Ha Huy Khoai, Vu Hoai An

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Value distribution problem for $p$-adic meromorphic functions and their derivatives

Ha Huy Khoai$^{(1)}$, Vu Hoai An$^{(2)}$

**Abstract.** — In this paper we discuss the value distribution problem for $p$-adic meromorphic functions and their derivatives, and prove a generalized version of the Hayman Conjecture for $p$-adic meromorphic functions.

**Résumé.** — Dans cet article on discute le problème de la distribution des valeurs pour des fonctions méromorphes p-adiques et ses dérivés, et démontre une version généralisée de la conjecture de Hayman pour des fonctions méromorphes p-adiques.

1. Introduction

In [11] Hayman proved the following well-known result:

**Theorem 1.1.** — Let $f$ be a meromorphic function on $\mathbb{C}$. If $f(z) \neq 0$ and $f^{(k)}(z) \neq 1$ for some fixed positive integer $k$ and for all $z \in \mathbb{C}$, then $f$ is constant.

Hayman also proposed the following conjecture (see [12]).

**Hayman Conjecture.** — If an entire function $f$ satisfies $f^n(z)f'(z) \neq 1$ for a positive integer $n$ and all $z \in \mathbb{C}$, then $f$ is a constant.

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$^{(1)}$ Institute of Mathematics, 18 Hoang Quoc Viet, 10307, Hanoi, Viet Nam
hhkhoai@math.ac.vn

$^{(2)}$ Hai Duong Pedagogical College, Hai Duong, Viet Nam
vuhoaianmai@yahoo.com
It has been verified for transcendental entire functions by Hayman himself for \( n > 1 \) ([12]), and by Clunie for \( n \geq 1 \) ([5]). These results and some related problems have become to be known as Hayman’s Alternative, and caused increasingly attentions (see [1], [2], [4], [14], [15], [17]).

In recent years the similar problems are investigated for functions in a non-Archimedean fields. In [16] J. Ojeda proved that for a transcendental meromorphic function \( f \) in an algebraically closed fields of characteristic zero, complete for a non-Archimedean absolute value \( K \), the function \( f'f^n - 1 \) has infinitely many zeros, if \( n \geq 2 \).

The aim of this paper is to establish a similar results for a differential monomial of the form \( f^n(f^{(k)})^m \), where \( f \) is a meromorphic function in \( \mathbb{C}_p \). Namely, we prove the following theorem.

**Theorem 1.2 (A generalized version of the Hayman Conjecture for \( p \)-adic meromorphic functions).** — Let \( f \) be a meromorphic function on \( \mathbb{C}_p \), satisfying the condition \( f^n(f^{(k)})^m(z) \not= 1 \) for all \( z \in \mathbb{C}_p \) and for some non-negative integers \( n, k, m \). Then \( f \) is a polynomial of degree \( < k \) if one of the following conditions holds:

1. \( f \) is an entire function.
2. \( k > 0 \), and either \( m = 1 \), \( n > \frac{1+\sqrt{1+4k}}{2} \), or \( m > 1, n \geq 1 \).
3. \( n \geq 0, m > 0, k > 0 \), and there are constants \( C, r_0 \) such that \( |f|_r < C \) for all \( r > r_0 \).

In the next section we first recall some facts of the \( p \)-adic Nevanlinna theory ([6-10], [13]) for later use. Theorem 1.2 is proved in Section 3 by using several Lemmas.

**2. Value distribution of \( p \)-adic meromorphic functions**

Let \( f \) be a non-constant holomorphic function on \( \mathbb{C}_p \). For every \( a \in \mathbb{C}_p \), expanding \( f \) around \( a \) as \( f = \sum P_i(z - a) \) with homogeneous polynomials \( P_i \) of degree \( i \), we define

\[
v_f(a) = \min \{i : P_i \not\equiv 0\}.
\]

For a point \( d \in \mathbb{C}_p \) we define the function \( v_f^d : \mathbb{C}_p \to \mathbb{N} \) by

\[
v_f^d(a) = v_{f-d}(a).
\]
Fix a real number $\rho$ with $0 < \rho \leq r$. Define

$$N_f(a, r) = \frac{1}{\ln p} \int_\rho^r \frac{n_f(a, x)}{x} dx,$$

where $n_f(a, x)$, as usually, is the number of the solutions of the equation $f(z) = a$ (counting multiplicity) in the disk $D_x = \{z \in \mathbb{C}_p : |z| \leq x\}$.

If $a = 0$, then set $N_f(r) = N_f(0, r)$.

