overline{D}$, is quasi-hyperconvex since for $\varepsilon > 0$ small enough, the function $\rho := \varepsilon\, r$ is $\omega-$plurisubharmonic on a neighbourhood of $\overline{D}$ and is a bounded exhaustion for D. Observe that such a domain can be pseudoconcave.

Here we will consider only quasi-hyperconvex domains D satisfying

$$\int_D \omega^n < \int_X \omega^n. \tag{3.4}$$

Definition 3.3. — *Given a quasi-hyperconvex domain D, we define the class of test functions $\mathcal{P}_0(D,\omega)$ to be the class of functions $\varphi \in PSH^-(D,\omega) \cap L^\infty(D)$ such that $\lim_{z\to\partial D} \varphi = 0$ and $\int_D (\omega + dd^c\varphi)^n < +\infty$.*

Observe that for any negative smooth function h with compact support in D, the function εh is in $\mathcal{P}_0(D,\omega)$ for $\varepsilon > 0$ small enough. Moreover, if ρ is an $\omega-$plurisubharmonic defining function for D then for any $0 \leqslant t \leqslant 1$, $t\rho \in \mathcal{P}_0(D,\omega)$.

Lemma 3.4. — *Let T be a closed positive current of bidimension (p,p) $(1 \leqslant p \leqslant n)$ and $\varphi, \psi \in PSH(D,\omega) \cap L^\infty(D)$ be such that $(\varphi - \psi)_\star \geqslant 0$ on ∂D. Assume that $\int_D (dd^c\varphi)^p \wedge T < +\infty$. Then we have*

$$\int_{\{\varphi<\psi\}} \omega_\psi^p \wedge T \leqslant \int_{\{\varphi<\psi\}} \omega_\varphi^p \wedge T,$$

and

$$\int_{\{\varphi\leqslant\psi\}} \omega_\psi^p \wedge T \leqslant \int_{\{\varphi\leqslant\psi\}} \omega_\varphi^p \wedge T.$$

and if $\varphi \leqslant \psi$ on D then

$$\int_D \omega_\psi^p \wedge T \leqslant \int_D \omega_\varphi^p \wedge T.$$

In particular if $\varphi \in PSH^-(D,\omega) \cap L^\infty(D)$ and $\varphi \to 0$ at the boundary, then

$$\int_D \omega^p \wedge T \leqslant \int_D \omega_\varphi^p \wedge T.$$

Proof. — Recall that the condition $(\varphi - \psi)_\star \geqslant 0$ means that for any $\varepsilon > 0$, $\{\varphi < \psi - \varepsilon\} \Subset D$. So replacing ψ by $\psi - \varepsilon$ and letting $\varepsilon \searrow 0$, we can assume that $\{\varphi < \psi\} \Subset D$. Then the function $\vartheta := \sup\{\varphi, \psi\} \in PSH(D,\omega) \cap L^\infty(D)$ coincides with φ near the boundary of D. This implies that

$$\int_D (\omega + dd^c\vartheta)^p \wedge T = \int_D (\omega + dd^c\varphi)^p \wedge T. \tag{3.5}$$

Indeed, using local regularization of plurisubharmonic functions, we see that $(\omega + dd^c\vartheta)^p \wedge T - (\omega + dd^c\varphi)^p \wedge T = dS$, in the sense of currents in D, where

$S := d^c(\vartheta - \varphi)\left((\omega + dd^c\,\vartheta)^{p-1} + \cdots + (\omega + dd^c\varphi)^{p-1}\right) \wedge T$ is a well defined current with measure coefficients and with compact support in D. Therefore, by definition of the differential of a current, we get $\int_D \chi dS = 0$ for any test function χ which is identically 1 in a neignbourhood of the support of S. This implies the identity (3.5).

Now by Proposition 3.1, we get

$$\int_{\{\varphi<\psi\}} \omega_\psi^p \wedge T = \int_{\{\varphi<\psi\}} \omega_\vartheta^p \wedge T.$$

Then using the identity (3.5) and again Proposition 3.1, we deduce

$$\begin{aligned}\int_{\{\varphi<\psi\}} \omega_\psi^p \wedge T &= \int_D \omega_\vartheta^p \wedge T - \int_{\{\varphi\geqslant\psi\}} \omega_\vartheta^p \wedge T\\ &\leqslant \int_D \omega_\varphi^p \wedge T - \int_{\{\varphi>\psi\}} \omega_\vartheta^p \wedge T\\ &= \int_D \omega_\varphi^p \wedge T - \int_{\{\varphi>\psi\}} \omega_\varphi^p \wedge T,\end{aligned}$$

which implies

$$\int_{\{\varphi<\psi\}} \omega_\psi^p \wedge T \leqslant \int_{\{\varphi\leqslant\psi\}} \omega_\varphi^p \wedge T.$$

Applying this result to $\varphi+\varepsilon$ and ψ and letting $\varepsilon \to 0$, we obtain the required inequality.

To obtain the second inequality, we can assume $\varphi, \psi < 0$ on D. Now apply the above inequality to φ and $t\psi$ with $0 < t < 1$ and observe that $(dd^c(t\psi) + \omega)^n \geqslant t^n \omega_\psi^n$. Then letting $t \to 1$, we obtain the required inequality. □

If $m = 0$ we set $T_0 = 1$ and for $m \geqslant 1$ we set $T_m := \omega_{u_1} \wedge \cdots \wedge \omega_{u_m}$, where $u_1, \cdots u_m \in \mathcal{P}_0(D,\omega)$. Thus T_m is a closed positive current on D. Then we have the following important result.

