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Sets in $\mathbb{C}^N$ with vanishing global extremal function and polynomial approximation

Józef Siciak[(1)]

Abstract. — Let Γ be a non-pluripolar set in $\mathbb{C}^N$. Let f be a function holomorphic in a connected open neighborhood G of Γ. Let $\{P_n\}$ be a sequence of polynomials with $\deg P_n \leqslant d_n$ $(d_n < d_{n+1})$ such that

$$\limsup_{n\to\infty} |f(z) - P_n(z)|^{1/d_n} < 1, \; z \in \Gamma.$$

We show that if

$$\limsup_{n\to\infty} |P_n(z)|^{1/d_n} \leqslant 1, \; z \in E,$$

where E is a set in $\mathbb{C}^N$ such that the global extremal function $V_E \equiv 0$ in $\mathbb{C}^N$, then the maximal domain of existence G_f of f is one-sheeted, and

$$\limsup_{n\to\infty} \|f - P_n\|_K^{\frac{1}{d_n}} < 1$$

for every compact set $K \subset G_f$. If, moreover, the sequence $\{d_{n+1}/d_n\}$ is bounded then $G_f = \mathbb{C}^N$.
If E is a closed set in $\mathbb{C}^N$ then $V_E \equiv 0$ if and only if each series of homogeneous polynomials $\sum_{j=0}^{\infty} Q_j$, for which some subsequence $\{s_{n_k}\}$ of partial sums converges point-wise on E, possesses Ostrowski gaps relative to a subsequence $\{n_{k_l}\}$ of $\{n_k\}$.
In one-dimensional setting these results are due to J. Müller and A. Yavrian [5].

Résumé. — Soit Γ un sous-ensemble non pluripolaire de $\mathbb{C}^N$. Soit f une fonction holomorphe sur un voisinage ouvert connexe G de Γ. Soit $\{P_n\}$ une suite de polynômes de degré $\deg P_n \leqslant d_n$ $(d_n < d_{n+1})$ telle que

$$\limsup_{n\to\infty} |f(z) - P_n(z)|^{1/d_n} < 1, \; z \in \Gamma.$$

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On démontre que si

$$\limsup_{n\to\infty} |P_n(z)|^{1/d_n} \leqslant 1, \ z \in E,$$

où E is est un sous-ensemble de $\mathbb{C}^N$ tel que la fonction extrémale globale $V_E \equiv 0$ sur $\mathbb{C}^N$, alors le domaine maximal d'existence G_f de f est uniforme, et

$$\limsup_{n\to\infty} \|f - P_n\|_K^{\frac{1}{d_n}} < 1$$

pour tout compact $K \subset G_f$. Si, de plus, la suite $\{d_{n+1}/d_n\}$ est bornée alors $G_f = \mathbb{C}^N$.
Si E est un sous-ensemble fermé de $\mathbb{C}^N$ alors $V_E \equiv 0$ si et seulement si chaque série de polynômes homogènes $\sum_{j=0}^{\infty} Q_j$, ayant une sous-suite $\{s_{n_k}\}$ de sommes partielles convergeant ponctuellement sur E, admet des lacunes de type Ostrowski relativement à une sous-suite $\{n_{k_l}\}$ de $\{n_k\}$.
En dimension 1, ces résultats sont dûs à J. Müller and A. Yavrian [5].

1. Introduction

Given an open set Ω in $\mathbb{C}^N$, let $PSH(\Omega)$ denote the set of all plurisubharmonic (PSH) functions in Ω. Let $\mathcal{L}$ be the class of PSH functions in $\mathbb{C}^N$ with minimal growth, i.e. $u \in \mathcal{L}$ if and only if $u \in PSH(\mathbb{C}^N)$ and $u(z) - \log(1 + \|z\|) \leqslant \beta$ on $\mathbb{C}^N$, where β is a real constant depending on u.

If E is a subset of $\mathbb{C}^N$, the *global extremal function* V_E associated with E is defined as follows.

If E is bounded, we put

$$V_E(z) := \sup\{u(z);\ u \in \mathcal{L},\ u \leqslant 0 \text{ on } E\}, \ \ z \in \mathbb{C}^N.$$

If E is unbounded, we put (see [7])

$$V_E(z) := \inf\{V_F(z);\ F \subset E,\ F \text{ is bounded}\}, \quad \mathrm{z} \in \mathbb{C}^{\mathrm{N}}.$$

It is known (see e.g. [6, 7]) that V_E^* (the upper semicontinuous regularization) is a member of $\mathcal{L}$ iff E is non-pluripolar (non-plp). $V_E^* \equiv +\infty$ iff E is pluripolar (plp).

If $N = 1$ and E is a compact non-polar subset of $\mathbb{C}$, then $V_E^*(z) \equiv g_E(z, \infty)$ for $z \in D_\infty$, where D_∞ is the unbounded component of $\mathbb{C} \setminus E$, and g_E is the Green function of D_∞ with the logarithmic pole at infinity.

If $N \geqslant 2$ and E is non-pluripolar, the function V_E^* is called *pluricomplex Green function* (with pole at infinity).

By [5] a closed subset E of $\mathbb{C}$ is non-thin at ∞ if and only if $V_E^* \equiv 0$. One can check that for all $E \subset \mathbb{C}^N,\ N \geqslant 1$, we have $V_E^* \equiv 0$ if and only if $V_E \equiv 0$. Therefore, one can agree with the author of [9] that it is reasonable to say that a set $E \subset \mathbb{C}^N$ is *non-thin at infinity* (resp., *thin at infinity*), if $V_E \equiv 0$ (resp., $V_E \not\equiv 0$). In particular, if $V_E^* \equiv \infty$ the set E is thin at infinity.

In chapter 2 of this paper we discuss properties of sets E in $\mathbb{C}^N$ with $V_E \equiv 0$. Similarly, as in [5] and [9] , very important role in our applications is played by the necessary and sufficient conditions stated in section 2.18 (which are a slightly modified version of the conditions of Tuyen Trung Truong's Theorem 2 in [9]).

In chapters 3 and 4 we prove an N-dimensional version of the classical Ostrowski Gap Theorems for power series of a complex variable.

In chapters 5 an 6 we show that properties of sets $E \subset \mathbb{C}^N$ with $V_E \equiv 0$ $(N \geqslant 1)$ may be applied to obtain results in N-dimensional setting analogous to those obtained earlier by J. Müller and A. Yavrian [5] in the one-dimensional case.

2. Sets in $\mathbb{C}^N$ with $V_E \equiv 0$

Now we shall state several properties of the global extremal function. Most of the properties are known and follow either from the elementary theory of the Lelong class $\mathcal{L}$ and from the definition of the extremal function, or from the Bedford-Taylor theorem on negligible sets in $\mathbb{C}^N$.

In the sequel F, E, E_n (resp., K, K_n) are arbitrary (resp., compact) subsets of $\mathbb{C}^N$.

2.1. Monotonicity property of the extremal function. $V_F \leqslant V_E$, if $E \subset F$.

2.2. $V_E = \lim_{R\to\infty} V_{E_R}$, where $E_R := E \cap B(0,R)$, and $B(0,R) := \{z \in \mathbb{C}^N;\ \|z\| < R\}$ (resp., $B(0,R) := \{\|z\| \leqslant R\}$).