For $l$ a positive integer, set

$$N_{l,f}(a, r) = \frac{1}{\ln p} \int_\rho^r \frac{n_{l,f}(a, x)}{x} dx,$$

where

$$n_{l,f}(a, r) = \sum_{|z| \leq r} \min \{v_{f-a}(z), l\}.$$

Let $k$ be a positive integer. Define the function $v_f^{\leq k}$ from $\mathbb{C}_p$ into $\mathbb{N}$ by

$$v_f^{\leq k}(z) = \begin{cases} 0 & \text{if } v_f(z) > k \\ v_f(z) & \text{if } v_f(z) \leq k, \end{cases}$$

and

$$n_f^{\leq k}(r) = \sum_{|z| \leq r} v_f^{\leq k}(z), \quad n_f^{\leq k}(a, r) = n_{f-a}^{\leq k}(r).$$

Define

$$N_f^{\leq k}(a, r) = \frac{1}{\ln p} \int_\rho^r \frac{n_f^{\leq k}(a, x)}{x} dx.$$

If $a = 0$, then set $N_f^{\leq k}(r) = N_f^{\leq k}(0, r)$.

Set

$$N_{l,f}^{\leq k}(a, r) = \frac{1}{\ln p} \int_\rho^r \frac{n_{l,f}^{\leq k}(a, x)}{x} dx,$$

where

$$n_{l,f}^{\leq k}(a, r) = \sum_{|z| \leq r} \min \{v_{f-a}^{\leq k}(z), l\}.$$

In a like manner to used for holomorphic functions we define

$$N_f^{< k}(a, r), N_{l,f}^{< k}(a, r), N_f^{> k}(a, r), N_{l,f}^{> k}(a, r), N_{l,f}^{\geq k}(a, r), N_{l,f}^{\leq k}(a, r).$$
Recall that for a holomorphic function $f(z)$ in $\mathbb{C}_p$, represented by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

for each $r > 0$, we define $|f|_r = \max\{|a_n|r^n, 0 \leq n < \infty\}$.

Now let $f = \frac{f_1}{f_2}$ be a non-constant meromorphic function on $\mathbb{C}_p$, where $f_1, f_2$ be holomorphic functions on $\mathbb{C}_p$ having no common zeros, we set $|f|_r = |f_1|_r / |f_2|_r$. For a point $d \in \mathbb{C}_p \cup \{\infty\}$ we define the function $v^d_f : \mathbb{C}_p \to \mathbb{N}$ by

$$v^d_f(a) = v_{f_1} - df_2(a)$$

with $d \neq \infty$, and

$$v^\infty_f(a) = v_{f_2}(a).$$

For a point $a \in \mathbb{C}$ define:

$$m_f(\infty, r) = \max\{0, \log |f|_r\}, m_f(a, r) = m_{\frac{f_1}{f_2}}(\infty, r),$$

$$N_f(a, r) = N_{f_1-a f_2}(r), N_f(\infty, r) = N_{f_2}(r),$$

$$T_f(r) = \max_{1 \leq i \leq 2} \log |f_i|_r.$$

In a like manner we define

$$N_{i,f}(a, r), N_{i,f}^{<k}(a, r), N_{i,f}^{<k}(a, r), N_{f}^{<k}(a, r), N_{i,f}^{>k}(a, r),$$

$$N_{f}^{>k}(a, r), N_{i,f}^{>k}(a, r), N_{i,f}^{>k}(a, r),$$

with $a \in \mathbb{C}_p \cup \{\infty\}$.

Then we have (see [11])

$$N_f(a, r) + m_f(a, r) = T_f(r) + O(1)$$

with $a \in \mathbb{C}_p \cup \{\infty\}$,

$$T_f(r) = T_{\frac{f_1}{f_2}}(r) + O(1),$$

$$|f^{(k)}|_r \leq \left|\frac{f}{r^k}\right|_r,$$

$$m_{\frac{f}{r^k}}(\infty, r) = O(1).$$

The following two lemmas were proved in [11] (see also [3], [6]).
Lemma 2.1. — Let $f$ be a non-constant holomorphic function on $\mathbb{C}_p$. Then
\[ T_f(r) - T_f(\rho) = N_f(r), \]
where $0 < \rho \leq r$.

Notices that $N_f(r)$ depends on fixed $\rho$.

Lemma 2.2. — Let $f$ be a non-constant meromorphic function on $\mathbb{C}_p$ and let $a_1, a_2, \ldots, a_q$ be distinct points of $\mathbb{C}_p$. Then
\[ (q - 1)T_f(r) \leq N_{1,f}(\infty, r) + \sum_{i=1}^{q} N_{1,f}(a_i, r) - N_{0,f'}(r) - \log r + O(1), \]
where $N_{0,f'}(r)$ is the counting function of the zeros of $f'$ which occur at points other than roots of the equations $f(z) = a_i, i = 1, \ldots, q$, and $0 < \rho \leq r$.