Corollary 3.5. — *1) The class $\mathcal{P}_0(D,\omega)$ is convex and satisfies the lattice condition:*

$$\varphi \in \mathcal{P}_0(D,\omega), u \in PSH^-(D,\omega) \Longrightarrow \sup\{\varphi, u\} \in P_0(D,\omega).$$

2) Let $1 \leqslant p, q$ be integers such that $p+q \leqslant n$ and denote by $m := n-p-q$. Then for any $\varphi, \psi \in P_0(D,\omega)$,

$$\int_D \omega_\varphi^p \wedge \omega_\psi^q \wedge T_m \leqslant \int_D \omega_\varphi^{p+q} \wedge T_m + \int_D \omega_\psi^{p+q} \wedge T_m. \tag{3.6}$$

3) If $\varphi_1, \cdots, \varphi_n \in \mathcal{P}_0(D,\omega)$, *then*

$$\int_D \omega_{\varphi_1} \wedge \cdots \wedge \omega_{\varphi_n} \leqslant 2^{n-1} \sum_{j=1}^{n} \int_D \omega_{\varphi_j}^n .$$

Proof. — Let $\varphi \in \mathcal{P}_0(D,\omega)$ and $u \in PSH^-(D,\omega)$ and denote by $\sigma(\varphi,u) := \sup\{\varphi, u\}$. Since $\varphi \leqslant \sigma(\varphi,u) \leqslant 0$, it is clear from the lemma above that

$$\int_D \omega^n \leqslant \int_D (\omega + dd^c \sigma(\varphi,u))^n \leqslant \int_D \omega_\varphi^n,$$

which implies that $\sigma(\varphi,u) \in P_0(D,\omega)$.

We first prove the inequality (3.6) when $m = 0$, $T_m = 1$ and $p + q = n$. Indeed, by Lemma 3.4 we get

$$\int_{\{\varphi+\epsilon<\psi\}} \omega_\varphi^p \wedge \omega_\psi^q \leqslant \int_{\{\varphi<\psi\}} \omega_\varphi^{p+q} \leqslant \int_D \omega_\varphi^{p+q}.$$

Applying this result with $\psi = 0$ we deduce that

$$\int_D \omega_\varphi^p \wedge \omega^q \leqslant \int_D \omega_\varphi^{p+q}. \tag{3.7}$$

In the same way we obtain

$$\int_{\{\psi<\varphi\}} \omega_\varphi^p \wedge \omega_\psi^q \leqslant \int_D \omega_\psi^{p+q}.$$

Therefore

$$\int_D \omega_\varphi^p \wedge \omega_\psi^q \leqslant \int_D \omega_\varphi^{p+q} + \int_D \omega_\psi^{p+q},$$

if we choose $\epsilon > 0$ such that $\int_{\{\psi+\epsilon=\varphi\}} \omega_\varphi^p \wedge \omega_\psi^q = 0$ and let ϵ decrease to 0.

Then the convexity of $\mathcal{P}_0(D,\omega)$ follows since for $\varphi, \psi \in \mathcal{P}_0(D,\omega)$ and $0 < t < 1$, we have

$$(\omega + dd^c(t\varphi + (1-t)\psi))^n = \sum_{p=0}^{n} \binom{n}{p} t^p (1-t)^{n-p} \omega_\varphi^p \wedge \omega_\psi^{n-p},$$

which implies by the previous inequality that

$$\int_D (\omega + dd^c(t\varphi + (1-t)\psi))^n \leqslant \int_D \omega_\varphi^n + \int_D \omega_\psi^n,$$

which is finite and thus $t\varphi + (1-t)\psi \in \mathcal{P}_0(D,\omega)$. From this, it follows that for any $\varphi_1, \cdots, \varphi_n \in \mathcal{P}_0(D)$, we have $u := (\varphi_1 + \cdots + \varphi_n)/n \in \mathcal{P}_0(D)$ and from the last inequality we deduce that

$$\int_D (\omega + dd^c\varphi_1) \wedge \cdots \wedge (\omega + dd^c\varphi_n) \leqslant n^n \int_D (\omega + dd^c u)^n < +\infty.$$

Therefore we can apply Lemma 3.4 for T_m and use the same argument as before to get the inequality (3.6) in the general case. □

3.2. Integration by parts formula

To prove the integration by parts formula (IBP) which will be crucial for our considerations, we need a semi-global version of the classical (local) convergence theorem of Bedfod and Taylor for our class $\mathcal{P}_0(D,\omega)$.

PROPOSITION 3.6. — *Let $(\varphi_j^0), \cdots (\varphi_j^n)$ be sequences of locally uniformly bounded ω−plurisubharmonic functions in the class $\mathcal{P}_0(D,\omega)$ converging monotonically to $\varphi^0, \cdots, \varphi^n \in \mathcal{P}_0(D,\omega)$ respectively. Then the positive currents $S_j := (dd^c\varphi_j^1 + \omega) \wedge \cdots \wedge (dd^c\varphi_j^n + \omega)$ and $S := (dd^c\varphi^1 + \omega) \wedge \cdots \wedge (dd^c\varphi^n + \omega)$ have uniformly bounded total masses in D and*

$$\lim_{j\to+\infty} \int_D (-\varphi_j^0)(dd^c\varphi_j^1 + \omega) \wedge \cdots \wedge (dd^c\varphi_j^n + \omega) =$$

$$\int_D (-\varphi^0)(dd^c\varphi^1 + \omega) \wedge \cdots \wedge (dd^c\varphi^n + \omega).$$

Proof. — Observe first that the local theory of Bedford and Taylor implies that $(-\varphi_j^0)S_j \to (-\varphi^0)S$ weakly on D (see [BT2]). It follows from our hypothesis that given $\varepsilon > 0$, there exists an open set $D' \Subset D$ such that $-\varepsilon \leqslant \varphi_j^0 \leqslant 0$ and $-\varepsilon \leqslant \varphi^0 \leqslant 0$ on $D \setminus D'$. Then

$$\int_D (-\varphi_j^0)S_j - \int_D (-\varphi^0)S = \int_{D'} (-\varphi_j^0)S_j - \int_{D'} (-\varphi^0)S_0 \ + \ O(\varepsilon), \quad (3.8)$$

uniformly in $j \in \mathbb{N}$. Here we have used the fact that the currents S_j have uniformly bounded mass on D by Lemma 3.4. Now observe that we can always choose the domain D' so that the positive measure $\mu_0 := (-\varphi^0)S$ puts no mass on its boundary $\partial D'$. Then since the positive measures $\mu_j := (-\varphi_j^0)S_j$ converge weakly to μ_0 in D, it follows that

$$\mu_0(D') \leqslant \liminf_j \mu_j(D') \leqslant \limsup_j \mu_j(\overline{D'}) \leqslant \mu_0(\overline{D'}) = \mu_0(D'),$$

which proves that the first integral on the right hand side converges to 0 and the proposition is proved.