2.3. $V_E^*(z) = \lim_{R\to\infty} V_{E_R}^*(z) = \sup\{u(z);\ u \in \mathcal{L}, u \leqslant 0 \quad \text{q.a.e. on } E\}$, where "q.a.e. on E" means that the corresponding property holds quasi-almost everywhere on E, i.e. on $E \setminus A$, where A is a pluripolar set.

Hence, *if E is non-pluripolar then the pluricomplex Green function V_E^* is the unique maximal element of the set $\mathcal{W}^*(E) := \{u \in \mathcal{L}, u \leqslant 0 \text{ q.a.e. on } E\}$ ordered by the condition: if $u_1, u_2 \in \mathcal{W}^*(E)$ then $u_1 \preceq u_2$ if $u_1(z) \leqslant u_2(z)$ for all $z \in \mathbb{C}^N$.*

2.4. $V_{K_n} \uparrow V_K$, if $K_{n+1} \subset K_n$, $K = \cap K_n$.

2.5. $V^*_{E_n} \downarrow V^*_E$, if $E_n \subset E_{n+1}$, $E = \cup E_n$.

2.6. $(\lim V_{E_n})^* = (\lim V^*_{E_n})^* = V^*_E$, if $E_{n+1} \subset E_n, E = \cap E_n$.

2.7. If $E,\ A$ are subsets of $\mathbb{C}^N$ and A is pluripolar then $V^*_{E\cup A} \equiv V^*_E \equiv V^*_{E\setminus A}$.

2.8. Product property of the extremal function [1]. If $E \subset \mathbb{C}^M$, $F \subset \mathbb{C}^N$ then

$$V^*_{E\times F}(z,w) = \max\{V^*_E(z)\,,\, V^*_F(w)\},\ (z,w) \in \mathbb{C}^M \times \mathbb{C}^N.$$

Hence, *a product $E \times F$ is non-thin at infinity if and only if the both factors are non-thin at infinity* (a different proof of this property was given in [9]) .

In the sequel we shall omit "at infinity" while speaking about non-thin (resp., thin) sets at infinity.

2.9. *A set E in $\mathbb{C}^N$ is non-thin if and only if the set $E\setminus B$ (resp., $E\cup B$) is non-thin for every bounded set B.*

Without loss of generality we may assume that B is a ball $B(0,R)$. If $E \setminus B$ is non-thin then E is non-thin by the monotonicity property.

Now assume that E is non-thin. Then $E \setminus B$ is non-pluripolar because otherwise we would have $\log^+ \frac{\|z\|}{R} \equiv V^*_B(z) \equiv V^*_{B\cup(E\setminus B)}(z) \equiv V^*_E(z) \equiv 0$. A contradiction. Therefore $V^*_{E\setminus B} \in \mathcal{L}$. Put $M = \max_{\|z\|=R} V^*_{E\setminus B}(z)$. Then $u := V^*_{E\setminus B} - M \in \mathcal{L}$ and $u \leqslant 0$ q.a.e. on E. Hence $u \leqslant V^*_E \equiv 0$ in $\mathbb{C}^N$ which implies that $E \setminus B$ is non-thin.

It is obvious that $E \cup B$ is non-thin if E is non-thin. In order to show the inverse implication, it sufficient to observe that $E \setminus B = (E \cup B) \setminus B$.

2.10. *If E is non-pluripolar then the limit*

$$\sigma := \lim_{R\uparrow\infty} \max_{\|z\|=R} V^*_E(z)/\log R$$

exists and σ either equals 0 *(if and only if E is non-thin), or $\sigma = 1$ (if and only if E is thin).*

The function V^*_E is a member of the class $\mathcal{L}$. Therefore the limit exists and $0 \leqslant \sigma \leqslant 1$. One can check that $\sigma = 0$ if and only if E is non-thin.

We should show that the case $0 < \sigma < 1$ is excluded. Indeed, the function $u := \frac{1}{\sigma}V_E^*$ is a member of $\mathcal{L}$, and $u \leqslant 0$ q.a.e. on E. Hence, $\frac{1}{\sigma}V_E^* \leqslant V_E^*$ on $\mathbb{C}^N$. It follows that $\sigma \geqslant 1$. Consequently, $\sigma = 1$.

2.11. Robin function, Robin constant and logarithmic capacity. If E is non-pluripolar then there exists a uniquely determined homogeneous PSH function $\tilde{V}_E(\lambda, z)$ of $1 + N$ variables $(\lambda, z) \in \mathbb{C} \times \mathbb{C}^N$ such that $\tilde{V}_E(1, z) = V_E^*(z)$ on $\mathbb{C}^N$. One may check that $\tilde{V}(\lambda, z) = \log|\lambda| + V_E^*(z/\lambda)$ if $\lambda \neq 0$, and $\tilde{V}_E(0, z) = \limsup_{(\lambda,\zeta)\to(0,z)}(\log|\lambda| + V_E(\zeta/\lambda))$.

The homogeneous PSH function $\tilde{V}_E(0\,,\,z)$ is called *Robin function of* E, and the set function $\gamma(E) := \max_{\|z\|=1} \tilde{V}_E(0, z)$ - *Robin constant of* E. The set function $c(E) := e^{-\gamma}$ is called *logarithmic capacity of* E. It is clear that the Robin constant and the logarithmic capacity of E depend on the choice of the norm $\|\cdot\|$ in $\mathbb{C}^N$.

2.12. A necessary condition for non-thinness. *If* E *is non-thin then* $c(E) = \infty$.

Indeed, if $V_E \equiv 0$ then $\tilde{V}_E(\lambda\,,\,z) \equiv \log|\lambda|$. Hence, $\tilde{V}_E(0, z) \equiv -\infty$ which implies that $\gamma(E) = -\infty$, i.e. $c(E) = +\infty$.

It is known that the condition 2.12 is not sufficient for closed subsets of the complex plane (and, consequently, for subsets of $\mathbb{C}^N$ with $N \geqslant 2$). We shall give a simple example.

2.13. An example of a closed set $E \subset \mathbb{C}$ with $V_E \not\equiv 0$ and $c(E) = \infty$.

Let $\{a_n\}$, $\{\epsilon_n\}$ be two sequences of real numbers such that:

$$a_{n+1} > a_n > 0, \quad \epsilon_n > 0, \quad \sum_1^\infty \epsilon_n = 1, \quad \lim_{n\to\infty} \sum_1^n \epsilon_k \log a_k = +\infty,$$

e.g. $\epsilon_n = 2^{-n}$, $a_n = e^{2^n}$.

Put

$$U(z) := \sum_1^\infty \epsilon_n \log \frac{|z - a_n|}{1 + a_n}, \quad E := \{z;\, U(z) \leqslant 0\}.$$

It is clear that E is closed and unbounded. It remains to check that $c(E) = +\infty$ and $V_E(z) = U^+(z)$, where $U^+(z) := \max\{0\,,\,U(z)\}$. To this order we put

$$U_n(z) := (\sum_1^n \epsilon_k)^{-1} \sum_1^n \epsilon_k \log \frac{|z - a_k|}{1 + a_k}, \quad E_n := \{z; U_n(z) \leqslant 0\}.$$

One can easily check that E_n is compact and regular (E_n is a finite union of non-trivial continua), $E_n \subset E_{n+1}$, $V_{E_n}(z) \equiv U_n^+(z) \downarrow U^+(z) \equiv V_E(z)$, $\tilde{V}_{E_n}(\lambda, z) = (\sum_1^n \epsilon_k)^{-1} \sum_1^n \epsilon_k \log \frac{|z-\lambda a_k|}{1+a_k}$ if $|z/\lambda| \geqslant R = R(n) = const > 0$, $\tilde{V}_{E_n}(0, z) \equiv \log \|z\| + \gamma(E_n)$ for all $z \in \mathbb{C}$, and hence $\log c(E_n) = -\gamma(E_n) = (\sum_1^n \epsilon_k)^{-1} \sum_1^n \epsilon_k \log(1 + a_k)$ for all $n \geqslant 1$, which gives the required result.