3. A Generalized Hayman-Conjecture for $p$-adic meromorphic functions

We are going to prove Theorem 1.2. We need the following Lemmas.

Lemma 3.1. — Let $f$ be a non-constant meromorphic function on $\mathbb{C}_p$ such that $f^{(k)} \not\equiv 0$ and $n, k, m$ be positive integers. Then
1. $T_f(r) \leq T_{f^n(f^{(k)})^{m-1}}(r) + O(1)$,
2. $T_f(r) \leq T_{f^n(f^{(k)})^m}(r) + O(1)$,

In particular $f^n(f^{(k)})^m$ is non-constant.

Proof. — 
1. Set $A = f^n(f^{(k)})^m - 1$. Then we have
\[ A + 1 = f^n(f^{(k)})^m, \]
\[ N_f(0, r) \leq N_{A+1}(0, r), \]
\[ \frac{1}{f^{n+m}} = \frac{1}{A+1}\left(\frac{f^{(k)}}{f}\right)^m. \]
Moreover
\[ m_{f^{(k)}}(\infty, r) = O(1). \]
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Therefore

$$m_f(0, r) \leq (n + m)m_f(0, r) = m_{f_{n+m}}(0, r) \leq m_{A+1}(0, r) + O(1).$$

Thus

$$T_f(r) = N_f(0, r) + m_f(0, r) \leq N_{A+1}(0, r) + m_{A+1}(0, r) = T_{f_{n(f(k))}}^{m-1} + O(1).$$

2. Since $T_{f_{n(f(k))}}^{m}(r) = T_{f_{n(f(k))}}^{m-1}(r) + O(1)$ we have

$$T_f(r) \leq T_{f_{n(f(k))}}^{m}(r) + O(1).$$

From this it follows that $f^n(f^{(k)})^m$ is non-constant.

Lemma 3.1 is proved. □

**Lemma 3.2.** — Let $f$ be a non-constant meromorphic function on $\mathbb{C}_p$ such that $f^{(k)} \not\equiv 0$, and let $m, n > 1, k > 0$ be integers, $a \in \mathbb{C}_p$, $a \neq 0$. Then we have:

1. $\frac{n(n - 2) + k(mn - m - n) + m(n - 1)}{(n + k)(n + m + km)}T_f(r) \leq N_{1,f^n(f^{(k)})^m}(a, r) - \log r + O(1),$

2. If $n^2 - n - k > 0$,

$$\frac{n^2 - n - k - 1}{(n + k)(n + 1 + k)}T_f(r) \leq N_{1,f^n(f^{(k)})^m}(a, r) - \log r + O(1).$$

**Proof.** —

1. Since $m, n > 1$ we have $n(n - 2) + k(mn - m - n) + m(n - 1) \geq 0$.

Because $f^{(k)} \not\equiv 0$, from Lemma 3.1 it follows that $f^n(f^{(k)})^m$ is not constant.

Applying Lemma 2.2 to $f^n(f^{(k)})^m$ with the values $\infty$, 0 and $a$, we obtain

$$T_{f^n(f^{(k)})^m}(r) \leq N_{1,f^n(f^{(k)})^m}(\infty, r) + N_{1,f^n(f^{(k)})^m}(0, r) + N_{1,f^n(f^{(k)})^m}(a, r) - \log r + O(1).$$

Denote by $N_{f^{(k)}}(0, r; f \neq 0)$ the counting function of those zeros of $f^{(k)}$ which are not the zeros of $f$, where a zero of $f^{(k)}$ is counted according to
its multiplicity. Then we get

\[ N_{f(k)}(0, r; f \neq 0) = N_{f(k)}(0, r) \]
\[ \leq N_{f(k)}(\infty, r) + m_{f(k)}(\infty, r) + O(1) \]
\[ \leq kN_{1,f}(\infty, r) + N_{f}^{\leq k}(0, r) + kN_{1,f}^{>k}(0, r) + O(1). \]

Therefore,

\[ N_{f(k)}(0, r; f \neq 0) \leq kN_{1,f}(\infty, r) + N_{f}^{\leq k}(0, r) + kN_{1,f}^{>k}(0, r) + O(1). \]

From this it follows

\[ N_{1,f^{n}(f(k))^{m}}(0, r) \leq N_{1,f}(0, r) + N_{f(k)}(0, r; f \neq 0) \]
\[ \leq kN_{1,f}(\infty, r) + N_{f}^{\leq k}(0, r) + kN_{1,f}^{>k}(0, r) + O(1) \]
\[ \leq (k + 1)N_{1,f}(0, r) + kN_{1,f}(\infty, r) \]

(3.3)