Now we can prove the following integration by parts formula which will be useful in the sequel.

LEMMA 3.7. — *Let* $T := (\omega + dd^c u_1) \wedge \cdots \wedge (\omega + dd^c u_{n-1})$, *where* $u_1, \cdots u_{n-1} \in \mathcal{P}_0(D, \omega)$. *Let* $u, v \in \mathcal{P}_0(D, \omega)$. *Then*

$$\int_D u dd^c v \wedge T = \int_D v dd^c u \wedge T, \tag{3.9}$$

and

$$\int_D u\omega_v \wedge T - \int_D v\omega_u \wedge T = \int_D (u - v)\omega \wedge T. \tag{3.10}$$

Proof. — Denote by $H(u, v) := udd^c v \wedge T - vdd^c u \wedge T$. Then by Proposition 3.1 the current $H(u, v)$ has finite total mass in D. It follows from Stokes formula that if $\bar{u}, \bar{v}$ are bounded $\omega-$plurisubharmonic functions on D such that $\bar{u} = u$ and $\bar{v} = v$ near the boundary ∂D then

$$\int_D H(\bar{u}, \bar{v}) = \int_D H(u, v).$$

Indeed observe that since $u, v, \bar{u}, \bar{v}$ are bounded $\omega-$quasiplurisubharmonic functions on D, it follows from the local theory that the currents $S := ud^c v \wedge T - vd^c u \wedge T$ and $\bar{S} :=:= \bar{u}d^c\bar{v} \wedge T - \bar{v}d^c\bar{u} \wedge T$ are well defined currents with measure coefficients on D such that $dS = udd^c v \wedge T - vdd^c u \wedge T = H(u, v)$ and $d\bar{S} = \bar{u}dd^c\bar{v} \wedge T - \bar{v}dd^c\bar{u} \wedge T = H(\bar{u}, \bar{v})$ in the weak sense of currents on D. Now since $S - \bar{S}$ is of compact support in D, it follows that $\int_D d(S - \bar{S}) = 0$ and then

$$\int_D H(u, v) = \int_D H(\bar{u}, \bar{v}).$$

Now for $\varepsilon > 0$ small enough, set $u_\varepsilon := \sup\{u, v - \varepsilon\}$ and $v_\varepsilon := \sup\{v, u - \varepsilon\}$ and observe that $u_\varepsilon = u$ and $v_\varepsilon = v$ near ∂D. Thus by the previous remark, we have for $\varepsilon > 0$ small enough

$$\int_D H(u_\varepsilon, v_\varepsilon) = \int_D H(u, v). \tag{3.11}$$

We want to pass to the limit. Here we must use the fact that $u = v = 0$ on ∂D, which implies that $u_\varepsilon = v_\varepsilon = 0$ on ∂D. Now for $\varepsilon > 0$ small enough, we have

$$H(u_\varepsilon, v_\varepsilon) = u_\varepsilon dd^c v_\varepsilon \wedge T - v_\varepsilon dd^c u_\varepsilon \wedge T.$$

Since $u_\varepsilon \nearrow g := \max\{u, v\}$ and $v_\varepsilon \nearrow g$, it follows from Proposition 3.1 that

$$\lim_{\varepsilon \to 0} \int_D u_\varepsilon dd^c v_\varepsilon \wedge T = \int_D g dd^c g \wedge T = \lim_{\varepsilon \to 0} \int_D v_\varepsilon dd^c u_\varepsilon \wedge T,$$

which implies the required integration by parts formula. □

4. Subextension of quasi-plurisubharmonic functions

4.1. Weighted Monge-Ampère energy classes

In the contrast to the local case, the domain of definition of the complex Monge-Ampère operator is not well understood in the global case. Interesting classes have been investigated in [GZ2] and [CGZ]. We are going to introduce similar classes in the semi-global case where the complex Monge-Ampère operator is well defined and continuous under deacreasing sequences. The first class is modeled on the class defined by Cegrell in ([Ce2]) as follows.

Definition 4.1. — *We say that $\varphi \in \mathcal{F}(D, \omega)$ if there exists a decreasing sequence (φ_j) from the class $\mathcal{P}_0(D, \omega)$ which converges to φ on D such that*

$$\sup_j \int_D \omega_{\varphi_j}^n < +\infty.$$

Observe that $\mathcal{F}(D, \omega)$ is a convex set and $\mathcal{P}_0(D, \omega) \subset \mathcal{F}(D, \omega)$. The class $\mathcal{F}(D, \omega)$ is the counterpart of the class defined by Cegrell in [Ce2]. Let D be a hyperconvex domain where the form ω has a plurisubharmonic potential q on D with boundary values 0 and let $\mathcal{F}(D)$ the class defined in [Ce2]. Then if $\varphi \in \mathcal{F}(D, \omega)$ iff $u := \varphi + q \in \mathcal{F}(D)$.

We do not know at the moment if the Monge-Ampère operator is well defined on the class $\mathcal{F}(D, \omega)$ but we can define the Monge-Ampère mass of a function $\varphi \in \mathcal{F}(D, \omega)$ thanks to the following lemma.

Lemma 4.2. — *Let $\varphi \in \mathcal{F}(D, \omega)$ be a fixed function. Then the constant*

$$M_D(\varphi) := \lim_j \int_D (\omega + dd^c \varphi_j)^n = \sup_j \int_D (\omega + dd^c \varphi_j)^n$$

is independant of the decreasing sequence (φ_j) from $\mathcal{P}_0(D, \omega)$ converging to φ.

Moreover if $\psi \in PSH(D, \omega)$ and $\varphi \leqslant \psi \leqslant 0$ then $\psi \in \mathcal{F}(D, \omega)$.