Taking $E \times F$ with E in $\mathbb{C}$ as just above , and with a non-thin subset F of $\mathbb{C}^{N-1}$ ($N \geqslant 2$), one gets a thin subset of $\mathbb{C}^N$ with $c(E \times F) = \infty$.

2.14. A sufficient condition for non-thinness. Using an inequality due to B. A. Taylor [8] one can show (see [9] for details) that a sufficient condition for E to be non-thin is

$$\limsup_{R\uparrow\infty} \frac{\log c(E_R)}{\log R} > 1 - \frac{1}{C_N},$$

where C_N is a constant depending only on the dimension N with $C_N > 1$ for $N \geqslant 2$, and $C_1 = 1$.

2.15. Example. Let $\{a_n\}$ be a sequence of distinct points in $\mathbb{C}^N$ with $a_n \neq 0$ ($n \geqslant 1$). Let ϵ_n be a sequence of positive real numbers such that $\sum_1^\infty \epsilon_n = 1$. Let u be the function defined by

$$u(z) = \sum_1^\infty \epsilon_n \log \frac{\|z - a_n\|}{1 + \|a_n\|}, \quad z \in \mathbb{C}^N.$$

Then u is a non-constant ($u(0) \geqslant -\log 2$, $u(a_n) = -\infty$ for every $n \geqslant 1$) member of the class $\mathcal{L}$ such that $E := \{z\,; u(z) < 0\}$ is an open set containing the unit ball and all points a_n. It is clear that E is thin. Moreover, if the sequence $\{a_n\}$ is dense in $\mathbb{C}^N$ then E is a thin unbounded open set dense everywhere.

2.16. Example. Every non-pluripolar real cone E in $\mathbb{C}^N$ (without loss of generality, we assume that E has its vertex at the origin, so that $tz \in E$, if $t \in \mathbb{R}$, $t \geqslant 0$, $z \in E$) is non-thin. Indeed, one can check that the sets $E_R := E \cap \{\|z\| \leqslant R\}$ are non-pluripolar, and $E_R = RE_1$ for all $R \geqslant 1$. Observe that $V_E(z) \leqslant V_{E_R}(z) \equiv V_{E_1}(\frac{1}{R}z)$ for all z in $\mathbb{C}^N$ and for $R \geqslant 1$. It follows that $V_E(z) \leqslant V_{E_1}(0)$ for all z which gives the required result.

2.17. Example. It follows from *Wiener Criterion* [3] that if E is a countable union of closed (or open) discs $\{z \in \mathbb{C};\, |z - a_n| \leqslant r\}$, where $r = const > 0$, $a_n \in \mathbb{C}$ and $a_n \to \infty$ as $n \to \infty$, then E is non-thin at infinity.

We shall show that analogous property is no more true in $\mathbb{C}^N$ with $N \geqslant 2$. Put $E := \cup_1^\infty B_n$ where $B_n := \{(z_1\,, z_2);\, |z_1 - a_n|^2 + |z_2|^2 \leqslant 1\}$, $a_n \in \mathbb{C}$

and $a_n \to \infty$ as $n \to \infty$. It is sufficient to prove that $V_E(z_1\,,\,z_2) = \log^+|z_2|$ for all $(z_1\,,\,z_2)$. It is clear that $\log^+|z_2| \leqslant V_E(z_1, z_2)$ on E and hence in the whole space $\mathbb{C}^2$. Now let u be a function of the class $\mathcal{L}$ with $u \leqslant 0$ on E. We want to show that $u(z_1, z_2) \leqslant \log^+|z_2|$ in $\mathbb{C}^2$. Without loss of generality we may assume (by taking $\max[u\,,\,0]$) that $u = 0$ on E. Fix z_2° with $|z_2^\circ| \leqslant 1$. Then $u(z_1, z_2^\circ) = 0$ for all z_1 in the union of the discs $\{|z_1 - a_n| \leqslant 1\}$. Therefore $u(z_1, z_2) = 0$ for all (z_1, z_2) with $z_1 \in \mathbb{C}$ and $|z_2| \leqslant 1$. Hence $u(z_1, z_2) \leqslant \log^+|z_2|$ in $\mathbb{C}^2$.

2.18. Necessary and sufficient conditions for non-thinness. For a non-pluripolar set $E \subset \mathbb{C}^N$ the following conditions are equivalent.

(1) *If* $u \in \mathcal{L}$, $u \leqslant 0$ *q.a.e. on* E *then* $u = const \leqslant 0$;

(2) $V_E \equiv 0$;

(3) $V_E^* \equiv 0$;

(4) *If* $u_k \in \mathcal{L}$ $(k \geqslant 1)$ *and* $u(z) := \limsup_{k\to\infty} u_k(z) \leqslant 0$ *q.a.e. on* E *then* $u^* = const \leqslant 0$;

(5) *If* $\{p_k\}$ *is a sequence of polynomials of* N *complex variables and* $\{n_k\}$ *is a sequence of natural numbers such that* $\deg p_k \leqslant n_k$ *and* $v := \limsup_{k\to\infty} \frac{1}{n_k}\log|p_k| \leqslant 0$ *q.a.e. on* E *then* $v^* = const \leqslant 0$.

Proof. — The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are easy to check. In order to show the implication (5) $\Rightarrow$ (1) fix $u \in \mathcal{L}$ with $u \leqslant 0$ q.a.e. on E. Assuming (5) holds, we need to show that $u = const \leqslant 0$.

It is known [6, 7] that there exits a sequence of holomorphic polynomials $\{p_n\}$ such that $\deg p_n \leqslant n$ and $u = v^*$ where $v := \limsup_{n\to\infty} \frac{1}{n}\log|p_n|$. By theorem on negligible sets [4], we know that $u = v^* \leqslant 0$ q.a.e. on E. By (5) it follows that $u = v^* = const \leqslant 0$. ☐

2.19. Remark. Consider the following property (1') of E

(1') *If* $u \in \mathcal{L}, u \leqslant 0$ *on* E *then* $u = const \leqslant 0$.

It is obvious that if E has the property (1) then E satisfies (1'). The inverse implication does not hold for $N \geqslant 2$ (we do not know if it is true for arbitrary sets on the complex plane). Namely, by Example 1.1. of [2], the set $E := \{(z_1, z_2) \in \mathbb{C}^2; (z_1 \in \mathbb{C}, |z_2| \leqslant 1) \text{ or } (z_1 = 0, z_2 \in \mathbb{C})\}$ satisfies (1') but it does not satisfy (1), because $V_E^*(z_1, z_2) \equiv \log^+|z_2|$.

3. Power series with Ostrowski gaps

Let

$$f(z) = \sum_0^\infty Q_j(z), \quad \text{where} \quad Q_j(z) = \sum_{|\alpha|=j} a_\alpha z^\alpha, \tag{3.1}$$

be a *power series* in $\mathbb{C}^N$, i.e. a series of homogeneous polynomials Q_j of N complex variables of degree j.