Again, we see that

\[ N_{f^{n}(f(k))^{m}}(0, r) - N_{1,f^{n}(f(k))^{m}}(0, r) \]
\[ \geq ((1 + k)n + m - 1)N_{1,f}^{(k+1)}(0, r) + (n - 1)N_{1,f}^{<k}(0, r). \]

(3.4)

On the other hand,

\[ N_{1,f}(0, r) = N_{1,f}^{<k}(0, r) + N_{1,f}^{(k+1)}(0, r). \]

From this and (3.3), (3.4) we obtain

\[ N_{f^{n}(f(k))^{m}}(0, r) \leq (k + 1)N_{1,f}^{(k+1)}(0, r) + kN_{1,f}(\infty, r) \]
\[ + \frac{k + 1}{n - 1} (N_{f^{n}(f(k))^{m}}(0, r) - N_{1,f^{n}(f(k))^{m}}(0, r)) \]
\[ - ((k + 1)n + m - 1)N_{1,f}^{(k+1)}(0, r)) + O(1). \]

Thus

\[ \frac{n + k}{n - 1}N_{1,f^{n}(f(k))^{m}}(0, r) \leq \frac{k + 1}{n - 1} N_{f^{n}(f(k))^{m}}(0, r) + kN_{1,f}(\infty, r) \]
\[ + (k+1-\frac{(k+1)((k+1)n+m-1)}{n-1})N_{1,f}^{(k+1)}(0, r) + O(1). \]

Note that

\[ k + 1 - \frac{(k+1)((k+1)n+m-1)}{n-1} < 0, \]

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we have

$$N_{1,f^n(f(k))m}(0,r) \leq \frac{k+1}{n+k} N_{f^n(f(k))m}(0,r) + \frac{k(n-1)}{n+k} N_{1,f}(\infty,r) + O(1).$$

Moreover if $a$ is a pole of $f$ with multiplicity $t$ then $a$ is a pole of $f^n(f(k))^m$ with multiplicity $nt + (t + k)m \geq n + (1 + k)m$. Thus

$$N_{f^n(f(k))m}(\infty,r) \geq (n + (k + 1)m) N_{1,f}(\infty,r),$$

and

$$N_{1,f^n(f(k))m}(\infty,r) = N_{1,f}(\infty,r).$$

Therefore,

$$T_{f^n(f(k))m}(r) \leq \frac{k+1}{n+k} N_{f^n(f(k))m}(0,r) + (1 + \frac{k(n-1)}{n+k}) N_{1,f^n(f(k))m}(\infty,r) + N_{1,f^n(f(k))m}(a,r) - \log r + O(1),$$

$$T_{f^n(f(k))m}(r) \leq \frac{k+1}{n+k} N_{f^n(f(k))m}(0,r) + \frac{n(k+1)}{(n+k)(n+(k+1)m)} N_{f^n(f(k))m}(\infty,r) + N_{1,f^n(f(k))m}(a,r) - \log r + O(1).$$

From this and by Lemma 2.1, we have

$$\frac{n(n-2) + k(mn - m - n) + m(n-1)}{(n+k)(n+m+km)} T_{f^n(f(k))m}(r) \leq N_{1,f^n(f(k))m}(a,r) - \log r + O(1).$$

By Lemma 3.1

$$\frac{n(n-2) + k(mn - m - n) + m(n-1)}{(n+k)(n+m+km)} T_f(r) \leq N_{1,f^n(f(k))m}(a,r) - \log r + O(1).$$

Applying the above arguments to case $m = 1$, and using $n^2 - n - k > 0$, we obtain 2.

Lemma 3.2 is proved. \(\Box\)

For simplicity we denote:

$$B = f(f(k))^m, b = f(k), c = f(f(k))^m - 1, v = \frac{1}{(f(k))^m},$$

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\[ a_0 = \frac{v^{(k+1)} - \frac{b'}{b} v^{(k)}}{v} \], \quad a_i = \frac{(k+1)^{v^{(k+1-i)}} - \frac{b'}{b} v^{(k-i)}}{v}, \\
i = 1, 2, \ldots, k, a_{k+1} = 1.

Then we have the following lemma.

**Lemma 3.3.** — Let \( f \) be a non-constant meromorphic function on \( \mathbb{C}_p \) such that \( f^{(k)} \neq 0 \), and let \( k > 0, m > 1 \) be integers. Then we have

\[ B^{(k+1)} + a_k B^{(k)} + \ldots + a_1 B^{(1)} + a_0 B \equiv 0. \]

**Proof.** — We first prove that

\[ (Bv)^{(j)} \equiv \sum_{i=0}^{j} \binom{j}{i} B^{(i)} v^{(j-i)}, \quad (3.5) \]

\( j = 1, 2, \ldots, k + 1 \), by induction.