Proof. — Take a defining sequence $(\varphi_j)_j$ for φ. By Lemma 3.4 we know that the sequence $\{\int_D(\omega + dd^c\varphi_j)^n\}_j$ is increasing and by definition it is bounded so the limit $M_D(\varphi)$ exists. We only need to show that it does not depend on the sequence. Let (ψ_j) another decreasing sequence of functions in the class $\mathcal{P}_0(D,\omega)$ converging to φ in D. Fix $\varepsilon > 0$ and j. Since by Bedford-Taylor continuity theorem ([BT2]), $(\omega + dd^c \sup\{\psi_j, \varphi_k\})^n \to (\omega + dd^c\psi_j)^n$ weakly on D as $k \to \infty$, it follows that there exists k_j such that

$$\int_D (\omega + dd^c \sup\{\psi_j, \varphi_{k_j}\})^n > \int_D (\omega + dd^c\psi_j)^n - \varepsilon.$$

By Lemma 3.4, we have

$$\int_D (\omega + dd^c \sup\{\psi_j, \varphi_{k_j}\})^n \leqslant \int_D (\omega + dd^c\varphi_{k_j})^n \leqslant M_D(\varphi).$$

Therefore it follows that $\int_D(\omega + dd^c\psi_j)^n - \varepsilon \leqslant M_D(\varphi)$, which implies that $\sup_j \int_D(\omega + dd^c\psi_j)^n \leqslant M_D(\varphi)$ and proves the first part of the lemma.

Now set $\psi_j := sup\{\psi, \varphi_j\}$. Then by Lemma 3.4, $\psi_j \in \mathcal{P}_0(D)$ and $\int_D(\omega + dd^c\psi_j)^n \leqslant \int_D(\omega + dd^c\varphi_j)^n \leqslant M_D(\varphi)$. Since (ψ_j) decreases to ψ, it follows that $\psi \in \mathcal{F}(D,\omega)$ and from the first part of the proof we deduce that $M_D(\psi) \leqslant M_D(\varphi)$. □

Let us introduce the following classes of finite weighted Monge-Ampère energy (see [Ce1], [GZ2], [BGZ]). A weight function is by definition an increasing function $\chi : \mathbb{R} \longmapsto \mathbb{R}$ such that $\chi(t) = t$ is $t \geqslant 0$ and $\chi(-\infty) = -\infty$. To any weight function we associate the class $\mathcal{E}_\chi(D,\omega)$ of of $\omega-$plurisubharmonic functions $\varphi \in PSH(D,\omega)$ for which there exists a sequence $(\varphi_j) \in \mathcal{P}_0(D,\omega)$, $\varphi_j \searrow \varphi$ such that

$$\sup_j \int_D |\chi(\varphi_j)|\omega_{\varphi_j}^n < +\infty.$$

In our case the weight function χ will be convex. From the (IBP) formula, we can derive the following fundamental inequality which will be useful (see [GZ2]).

Proposition 4.3 *Let $\chi : \mathbb{R} \longmapsto \mathbb{R}$ be a convex weight function. Then for any $\varphi, \psi \in \mathcal{P}_0(D,\omega)$ with $\varphi \leqslant \psi$, we have*

$$\int_D |\chi(\psi)|\omega_\psi^n \leqslant 2^n \int_D |\chi(\varphi)|\omega_\varphi^n. \tag{4.1}$$

We can prove that the complex Monge-Ampère operator is well defined and continuous on decreasing sequences in the class $\mathcal{E}_\chi(D,\omega)$, where χ is a convex increasing functions $\mathbb{R} \longmapsto \mathbb{R}$ (see [GZ2], [CGZ]).

Proposition 4.4. — *The complex Monge-Ampère operator is well defined on the class $\mathcal{E}_\chi(D,\omega)$. Moreover if (φ_j) is a decreasing sequence from the class $\mathcal{E}_\chi(D,\omega)$ which converges to $\varphi \in \mathcal{E}_\chi(D,\omega)$, then the Monge-Ampère measures $(\omega^n_{\varphi_j})$ converge to ω^n_φ weakly on D. Moreover for any $h \in PSH(D,\omega) \cap L^\infty(D)$*

$$\lim_j \int_D h\omega^n_{\varphi_j} = \int_D h\omega^n_\varphi.$$

Using the integration by parts formula, the fundamental inequality and following the same arguments as [GZ2], it is possible to prove the following result.

Proposition 4.5. — *Let $\varphi \in \mathcal{P}SH(D,\omega)$. Assume there exists a decreasing sequence $(\varphi)_{j\in\mathbb{N}}$ in $\mathcal{P}_0(D,\omega)$ which converges to $\varphi \in PSH(D,\omega)$ and satisfies $\sup_j \int_D |\chi(\varphi_j)|\omega^n_{\varphi_j} < +\infty$. Then $\varphi \in \mathcal{E}_\chi(D,\omega)$ and*

$$\lim_{j\to+\infty} \int_D |\chi(\varphi_j)|\omega^n_{\varphi_j} = \int_D |\chi(\varphi)|\omega^n_\varphi.$$

4.2. A general subextension theorem

We now prove the following general subextension result which generalizes our previous result with a new proof (see [CKZ]).

Theorem 4.6. — *Let $D \subset X$ be a quasi-hyperconvex domain satisfying the condition (3.4). Let $\varphi \in \mathcal{F}(D,\omega)$ such that $M_D(\varphi) \leqslant \int_X \omega^n$. Then there exists a function $\varphi \in PSH(X,\omega)$ such that $\varphi \leqslant \varphi$ on D.*