The set $\mathcal{D}$ given by the formula $\mathcal{D} := \{a \in \mathbb{C}^N;$ the sequence (3.1) is convergent in a neighborhood of a $\}$ is called a *domain of convergence of* (3.1).

It is known that

$$\mathcal{D} = \{z \in \mathbb{C}^N; \psi^*(z) < 0\},$$

where

$$\psi(z) := \limsup_{j\to\infty} \sqrt[j]{|Q_j(z)|}.$$

If ψ^* is finite then it is PSH and absolutely homogeneous (i.e. $\psi^*(\lambda z) = |\lambda|\,\psi^*(z)$, $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^N$). Therefore the domain of convergence $\mathcal{D}$ is either empty, or it is a *balanced* (i.e. $\lambda z \in \mathcal{D}$ for all $\lambda \in \mathbb{C}$ with $|\lambda| \leqslant 1$ and $z \in \mathcal{D}$) domain of holomorphy. Every balanced domain of holomorphy is a domain of convergence of a series (3.1).

The number

$$\rho := 1/\limsup_{j\to\infty} \sqrt[j]{\|Q_j\|_{\mathbb{B}}},$$

where $\mathbb{B} := \{z \in \mathbb{C}^N; \|z\| \leqslant 1\}$, is called a *radius of convergence* of series (3.1) (with respect to a given norm $\|\cdot\|$).

If $N = 1$ then $\mathcal{D} = \rho\mathbb{B}$. If $N \geqslant 2$ then $\rho\mathbb{B} \subset \mathcal{D}$ but, in general, $\mathcal{D} \neq \rho\mathbb{B}$.

Series (3.1) is *normally convergent* in $\mathcal{D}$, i.e.

$$\limsup_{j\to\infty} \sqrt[j]{\|Q_j\|_K} < 1, \quad \limsup_{n\to\infty} \sqrt[n]{\|f - s_n\|_K} < 1,$$

for all compact sets $K \subset \mathcal{D}$, where $s_n := Q_0 + \cdots + Q_n$ is the *nth partial sum of* (3.1).

For a strictly increasing sequence $\{n_k\}$ of positive integers we say that a power series (3.1) possesses *Ostrowski gaps relative to* $\{n_k\}$ if there exists a sequence of real numbers $q_k > 0$ such that $\lim q_k = 0$ and

$$\lim_{j\to\infty,\, j\in I} \|Q_j\|^{1/j} = 0 \tag{3.2}$$

where $\mathbb{B}$ is the unit ball in $\mathbb{C}^N$, and $I := \cup_k [q_k n_k \,,\, n_k] \cap \mathbb{N}$.

We say that a series (3.1) is *overconvergent*, if a subsequence $\{s_{n_k}\}$ of its partial sums is uniformly convergent in a neighborhood of some point $a \in \mathbb{C}^N \setminus \mathcal{D}$.

Example. — Consider the function $f(z) = \sum_0^\infty \left(\frac{z(z+1)}{r}\right)^{2^{k^2}} = \sum_0^\infty r^{-2^{k^2}} (z^{2^{k^2}} + \cdots + z^{2^{k^2+1}}) = \sum_0^\infty c_j z^j$, where $c_j = 0$, when $2^{(k-1)^2+1} + 1 \leqslant j \leqslant 2^{k^2+1} - 1$, $k \geqslant 1$. The function f is given by a power series with Ostrowski gaps relative to the sequence $n_k = 2^{k^2+1} - 1$ (with $q_k := (2^{(k-1)^2+1}+1)/(2^{k^2+1}-1)$). The sequence $s_{n_k}(z) = \sum_0^{2^{k^2+1}-1} c_j z^j = \sum_0^{2^{(k-1)^2+1}} c_j z^j = \sum_0^{k-1} \left(\frac{z(z+1)}{r}\right)^{2^{j^2}}$ is normally convergent to $f(z)$ in the lemniscate $\mathcal{E}_r = \{z;\, |z(z+1)| < r\}$, $r > 0$.

The radius of convergence of our power series is given by the formula $\rho = dist(0\,, \partial\mathcal{E}_r)$. If $0 < r \leqslant \frac{1}{4}$ then $\mathcal{E}_r$ has two disjoint components. If $r > \frac{1}{4}$ the lemniscate $\mathcal{E}_r$ is connected. Our power series is overconvergent at every point of $\mathcal{E}_r \setminus \{|z| \leqslant \rho\}$. If G is a a connected component of $\mathcal{E}_r$ then the function $f|_G$ is holomorphic in G and it has analytic continuation across no boundary point of G.

4. Two Ostrowski Gap Theorems in $\mathbb{C}^N$

We say that a compact subset K of $\mathbb{C}^N$ is *polynomially convex* if K is identical with its *polynomially convex hull* $\hat{K} := \{a \in \mathbb{C}^N;\, |P(a)| \leqslant \|P\|_K$ for every polynomial P of N complex variables $\}$.

We say that an open set Ω in $\mathbb{C}^N$ is *polynomially convex*, if for every compact subset K of Ω the polynomially convex hull $\hat{K}$ of K is contained in Ω.

The aim of this section is to prove the two fundamental Ostrowski gap theorems in N-dimensional setting, $N \geqslant 1$.

Let f be a function holomorphic in a neighborhood of the origin of $\mathbb{C}^N$ whose Taylor series development (3.1) possesses Ostrowski gaps relative to a sequence $\{n_k\}$.

Let Ω be the set of points a in $\mathbb{C}^N$ such that the sequence $\{s_{n_k}\}$ is uniformly convergent in a neighborhood of a. By classical theory of envelops of holomorphy, each connected component of Ω is a polynomially convex domain. Let G be a connected component of Ω with $0 \in G$.

THEOREM 1. — *G is the maximal domain of existence of f. Moreover, G is polynomially convex and*

$$\limsup_{k\to\infty} \|f - s_{n_k}\|_K^{1/n_k} < 1$$

for every compact subset K of G.

COROLLARY 4.1. — *The maximal domain of existence G of a function f holomorphic in a neighborhood of the origin of $\mathbb{C}^N$ with Taylor series development possessing Ostrowski gaps relative to a sequence $\{n_k\}$ is a one-sheeted polynomially convex domain of holomorphy.*

COROLLARY 4.2. — *If a function f holomorphic in a neighborhood of $0 \in \mathbb{C}^N$ has Taylor series development of the form*

$$f(z) = \sum_0^\infty Q_{m_k}(z), \quad \text{where} \quad m_k < m_{k+1}\ , \frac{m_{k+1}}{m_k} \to \infty,$$

then the domain of convergence of the series is identical with the maximal domain of existence of f.

We need the following lemma (known for $N = 1$, see e.g. [5], Lemma 3).

LEMMA 4.3. — *If a power series (3.1) with positive radius of convergence possesses Ostrowski gaps relative to a sequence $\{n_k\}$ then for every $R > 0$ we have*

$$\limsup_{k\to\infty} \|s_{n_k}\|_{B_R}^{1/n_k} \leqslant 1, \tag{4.0}$$

where $B_R := B(0,R)$ is a ball with center 0 and radius R.