For \( j = 1 \), we have

\[ (Bv)^{(1)} \equiv \sum_{i=0}^{1} \binom{1}{i} B^{(i)} v^{(1-i)}. \]

Assume

\[ (Bv)^{(j)} \equiv \sum_{i=0}^{j} \binom{j}{i} B^{(i)} v^{(j-i)}, \]

we will prove that

\[ (Bv)^{(j+1)} \equiv \sum_{i=0}^{j+1} \binom{j+1}{i} B^{(i)} v^{(j+1-i)}. \]

Indeed, we have

\[ (Bv)^{(j+1)} \equiv ((Bv)^{(j)})^{(1)} \equiv \sum_{i=0}^{j} \binom{j}{i} (B^{(i)} v^{(j-i)})^{(1)} \]

\[ \equiv \sum_{i=0}^{j} \binom{j}{i} (B^{(i+1)} v^{(j-i)} + B^{(i)} v^{(j+1-i)}) \equiv \sum_{i=0}^{j+1} \binom{j+1}{i} B^{(i)} v^{(j+1-i)}. \]

Returning to the proof of Lemma 3.3, from \( b = f^{(k)} \), we have \( b' = f^{(k+1)} \). Therefore

\[ f^{(k+1)} - \frac{b'}{b} f^{(k)} \equiv 0 \quad (3.6) \]

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Because $B = f(f^{(k)})^m$, $v = \frac{1}{(f^{(k)})^m}$, we obtain $f \equiv Bv$. Since (3.6) we have

$$ (Bv)^{(k+1)} - \frac{b'}{b}(Bv)^{(k)} \equiv 0 \quad (3.7) $$

From (3.5), (3.7) we obtain

$$ \sum_{i=0}^{k+1} \binom{k+1}{i} B^{(k)} v^{(k+1-i)} - \sum_{i=0}^{k} \binom{k}{i} B^{(i)} v^{(k-i)} \equiv 0. $$

Thus

$$ Bu^{(k+1)} + \binom{k+1}{1} B^{(1)} v^{(k)} + \binom{k+1}{2} B^{(2)} v^{(k-1)} + \ldots + \binom{k+1}{k} B^{(k)} v^{(1)} + B^{(k+1)} v $$

$$ - \frac{b'}{b} (Bu^{(k)} + \binom{k}{1} B^{(1)} v^{(k-1)} + \binom{k}{2} B^{(2)} v^{(k-2)} + \ldots + \binom{k}{k-1} B^{(k-1)} v^{(1)} + B^{(k)} v) $$

$$ \equiv 0. $$

Dividing the left hand side by $v$, we get

$$ \frac{v^{(k+1)} - \frac{b'}{b} v^{(k)}}{v} B + \binom{k+1}{1} \frac{v^{(k)}}{v} - \binom{k}{1} \frac{b'}{b} \frac{v^{(k-1)}}{v} B^{(1)} $$

$$ + \ldots + \binom{k+1}{k} \frac{v^{(1)}}{v} - \binom{k}{k} \frac{b'}{b} \frac{v^{(0)}}{v} B^{(k)} + B^{(k+1)} \equiv 0. $$

So

$$ B^{(k+1)} + a_k B^{(k)} + \ldots + a_1 B^{(1)} + a_0 B \equiv 0. \quad (3.8) $$

□

**Lemma 3.4.** — Let $f$ be a non-constant meromorphic function on $\mathbb{C}_p$ such that $f^{(k)} \neq 0$, and let $k > 0, m > 1$ be integers. Suppose that $f$ is not a polynomial of degree $k$. Then we have $a_0 \neq 0$, and

$$ \frac{m^2k + m^2 - 2mk - m - 1}{m(k+1)(mk + m + 1)} T_f(r) \leq N_{1,f(f^{(k)})^m}(1, r) + O(1). $$

**Proof.** — Suppose $a_0 \equiv 0$. Therefore $a_0 = \frac{v^{(k+1)} - \frac{b'}{b} v^{(k)}}{v}$, we get

$$ v^{(k+1)} \equiv \frac{b'}{b} v^{(k)}. \quad (3.9) $$

Consider following two cases.
Case 1. \(v^{(k)} \equiv 0\). We have \(v \equiv h\), a polynomial of degree \(< k\), and \(h \not\equiv 0\). Thus \((f^{(k)})^m h \equiv 1\). If \(z_0\) is a pole of \(f^{(k)}\), then \(z_0\) is a pole of \(f\) with multiplicity at least \(k + 1\). So \(z_0\) is a zero of \(h\) with multiplicity at least \(k + 1\), a contradiction. Thus \(f^{(k)}\) has no poles, and from \((f^{(k)})^m h \equiv 1\) it follows that \(f\) is a polynomial of degree \(k\), a contradiction.