Proof. — Let (φ_j) be a decreasing sequence from the class $\mathcal{P}_0(D,\omega)$ which converges to φ on D. By Lemma 4.2 we have

$$\int_D (\omega + dd^c\varphi_j)^n \leqslant M_D(\varphi).$$

First assume that $M_D(\varphi) < \int_X \omega^n$. Then by [GZ2] there exists $u_j \in \mathcal{E}^1(X,\omega)$ with $\sup_X u_j = -1$ such that

$$(\omega + dd^c u_j)^n = \mathbf{1}_D(\omega + dd^c\varphi_j)^n + \varepsilon_j\omega^n$$

on X, where $\varepsilon_j > 0$ is chosen so that the total mass of both sides are equal. Fix $j \in \mathbb{N}$. Since $\{\varphi_j < u_j\} := \{x \in D; \varphi_j < u_j\} \Subset D$, and φ_j is bounded, it follows that for $s > 1$ large enough, $\{\varphi_j < u_j^s\} = \{\varphi_j < u_j\} \Subset D$, where $u_j^s := \sup\{u_j, -s\}$. Then by the comparison principle (Lemma 3.4), it follows that

$$\int_{\{\varphi_j < u_j^s\}} (\omega + dd^c u_j^s)^n \leqslant \int_{\{\varphi_j < u_j^s\}} (\omega + dd^c \varphi_j)^n.$$

Recall that $\mathbf{1}_{\{u_j > -s\}}(\omega + dd^c u_j^s)^n = \mathbf{1}_{\{u_j > -s\}}(\omega + dd^c u_j)^n$ (see [GZ2]). Therefore

$$\int_{\{\varphi_j < u_j\}} (\omega + dd^c u_j)^n \leqslant \int_{\{\varphi_j < u_j\}} (\omega + dd^c \varphi_j)^n,$$

which implies that $Vol_\omega(\{\varphi_j < u_j\}) = 0$ and then $u_j \leqslant \varphi_j$ on D. Due to the normalization of u_j, the function $u := (\limsup_{j \to +\infty} u_j)^* \in PSH(X, \omega)$ and satisfies $u \leqslant \varphi$ on D and $\max_X u = -1$ (see [GZ1]).

Now assume $\varphi \in \mathcal{F}(D, \omega)$ with $M_D(\varphi) = \int_X \omega^n$ and consider a decreasing sequence (φ_j) in $\mathcal{P}_0(D, \omega)$ converging to φ with uniformly bounded Monge-Ampère masses. Then it follows that for any $0 < t < 1$ the function $t\varphi_j \in \mathcal{P}_0(D, \omega)$ and $\int_D (\omega + dd^c t\varphi_j)^n = \int_D (t\omega_{\varphi_j} + (1-t)\omega)^n$. By Lemma 3.4 we have $\int_D \omega_{\varphi_j}^p \wedge \omega^{n-p} \leqslant \int_D \omega_{\varphi_j}^n$. Therefore since $\int_D \omega^n < \int_X \omega^n$, it follows that $M_D(t\varphi_j) = \int_D (\omega + dd^c t\varphi_j)^n < \int_X \omega^n$. By the first part we can find a subextension $\psi_j^t \in PSH(X, \omega)$ of $t\varphi_j$ to X satisying $\max_X \psi_j^t = -1$. Therefore the function $\psi_j := (\limsup_{t \nearrow 1} {\psi_j}^t)^*$ is an $\omega-$plurisubharmonic subextension of φ_j to X with $\max_X \psi_j = -1$. Now observe that, as before, $\psi := (\limsup_{j \to +\infty} \psi_j)^* \in PSH(X, \omega)$ and satisfies $\max_X \psi = -1$ and $\psi \leqslant \varphi$ on D. $\square$

It follows from the above theorem that given $\varphi \in \mathcal{F}(D, \omega)$ such that $M_D(\varphi) \leqslant \int_X \omega^n$, the following function

$$\varphi = \varphi_D := \sup\{\psi \in PSH(X, \omega); \psi \leqslant \varphi \text{ on } D\}$$

is a well defined $\omega-$plurisubharmonic function on X and will be called the maximal subextension of φ from D to X.

The example below shows that in general the maximal subextension does not belong to the global domain of definition of the complex Monge-Ampère operator on X since it may have positive Lelong number along a hypersurface.

However if the given function has a finite weighted Monge-Ampère energy in the sense of [GZ2], we will prove that the maximal subextension satisfies the same property.

Theorem 4.7. — *Let $D \subset X$ be an quasi-hyperconvex domain satisfying the condition* (3.4) *and let $\varphi \in \mathcal{E}_\chi(D,\omega)$ be such that $\int_D \omega_\varphi^n \leqslant \int_X \omega^n$, where $\chi : \mathbb{R} \longmapsto \mathbb{R}$ is a convex weight function. Then the maximal subextension $\tilde{\varphi}$ of φ from D to X exists and has the following properties:*
(i) $\varphi \in \mathcal{E}_\chi(X,\omega)$ and $\int_X |\chi \circ \varphi|(\omega + dd^c\varphi)^n \leqslant \int_D |\chi \circ \varphi|(\omega + dd^c\varphi)^n$,
(ii) $\mathbf{1}_D(\omega + dd^c\varphi)^n \leqslant \mathbf{1}_D(\omega + dd^c\varphi)^n$ holds in the sense of measures on X,
(iii) the measure $(\omega + dd^c\varphi)^n$ is carried by the Borel set $\{\varphi = \varphi\} \cup \partial D$.

We will need the following lemma which can be proved using the argument from the first part of the proof of Theorem 2.1.

Lemma 4.8. — *Let D be as above and $\varphi \in \mathcal{P}_0(D,\omega)$ be such that $\int_D \omega_\varphi^n \leqslant \int_X \omega^n$, then $\varphi \in PSH(X,\omega) \cap L^\infty(X)$ and $\mathbf{1}_D(\omega + dd^c\varphi)^n \leqslant \mathbf{1}_D(\omega + dd^c\varphi)^n$ in the sense of measures on X. Moreover the measure $(\omega + dd^c\varphi)^n$ is carried by the Borel set $\{x \in \bar{D}; \tilde{\varphi}(x) = \varphi(x)\}$.*