If series (3.1) possesses Ostrowski gaps relative to $\{n_k\}$, then either $\lim q_k n_k = \infty$, or $\mathbb{N} \setminus I$ is finite and consequently the function f is entire. In the second case (4.0) is obvious. In the first case, we have

$$\epsilon_k := max\{\|Q_j\|_{\mathbb{B}}^{1/j};\ q_k n_k \leqslant j \leqslant n_k\} \to 0 \text{ as } k \to \infty.$$

Fix $R > 0$. Since the radius of convergence of the series (3.1) is positive, there exists $M > 1$ such that $RM > 1$, and

$$\|Q_j\|_{B_R} \leqslant (MR)^j, \quad j \geqslant 0,$$

because $|Q_j(z)| \leqslant \|Q_j\|_{\mathbb{B}}\|z\|^j \leqslant (M\|z\|)^j$, $j \geqslant 0$, where $M > 1$ is sufficiently large. Therefore $\|s_{n_k}\|_{B_R}^{1/n_k} \leqslant \sum_{j=0}^{\lceil q_k n_k \rceil - 1}(MR)^j + \sum_{j=\lceil q_k n_k \rceil}^{n_k}(\epsilon_k R)^j \leqslant \lceil q_k n_k \rceil (MR)^{q_k n_k} + (n_k - \lceil q_k n_k \rceil)(\epsilon_k MR)^{q_k n_k} \leqslant n_k (MR)^{q_k n_k}$, where $k \geqslant k_0$ and k_0 is so large that $\epsilon_k MR \leqslant 1$ for $k \geqslant k_0$, and $\|Q_j\|_{\mathbb{B}}^{1/j} \leqslant \epsilon_k$ for $k \geqslant k_0$, $q_k n_k \leqslant j \leqslant n_k$. Therefore

$$\limsup_{k\to\infty} \|s_{n_k}\|_{B(0,R)}^{1/n_k} \leqslant \limsup_{k\to\infty} n_k^{1/n_k}(MR)^{q_k} = 1.$$

Proof of the Lemma is completed.

Proof of Theorem 1. — In the component G of Ω the function f is a locally uniform limit of the sequence of polynomials $\{s_{n_k}\}$ of corresponding degrees $\leqslant n_k$.

The function

$$u_k := \frac{1}{n_k} \log |f - s_{n_k}|$$

is PSH in G. By (4.0), the sequence $\{u_k\}$ is locally uniformly upper bounded in G. Therefore, if $u := \limsup_{k\to\infty} u_k$, then $u^* \in PSH(G)$, $u^* \leqslant 0$ in G and $u^* < 0$ in a neighborhood of 0. Hence, by the maximum principle for PSH functions , we have $u^* < 0$ in G. Hence, by Hartogs Lemma,

$$\limsup_{k\to\infty} \|f - s_{n_k}\|_K^{1/n_k} < 1$$

for every compact subset K of G.

Suppose G is not a maximal domain of existence of f. Then, there exist a point $a \in G$, a real number $r > dist(a, \partial G) =: r_0$, and a function g holomorphic in the ball $B(a, r)$ such that $g = f$ on $B(a, r_0)$. Basing on the inequality (4.0), similarly as just above, we can show that

$$\limsup_{k\to\infty} \|g - s_{n_k}\|_K^{1/n_k} < 1$$

for every compact subset K of $B(a, r)$. It follows that $s_{n_k} \to g$ locally uniformly in $B(a, r)$ as $k \to \infty$. Therefore the sequence $\{s_{n_k}\}$ converges uniformly in a neighborhood of some boundary point of G which contradicts the definition of Ω. It follows that G is a polynomially convex maximal domain of existence of f. The proof of Theorem 1 is completed.

THEOREM 2. — *For every polynomially convex open set $\Omega \subset \mathbb{C}^N$ with $0 \in \Omega$ there exists a function f holomorphic in Ω whose Taylor series development around* 0

$$f(z) = \sum_0^\infty Q_j(z), \quad Q_j(z) := \sum_{|\alpha|=j} \frac{f^{(\alpha)}(0)}{\alpha!} z^\alpha, \tag{4.1}$$

possesses Ostrowski gaps relative to a sequence $\{n_k\}$ such that:

(i) Every connected component D of Ω is the maximal domain of existence of $f_{|D}$;

(ii) The subsequence $\{s_{n_k}\}$ of partial sums of (4.1) converges locally uniformly to f in Ω; in particular, Taylor series (4.1) is overconvergent at every point a of $\Omega \setminus \mathcal{D}$, where $\mathcal{D}$ is the domain of convergence of (4.1);

(iii) If G is the component of Ω with $0 \in G$ then

$$\limsup_{k\to\infty} \|f - s_{n_k}\|_K^{1/n_k} < 1$$

for every compact subset K of G.

Proof. — Let $\{\xi^{(\nu)}\}$ ($\xi^{(j)} \neq \xi^{(k)}$, $j \neq k$) be a countable dense subset of Ω. Put $B_\nu := B(\xi^{(\nu)}, r_\nu)$ with $r_\nu := dist\,(\xi^{(\nu)}, \partial\Omega)$. Let $c^{(\nu)}$ be a point of $\partial\Omega \cap \partial B_\nu$, and let $E_\nu = \{a^{(\mu\nu)}\}_{\mu\geqslant 1}$ be a sequence of points of the ball B_ν such that $a^{(\mu\nu)} \in (\xi^{(\nu)}, c^{(\nu)}) := \{\xi^\nu + t(c^{(\nu)} - \xi^{(\nu)});\ 0 < t \leqslant 1\}$ and

$$\|a^{(\mu\nu)} - c^{(\nu)}\| < \frac{1}{\mu\nu}, \quad \mu \geqslant 1.$$

Let $\{E_\nu^*\}$ denote the sequence

$$E_1; E_1, E_2; E_1, E_2, E_3; E_1, \cdots, E_\nu; \cdots \tag{4.2}$$

in which every set E_ν is repeated infinitely many times.

Since Ω is polynomially convex there exists a sequence of polynomially convex compact sets $\{\Delta_k\}$ such that Δ_k is contained in the interior of Δ_{k+1} and $\Omega = \cup_1^\infty \Delta_k$.

Taking, if necessary, a subsequence of $\{\Delta_k\}$, we may assume that $0 \in \Delta_1$ and

$$E_k^* \cap (\Delta_{k+1} \setminus \Delta_k) \neq \emptyset, \quad k \geqslant 1.$$

Let $a^{(k)}$ be an arbitrary fixed point of this intersection. Given $k \geqslant 1$, let W_k be a polynomial such that $d_k := \deg W_k \geqslant k$, and

$$\|W_k\|_{\Delta_k} < 1 < |W_k(a^{(k)})|. \tag{4.3}$$

Put $f_0(z) \equiv 0$, $\mu_0 = \nu_0 = 1$, and

$$f_k(z) = \left(\frac{\bar{a}_1^{(k)} z_1 + \cdots + \bar{a}_N^{(k)} z_N}{\|a^{(k)}\|^2} \right)^{\mu_k} (W_k(z))^{\nu_k}, \quad k \geqslant 1, \tag{4.4}$$

where μ_k, ν_k are positive integers. We claim that integers can be chosen in such a way that the following conditions are satisfied for all $k \geqslant 1$

(a) $\mu_{k-1} + \nu_{k-1} d_{k-1} < \mu_k / k$;

(b) $\|f_k\|_{\Delta_k} \leqslant 2^{-k}$;

(c) $|f_k(a^{(k)})| \geqslant k + |\sum_{j=0}^{k-1} f_j(a^{(k)})|$.