Case 2. \(v^{(k)} \not\equiv 0\). From (3.8), we have

\[
\frac{v^{(k+1)}}{v^{(k)}} \equiv \frac{b'}{b}.
\]

So \(v^{(k)} \equiv cb \equiv cf^{(k)}, c \not\equiv 0\). Solving this, we get

\[
v \equiv c(f + t), t^{(k)} \equiv 0.
\]

From this \(t\) we see that \(t\) is a polynomial of degree \(< k\), and \(\frac{1}{(f^{(k)})^m} \equiv c(f + t)\). Thus \(c(f + t)(f^{(k)})^m \equiv 1\). Set \(F = f + t\). Then \(F^{(k)} \equiv f^{(k)}\) and \(cF(F^{(k)})^m \equiv 1\). By Lemma 3.1, we get a contradiction, and then \(a_0 \not\equiv 0\).

Now we are going to prove the inequality in the lemma. Since \(k, m\) are positive integers and \(m \geq 2\), it is easy to see that \(m^2k + m^2 - 2mk - m - 1 \geq 0\). From (3.8) and \(B \equiv c + 1\) we get

\[
(c + 1)^{(k+1)} + a_k(c + 1)^{(k)} + ... + a_1(c + 1)^{(1)} + a_0(c + 1) \equiv 0,
\]

\[
c^{(k+1)} + a_k c^{(k)} + ... + a_1 c^{(1)} + a_0(c + 1) \equiv 0,
\]

\[
a_0 c + c^{(k+1)} + a_k c^{(k)} + ... + a_1 c^{(1)} \equiv -a_0,
\]

(3.10)

\[
\frac{1}{a_0} \left( \frac{c^{(k+1)}}{c} + a_k \frac{c^{(k)}}{c} + ... + a_1 \frac{c^{(1)}}{c} \right) + \frac{1}{c} + 1 \equiv 0.
\]

(3.11)

Since \(a_0 = \frac{v^{(k+1)} - \frac{b'}{b} v^{(k)}}{v}\), we see that any pole of \(a_0\) can occur only at poles or zeros of \(b\), and each pole of \(a_0\) has multiplicity at most \(k + 1\). So

\[
N_{a_0}(\infty, r) \leq (k + 1)(N_{1,b}(\infty, r) + N_{1,b}(0, r))
\]

\[
\leq (k + 1)(N_{1,f}(\infty, r) + N_{1,b}(0, r)).
\]

On the other hand, a zero of \(b\) of multiplicity \(s\) is a zero of \(c'\) of multiplicity at least \(ms - 1 \geq (m - 1)s\). Also, \(c + 1 \not\equiv 0\) at such a zero of \(b\).

\[
N_{1,b}(0, r) \leq \frac{1}{m - 1} N_{c'}(\infty, r)
\]

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\[
\begin{align*}
\leq & \frac{1}{m-1} H_{c'}(r) = \frac{1}{m-1} H_{c'}(r) \\
= & \frac{1}{m-1} \left( N_{c'}(\infty, r) + m_{c'}(\infty, r) \right) \\
= & \frac{1}{m-1} N_{c'}(\infty, r) + O(1) \\
= & \frac{1}{m-1} \left( N_{1,c}(\infty, r) + N_{1,c}(0, r) \right) + O(1) \\
= & \frac{1}{m-1} \left( N_{1,f}(\infty, r) + N_{1,c}(0, r) \right) + O(1).
\end{align*}
\]

Thus

\[
N_{a_0}(\infty, r) \leq (k + 1) \left( N_{1,b}(\infty, r) + N_{1,b}(0, r) \right) \\
\leq (k + 1) \left( N_{1,f}(\infty, r) + \frac{1}{m-1} \left( N_{1,f}(\infty, r) + N_{1,c}(0, r) \right) \right) + O(1) \\
= \frac{m(k+1)}{m-1} N_{1,f}(\infty, r) + \frac{k+1}{m-1} N_{1,c}(0, r).
\]

Note that $B \equiv c + 1 \equiv f(f(k))^m$. Therefore a pole of $f$ of multiplicity $s$ is a pole of $B$ of multiplicity $s + (s + k)m \geq 1 + (1 + k)m$. So

\[
N_{1,f}(\infty, r) \leq \frac{1}{1 + m(k+1)} N_B(\infty, r) \leq \frac{1}{1 + m(k+1)} T_B(r) + O(1).
\]