Proof of the theorem. — Let (φ_j) a sequence $(\varphi_j) \in \mathcal{P}_0(D,\omega)$ which decreases to φ on D. Define $\tilde{\varphi}_j$ to be the maximal subextension of φ_j from D to X. Then by the previous lemma $\tilde{\varphi}_j \in PSH(X,\omega) \cap L^\infty(X)$ and $(\omega + dd^c\varphi_j)^n$ is supported on the contact set $\{x \in \bar{D} : \tilde{\varphi}_j(x) = \varphi_j(x)\}$. Hence $(-\chi \circ \tilde{\varphi}_j)(\omega + dd^c\tilde{\varphi}_j)^n \leqslant \mathbf{1}_D(-\chi \circ \varphi_j)(\omega + dd^c\varphi_j)^n$ in the sense of measures on X. Therefore there is a uniform constant $C > 0$ such that for any $j \in \mathbb{N}$,

$$\int_X (-\chi \circ \tilde{\varphi}_j)(\omega + dd^c\tilde{\varphi}_j)^n \leqslant \int_D (-\chi \circ \varphi_j)(\omega + dd^c\varphi_j)^n \leqslant C.$$

Since $(\tilde{\varphi}_j) \searrow \varphi$ on X it follows from [GZ2] that $\varphi \in \mathcal{E}_\chi(X,\omega)$. Moreover by the convergence theorem ([GZ2], [CGZ]) it follows that $\mathbf{1}_D|\chi \circ \tilde{\varphi}|(\omega + dd^c\tilde{\varphi})^n \leqslant \mathbf{1}_D|\chi \circ \varphi|(\omega + dd^c\varphi)^n$ in the sense of measures on X.

The third part of the theorem is proved along the same lines as the last part of the proof of Theorem 2.1 using Lemma 4.8 and Proposition 4.2. □

Remark 4.9. — In contrast to the local case it may happen that a part of the Monge-Ampère measure of $\tilde{\varphi}$ lives on the boundary of D.

As we already said before, the example in the last section shows that the maximal subextension of a given function $\varphi \in \mathcal{F}(D,\omega)$ may have not a well defined Monge-Ampère measure. However the following property may be useful.

Proposition 4.10. — *Let $\varphi \in \mathcal{F}(D,\omega)$ be a given function. Then if (φ_j) is a decreasing sequence of functions in the class $\mathcal{P}_0(D,\omega)$ converging to φ then the sequence (φ_j) decreases to φ on X. Moreover any Borel measure μ on X which is a limit point of the sequence of measures $(\omega + dd^c\varphi_j)^n$ on X satisfies the inequality $\mathbf{1}_D\mu \leqslant \mathbf{1}_D(\omega + dd^c\varphi)^n$ in the sense of measures on X.*

Proof. — Observe that for each $j \in \mathbb{N}$, φ is a global subextension of φ_j to X and then $\varphi \leqslant \varphi_j$ on X. Therefore it is clear that the sequence (φ_j) decreases to an $\omega-$plurisubharmonic function ψ on X which satisties the inequality $\varphi \leqslant \psi$ on X. This shows that $\psi \in \mathcal{PSH}(X,\omega)$. On the other hand since $\psi \leqslant \varphi_j \leqslant \varphi_j$ on D we infer that $\psi \leqslant \varphi$ on D, which proves that ψ is a subextension of φ to X and then $\psi \leqslant \varphi$ on D. We conclude that $\psi = \varphi$ on X. We know from the last lemma that $\mathbf{1}_D(\omega + dd^c\varphi_j)^n \leqslant \mathbf{1}_D(\omega + dd^c\varphi_j)^n$ in the sense of measures on X, which implies the last statement of the proposition. □

4.3. Subextension in $\mathbb{C}^n$

Now we pass to subextensions from a hyperconvex domain $D \Subset \mathbb{C}^n$ to $\mathbb{C}^n$, considered as an open subset of the complex projective space $\mathbb{P}^n$. Recall that the Lelong class is defined by

$$\mathcal{L}(\mathbb{C}^n) := \{u \in PSH(\mathcal{C}^n); \sup\{u(z) - \log^+|z| < +\infty\}.$$

Let $\omega = \omega_{FS}$ be the normalized Fubini-Study metric on $\mathbb{P}^n$ defiend in affine coordinates by

$$\omega := dd^c \log|\zeta|,$$

where $\zeta := [\zeta_0 : \cdots : \zeta_n]$ are the homogeneous coordinates on $\mathbb{P}^n$. As usual we will consider $\mathbb{C}^n = \mathbb{P}^n \setminus \{\zeta_0 = 0\}$ with the affine coordinates defined as by $z_j := \zeta_j/\zeta_0$ $(1 \leqslant j \leqslant n)$. With these notations we have $\omega|\mathbb{C}^n = dd^c\ell$, where $\ell(z) := (1/2)\log(1 + |z|^2)$. Therefore given any $u \in \mathcal{L}(\mathbb{C}^n)$, the function defined by

$$\varphi(\zeta) := u(z) - (1/2)\log(1 + |z|^2), \zeta_0 \neq 0$$

is $\omega-$plurisubharmonic on $\mathbb{P}^n \setminus \{\zeta_0 = 0\}$ and locally upper bounded in a neighbourhood of the hyperplane at infinity $H_\infty := \{\zeta_0 = 0\}$ so that it extends to an $\omega-$plurisubharmonic function on $\mathbb{P}^n$ which we also denote by φ. It follows that the correspondance $u \longmapsto \varphi$ is a bijection between $\mathcal{L}(\mathbb{C}^n)$ and $PSH(\mathbb{P}^n,\omega)$ such that $\omega + dd^c\varphi = dd^c u$ on $\mathbb{C}^n$.

From the last theorem we can deduce a generalization of our earlier result (see [CKZ], Theorem 5.3).