Indeed, put $\mu_1 = 1$ and choose $\nu_1 \geqslant 1$ so large that $\|f_1\|_{\Delta_1} \leqslant \frac{1}{2}$. Then the conditions are satisfied for $k = 1$. Suppose that μ_j, ν_j are already chosen for $j = 0, 1, \cdots, k$ for a fixed $k \geqslant 1$. Observe that $|f_k(a^{(k)})| = |W_k(a^{(k)}|^{\nu_k}$ tends - by right hand side of (4.3) and (c) - to ∞ as $\nu_k \to \infty$ (here ν_k denotes a positive integer valued variable). It is clear that one can find an integer μ_{k+1} such that (a) is satisfied with k replaced by $k + 1$. Now, applying left hand side (respectively, right hand side) inequality of (4.3) one can find an integer ν_{k+1} so large that (b) (respectively,(c)) is satisfied for k replace by $k + 1$. By the induction principle, the claim is true.

We shall prove that the function f, given by the formula

$$f(z) = \sum_{j=0}^{\infty} f_j(z), \quad z \in \Omega,$$

where f_j are defined by (4.4), has the required properties.

It follows from (b) that the series is uniformly convergent on compact subsets of Ω. Hence $f \in \mathcal{O}(\Omega)$. Since for $\nu = 1, 2, ...$ the sequence $\{a^{(k)}\}$ contains a subsequence of the sequence $\{a^{(\mu\nu)}\}_{\mu \geqslant 1}$, we have

$$\limsup_{t \uparrow 1} |f(\xi^{(\nu)} + t(c^{(\nu)} - \xi^{(\nu)}))| = +\infty.$$

It follows that every connected component D of Ω is a maximal domain of existence of $f_{|D}$.

The function f_k is a polynomial given by

$$f_k(z) = \sum_{j=\mu_k}^{\mu_k + \nu_k d_k} Q_j(z),$$

where Q_j is a homogeneous polynomial of degree j. By the condition (a), the Taylor series development of f around 0 is given by

$$f(z) = \sum_0^\infty Q_j(z), \quad \|z\| < \rho, \tag{4.5}$$

where $\rho = dist(0, \partial\mathcal{D})$ and $Q_j = 0$ for $\mu_{k-1} + \nu_{k-1}d_{k-1} + 1 \leqslant j \leqslant \mu_k - 1$, $k \geqslant 1$.

Put $n_k := \mu_k - 1$, and $q_k := \frac{\mu_{k-1}+\nu_{k-1}+1}{\mu_k - 1}$. Then $q_k > 0$ and, by (a), $lim_{k\to\infty} q_k = 0$. It follows that the series (4.5) has Ostrowski gaps relative to the sequence $n_k := \mu_k - 1,\ k \geqslant 1$. It is clear that

$$s_{n_k}(z) = \sum_{j=0}^{n_k} Q_j(z) = \sum_{j=0}^{k} f_j(z).$$

Therefore the subsequence $\{s_{n_k}\}$ of partial sums of the Taylor series (4.5) converges locally uniformly to f in Ω. Moreover, by Theorem 1, we conclude that $\{s_{n_k}\}$ satisfies condition (iii), which completes the proof of Theorem 2.

5. Sets E in $\mathbb{C}^N$ with $V_E \equiv 0$ and power series with Ostrowski gaps

The following theorem is an N-dimensional version of Theorem 2 in [4].

THEOREM 3. — *Given a closed subset E of $\mathbb{C}^N$, the following two conditions are equivalent:*

(a) $V_E \equiv 0$.

(b) If a subsequence $\{s_{n_k}\}$ of partial sums of a power series (3.1) satisfies the inequality

$$\limsup_{k\to\infty} |s_{n_k}(z)|^{\frac{1}{n_k}} \leqslant 1, \quad \text{for every} \quad \mathrm{z} \in \mathrm{E}, \tag{5.1}$$

then series (3.1) possesses Ostrowski gaps relative to a subsequence $\{n_{k_\ell}\}$ of the sequence $\{n_k\}$.

Proof of Theorem 3. — Our proof is an adaptation of the proof in one-dimensional case presented in [5].

First we shall show that $(a) \Rightarrow (b)$. To this order observe that – by (a) – we have (5) of section 2.18 which implies – by Hartogs Lemma – that

$$\limsup_{k\to\infty} \|s_{n_k}\|_{B(0,R)}^{\frac{1}{n_k}} \leqslant 1, \quad \text{for every} \quad \mathrm{R} > 0. \tag{5.2}$$

The implication $(a) \Rightarrow (b)$ follows from

Lemma 5.1. — *If $\{s_{n_k}\}$ satisfies (5.2) then the power series (3.1) possesses Ostrowski gaps relative to a subsequence $\{n_{k_l}\}$ of $\{n_k\}$.*

Proof of Lemma 5.1. — By (5.2), for every $l \geqslant 1$, we can find $k_l \in \mathbb{N}$ such that $k_l < k_{l+1}$ and

$$\|s_{n_{k_l}}\|_{B(0\,,\,l)} \leqslant (1 + \frac{1}{l})^{n_{k_l}}, \quad l \geqslant 1.$$

Hence, by Cauchy inequalities, we get

$$\|Q_j\|_{\mathbb{B}}^{1/j} \leqslant \frac{1}{l}(1 + \frac{1}{l})^{l \cdot \frac{n_{k_l}}{lj}} \quad \leqslant \frac{e}{l}, \quad \frac{n_{k_l}}{l} \leqslant j \leqslant n_{k_l}, \quad l \geqslant 1,$$

which (with $q_l := \frac{1}{l}$) completes the proof of Lemma 5.1.

$(b) \Rightarrow (a)$. It is enough to prove that $non(a) \Rightarrow non(b)$. Let E be a thin closed set in $\mathbb{C}^N$. We shall construct a power series (3.1), for which a subsequence $\{s_{n_k}\}$ satisfies (5.1), but which does not possess Ostrowski gaps relative to any subsequence of $\{n_k\}$.

Our construction is based on the following useful known result.

Lemma 5.2. — *If K is a compact subset of $\mathbb{C}^N$ then*

$$V_K(z) = \sup\{\frac{1}{k}\log|P_k(z)|;\ \|P_k\|_K = 1,\ k \geqslant 1\},\ z \in \mathbb{C}^N,$$

where P_k is a polynomial of N complex variables of degree at most k.

Without loss of generality we may assume that $\bar{\mathbb{B}} \subset E$ (because, by property 2.9 we know that E is thin if and only if $E \cup \bar{\mathbb{B}}$ is thin).

Choose a point $a \in \mathbb{C}^N$ such that $R_0 := \|a\| > 1$ and $V_E(a) =: \eta > 0$. Put $\epsilon_k := \eta/k$, $R_k := R_0 + k$, and $E_k = E \cap \{\|z\| \leqslant R_k\}$ for $k \geqslant 0$. Then $V_{E_k}(a) \downarrow V_E(a)$.

Let p_0, $q_0 \geqslant 1$ be arbitrary integers, and let W_{q_0} be a polynomial of degree $\leqslant q_0$ such that $\|W_{q_0}\|_{E_0} = 1$, $|W_{q_0}(a)| > e^{(\eta-\epsilon_0)q_0}$, where $0 < \epsilon_0 < 1$.