Combining the above inequalities and note that $T_B(r) = T_c(r) + O(1)$ we obtain

\[
N_{a_0}(\infty, r) \leq \frac{m(k+1)}{(m-1)(1 + m(k+1))} T_c(r) + \frac{k+1}{m-1} N_{1,c}(0, r) + O(1).
\]

Since (3.10), a zero of $c$ of multiplicity $s > k + 1$ is a zero of $a_0$. From this and (3.11) we have

\[
N_c(0, r) \leq N_{a_0}(0, r) + (k + 1) N_{1,c}(0, r),
\]

\[
m_c(0, r) \leq m_{a_0}(0, r) + O(1).
\]

Then (3.8) and Lemma 2.1 give us

\[
T_c(r) = N_c(0, r) + m_c(0, r) + O(1) \\
\leq N_{a_0}(0, r) + (k + 1) N_{1,c}(0, r) + m_{a_0}(0, r) + O(1) \\
\leq T_{a_0}(r) + (k + 1) N_{1,c}(0, r) + O(1) \\
= N_{a_0}(\infty, r) + m_{a_0}(\infty, r) + (k + 1) N_{1,c}(0, r) + O(1)
\]

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\[
\begin{align*}
&= N_{a_0}(\infty, r) + m \frac{B^{(k+1)} + a_k B^{(k)} + \ldots + a_1 B^{(1)}}{B} (\infty, r) \\
&\quad + (k + 1) N_{1,c}(0, r) + O(1) \\
&= N_{a_0}(\infty, r) + (k + 1) N_{1,c}(0, r) + O(1) \\
&\leq \frac{m(k + 1)}{(m - 1)(1 + m(k + 1))} T_c(r) + \frac{k + 1}{m - 1} N_{1,c}(0, r)(0, r) \\
&\quad + (k + 1) N_{1,c} + O(1) \\
&\leq \frac{m(k + 1)}{(m - 1)(1 + m(k + 1))} T_c(r) + \frac{m(k + 1)}{m - 1} N_{1,c}(0, r) + O(1). \\
\end{align*}
\]

So
\[
(1 - \frac{m(k + 1)}{(m - 1)(1 + m(k + 1))}) T_c(r) \leq \frac{m(k + 1)}{m - 1} N_{1,c}(0, r) + O(1).
\]

From this and Lemma 3.1 we obtain
\[
\frac{m^2 k + m^2 - 2mk - m - 1}{m(k + 1)(mk + m + 1)} T_f(r) \leq N_{1,f(f^{(k)})^m}(1, r) + O(1).
\]

□

Now we use the above Lemmas to prove the main result of the paper.

**Proof of Theorem 1.2.** — Assume, on the contrary, that \( f \) is not a polynomial of degree \(< k \).

If \( f \) is an entire function, then from Lemma 3.1 it implies that \((f^n(f^{(k)})^m)\) is not constant. Then \((f^n(f^{(k)})^m(z) - 1)\) must have a zero, a contradiction.

Assume \( k > 0 \). If \( m > 1, n > 1 \) then the condition 1. in Lemma 3.2 holds, and we see that \((f^n(f^{(k)})^m(z) - 1)\) is not constant, so it must have a zero, a contradiction.

If \( m = 1, n = 1 \), the condition 2. in Lemma 3.2 is satisfied. Then \((f^n f^{(k)}(z) - 1)\) must have a zero, a contradiction.

Now let \( m > 1, n = 1 \). It is easy to see that in this case we have \( m^2 k + m^2 - 2mk - m - 1 > 0 \). If \( f \) is a polynomial of degree \(> k \), then by Lemma 3.3, we see that \((f(f^{(k)})^m(z) - 1)\) has a zero, a contradiction. On the other hand, if \( f \) is a polynomial of degree \( k \), or \( f \) is a transcendental function, then it is obviously that \((f(f^{(k)})^m(z) - 1)\) also has a zero, a contradiction.

It remains to consider the case when the condition 3. is satisfied. Then \( f^{(k)} \not\equiv 0 \). Write \( f = \frac{f_1}{f_2} \), where \( f_1 \) and \( f_2 \) are holomorphic functions,
having no common zeros, and \( f(k) = \frac{a_k}{f_2^k} \), where \( a_k \) is a polynomial of \( f_1, f_2, f'_1, f'_2, \ldots, f_1^{(k)}, f_2^{(k)} \). If \( f_2 \) is constant, then by \( |f_1|_r < C|f_2|_r \), we see that \( f_1 \) is constant, and therefore, \( f \) is constant, a contradiction. Suppose that \( f_2 \) is non-constant. Then \( f_2 \) has a zero. Let \( d \) denote the greatest common divisor of \( a_k \) and \( f_2^{k+1} \). Set \( h = \frac{a_k}{d} \) and \( l = \frac{f_2^{k+1}}{d} \). Let \( d_1 \) denote the greatest common divisor of \( h^m \) and \( f_2^{n} \). Set \( h_1 = \frac{h^m}{d_1} \) and \( l_1 = \frac{f_2^{n}}{d_1} \). Then

\[
 f^n(f(k))^m - 1 = \frac{f_1^n h^{m} - f_2^{n} l^{m}}{f_2^{n} l^{m}} = \frac{f_1^n h_1 - l_1 l^{m}}{l_1 l^{m}} \quad (3.11)
\]