Theorem 4.11. — *Let $D \Subset \mathbb{C}^n$ be a hyperconvex domain and let $u \in \mathcal{F}(D)$ be such that $(dd^c u)^n$ does not put any mass on pluripolar sets in D and $\int_D (dd^c u)^n \leqslant 1$. Then its maximal subextension $\tilde{u}$ from D to $\mathbb{C}^n$ belongs to $\mathcal{L}(\mathbb{C}^n)$ and has a well defined global Monge-Ampère measure $(dd^c \tilde{u})^n$ which is carried by the set $\{\tilde{u} = u\} \cup \partial D$ and satsifies the inequality $\mathbf{1}_D (dd^c \tilde{u})^n \leqslant \mathbf{1}_D (dd^c u)^n$.*

Proof. — Assume first that $D = B_R$ is an euclidean ball with center at the origin and radius $R > 0$. Then the function $q := (1/2)\log(1 + |z|^2) - (1/2)log(1 + R^2)$ is a potential of the normalized Fubini-Study form ω on $\mathbb{C}^n$ which vanishes on ∂D. In this case $\varphi := u - q \in \mathcal{F}(D, \omega)$. From our hypothesis $(\omega + dd^c \varphi)^n(\{\varphi = -\infty\}) = (dd^c u)^n(\{u = -\infty\}) = 0$. It follows from standard fact in measure theory that there exists a convex inceasing function $\chi :]-\infty, 0] \longrightarrow]-\infty, 0]$ such that $\int_D (-\chi \circ \varphi)(\omega + dd^c \varphi)^n < +\infty$ (see [GZ 2]. It easily follows that $\varphi \in \mathcal{E}_\chi(D, \omega)$ and then we can apply the last result to find a subextension $\tilde{\varphi} \in \mathcal{E}(\mathbb{P}^n, \omega)$ of φ to $\mathbb{P}^n$. Then $\tilde{u} := \tilde{\varphi} + q$ is the maximal subextension of u to $\mathbb{C}^n$.

Now in the general case consider an euclidean ball B such that $D \subset B$ and use Theorem 2.1 to produce a subextension $v \in \mathcal{F}(B)$ of u. Then by the previous case v has a subextension $\tilde{v}$ such that $\psi := \tilde{v} - q$ is a function in $\mathcal{E}(\mathbb{P}_n, \omega)$ which is a subextension of $\varphi := u - q$ from D to $\mathbb{P}^n$. Therefore the maximal subextension $\tilde{\varphi}$ of φ exists and since $\psi \leqslant \tilde{\varphi}$ it follows that $\tilde{\varphi} \in \mathcal{E}(\mathbb{P}^n, \omega)$. Thus $\tilde{u} := \varphi + q \in \mathcal{L}(\mathbb{C}^n)$ is the maximal subextension of u to $\mathbb{C}^n$. The other properties follow in the same way as in the proof of Theorem 4.8. □

Now we consider an arbitrary function $u \in \mathcal{F}(D)$ and a positive γ satisfying

$$\gamma^n \geqslant \int_D (dd^c u)^n.$$

Then from Theorem 4.6 the set of entire subextensions of logarithmic growth

$$\{v \in PSH(\mathbb{C}^n); v|_D \leqslant u, v(z) \leqslant a_v + \gamma log^+|z|\}$$

is not empty. Thus, using notation

$$\mathcal{L}_\gamma(\mathbb{C}^n) = \{v \in PSH(\mathbb{C}^n); v(z) \leqslant a_v + \gamma log^+|z|\}$$

one can choose the maximal subextension of u of logarithmic growth related to γ

$$\tilde{u}_\gamma = \sup\{v \in \mathcal{L}_\gamma(\mathbb{C}^n); v|_D \leqslant u\}.$$

As we shall see the Monge-Ampère measure of this subextension may not exist. If it exists however, one can deduce some information on the support of such measure.

Define

$$N_u = \{z \in \mathbb{C}^n ; \tilde{u}_\gamma < 0\}.$$

PROPOSITION 4.12. — *Assume that $u \in \mathcal{F}(D)$ and let $\gamma^n = \int_D (dd^c u)^n$. Then for any sequence $u_j \in \mathcal{E}_0(D) \cap C(\bar{D})$, decreasing to u if μ is an accumulation point of $(dd^c \tilde{u}_{j,\gamma})^n$ then $\mu = f(dd^c u)^n + \nu$ where $0 \leqslant f \leqslant 1$ is a function vanishing outside D and where ν is a positive measure,* $\operatorname{supp}\nu \subset \partial \mathrm{N_u}$.

Proof. — Assume first that $u \in \mathcal{E}_0(D) \cap C(\bar{D})$. Then $\tilde{u}_\gamma$ is continuous and the zero sublevel set of $\tilde{u}_\gamma, N_u$ is hyperconvex.

By definition, $D \subset N_u$ and by Theorem 5.1 in [CKZ] D is not relatively compact in N_u. There are two possibilities:

1) $D = N_u$.

2) $D \neq N_u \subset\subset \mathbb{C}^n$.

If 1) occurs then $\tilde{u}_\gamma$ extends u to a function in $\mathcal{L}_\gamma \cap L^\infty_{loc}$ and

$$\mathbf{1}_{N_u}(dd^c \tilde{u}_\gamma)^n = \mathbf{1}_D (dd^c \tilde{u}_\gamma)^n = \mathbf{1}_D (dd^c u)^n.$$

In particular, if $\gamma^n = \int_D (dd^c u)^n$ then $(dd^c \tilde{u}_\gamma)^n = \mathbf{1}_D (dd^c u)^n$ on $\mathcal{C}^n$.

Generically we have 2). Then on N_u, $\tilde{u}_\gamma$ is equal to $\tilde{u}$, the maximal local subextension of u from D to N_u. Consider $D_j \subset\subset D_{j+1} \subset\subset D$ an exhaustion sequence of D. Denote by $\tilde{u}_j$ the corresponding local maximal subextension to N_u of the solution $u_j \in \mathcal{E}_0(D)$ to $(dd^c u_j)^n = \mathbf{1}_{D_{j-1}}(dd^c u)^n$. Then $\tilde{u} \leqslant \tilde{u}_j$ and $(dd^c \tilde{u}_j)^n \leqslant \mathbf{1}_{D_{j-1}}(dd^c u)^n$ on N_u by Theorem 2.1 and so $(dd^c \tilde{u})^n \leqslant \mathbf{1}_D (dd^c u)^n$ on N_u.

Therefore, $(dd^c \tilde{u}_\gamma)^n = f(dd^c u)^n + \nu$ where $0 \leqslant f \leqslant 1$ is a function vanishing outside D and where ν is a positive measure, $\operatorname{supp}\nu \subset \partial \mathrm{N_u} \cap \partial \mathrm{D}$.