Suppose p_j, q_j, W_{q_j} $(j = 0, \ldots, k)$ are already chosen in such a way that W_{q_j} is a polynomial of degree $\leqslant q_j$ and

$$p_{j-1} + q_{j-1} < p_j < q_j/j, \tag{5.3}$$

$$\frac{R_j^{p_j}}{(1+\epsilon_j)^{q_j}} \leqslant \frac{1}{j^2}, \tag{5.4}$$

$$\|W_{q_j}\|_{E_j} = 1, \quad |W_{q_j}(a)| > e^{(\eta-\epsilon_j)q_j}. \tag{5.5}$$

Now, it is easy to find integers p_{k+1}, q_{k+1} and a polynomial $W_{q_{k+1}}$ such that (5.3), (5.4), (5.5) are satisfied for $j = k+1$.

First choose an arbitrary integer $p_{k+1} > p_k + q_k$, next choose an arbitrary integer $q_{k+1} > (k+1)p_{k+1}$ and a polynomial $W_{q_{k+1}}$ such that (5.4) and (5.5) are satisfied with $j = k+1$.

Consider the series

$$f(z) \quad = \quad \sum_{k=0}^{\infty} \left(\frac{\bar{a}_1 z_1 + \ldots + \bar{a}_N z_N}{\|a\|^2} \right)^{p_k} \frac{W_{q_k}(z)}{(1+\epsilon_k)^{q_k}}. \tag{5.6}$$

From (5.5) it follows that series (5.6) converges uniformly on every E_k, $k \geqslant 0$. In particular, its sum f is a holomorphic function in the unit ball. The k-th component of (5.6) is of the form $\sum_{j=p_k}^{p_k+q_k} Q_j$, where Q_j is a homogeneous polynomial of degree j. Hence $f(z) = \sum_{k=o}^{\infty} \left(\sum_{j=p_k}^{p_k+q_k} Q_j(z) \right)$, $z \in \mathbb{B}$. After removing the parentheses we get a power series with positive radius of convergence. Put $n_k = p_k + q_k$. It is clear that for every $k \geqslant 1$

$$|s_{n_k}(a)| \geqslant \frac{|W_{q_k}(a)|}{(1+\epsilon_k)^{q_k}} - |s_{n_{k-1}}(a)| \geqslant \frac{e^{q_k(\eta-\epsilon_k)}}{(1+\epsilon_k)^{q_k}} - \sum_{0}^{k-1} exp q_j V_{E_j}(a) \geqslant$$

$$\frac{e^{q_k(\eta-\epsilon_k)}}{(1+\epsilon_k)^{q_k}} - kM^{q_{k-1}},$$

where M is a positive constant. Taking into account that $\epsilon_k \to 0$, $(kM^{q_{k-1}})^{1/q_k} \to 1$ and $p_k/q_k \to 0$ as $k \to \infty$, we have

$$\liminf_{k\to\infty} \|s_{n_k}\|_{B(0\,,\,R_0)}^{\frac{1}{n_k}} \geqslant \liminf_{k\to\infty} |s_{n_k}(a)|^{\frac{1}{n_k}} \geqslant e^{\eta} > 1,$$

which by Lemma 4.3 gives the required result.

Remark. — The same idea of proof may be used to show that Theorem 3 remains true if $E \subset \mathbb{C}^N$ is of type F_σ. The implication $(a) \Rightarrow (b)$ holds for every set E with $V_E \equiv 0$.

6. Approximation by polynomials with restricted growth near infinity

Let E be a subset of $\mathbb{C}^N$ with $V_E \equiv 0$. Let Γ be a non-pluripolar subset of an open connected set G. Let f be a function holomorphic in G. The following theorem is an N-dimensional counterpart of Theorem 1 in [5].

Theorem 4.— If $\{P_n\}$ is a sequence of polynomials of N complex variables with $degP_n \leqslant d_n$ ($d_n < d_{n+1}$, d_n is an integer) such that

$$\limsup_{n\to\infty} |f(z) - P_n(z)|^{1/d_n} < 1, \quad z \in \Gamma, \tag{6.1}$$

$$\limsup_{n\to\infty} |P_n(z)|^{1/d_n} \leqslant 1, \quad z \in E, \tag{6.2}$$

then the maximal domain of existence G_f of f is a polynomially convex open subset of $\mathbb{C}^N$ such that

$$\limsup_{n\to\infty} \|f - P_n\|_K^{1/d_n} < 1 \tag{6.3}$$

for every compact subset K of G_f.

If, moreover, the sequence $\{d_{n+1}/d_n\}$ is bounded then $G_f = \mathbb{C}^N$.

Observe that the point-wise geometrical convergence (6.1) of $\{P_n\}$ to f on a non-pluripolar set Γ along with the restricted growth (6.2) of $\{P_n(z)\}$ at every point z of a non-thin set E imply the uniform geometrical convergence (6.3) of $\{P_n\}$ to f on every compact subset K of G_f.

In Theorem 1 of [5] the authors assume that Γ is a nontrivial continuum in $\mathbb{C}$, and $\limsup_{k\to\infty} \|f - P_n\|_\Gamma^{1/d_n} < 1$, which in the case of $\mathbb{N} = 1$ is more restrictive than (6.1).

Proof of Theorem 4.— 1^0. First we shall show that (6.3) is true for every compact subset K of G. To this order observe that the function

$$u_n(z) := \frac{1}{d_n} \log |f(z) - P_n(z)|$$

is PSH(G). The condition (6.2) and property (5) of the necessary and sufficient conditions 2.18 for non-thinness imply that for every compact subset K of G and for every $\epsilon > 0$ there exist a positive constant $M = M(K,\epsilon)$ and a positive integer $n_0 = n_0(K,\epsilon)$ such that $u_n(z) \leqslant \frac{1}{d_n}\log(M + M(1+\epsilon)^{d_n}) \leqslant \frac{1}{d_n}\log(2M) + \epsilon$, $n \geqslant n_0$, $z \in K$. Hence $u := \limsup_{n\to\infty} u_n \leqslant 0$ in G , and

$u < 0$ on Γ by (6.1). The function u^* is non-positive and plurisubharmonic in G, and , by the theorem on negligible sets, we have $u(z) = u^*(z) < 0$ on $\Gamma \setminus A$, where A is pluripolar. By the maximum principle $u^*(z) < 0$ in G which, by the Hartogs Lemma, implies the required inequality (6.3) for compact sets $K \subset G$.

2^0. Put $\Omega := \{a \in \mathbb{C}^N;$ *the sequence* $\{P_n\}$ *is uniformly convergent in a neighborhood of* $a\}$. It follows from 1^0 that $G \subset \Omega$. Let G_f denote the connected component of Ω containing G. It is clear that G_f is polynomially convex. We claim that G_f is the maximal domain of existence of f. It is clear that $\tilde{f}(z) := \lim_{n\to\infty} P_n(z)$, $z \in G_f$, is holomorphic in G_f, and $\tilde{f} = f$ in G. We need to show that G_f is the maximal domain of existence of $\tilde{f}$. By 1^0 we have (6.3) with G replaced by G_f and f by $\tilde{f}$.

Suppose, contrary to our claim, that there exist $a \in G_f, r > dist(a, \partial G_f) =: r_0$ and a function g holomorphic in the ball $B(a, r)$ such that $g(z) = \tilde{f}(z)$ if $\|z - a\| < r_0$. By 1^0 we have $\limsup_{n\to\infty} \|g - P_n\|_K^{1/d_n} < 1$ for every compact subset K of the ball $B(a\ r)$. Therefore the sequence $\{P_n\}$ converges locally uniformly in this ball which contains boundary points of G_f. This contradicts the definition of the last set.