Note that \( f^n(f(k))^m(z) \neq 1 \) for all \( z \in \mathbb{C}_p \). Thus \( f^n(f(k))^m \neq 1 \). If \( l \) is constant, then \( f(k) \) is an entire function. Thus \( f \) is an entire function, a contradiction. So \( l \) is non-constant. Therefore, \( l \) has a zero.

Next we are going to show by induction that \( |f_1^n|_r |a_k^n|_r < |f_2^{n+(k+1)}|_r \), for all \( r \) satisfying \( r > R_0, r > r_0 \), where \( R_0 \) is a some constant. For \( k = 1 \), we have \( a_1 = f'_1 f_2 - f'_2 f_1 \). Since \( |f'_1|_r \leq \frac{|f_1|^c}{r}, |f'_2|_r \leq \frac{|f_2|^c}{r} \) and \( |f_1|_r < C|f_2|_r \), we get \( |f'_1 f_2|_r \leq \frac{|f_1|^c |f_2|^c}{r} \), \( |f'_2 f_1|_r \leq \frac{|f_1|^c |f_2|^c}{r} \) and \( |f_1^n|_r |a_k^n|_r < |f_2^{n+2m}|_r \), for all \( r \) satisfying \( r > R_1, r > r_0 \), where \( R_1 \) is a some constant. Assume we have \( |f_1^n|_r |a_k^n|_r < |f_2^{n+(i+1)m}|_r \), for all \( r \) satisfying \( r > R_i, r > r_0 \), where \( R_i \) is a some constant. Now for \( k = i + 1 \) we get \( a_{i+1} = f'_i f_2 - f'_2 (i + 1) a_i \). By the induction hypothesis and \( |i + 1| \leq 1 \), \( |a_i'|_r \leq \frac{|a_i|_r}{r}, |f'_i|_r \leq \frac{|f_2|^c}{r} \), we have \( |f_2^n|_r |a_{i+1}^m|_r < |f_2^{n+(i+2)}|_r \), for all \( r \) satisfying \( r > R_{i+1}, r > r_0 \).

So \( |f_1^n|_r |a_k^n|_r < |f_2^{n+(k+1)m}|_r \), for all \( r \) satisfying \( r > R_0, r > r_0 \), where \( R_0 \) is a some constant. From this and (3.11) it follows \( N_{f^n(f(k))^m}(1, r) = N_{f_1^n h_1-l_1 l^{m}}(r) \) and \( |f'_1|_r |h^{m}|_r < |f'_2|_r |l^{m}|_r, |f_1^n|_r |h_1|_r < |l_1|_r |l^{m}|_r. \) Therefore \( |f_1^n h_1 - l_1 l^{m}|_r = |l_1 l^{m}|_r \). So \( T_{f_1^n h_1-l_1 l^{m}}(r) = T_{l_1 l^{m}}(r) \). By Lemma 2.1 we get \( N_{f_1^n h_1-l_1 l^{m}}(r) = N_{l_1 l^{m}}(r) + O(1) \). Because \( l \) has a zero. Thus \( l_1 l^{m} \) has a zero. Therefore, \( f^n(f(k))^m - 1 \) has a zero, a contradiction.

Theorem 1.2 is proved.

By taking \( k = 1 \) we have a differential monomial like in Hayman results, and from Theorem 1.2 it follows the following

**Corollary 3.5.** — Let \( f \) be a meromorphic function on \( \mathbb{C}_p \), satisfying the condition \( f^n(f')^m(z) \neq 1 \) for all \( z \in \mathbb{C}_p \) and for some positive integers \( n, m \). Then \( f \) is a constant function if one of the following conditions holds:

1. \( f \) is an entire function,
2. \( \max\{m, n\} > 1 \),
3. There exist constants \( C, r_0 \) such that \( |f|_r < C \) for all \( r > r_0 \).
Remark. — Indeed, in [16], Theorem 3 shows that $f' + f^4$ has at least one zero that is not a zero of $f$, hence setting $g(x) = \frac{1}{f(x)}$, we can check that $g^2 g'$ takes the value 1 at least one time. So the case $n = 2, m = k = 1$ of Theorem 1.2 has been established in [16].

Bibliography