Now consider the general case. Choose a decreasing sequence (u_j) in $\mathcal{E}_0(D) \cap C(\bar{D})$, decreasing to u. Then $\tilde{u}_{j,\gamma}$ decreases to $\tilde{u}_\gamma$ and $(dd^c \tilde{u}_{j,\gamma})^n = f_j (dd^c u_j)^n + \nu_j$ where $0 \leqslant f_j \leqslant 1$ is a function vanishing outside D and where ν_j is a positive measure, $\operatorname{supp}\nu_\mathrm{j} \subset \partial \mathrm{N_{u_j}}$. Also $\int (dd^c \tilde{u}_{j,\gamma})^n = \gamma^n$. So if μ is any weak limit of $(dd^c \tilde{u}_{j,\gamma})^n$, then $\mu = f(dd^c u)^n + \nu$ where $0 \leqslant f \leqslant 1$ is a function vanishing outside D and where ν is a positive measure carried by ∂N_u. □

Corollary 4.13. — *If, for $u \in \mathcal{F}(D)$, the set N_u is bounded then the Monge-Ampère measure of u_γ is well defined and equal to the limit of $(dd^c\tilde{u}_{j,\gamma})^n$.*

If N_u is not a bounded hyperconvex set, u_γ need not to be in the domain of definition of the Monge-Ampère operator. This is shown in the following example.

Example 4.14. — The maximal entire subextension of a function from the class $\mathcal{F}(\mathbb{B})$ may not have well defined global Monge-Ampère measure on $\mathbb{C}^2$.

Consider the Green function g in the ball $\mathbb{B}(0,2) \subset \mathbb{C}^2$ with two poles at $(-1,0)$ and $(1,0)$ of weight $\frac{1}{\sqrt{2}}$ each. Then

$$\int_{\mathbb{B}(0,2)} (dd^c g)^2 = 1.$$

So there exists the maximal entire subextension $\tilde{g} = \tilde{g}_t$ in the Lelong class $\mathcal{L}_t(\mathbb{C}^2), 1 \leqslant t < \sqrt{2}$. Note that $\frac{1}{\sqrt{2}} \log ||\frac{z_2}{2}||$ is a subextension. By the definition of the Green function we have for some $R \in (0,1),\ A > 0$ the following inequalities

$$|g(z) - \tfrac{1}{\sqrt{2}} \log ||(z_1+1, z_2)||\,| < A \ \text{ in } \ \mathbb{B}((-1,0),R)$$
$$|g(z) - \tfrac{1}{\sqrt{2}} \log ||(z_1-1, z_2)||\,| < A \ \text{ in } \ \mathbb{B}((1,0),R).$$

Let $0 < r < \frac{R}{16}$ be fixed and let $z_2 = w$ be fixed with $0 < |w| < r$.

Consider the restriction $\tilde{g}$: $\tilde{g}_w(z) = \tilde{g}(z,w)$. If $|z-1| \leqslant r$ or $|z+1| \leqslant r$ then $||(z,w)|| < 2$ so $\tilde{g}(z,w) \leqslant 0$ on $\{|z-1| \leqslant r\}$ and $\{|z+1| \leqslant r\}$.

If $-\infty \not\equiv \tilde{g}_w \in \mathcal{L}_t(\mathbb{C})$ one concludes that the total mass of $\frac{1}{2\pi}\Delta\tilde{g}_w$ does not exceed t. By symmerty one can assume that

$$\int_{\mathbb{B}(1,R)} \Delta\tilde{g}_w \leqslant t/2. \tag{4.2}$$

(Otherwise consider $\mathbb{B}(-1,R)$ in place of $\mathbb{B}(1,R)$.)

If $|z-1| \leqslant |w|$ we then have

$$\tilde{g}_w(z) = \tilde{g}(z,w) \leqslant \frac{1}{\sqrt{2}} \log ||(z-1,w)|| + A \leqslant \frac{1}{\sqrt{2}} \log |w| + A + 1. \tag{4.3}$$

Let z be any point on $\{|w| < |z-1| \leqslant r\}$. Denote by $\mathbb{B}_1$ the disk $\mathbb{B}(z, 2r)$ and by $\mathbb{B}_2$ the disk $\mathbb{B}(1, r)$. Then $\mathbb{B}_2 \subset \mathbb{B}_1 \subset \mathbb{B}(1, R)$. If $J(K)$ denotes the average value of $\tilde{g}_w$ over a set $K \subset \mathbb{C}$ then

$$\tilde{g}_w(z) \leqslant J(\mathbb{B}_1) \leqslant \frac{1}{4} J(\mathbb{B}_2). \tag{4.4}$$

Since $J(\mathbb{B}_2)$ is dominated by the average of $\tilde{g}_w$ over the boundary of $\mathbb{B}_2$ one obtains from Riesz representation formula, using that $\tilde{g}_w \leqslant 0$ and (4.2), (4.3) :

$$\begin{aligned} J(\mathbb{B}_2) \quad &\leqslant \max_{\mathbb{B}(1,|w|)} \tilde{g}_w - \int_{\{|w|<|x-1|<r\}} \log|x-1| \Delta \tilde{g}_w \\ &\leqslant \tfrac{1}{\sqrt{2}} \log|w| + A + 1 - \tfrac{t}{2} \log|w| \\ &\leqslant (\tfrac{1}{\sqrt{2}} - \tfrac{t}{2}) \log|w| + A + 1. \end{aligned}$$

Therefore

$$\tilde{g}(z, w) \leqslant \frac{\sqrt{2} - t}{8} \log|w| + A + 1$$

for $||(z-1, w)|| < r$. Since the Monge-Ampère operator cannot be defined for $v(z, w) = \log|w|$ it follows that the same goes for the function $\tilde{g}$. □

Remark 4.15. — The above example relies on a geometrical effect which is also responsible for nonexistence of solutions to the Monge-Ampère equations in $\mathbb{C}P^n$ where we have on the right hand side a generic combination of Dirac measures (cf. [Co]).

Remark 4.16. — It follows from [S] that $\tilde{g}(z, w) - \frac{\sqrt{2}-t}{8} log|w|$ is plurisubharmonic on $\mathbb{C}^2$. For an elementary proof, see [Ce4].

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