3^0. Let us assume that the sequence $\{\frac{d_{n+1}}{d_n}\}$ is bounded, say $d_{n+1}/d_n \leqslant \alpha$,
$n \geqslant 1$. By 2^0, it is sufficient to show that in this case $\Omega = \mathbb{C}^N$. Consider the following sequence of elements of the Lelong class $\mathcal{L}$

$$u_n(z) := \frac{1}{d_{n+1}} \log |P_{n+1}(z) - P_n(z)|, \quad z \in \mathbb{C}^N.$$

Put $u(z) := \limsup_{n\to\infty} u_n(z)$, $z \in \mathbb{C}^N$. It follows from (6.1) that for every $z \in \Gamma$ there exit $\epsilon > 0$ and $M > 0$ such that $u_n(z) \leqslant \frac{1}{d_{n+1}} \log[Me^{-\epsilon d_{n+1}} + Me^{-\epsilon d_n}] \leqslant \frac{1}{d_{n+1}} \log(2M) - \frac{1}{\alpha}\epsilon$, $n \geqslant 1$. Hence, $u(z) < 0$ for every $z \in \Gamma$.

One can easily check that if $z \in E$, then by (6.2) $u(z) \leqslant 0$. Therefore $u^* \in \mathcal{L}$ and $u^*(z) \leqslant 0$ for all $z \in E \setminus A$, where A is pluripolar. It follows that $u^* \leqslant V_E^* = 0$ in $\mathbb{C}^N$. Hence $u^* = c = const \leqslant 0$. But, by the theorem on negligible sets, $u^*(z) < 0$ on a non-empty subset of Γ which implies that $c < 0$. Hence, by Hartogs Lemma, for every compact subset K of $\mathbb{C}^N$ and for $0 < \epsilon < -c$ there exists $n_0 = n_0(K, \epsilon)$ such that $u_n(z) \leqslant -\epsilon$ for all $z \in K$ and $n \geqslant n_0$. It follows that the sequence $\{P_n\}$ is uniformly convergent on K. By the arbitrary property of K we get $\Omega = \mathbb{C}^N$.

The method of proof of Theorem 4 may be used to show that the following corollaries are true.

Corollary 6.1. — *Let E be a subset of $\mathbb{C}^N$ with $V_E \equiv 0$. Let Γ be a non-pluripolar subset of $\mathbb{C}^N$. Let $\{d_n\}$ be a strictly increasing sequence of positive integers such that $d_{n+1}/d_n \leqslant \alpha$, $n \geqslant 1$, with $\alpha = const > 1$.*

If $f : \Gamma \to \mathbb{C}$ is a function such that there exists a sequence of polynomials $\{P_n\}$ with $\deg P_n \leqslant d_n$ such that

$$\limsup_{n\to\infty} |f(z) - P_n(z)|^{\frac{1}{d_n}} < 1, \quad z \in \Gamma, \tag{6.4}$$

$$\limsup_{n\to\infty} |P_n(z)|^{\frac{1}{d_n}} \leqslant 1, \quad z \in E, \tag{6.5}$$

then f extends to an entire function $\tilde{f}$ such that for every compact set $K \subset \mathbb{C}^N$ we have

$$\limsup_{n\to\infty} \|\tilde{f} - P_n\|_K^{\frac{1}{d_n}} < 1.$$

Indeed, by (6.4), given $z \in \Gamma$, there are $M > 0$ and $0 < \theta = \theta(z) < 1$ such that $|f(z) - P_n(z)| \leqslant M\theta^{d_n}$, $n \geqslant 1$. Hence $|P_{n+1}(z) - P_n(z)| \leqslant 2M\theta^{\frac{1}{\alpha}d_{n+1}}$ which implies

$$\limsup_{n\to\infty} |P_{n+1}(z) - P_n(z)|^{\frac{1}{d_{n+1}}} < 1, \quad z \in \Gamma.$$

By (6.5), given $z \in E$ and $\epsilon > 0$, there is $M > 0$ such that $|P_{n+1}(z) - P_n(z)| \leqslant |P_{n+1}(z)| + |P_n(z)| \leqslant Me^{d_{n+1}\epsilon} + e^{d_n\epsilon} \leqslant 2Me^{\alpha\epsilon d_n}$, $n \geqslant 1$, which implies that

$$\limsup_{n\to\infty} \sqrt[d_{n+1}]{|P_{n+1}(z) - P_n(z)|} \leqslant 1, \quad z \in E.$$

Put $u(z) := \limsup \frac{1}{d_{n+1}} \log |P_{n+1}(z) - P_n(z)|, \quad z \in \mathbb{C}^N$. Then $u^* \in \mathcal{L}$, $u^* \leqslant 0$ on E and $u^* < 0$ on $\Gamma \setminus A$, where A is pluripolar. Therefore $u^* = const < 0$. Hence, by Hartogs Lemma, we have $\limsup \|P_{n+1} - P_n\|_K^{1/d_{n+1}} < 1$ for every compact subset K of $\mathbb{C}^N$. It follows that $\tilde{f} := P_1 + \sum_1^\infty (P_{n+1} - P_n)$ is an entire function with the required properties.

In the sequel P_n denotes polynomials with $deg P_n \leqslant d_n$, where d_n are integers with $1 \leqslant d_n < d_{n+1} \leqslant \alpha d_n$, $\alpha = const > 1$, Γ is a non-pluripolar, subset of $\mathbb{C}^N$, and f is a complex valued function defined on Γ.

Corollary 6.2. — *If f is holomorphic in an open connected set G containing Γ such that*

$$\limsup_{n\to\infty} |f(z) - P_n(z)|^{\frac{1}{d_n}} < 1, \quad z \in \Gamma, \tag{6.6}$$

$$\limsup_{n\to\infty} |P_n(z)|^{\frac{1}{d_n}} \leqslant 1, \quad z \in G, \tag{6.7}$$

then f has a holomorphic extension $\tilde{f}$ to G such that

$$\limsup_{n\to\infty} \|\tilde{f} - P_n\|_K^{\frac{1}{d_n}} < 1\,, \limsup_{n\to\infty} \|P_{n+1} - P_n\|_K^{\frac{1}{d_{n+1}}} < 1, \tag{6.8}$$

for every compact set $K \subset G$. If, moreover, G is non-thin at infinity then there is an entire function $\tilde{f}$ satisfying (6.8) for $G = \mathbb{C}^N$ such that $\tilde{f} = f$ on Γ.

Corollary 6.3. — *If*

$$\limsup_{n\to\infty} |f(z) - P_n(z)|^{\frac{1}{d_n}} = 0, \quad z \in \Gamma, \tag{6.9}$$

then f extends to a unique entire function

$$\tilde{f}(z) = P_1(z) + \sum_{j=1}^{\infty}(P_{n+1}(z) - P_n(z)), \quad z \in \mathbb{C}^N,$$

and (6.8) is satisfied.

In order to show the last two corollaries, define

$$u(z) := \limsup_{n\to\infty} \frac{1}{d_{n+1}} \log |P_{n+1}(z) - P_n(z)|,$$

observe that $u^* \in \mathcal{L}$, and check that $u^*(z) < 0$ on G in the case of Corollary 6.2 (resp., $u^*(z) = -\infty$ on $\mathbb{C}^N$ in the case of Corollary 6.3) which, by Hartogs Lemma, implies Corollary 6.2 (resp., Corollary 6.3).

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